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GLOBAL EXISTENCE OF SOLUTIONS OF THE YANG–MILLS EQUATIONS ON GLOBALLY HYPERBOLIC FOUR DIMENSIONAL LORENTZIAN MANIFOLDS*

PIOTR T. CHRUŚCIEL† AND JALAL SHATAH‡

Abstract. We prove global solvability of the Cauchy problem for the Yang–Mills equations on smooth globally hyperbolic four dimensional Lorentzian manifolds.

1. Introduction. In a classical paper [9], Eardley and Moncrief have shown that solutions of the Yang–Mills equations on Minkowski space–time (with a compact gauge group) do not develop singularities in finite time, provided the initial data are sufficiently regular. This result has been generalized[1] to the Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$ in [6, 8, 4], and to anti–de–Sitter space–time in [1], making use of the conformal invariance of the Yang–Mills equations. It is of interest to enquire whether or not this result remains true for general globally hyperbolic Lorentzian manifolds; such a result in dimensions $1+1$ and $2+1$ has been established, for quite a large class of manifolds, via direct energy methods, in [3]. It is the purpose of this paper to show that global existence holds true in $3+1$ dimensions: we show global solvability of the Cauchy problem for the Yang–Mills equations on any globally hyperbolic Lorentzian manifold[2] (M, g).

It is convenient to start with a description of the geometric context. Recall that a Lorentzian manifold is globally hyperbolic if it admits a Cauchy surface Σ, i.e. a hypersurface $\Sigma \subset M$ that is intersected precisely once by every inextendible causal curve. For such manifolds there exists a smooth time function t on M such that $\Sigma = \{t = 0\}$, with each level set Σ_t of t being a Cauchy surface [11, 16]. Moreover, flowing along the gradient of t one obtains a diffeomorphism between M and $\mathbb{R} \times \Sigma$.

Let G be any connected Lie group with Lie algebra $\mathfrak{g}$. Throughout we shall assume that 1) $\mathfrak{g}$ is compact, that is, $\mathfrak{g}$ admits an Ad–invariant positive scalar product $k(\cdot, \cdot)$ (that will be the case if, *e.g.*, G is compact). 2) $\mathfrak{g}$ has a faithful representation as a subalgebra of the algebra of matrices over some finite dimensional vector space V, so that the bracket operation corresponds to the commutator of matrices, with the adjoint representation of G acting on $\mathfrak{g}$ as an appropriate product of matrices. Let P be a G–principal bundle over M and let P^Σ be the pull–back of P to a Cauchy surface Σ Choose any smooth connection on P, let X denote the horizontal lift of the gradient of t to P, we can identify P with $\mathbb{R} \times P^\Sigma$ by flowing along the integral curves of X. This leads to the commutative diagram:

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†Département de Mathématiques, Faculté des Sciences, Parc de Grandmont, F37200 Tours, France (Chrusciel@Univ-Tours.Fr). Supported in part by a Polish Research Council grant number 2 P301 050 07.

‡Courant Institute, New York University, 251 Mercer Street, New York NY 10012, USA (shatah@cims.nyu.edu). Supported in part by N.S.F. DMS 9401558.

[1] Some further improvements of the Eardley–Moncrief theorem can be found in [12, 14].

[2] We use the signature $-+++$, and we assume that both the manifold and the metric are smooth. M is moreover assumed to be paracompact, Hausdorff, and time oriented.