Asian J. Math.
Vol. 6, No. 3, pp. 399–408, September 2002

002

ON RELATIONS BETWEEN ENTROPY AND HAUSDORFF DIMENSION OF MEASURES *

ATHANASIOS BATAKIS† AND YANICK HEURTEAUX‡

Abstract. We characterize probability measures whose Hausdorff dimension or packing dimension can be calculated by an entropy formula. In particular, we prove that such measures are unidimensional. We also construct examples of unidimensional measures for which entropy does not calculate the dimension.

Let D be an integer greater than 1 and m be a probability measure in $[0,1)^D$. Fix $\ell \geq 1$ and denote by $\mathcal{F}_n$ the family of ℓ-adic cubes of the n^{th} generation, that is

$$\mathcal{F}_n = \left\{ I = \prod_{i=1}^{D} [k_i/\ell^n, (k_i+1)/\ell^n) \ ; \ 0 \leq k_i < \ell^n \right\} .$$

For an arbitrary point x in $[0,1)^D$, let $I_n(x)$ be the unique cube $I \in \mathcal{F}_n$ such that $x \in I$.

We want to give estimates for the lower and the upper dimension of the measure m. These are respectively defined by

$$\begin{cases} \dim_*(m) = \inf(\dim(E) \ ; \ m(E) > 0) \\ \dim^*(m) = \inf(\dim(E) \ ; \ m(E) = 1) \end{cases} .$$

It is well known that there exist some relations between these quantities and the function τ which appears in the multifractal formalism. More precisely, if we let :

$$\tau_n(t) = \frac{1}{n \log \ell} \log \left(\sum_{I \in \mathcal{F}_n} m(I)^t \right) \quad \text{and} \quad \tau(t) = \limsup_{n \to +\infty} \tau_n(t) ,$$

it is proved in [11] that

$$-\tau'_+(1) \leq \dim_*(m) \leq h_*(m) \tag{1}$$

where $h_*(m)$ is the lower entropy (also called lower Rényi dimension) of the measure m, defined as

$$h_*(m) = \liminf_{n \to +\infty} h_n(m) \quad \text{where} \quad h_n(m) = -\tau'_n(1) = \frac{-1}{n \log \ell} \sum_{I \in \mathcal{F}_n} m(I) \log(m(I)) .$$

In [11], we also give sufficient conditions for the equality $-\tau'_+(1) = \dim_*(m)$ to hold.

In this paper we are interested in describing the measures m satisfying $\dim_*(m) = h_*(m)$. Theorem 1.1 states that $\dim_*(m) = h_*(m)$ holds if and only if there exists a subsequence n_k such that

$$\lim_{k \to +\infty} \frac{\log m(I_{n_k}(x))}{-n_k \log \ell} = \dim_*(m) \quad dm\text{-almost surely} . \tag{2}$$

*Received August 19, 2001; accepted for publication May 7, 2002.
†Département de Mathématiques, Université d'Orléans, B.P. 6759, F-45067 Orléans cedex 2, France (Athanasios.Batakis@labomath.univ-orleans.fr).
‡Laboratoire de Mathématiques pures, Université Blaise Pascal, F-63177 Aubière cedex, France (Yanick.Heurteaux@math.univ-bpclermont.fr).