Generalization of Lawson and Simons' result to quaternion and octonion geometry

Siu-Cheong Lau and Naichung Conan Leung

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Abstract

A theorem of Lawson and Simons[8] states that the only stable minimal submanifolds in $\mathbb{CP}^n$ are complex submanifolds. We generalize their result to the cases of $\mathbb{HP}^n$ and $\mathbb{OP}^2$. The treatment is given in the context of Jordan algebra, so that the seemingly unrelated case of $\mathbb{S}^m$ is unified naturally with the above projective spaces.

1 Introduction

Complex geometry is a very rich subject. Some of its beautiful theorems have natural generalizations to quaternion geometry or even octonion geometry. This paper gives one such generalization.

In the seventies, Lawson and Simons [8] showed that the average second variation of any submanifold S in $\mathbb{CP}^n$ is negative unless S is complex, where the average is taken over all holomorphic vector fields in $\mathbb{CP}^n$. As a corollary, complex submanifolds are the only stable minimal submanifolds in $\mathbb{CP}^n$. (Here stability means that the submanifold has non-negative second variation along every vector field.) We generalize this result to $\mathbb{HP}^n$ and $\mathbb{OP}^2$, leading to the following theorem:

> **Main Theorem:** Let $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. For any submanifold S (or more generally rectifiable current) in $\mathbb{AP}^n$, the average second variation of S is given by
>
> $$-\int_S \left(\sum_{l=1}^{\Lambda} \|\mathbb{J}_l \cdot S_x\|^2 \right) dx \leq 0$$
>
> where S_x denotes the unit simple vector representing the oriented tangent space of S at each $x \in S$, and $\{\mathbb{J}_l, l \in 1, \dots, \Lambda\}$ is an orthonormal basis of the space of linear complex structures of $\mathbb{AP}^n$ at x, each acting on S_x as derivation.

> As a consequence, complex submanifolds are the only stable minimal submanifolds in $\mathbb{CP}^n$; quaternionic submanifolds are the only stable minimal submanifolds in $\mathbb{HP}^n$; octonionic submanifolds are the only stable minimal submanifolds in $\mathbb{OP}^2$.

The term '*average*' second variation appeared in the above theorem will be explained in detail in Section 3.

In the same paper, Lawson and Simons showed that the average second variation of any submanifold S in $\mathbb{S}^m$ is negative unless S is of dimension 0 or m, where the average is taken over all conformal vector fields in $\mathbb{S}^m$. As a corollary, there are no stable minimal submanifolds in $\mathbb{S}^m$ other than points and $\mathbb{S}^m$ itself. At first glance, the conformal geometry of $\mathbb{S}^m$ may seem differ a lot from the complex geometry of $\mathbb{CP}^n$.

Surprisingly, under the notion of Jordan algebra, these different kinds of geometries can be treated in a unified manner. To see this, we recall that for any normed algebra $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, its projective space $\mathbb{AP}^n$ can be identified as the subset of all rank one projections in the space $\mathfrak{h}_{n+1}(\mathbb{A})$ of $(n+1)\times(n+1)$ Hermitian matrices over $\mathbb{A}$. These $\mathfrak{h}_{n+1}(\mathbb{A})$ are simple formally real Jordan algebras. When $\mathbb{A} = \mathbb{O}$ is non-associative, we can only take $n \leq 2$. From the classification result [7], there is one more family of simple Jordan algebra: This corresponds to $\mathbb{A} = \mathbb{R}^m$, which is not even an algebra, and we can only take $n = 1$. In this case, $\mathbb{AP}^1$ is the standard sphere $\mathbb{S}^m$.

We give a uniform treatment to $\mathbb{AP}^n \subset \mathfrak{h}_{n+1}(\mathbb{A})$ to achieve our main theorem. First we give a brief introduction to Jordan algebras in Section 2. Then in Section 3 we derive a formula for average second variation of cycles in a compact symmetric space G/K which is a G-orbit in an orthogonal representation of G. This formula is applied to the projective spaces in Section 4 to show the main theorem. The results of Lawson and Simons in the complex and conformal cases are recovered when $\mathbb{A}$ equals to $\mathbb{C}$ and $\mathbb{R}^m$. On the one hand, we have generalized their results for submanifolds in $\mathbb{CP}^n$ to $\mathbb{HP}^n$ and $\mathbb{OP}^2$; On the other hand, our approach unifies the conformal case with the others.

2 Projective spaces and simple Jordan algebra

Let's begin by recollecting some facts about our working platform: projective spaces inside simple Jordan algebras. A formally real Jordan algebra is an algebra over $\mathbb{R}$ whose multiplication $\circ$ is commutative and power associative (that is, $(a \circ a) \circ a = a \circ (a \circ a)$), together with

$$a_1 \circ a_1 + \ldots + a_n \circ a_n = 0 \Rightarrow a_1 = \ldots = a_n = 0$$

This notion is invented by Jordan [6] in 1932 to describe the algebra of observables in quantum mechanics. These algebras are classified by Jordan, von Neumann and Wigner [7]: Every formally real Jordan algebra can be written as direct sum of simple ones, which are listed completely below as sets of Hermitian matrices. Inside each of them we get a projective space consisting of all rank one projections, which are exactly those matrices p with $p \circ p = p$ and $\operatorname{tr} p = 1$. Multiplication is defined as symmetrization of the ordinary matrix multiplication:

$$A \circ B = \frac{AB + BA}{2}$$

1. $\mathfrak{h}_{n+1}(\mathbb{R}) := \{\text{Hermitian real } (n+1) \times (n+1) \text{ matrices}\} \supset \mathbb{RP}^n$.

   $\mathfrak{b} = \mathfrak{g} \oplus \mathfrak{h}^0_{n+1}(\mathbb{R})$, where $\mathfrak{b}$ is the Lie algebra of $\mathcal{B} = SL(n+1, \mathbb{R})$, and $\mathfrak{g}$ is the Lie algebra of its maximal compact subgroup $G = SO(n+1)$, who acts on $\mathfrak{h}^0_{n+1}(\mathbb{R})$, the subspace of trace-free matrices in $\mathfrak{h}_{n+1}(\mathbb{R})$, as automorphisms by adjoint action.

2. $\mathfrak{h}_{n+1}(\mathbb{C}) := \{\text{Hermitian complex } (n+1) \times (n+1) \text{ matrices}\} \supset \mathbb{CP}^n$.

   $\mathfrak{b} = \mathfrak{g} \oplus \mathfrak{h}^0_{n+1}(\mathbb{C})$, where $\mathfrak{b}$ is the Lie algebra of $\mathcal{B} = SL(n+1, \mathbb{C})$, and $\mathfrak{g}$ is the Lie algebra of its maximal compact subgroup $G = SU(n+1)$, who acts on $\mathfrak{h}^0_{n+1}(\mathbb{C})$, the subspace of trace-free matrices in $\mathfrak{h}_{n+1}(\mathbb{C})$, as automorphisms by adjoint action.

3. $\mathfrak{h}_{n+1}(\mathbb{H}) := \{\text{Hermitian quaternion } (n+1) \times (n+1) \text{ matrices}\} \supset \mathbb{HP}^n$.

   $\mathfrak{b} = \mathfrak{g} \oplus \mathfrak{h}^0_{n+1}(\mathbb{H})$, where $\mathfrak{b}$ is the Lie algebra of $\mathcal{B} = SL(n+1, \mathbb{H})$, and $\mathfrak{g}$ is the Lie algebra of its maximal compact subgroup $G = Sp(n+1)$, who acts on $\mathfrak{h}^0_{n+1}(\mathbb{H})$, the subspace of trace-free matrices in $\mathfrak{h}_{n+1}(\mathbb{H})$, as automorphisms by adjoint action.

4. $\mathfrak{h}_3(\mathbb{O}) := \{\text{Hermitian } 3 \times 3 \text{ matrices with octonion entries}\} \supset \mathbb{OP}^2$.

   $\mathfrak{b} = \mathfrak{g} \oplus \mathfrak{h}^0_3(\mathbb{O})$, where $\mathfrak{b}$ is the Lie algebra of $\mathcal{B} = E_6^{-26}$ [1], and $\mathfrak{g}$ is the Lie algebra of its maximal compact subgroup $G = F_4$, who acts on $\mathfrak{h}^0_3(\mathbb{O})$, the subspace of trace-free matrices in $\mathfrak{h}_3(\mathbb{O})$, as automorphisms by adjoint action [3].

5. Spin factor.

$$\mathfrak{h}_2(\mathbb{R}^m) := \left\{ \begin{pmatrix} a-b & v \\ v & a+b \end{pmatrix} : a, b \in \mathbb{R}, v \in \mathbb{R}^m \right\} \quad \cong \quad \mathbb{R}^m \oplus \mathbb{R} \oplus \mathbb{R}$$

$$\begin{pmatrix} a-b & v \\ v & a+b \end{pmatrix} \quad \leftrightarrow \quad \begin{pmatrix} v \\ b \\ a \end{pmatrix}$$

   where we define $v \cdot w = v^t w$ for $v, w \in \mathbb{R}^m$ to carry out matrix multiplication. The embedded projective space is

$$\left\{ \left(\begin{array}{c} v \\ b \\ a \end{array} \right) : \|v\|^2 + b^2 = \frac{1}{4} \right\} \cong \mathbb{S}^m$$

$\mathfrak{b} = \mathfrak{g} \oplus \mathfrak{h}_2^0(\mathbb{R}^m)$, where $\mathfrak{b}$ is the Lie algebra of $\mathcal{B} = O(m+1,1)$, and $\mathfrak{g}$ is the Lie algebra of its maximal compact subgroup $G = O(m+1)O(1)$, who acts on $\mathfrak{h}_2^0(\mathbb{R}^m)$, the subspace of trace-free matrices in $\mathfrak{h}_2^0(\mathbb{R}^m)$, as automorphisms by adjoint action.

Now we fix a simple formally real Jordan algebra $\mathfrak{h}_{n+1}(\mathbb{A})$, where it is understood that $n \in \mathbb{N}$ for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$; $n = 2$ for $\mathbb{A} = \mathbb{O}$, and $n = 1$ for $\mathbb{A} = \mathbb{R}^m$. The affine space $\mathfrak{h}_{n+1}^1(\mathbb{A})$ of all trace-one matrices is invariant under G because automorphisms preserve trace. The projective space $\mathbb{AP}^n$ sits *symmetrically* inside $\mathfrak{h}_{n+1}^1(\mathbb{A})$: $\mathbb{AP}^n$ is the orbit under G of the matrix $\mathbb{E}_{n+1,n+1} \in \mathfrak{h}_{n+1}^1(\mathbb{A})$ with value 1 at the $(n+1, n+1)$ position and all other entries zero. We have an G-invariant metric on $\mathfrak{h}_{n+1}^1(\mathbb{A})$:

$$\langle A, B \rangle := 2\,\mathrm{Re}(\mathrm{tr}\, AB) = 2\,\mathrm{Re}(\mathrm{tr}\, A \circ B)$$

inducing a G-invariant metric on $\mathbb{AP}^n$. With this metric $\mathbb{AP}^n$ is in fact a Riemannian symmetric space:

$$\mathbb{RP}^n = \frac{\mathrm{O}(n+1)}{\mathrm{O}(n)\mathrm{O}(1)}, \mathbb{CP}^n = \frac{\mathrm{U}(n+1)}{\mathrm{U}(n)\mathrm{U}(1)}, \mathbb{HP}^n = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\mathrm{Sp}(1)}, \mathbb{S}^m = \frac{\mathrm{O}(m+1)}{\mathrm{O}(m)}$$

are classical symmetric spaces, and

$$\mathbb{OP}^2 = \frac{\mathrm{F}_4}{\mathrm{Spin}(9)}$$

is an exceptional symmetric space, the Cayley plane. (Page 292 of [4].)

The group G of symmetries is extremely helpful for investigating the variational behaviour of cycles in $\mathbb{AP}^n$, as we shall see in the next section.

3 Average second variation in symmetric orbits

In this section we consider the following situation:
Setting: Assume that $M = G/K$ is a compact symmetric space which is a G-orbit of an orthogonal representation $\mathbb{V}$ of G. Such an M is called a symmetric orbit (Chapter 2 of [2]).

The projection of each $u \in \mathbb{V}$ determines a vector field V_u, or simply denoted by V, on M. Our main result in this section is the following simple formula for the average second variation of any p-frame in M, where the average is taken over $\{V_u : u \in \mathbb{V}\}$. By setting $\mathbb{V} = \mathfrak{h}_{n+1}^1(\mathbb{A})$, G to be the group of automorphisms of $\mathfrak{h}_{n+1}(\mathbb{A})$ (which is the notation we use in the last section) and $M = \mathbb{AP}^n$, this formula will be applied to projective spaces in the next section.

Theorem 1 *In the above setting, the average second variation of an oriented orthonormal p-frame $\xi = e_1 \wedge \ldots \wedge e_p$ at $x \in M$ is given by*

$$\operatorname{tr} \mathcal{Q}_\xi = \sum_{j,k=1}^{p,q} \left(2 \, \| \, \mathrm{II}(e_j, n_k)\|^2 - \langle \, \mathrm{II}(e_j, e_j), \, \mathrm{II}(n_k, n_k)\rangle\right)$$

where II *is the second fundamental form of $M \subset \mathbb{V}$ at x, and $\{e_j\}_{j=1}^p \cup \{n_k\}_{k=1}^q$ is an orthonormal basis of $T_x M$.*

The method of proof is similar to [8]. Let us first have a quick review on the general setup given in [8]. For a Riemannian manifold M with Levi-Civita connection ∇, the second variation of a rectifiable current S under a flow $\{\phi_t : M \to M\}_{t \in \mathbb{R}}$ generated by a global gradient vector field V on M is given by

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathbf{M}((\phi_t)_* S) = \int_M \langle \mathcal{A}_{V,V} S_x, S_x \rangle + 2\|\mathcal{A}_V S_x\|^2 - (\langle \mathcal{A}_V S_x, S_x \rangle)^2 \, \mathrm{d}\|S\|(x)$$

where $\mathbf{M}$ denotes the volume of a current, S_x is the unit simple vector representing the oriented tangent space of S at x, and $\|S\|$ denotes the Borel measure associated with S. Given vector fields V, W on M, $\mathcal{A}_V(u)$, $\mathcal{A}_{V,W}$ are endomorphisms of TM defined by

$$\begin{aligned} \mathcal{A}_V X &:= & & \nabla_X V; \\ \mathcal{A}_{V,W} X &:= & (\nabla_V \mathcal{A}_W) X = & \nabla_V \nabla_{\tilde{X}} W - \nabla_{\nabla_V \tilde{X}} W \end{aligned} \tag{1}$$

where $\tilde{X}$ is a smooth local extension of $X \in TM$. Each endomorphism L of TM is extended to operate on $\bigwedge^p TM$ by Leibniz rule:

$$L(e_1 \wedge \ldots \wedge e_p) = \sum_{j=1}^p e_1 \wedge \ldots \wedge L e_j \wedge \ldots \wedge e_p$$

A rectifiable current is said to be stable if its second variation along every vector fields on M are non-negative. We will denote the integrand in the above formula by $\mathcal{Q}_{S_x}(V)$, the second variation of S_x under V. For each oriented orthonormal p-frame $\xi = e_1 \wedge \ldots \wedge e_p$ at $x \in M$, the second variation $\mathcal{Q}_\xi$ is a quadratic form on the space of smooth vector fields on M, and it can be rewritten as

$$\mathcal{Q}_\xi(V) = \left(\sum_{j=1}^p \langle \mathcal{A}_V e_j, e_j \rangle\right)^2 + 2 \sum_{j=1}^p \sum_{k=1}^q (\langle \mathcal{A}_V e_j, n_k \rangle)^2 + \sum_{j=1}^p \langle \mathcal{A}_{V,V} e_j, e_j \rangle \tag{2}$$

where $\{e_1, \ldots, e_p, n_1, \ldots, n_q\}$ forms an orthonormal basis of $T_x M$.

Now let's take a closer look on our setting introduced in the beginning of this section. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, which acts linearly on an Euclidean space $\mathbb{V}$ with a G-invariant metric $\langle \cdot, \cdot \rangle$. Fix a point $0 \neq x_0 \in \mathbb{V}$, consider the orbit $M = G \cdot x_0$ with the induced G-invariant metric from $\mathbb{V}$. G acts on the space of orthonormal p-frames on M by

$$g \cdot (e_1 \wedge \ldots \wedge e_p) := (g_* \cdot e_1) \wedge \ldots \wedge (g_* \cdot e_p)$$

For $u \in \mathbb{V}$, the projection of u on $T_x M$ at each $x \in M$ gives a vector field V_u on M, which may also be regarded as a gradient vector field of the function $\langle \cdot, u \rangle$ on M. $\mathcal{Q}_\xi$ becomes a quadratic form on $\mathbb{V}$ by sending each $u \in \mathbb{V}$ to $\mathcal{Q}_\xi(V_u)$, which may be identified by the metric as a self-adjoint operator on $\mathbb{V}$. Our aim is to compute the average second variation $\operatorname{tr} \mathcal{Q}_\xi$.

Lemma 2 *Suppose ξ, η are two unit simple vectors on M. If $\eta = g \cdot \xi$ for some $g \in G$, then the average second variation of ξ is the same as that of η.*
Proof. *Since the metric is G-invariant, the Levi-Civita connection ∇ is G-equivariant, that is,*

$$\nabla_{g_* \cdot X}(g_* \cdot V) = g_* \cdot (\nabla_X V)$$

Apply this to equation (1)*,*

$$\mathcal{A}_V(g \cdot \xi) = g \cdot (\mathcal{A}_{g_*^{-1} V} \cdot \xi); \; \mathcal{A}_{V,W}(g \cdot \xi) = g \cdot (\mathcal{A}_{g_*^{-1} V, \, g_*^{-1} W} \cdot \xi)$$

Apply to equation (2)*, we get*

$$\mathcal{Q}_{g \cdot \xi}(V_u) = \mathcal{Q}_\xi(g_*^{-1} V_u) = \mathcal{Q}_\xi(V_{g_*^{-1} u})$$

the last equality is due to G-invariance of metric. And so

$$\operatorname{tr} \mathcal{Q}_\eta = \sum_u \mathcal{Q}_\eta(u) = \sum_u \mathcal{Q}_\xi(g_*^{-1} u) = \operatorname{tr} \mathcal{Q}_\xi$$

where u, and hence $g_*^{-1} u$, run through an orthonormal basis of $\mathbb{V}$. The last equality follows from the fact that trace is independent of choice of orthonormal basis. ■

By the above lemma, it suffices to consider average second variation of each p-frame $\xi = e_1 \wedge \ldots \wedge e_p$ at x_0, because p-frames at another point can be moved to x_0 by some $g \in G$, who acts transitively on M.

Denote the isotropy subgroup of x_0 by K, so that $M = G/K$. This gives a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \text{ where } \mathfrak{m} := \mathfrak{k}^\perp$$

Now suppose that M is at the same time a symmetric space, that is, we have an involutive isometry $\sigma : M \to M$ with x_0 as an isolated fixed point. On G we have a natural G-invariant metric given by negative of the Killing form. We may scale this metric such that $\mathfrak{m}$ is isometric to $T_{x_0} M$.

Let's complete $\xi = e_1 \wedge \ldots \wedge e_p$ to an orthonormal basis $\{e_1, \ldots, e_p, n_1, \ldots, n_q\}$ of $T_{x_0} M \cong \mathfrak{m}$, and further take an orthonormal basis $\{\beta_1, \ldots, \beta_r\}$ of $\mathfrak{k}$, so that $\{\beta_1, \ldots, \beta_r, e_1, \ldots, e_p, n_1, \ldots, n_q\}$ forms an orthonormal basis of $\mathfrak{g}$.

We now express the projection $V = V_u$ of $u \in \mathbb{V}$ in terms of Killing vector fields induced by $\mathfrak{g}$ on M. We shall use the same symbol to denote an element of $\mathfrak{g}$, its induced vector field on $\mathbb{V}$, and the restricted Killing vector field on M. Recall that

$$[g_1, g_2]_M = -[g_1, g_2] \tag{3}$$

where $[\cdot,\cdot]_M$ is the Lie bracket for vector fields on M, and $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{g}$. On the right hand side g_1, g_2 denote elements in $\mathfrak{g}$, while on the left hand side they denote their induced Killing vector fields on M. This fact will be used repeatedly to compute the Lie bracket, and hence covariant derivatives of Killing vector fields.

Lemma 3

$$V = \sum_{\mu=1}^{r} \left\langle u, \beta_\mu \right\rangle \beta_\mu + \sum_{\nu=1}^{p} \left\langle u, e_\nu \right\rangle e_\nu + \sum_{\gamma=1}^{q} \left\langle u, n_\gamma \right\rangle n_\gamma$$

Proof. Denote the basis $\{\beta_1, \ldots, \beta_r, e_1, \ldots, e_p, n_1, \ldots, n_q\}$ of $\mathfrak{g}$ by A.

At $x_0 \in M$ the above equation is obvious, because $\beta_\mu(x_0) = 0$,

and $\{e_1, \ldots, e_p, n_1, \ldots, n_q\}$ forms an orthonormal basis of $T_{x_0}M$.

At another point $x \in M$, let $\{\tilde{e_1}, \ldots, \tilde{e_p}, \tilde{n_1}, \ldots, \tilde{n_q}\}$ be an orthonormal basis of $T_xM \cong \mathfrak{m}$, and complete it to an orthonormal basis

$$B = \{\tilde{\beta}_1, \ldots, \tilde{\beta}_r, \tilde{e}_1, \ldots, \tilde{e}_p, \tilde{n}_1, \ldots, \tilde{n}_q\}$$

of $\mathfrak{g}$. Both A, B are orthonormal basis of $\mathfrak{g}$, so $B = AT$, where T is an orthogonal matrix.

$$V(x) = \sum_j \left\langle u, B_j \right\rangle B_j = \sum_j \left\langle u, A_k T_j^k \right\rangle A_i T_j^i = \sum_j \left\langle u, A_j \right\rangle A_j$$

since $\sum_j T_j^k T_j^i = \delta^{ki}$. ■

Proof to Theorem 1: By Lemma 2 we may assume ξ is a frame at x_0. From equation (2) the average second variation of ξ is

$$\sum_u \left(\sum_{j=1}^{p} \left\langle \mathcal{A}_V e_j, e_j \right\rangle \right)^2 + 2 \sum_u \sum_{j=1,k=1}^{p,q} \left(\left\langle \mathcal{A}_V e_j, n_k \right\rangle \right)^2 + \sum_u \sum_{j=1}^{p} \left\langle \mathcal{A}_{V,V} e_j, e_j \right\rangle$$

where u runs through an orthonormal basis of $\mathbb{V}$, each gives a vector field $V = V_u$ on M by projection. We compute term by term for the three terms appeared in the above expression.

Recall [5] that for a symmetric space,

$$\nabla_{K_1} K_2 = \frac{1}{2} [K_1, K_2]_M$$

for Killing vector fields K_1 and K_2 on M. Applying this to the expression of V given in Lemma 3,

$$\begin{aligned}\nabla_{e_j} V &= \left\langle u, \partial_{e_j}\beta_\mu \right\rangle \beta_\mu + \frac{1}{2}\left\langle u, \beta_\mu \right\rangle [e_j, \beta_\mu]_M + \left\langle u, \partial_{e_j} e_\nu \right\rangle e_\nu + \frac{1}{2}\left\langle u, e_\nu \right\rangle [e_j, e_\nu]_M \\ &\quad + \left\langle u, \partial_{e_j} n_\gamma \right\rangle n_\gamma + \frac{1}{2}\left\langle u, n_\gamma \right\rangle [e_j, n_\gamma]_M \end{aligned} \tag{4}$$

where ∂ is the trivial connection of $\mathbb{V}$, and so ∂_v is the usual directional derivative along $v \in T_{x_0}\mathbb{V} \cong \mathbb{V}$. (Recall that β_μ, e_ν, n_γ can be regarded as vector fields on $\mathbb{V}$, and so the above directional derivatives make sense.)

To simplify the above expression at x_0, notice that $\mathfrak{k}$ induces zero vectors at x_0, and hence $\beta_\mu \in \mathfrak{k}$ vanishes at x_0. Together with equation (3) and the fact that

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k} \tag{5}$$

we have

$$\nabla_{e_j} V(x_0) = \left\langle u, \partial_{e_j} e_\nu \right\rangle e_\nu + \left\langle u, \partial_{e_j} n_\gamma \right\rangle n_\gamma$$

and hence

$$\begin{aligned}\langle \mathcal{A}_V e_j, e_j \rangle &= \left\langle \nabla_{e_j} V(x_0), e_j \right\rangle = \left\langle u, \partial_{e_j} e_j \right\rangle \\ \langle \mathcal{A}_V e_j, n_k \rangle &= \left\langle \nabla_{e_j} V(x_0), n_k \right\rangle = \left\langle u, \partial_{e_j} n_k \right\rangle \end{aligned}$$

The first term is

$$\begin{aligned}\sum_u \left(\sum_{j=1}^{p} \langle \mathcal{A}_V e_j, e_j \rangle \right)^2 &= \sum_u \sum_{j,k=1}^{p} \left\langle u, \partial_{e_j} e_j \right\rangle \langle u, \partial_{e_k} e_k \rangle \\ &= \sum_{j,k=1}^{p} \left\langle \partial_{e_j} e_j, \partial_{e_k} e_k \right\rangle \\ &= \left\| \sum_{j=1}^{p} \mathrm{II}(e_j, e_j) \right\|^2 \end{aligned}$$

where $\partial_{e_j} e_j = \mathrm{II}(e_j, e_j)$ because $\nabla_{e_j} e_j = [e_j, e_j]_M / 2 = 0$.

The second term is

$$\begin{aligned} 2 \sum_u \sum_{j=1,k=1}^{p,q} \left(\langle \mathcal{A}_V e_j, n_k \rangle \right)^2 &= 2 \sum_u \sum_{j=1,k=1}^{p,q} \left(\left\langle u, \partial_{e_j} n_k \right\rangle \right)^2 \\ &= 2 \sum_{j,k=1}^{p,q} \| \partial_{e_j} n_k \|^2 \\ &= 2 \sum_{j,k=1}^{p,q} \| \mathrm{II}(e_j, n_k) \|^2 \end{aligned}$$

where $\partial_{e_j} n_k = \mathrm{II}(e_j, n_k)$ at x_0 because $\nabla_{e_j} n_k(x_0) = [e_j, n_k]_M / 2 = 0$.

Now we turn to compute the third term $\sum_u \sum_{j=1}^p \langle \mathcal{A}_{V,V} e_j, e_j \rangle$, which is more complicated. At x_0,

$$\begin{aligned}
\langle \mathcal{A}_{V,V} e_j, e_j \rangle &= \left\langle \nabla_V \nabla_{e_j} V - \nabla_{\nabla_V e_j} V, e_j \right\rangle \\
&= \left\langle \nabla_V \nabla_{e_j} V, e_j \right\rangle \\
&= \sum_{\nu=1}^{p} \langle u, e_\nu \rangle \left\langle \nabla_{e_\nu} \nabla_{e_j} V, e_j \right\rangle + \sum_{\gamma=1}^{q} \langle u, n_\gamma \rangle \left\langle \nabla_{n_\gamma} \nabla_{e_j} V, e_j \right\rangle
\end{aligned}$$

where $\nabla_{\nabla_V e_j} V(x_0) = 0$ because

$$\nabla_V e_j(x_0) = \sum_{\nu=1}^{p} \langle u, e_\nu \rangle \frac{[e_\nu, e_j]_M}{2} + \sum_{\gamma=1}^{q} \langle u, n_\gamma \rangle \frac{[n_\gamma, e_j]_M}{2} = 0$$

We now compute the first part $\sum \langle u, e_\nu \rangle \left\langle \nabla_{e_\nu} \nabla_{e_j} V, e_j \right\rangle$ of the third term. Differentiating equation (4) along e_ν, we get

$$\begin{aligned}
\nabla_{e_\nu} \nabla_{e_j} V(x_0) &= \frac{1}{2} \left\langle u, \partial_{e_j} \beta_\mu \right\rangle [e_\nu, \beta_\mu]_M + \frac{1}{2} \left\langle u, \partial_{e_\nu} \beta_\mu \right\rangle [e_j, \beta_\mu]_M \\
&\quad + \left\langle u, \partial_{e_\nu} \partial_{e_j} e_\alpha \right\rangle e_\alpha + \frac{1}{4} \langle u, e_\alpha \rangle [e_\nu, [e_j, e_\alpha]_M]_M \\
&\quad + \left\langle u, \partial_{e_\nu} \partial_{e_j} n_\gamma \right\rangle n_\gamma + \frac{1}{4} \langle u, n_\gamma \rangle [e_\nu, [e_j, n_\gamma]_M]_M
\end{aligned}$$

Using the identity $\langle [X, Y]_M, Z \rangle = - \langle Y, [X, Z]_M \rangle$ for Killing vector fields X, Y, Z, together with the relation (5) repeatedly, we get

$$\left\langle \nabla_{e_\nu} \nabla_{e_j} V(x_0), e_j \right\rangle = \left\langle u, \partial_{e_\nu} \partial_{e_j} e_j \right\rangle$$

and so

$$\begin{aligned}
&\sum_u \sum_{j=1}^{p} \sum_{\nu=1}^{p} \langle u, e_\nu \rangle \left\langle \nabla_{e_\nu} \nabla_{e_j} V, e_j \right\rangle \\
=\; &\sum_u \sum_{j=1}^{p} \sum_{\nu=1}^{p} \langle u, e_\nu \rangle \left\langle u, \partial_{e_\nu} \partial_{e_j} e_j \right\rangle \\
=\; &\sum_{j,\nu=1}^{p} \left\langle \partial_{e_\nu} \partial_{e_j} e_j, e_\nu \right\rangle \\
=\; &- \left\| \sum_{j=1}^{p} \mathrm{II}(e_j, e_j) \right\|^2 \qquad (6)
\end{aligned}$$

Now proceed to compute the second part $\sum \langle u, n_\gamma \rangle \left\langle \nabla_{n_\gamma} \nabla_{e_j} V, e_j \right\rangle$ of the third term. Differentiating the equation (4) along n_γ, we get

$$\begin{aligned}\nabla_{n_\gamma}\nabla_{e_j}V(x_0) &= \frac{1}{2}\left\langle u, \partial_{e_j}\beta_\mu\right\rangle [n_\gamma, \beta_\mu]_M + \frac{1}{2}\left\langle u, \partial_{n_\gamma}\beta_\mu\right\rangle [e_j, \beta_\mu]_M \\ &\quad + \left\langle u, \partial_{n_\gamma}\partial_{e_j}e_\nu\right\rangle e_\nu + \frac{1}{4}\left\langle u, e_\nu\right\rangle [n_\gamma, [e_j, e_\nu]_M]_M \\ &\quad + \left\langle u, \partial_{n_\gamma}\partial_{e_j}n_\alpha\right\rangle n_\alpha + \frac{1}{4}\left\langle u, n_\alpha\right\rangle [n_\gamma, [e_j, n_\alpha]_M]_M\end{aligned}$$

and so

$$\left\langle \nabla_{n_\gamma}\nabla_{e_j}V(x_0), e_j\right\rangle = \left\langle u, \partial_{n_\gamma}\partial_{e_j}e_j\right\rangle$$

$$\begin{aligned}&\sum_u\sum_{j=1}^p\sum_{\gamma=1}^q \langle u, n_\gamma\rangle \left\langle \nabla_{n_\gamma}\nabla_{e_j}V, e_j\right\rangle \\ = \;& \sum_u\sum_{j=1}^p\sum_{\gamma=1}^p \langle u, n_\gamma\rangle \left\langle u, \partial_{n_\gamma}\partial_{e_j}e_j\right\rangle \\ = \;& \sum_{j,\gamma=1}^{p,q}\left\langle \partial_{n_\gamma}\partial_{e_j}e_j, n_\gamma\right\rangle \\ = \;& -\sum_{j,\gamma=1}^{p,q}\langle \mathrm{II}(e_j,e_j), \mathrm{II}(n_\gamma,n_\gamma)\rangle \end{aligned} \tag{7}$$

Adding up equations (6) and (7), we get the third term

$$-\left\|\sum_{j=1}^p \mathrm{II}(e_j,e_j)\right\|^2 - \sum_{j,\gamma=1}^{p,q}\langle \mathrm{II}(e_j,e_j), \mathrm{II}(n_\gamma,n_\gamma)\rangle$$

Adding up all the three terms, the average second variation is

$$\sum_{j,k=1}^{p,q}\left(2\,\|\mathrm{II}(e_j,n_k)\|^2 - \langle \mathrm{II}(e_j,e_j), \mathrm{II}(n_k,n_k)\rangle\right) \blacksquare$$

4 Proof of the main theorem

Our projective spaces fit into the scenario introduced in the last section, with $\mathbb{V} = \mathfrak{h}^1_{n+1}(\mathbb{A})$, $G = \mathrm{Aut}(\mathfrak{h}_{n+1}(\mathbb{A}))$, $x_0 = \mathbb{E}_{n+1,n+1} \in \mathfrak{h}^1_{n+1}(\mathbb{A})$, which is the matrix with value 1 at the $(n+1, n+1)$ position and all other entries zero, and $M = \mathbb{AP}^n$. To apply the average second variation formula we need to compute the second fundamental form for $\mathbb{AP}^n \subset \mathfrak{h}^1_{n+1}(\mathbb{A})$ at x_0.

Let's take the following coordinates around x_0 for $\mathbb{AP}^n$:

$$\begin{aligned}\mathbb{A}^n &\to \mathbb{AP}^n \subset \mathfrak{h}^1_{n+1}(\mathbb{A}) \\ Q &\mapsto \frac{1}{1+\|Q\|^2}\begin{pmatrix} Q \\ 1\end{pmatrix}\begin{pmatrix} Q^* & 1\end{pmatrix}\end{aligned}$$

Here we adopt the following notations:

$$Q = \sum_{l=0}^{\Lambda} \mathbf{i}_l X_l$$

where X_l are column n-vectors, $\mathbf{i}_0 := 1$, and for $1 \le l \le \Lambda$, $\mathbf{i}_l$ are the linearly independent imaginary square roots of unity in $\mathbb{A}$. Recall that for the case $\mathbb{A} = \mathbb{R}^m$, $n = 1$, $\Lambda = 0$, $Q = X_0$ is an element in $\mathbb{R}^m$ with $Q^* = Q$ and $Q \cdot Q := \langle Q, Q \rangle$. For the other four cases, the entries of X_l are real numbers.

The basis of coordinate tangent vector fields is $\{\frac{\partial}{\partial x_l^j} : 0 \le l \le \Lambda, 1 \le j \le N\}$, where $\frac{\partial}{\partial x_l^j}$ denote the $\mathbf{i}_l$-directions. $N = m$ in the case of $\mathbb{A} = \mathbb{R}^m$, and $N = n$ for all the other four cases. Using product rule (which is valid for multiplication in $\mathbb{A}$),

$$\begin{aligned}
\left.\frac{\partial}{\partial x_l^j}\right|_Q &= \frac{1}{1+\|Q\|^2} \begin{pmatrix} \mathbf{i}_l w_j \\ 0 \end{pmatrix} \begin{pmatrix} Q^* & 1 \end{pmatrix} \\
&+ \frac{1}{1+\|Q\|^2} \begin{pmatrix} Q \\ 1 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{i}_l} w_j^T & 0 \end{pmatrix} \\
&- \frac{2X_l^T w_j}{(1+\|Q\|)^2} \begin{pmatrix} Q \\ 1 \end{pmatrix} \begin{pmatrix} Q^* & 1 \end{pmatrix}
\end{aligned}$$

where w_j stands for the column n-vector with j-th coordinate 1 and other coordinates zero, and T stands for transpose. Recall that when $\mathbb{A} = \mathbb{R}^m$, $n = 1$, and so transpose of an element is just itself. Differentiating both sides along $\frac{\partial}{\partial x_r^k}$ at $0 \in \mathbb{A}^n$,

$$\begin{aligned}
&\left.\frac{\partial}{\partial x_r^k}\right|_0 \left(\frac{\partial}{\partial x_l^j}\right) \\
= &\begin{pmatrix} \mathbf{i}_l w_j \\ 0 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{i}_r} w_k^T & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{i}_r w_k \\ 0 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{i}_l} w_j^T & 0 \end{pmatrix} \\
&-(\mathbf{i}_r \overline{\mathbf{i}_l} + \mathbf{i}_l \overline{\mathbf{i}_r}) w_k^T w_j \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \\
= &\begin{cases} \begin{pmatrix} 2\delta_{jk} & 0 \\ 0 & -2\delta_{jk} \end{pmatrix} & \text{for } \mathbb{A} = \mathbb{R}^m \\ \\ \begin{pmatrix} \mathbf{i}_r \overline{\mathbf{i}_l} \mathbb{E}_{kj} + \mathbf{i}_l \overline{\mathbf{i}_r} \mathbb{E}_{jk} & 0 \\ 0 & -(\mathbf{i}_r \overline{\mathbf{i}_l} + \mathbf{i}_l \overline{\mathbf{i}_r}) \delta_{jk} \end{pmatrix} & \text{for } \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \end{cases}
\end{aligned}$$

which is already perpendicular to $T_{x_0} \mathbb{AP}^n$, because

$$\left.\frac{\partial}{\partial x_l^j}\right|_0 = \begin{pmatrix} 0 & \mathbf{i}_l w_j \\ \overline{\mathbf{i}_l} w_j^T & 0 \end{pmatrix}$$

Under the metric $\langle A, B\rangle = 2\,\mathrm{Re}\,\mathrm{tr}\,(AB)$, our coordinate vectors are pairwise orthogonal, each has length 2. We scale them to get an orthonormal basis $\{\frac{1}{2}\frac{\partial}{\partial x_l^j} : 1 \le j \le n, 0 \le l \le \Lambda\}$.

We conclude that

Lemma 4

$$\mathrm{II}(\frac{1}{2}\frac{\partial}{\partial x_l^j}, \frac{1}{2}\frac{\partial}{\partial x_r^k}) = \begin{cases} \frac{1}{2}\begin{pmatrix} \delta_{jk} & 0 \\ 0 & -\delta_{jk} \end{pmatrix} & \text{for } \mathbb{A} = \mathbb{R}^m \\ \frac{1}{4}\begin{pmatrix} \mathbf{i}_r\overline{\mathbf{i}_l}\mathbb{E}_{kj} + \mathbf{i}_l\overline{\mathbf{i}_r}\mathbb{E}_{jk} & 0 \\ 0 & -(\mathbf{i}_r\overline{\mathbf{i}_l} + \mathbf{i}_l\overline{\mathbf{i}_r})\delta_{jk} \end{pmatrix} & \text{for } \mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\} \end{cases}$$

Now we are ready to compute $\mathrm{tr}\,\mathcal{Q}_\xi$ for an orthonormal p-frame $\xi = e_1 \wedge \ldots \wedge e_p$ at $x_0 \in \mathbb{AP}^n$. Complete $B = \{e_j\}_{j=1}^p$ to an orthonormal basis $\{e_j, n_k\}$ in the form

$$\begin{Bmatrix} v_1, & \mathbb{J}_1 v_1, & \ldots & \mathbb{J}_\Lambda v_1 \\ v_2, & \mathbb{J}_1 v_2, & \ldots & \mathbb{J}_\Lambda v_2 \\ \vdots & \vdots & & \vdots \\ v_N, & \mathbb{J}_1 v_N, & \ldots & \mathbb{J}_\Lambda v_N \end{Bmatrix}$$

where $\mathbb{J}_l : T_{x_0}\mathbb{AP}^n \to T_{x_0}\mathbb{AP}^n$ is the differential of left multiplication of $\mathbf{i}_l$ on $\mathbb{A}^n \subset \mathbb{AP}^n$.

In case of $\mathbb{AP}^n$, Such an orthonormal basis can be brought to the basis of normalized coordinate vectors by action of the isotropy group $K < G$. This is easy for $\mathbb{RP}^n$, $\mathbb{CP}^n$ and $\mathbb{HP}^n$: $SO(n)$, $SU(n)$ and $Sp(n)$ acts transitively on orthonormal frames, unitary frames and quaternionic unitary frames respectively. For $\mathbb{OP}^2$, $K = \mathrm{Spin}(9) < \mathrm{F}_4$, we argue as follows: $T_{x_0}\mathbb{OP}^2$ is the spinor representation of Spin(9). Under this action

$$T_{x_0}\mathbb{OP}^2 \supset \mathbb{S}^{15} \cong \mathrm{Spin}(9)/\mathrm{Spin}(7)$$

(see P.283 of [4]). Hence we can use $\sigma \in \mathrm{Spin}(9)$ to bring $\frac{1}{2}\frac{\partial}{\partial x_0^1}$ to v_1. Spin(7) fixes v_1 and hence acts on $T_{v_1}\mathbb{S}^{15}$, which splits into the vector representation V_7 and spinor representation of Spin(7). $\left\{\sigma\left(\frac{1}{2}\frac{\partial}{\partial x_l^1}\right)\right\}_{l=1}^7$ and $\{\mathbb{J}_l v_1\}_{l=1}^7$ form two base of V_7 having the same orientation. Then we can bring $\left\{\sigma\left(\frac{1}{2}\frac{\partial}{\partial x_l^1}\right)\right\}_{l=1}^7$ to $\{\mathbb{J}_l v_1\}_{l=1}^7$ by an element in Spin(7). $\left\{\sigma\left(\frac{1}{2}\frac{\partial}{\partial x_l^2}\right)\right\}_{l=1}^7$ can be brought to $\{\mathbb{J}_l v_2\}_{l=0}^7$ by Spin(7) using similar reasoning, because

$$\mathrm{Spin}(7)/G_2 \cong \mathbb{S}^7; \mathrm{G}_2/\mathrm{SU}(3) \cong \mathbb{S}^6$$

and SU(3) acts transitively on the collection of unitary bases.

Hence by Lemma 2 we may assume

$$\mathbb{J}_l v_j = \frac{1}{2}\frac{\partial}{\partial x_l^j}$$

so that we can apply Lemma 4 directly.

For $\mathbb{A} = \mathbb{R}^m$, $\mathbb{AP}^1 = \mathbb{S}^m$. Lemma 4 gives

$$\left\| \mathrm{II}(\frac{1}{2}\frac{\partial}{\partial x^j}, \frac{1}{2}\frac{\partial}{\partial x^k}) \right\|^2 = \delta_{jk}$$

which is the usual formula for the second fundamental form of $\mathbb{S}^m \subset \mathbb{R}^{m+1}$. Together with Theorem 1, the result of Lawson and Simons [8] is reproduced:

$$\mathrm{tr}\,\mathcal{A}_\xi = \sum_{j,k=1}^{p,q} (-1) = -pq \leq 0$$

implying that the average second variation of a rectifiable current of non-zero volume in $\mathbb{S}^n$ is negative for $0 < p < m$, and hence cannot be stable.

Now let's turn to the other four cases. Lemma 4 gives

$$\| \mathrm{II}(e_j, n_k)\|^2 = \begin{cases} 0 & \text{for } n_k = \pm\mathbb{J}_l e_j \text{ for some } 1 \leq l \leq \Lambda \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

and

$$\langle \mathrm{II}(e_j, e_j),\, \mathrm{II}(n_k, n_k)\rangle = \begin{cases} 1 & \text{for } n_k = \pm\mathbb{J}_l e_j \text{ for some } 1 \leq l \leq \Lambda \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

so the summand appeared in Theorem 1 is

$$2\| \mathrm{II}(e_j, n_k)\|^2 - \langle \mathrm{II}(e_j, e_j),\, \mathrm{II}(n_k, n_k)\rangle = \begin{cases} -1 & \text{for } n_k = \pm\mathbb{J}_l e_j \text{ for some } 1 \leq l \leq \Lambda \\ 0 & \text{otherwise} \end{cases}$$

meaning that for each e_j, every $\mathbb{J}_l e_j$-direction normal to ξ contributes -1 to $\mathrm{tr}\,\mathcal{Q}_\xi$, and all other normal directions make no effect. Hence

$$\begin{aligned} \mathrm{tr}\,\mathcal{Q}_\xi &= -\sum_{j=1}^{p} (\text{number of } l \text{ such that } \pm\mathbb{J}_l e_j \notin B) \\ &= -\sum_{j=1}^{p}\sum_{l=1}^{\Lambda} \|e_1 \wedge \ldots \wedge \mathbb{J}_l e_j \wedge \ldots \wedge e_p\|^2 \\ &= -\sum_{l=1}^{\Lambda} \|\mathbb{J}_l \cdot \xi\|^2 \leq 0 \end{aligned}$$

(Here $\mathbb{J}$ acts on ξ by Leibniz rule.) Equality holds if and only if $\|\mathbb{J}_l \cdot \xi\|^2 = 0$ for all $1 \leq l \leq \Lambda$, meaning that ξ is invariant under each $\mathbb{J}_l$, and hence invariant under the $\mathbb{S}^{\Lambda-1}$-family of complex structures. In the case of $\mathbb{RP}^n$, $\Lambda = 0$, the above formula says average second variation of S under those projections of constant vector in $\mathfrak{h}_n^0(\mathbb{R})$ is always zero, which is not a useful information. For the case that $\mathbb{A} = \mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, the result is interesting:

Theorem 5 *In $\mathbb{CP}^n$, a rectifiable current S is stable if and only if S is complex, that is, $T_x S$ is invariant under $\mathbb{J}$ for almost every $x \in S$;*

In $\mathbb{HP}^n$, S is stable if and only if S is quaternionic, that is, $T_x S$ is invariant under the $\mathbb{S}^2$-family of linear complex structures at x for almost every $x \in S$;

In $\mathbb{OP}^2$, S is stable if and only if S is octonionic, that is, $T_x S$ is invariant under the $\mathbb{S}^6$-family of linear complex structures at x for almost every $x \in S$.

In $\mathbb{S}^n$, S is unstable unless S consists of points or S is $\mathbb{S}^n$ itself.

The result of Lawson and Simons provides a similarity between the conformal geometry on $\mathbb{S}^n$ and the complex geometry on $\mathbb{CP}^n$. Our result gives a unified treatment from the viewpoint of Jordan algebra. This provides a hint on the reason behind the similarity of conformal geometry and complex geometry. We hope that our approach maybe helpful for generalizing other beautiful results in complex geometry to quaternion and octonion geometry.

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