Intersection theory of coassociative submanifolds in G_2-manifolds and Seiberg-Witten invariants

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Abstract

We study the problem of counting instantons with coassociative boundary condition in (almost) G_2-manifolds. This is analog to the open Gromov-Witten theory for counting holomorphic curves with Lagrangian boundary condition in Calabi-Yau manifolds. We explain its relationship with the Seiberg-Witten invariants for coassociative submanifolds.

Intersection theory of Lagrangian submanifolds is an essential part of the symplectic geometry. By counting the number of holomorphic disks bounding intersecting Lagrangian submanifolds, Floer and others defined the celebrated Floer homology theory. It plays an important role in mirror symmetry for Calabi-Yau manifolds and string theory in physics. In M-theory, Calabi-Yau threefolds are replaced by seven dimensional G_2-manifolds M (i.e. oriented Octonion manifolds [20]). The analog of holomorphic disks (resp. Lagrangian submanifolds) are instantons or associative submanifolds (resp. coassociative submanifolds or branes) in M [19]. An important project is to count the number of instantons with coassociative boundary conditions. In particular we want to study the following problem.

Problem: Given two nearby coassociative submanifolds C and C' in a (almost) G_2-manifold M. Relate the number of instantons in M bounding $C \cup C'$ to the Seiberg-Witten invariants of C.

The basic reason is a coassociative submanifold C' which is infinitesimally close to C corresponds to a symplectic form on C which degenerates along $C \cap C'$. Instantons bounding $C \cup C'$ would become holomorphic curves on C modulo bubbling. By the work of Taubes, we expect that the number of such instantons is given by the Seiberg-Witten invariant of C.

In this paper we treat the special case when C and C' are disjoint, i.e. C is a symplectic four manifold. Recall that Taubes showed that the Seiberg-Witten invariants of such a C is given by the Gromov-Witten invariants [25] of C. Our main result is following theorem.

Theorem 1 *Suppose that M is an (almost) G_2-manifold and $\{C_t\}$ is an one parameter family of coassociative submanifolds in M. Suppose that the self-dual two form $\eta = dC_t/dt|_{t=0} \in \Omega^2_+(C)$ is nonvanishing, then it defines an almost complex structure J on C_0.*

If $\{A_t\}$ is any one parameter family of instantons in M satisfying

$$\partial A_t \subset C_t \cup C_0, \ \lim_{t\to 0} A_t \cap C_0 = \Sigma \text{ in } C^1\text{-topology,(c.f. Proposition 5)}$$

then Σ is a J-holomorphic curve in C_0.

Conversely, suppose that Σ is a regular *J-holomorphic curve in C_0, then it is the limit of a family of instantons A_t's as described above.*

A few remarks are in order: First, counting such small instantons is basically a problem in four manifold theory because of Bryant's result [4] which says that the zero section C in $\Lambda^2_+(C)$ is always a coassociative submanifold for some incomplete G_2-metric on its neighborhood provided that the bundle $\Lambda^2_+(C)$ is topologically trivial. Second, when C and C' are not disjoint, the above theorem should still hold true. However using the present approach to prove it would require a good understanding of the Seiberg-Witten theory on any four manifold with a degenerated symplectic form as in Taubes program. Third, when C and C' are not close to each other then we have to take into account the bubbling phenomenon which has not been established yet. Nevertheless, one would expect that if the volume of A_t's are small, then bubbling cannot occur, thus they would converge to a holomorphic curve in C_0.

1 Review of Symplectic Geometry

Given any symplectic manifold (X, ω) of dimension $2n$, there exists a compatible metric g so that the equation

$$\omega(u, v) = g(Ju, v)$$

defines a Hermitian almost complex structure

$$J : T_X \to T_X,$$

that is $J^2 = -id$ and $g(Ju, Jv) = g(u, v)$.

A *holomorphic curve*, or *instanton*, is a two dimensional submanifold Σ in X whose tangent bundle is preserved by J. Equivalently Σ is calibrated by ω, i.e. $\omega|_\Sigma = vol_\Sigma$. By *counting* the number of instantons in X, one can define a highly nontrivial invariant for the symplectic structure on X, called the Gromov-Witten invariant.

When the instanton Σ has nontrivial boundary, then the corresponding free boundary value problem would require $\partial\Sigma$ to lie on a *Lagrangian submanifold* L in X, i.e. $\dim L = n$ and $\omega|_L = 0$. Floer studied the intersection theory of Lagrangian submanifolds and defined the Floer homology group $HF(L, L')$ under certain assumptions.

Suppose that X is a Calabi-Yau manifold, i.e. the holonomy group of the Levi-Civita connection is inside $SU(n)$, equivalently J is an integrable complex structure on X and there exists a holomorphic volume form $\Omega_X \in \Omega^{n,0}(X)$ on X satisfying $\Omega_X \bar{\Omega}_X = C_n \omega^n$. Under the mirror symmetry transformation, $HF(L, L')$ is expected to correspond to the Dolbeault cohomology group of coherent sheaves in the mirror Calabi-Yau manifold.

A Lagrangian submanifold L in X is called a *special Lagrangian submanifold* with phase zero (resp. $\pi/2$) if $\operatorname{Im} \Omega_X|_L = 0$ (resp. $\operatorname{Re} \Omega_X|_L = 0$). Such a L is calibrated by $\operatorname{Re} \Omega_X|_L$ (resp. $\operatorname{Im} \Omega_X|_L$). They play important roles in the Strominger-Yau-Zaslow mirror conjecture for Calabi-Yau manifolds [24].

When X is a Calabi-Yau threefold, there are conjectures of Vafa and others (e.g. [2][11]) that relates the (partially defined) open Gromov-Witten invariant of the number of instantons with Lagrangian boundary condition to the large N Chern-Simons invariants of knots in three manifolds.

2 Counting Instantons in (almost) G_2-manifolds

Notice that a real linear homomorphism $J : \mathbb{R}^m \to \mathbb{R}^m$ being a Hermitian complex structure on $\mathbb{R}^m$ is equivalent to the following conditions: for any vector $v \in \mathbb{R}^m$ we have (i) Jv is perpendicular to both v and (ii) $|Jv| = |v|$. We can generalize J to involve more than one vector. We call a skew symmetric homomorphism

$$\times : \mathbb{R}^m \otimes \mathbb{R}^m \to \mathbb{R}^m$$

a (2-fold) *vector cross product* if it satisfies

(i) $(u \times v)$ is perpendicular to both u and v, and
(ii) $|u \times v| =$ Area of parallelogram spanned by u and v.

The obvious example of this is the standard vector product on $\mathbb{R}^3$. By identifying $\mathbb{R}^3$ with $\operatorname{Im} \mathbb{H}$, the imaginary part of the quaternion numbers, we have

$$u \times v = \operatorname{Im} u\bar{v}.$$

The same formula defines a vector cross product on $\mathbb{R}^7 = \operatorname{Im} \mathbb{O}$, the imaginary part of the octonion numbers. Brown and Gray [12] showed that these two are the only possible vector cross product structures on $\mathbb{R}^m$ up to isomorphisms.

Suppose that M is a seven dimensional Riemannian manifold with a vector cross product $\times$ on each of its tangent spaces. The analog of the symplectic form is a degree three differential form Ω on M defined as follow:

$$\Omega(u, v, w) = g(u \times v, w).$$

Definition 2 *Suppose that (M, g) is a Riemannian manifold of dimension seven with a vector cross product $\times$ on its tangent bundle. Then (1) M is called an* ***almost G_2-manifold*** *if $d\Omega = 0$ and (2) M is called a* ***G_2-manifold*** *if $\nabla \Omega = 0$.*

It can be proven that if M is compact then the condition $\nabla\Omega = 0$ is equivalent to Ω being a harmonic form, i.e. $\Delta\Omega = 0$. Furthermore M is a G_2-manifold if and only if its holonomy group is inside the exceptional Lie group $G_2 = Aut(\mathbb{O})$. The geometry of G_2-manifolds can be interpreted as the symplectic geometry on its knot space (see e.g. [19], [23]).

For example, if (X, ω_X) is a Calabi-Yau threefold with a holomorphic volume form Ω_X, then the product manifold $M = X \times S^1$ is a G_2-manifold with

$$\Omega = \mathrm{Re}\,\Omega_X + \omega_X \wedge d\theta.$$

Next we define the analogs of holomorphic curves and Lagrangian submanifolds in the G_2 setting.

Definition 3 *Suppose that A is a three dimensional submanifold of an almost G_2-manifold M. We call A an* ***instanton*** *or* ***associative submanifold****, if A is preserved by the vector cross product $\times$.*

Harvey and Lawson [13] showed that $A \subset M$ is an instanton if and only if A is calibrated by Ω, i.e. $\Omega|_A = vol_A$.

In M-theory, associative submanifolds are also called *M2-branes*. For example when $M = X \times S^1$ with X a Calabi-Yau threefold, $\Sigma \times S^1$ (resp. $L \times \{p\}$) is an instanton in M if and only if Σ (resp. L) is a holomorphic curve (resp. special Lagrangian submanifold with zero phase) in X.

A natural interesting question is to count the number of instantons in M. In the special case of $M = X \times S^1$, these numbers are reduced to the conjectural invariants proposed by Joyce [17] by counting special Lagrangian submanifolds in Calabi-Yau threefolds. This problem has been discussed by many physicists. For example Harvey and Moore discussed in [14] the mirror symmetry aspects of these invariants; Aganagic and Vafa in [2] related these invariants to the open Gromov-Witten invariants for local Calabi-Yau threefolds; Beasley and Witten argued in [3] that when there is a moduli of instantons, then one should count them using the Euler characteristic of the moduli space. In this paper we count the number of instantons with boundary lying on a coassociative submanifold in M. The compactness issues of the moduli of instantons is a very challenging problem because the dimension of an instanton is bigger than two. This makes it very difficult to define an honest invariant by counting instantons.

When an instanton A has a nontrivial boundary, $\partial A \neq \phi$, one should require it to lie inside a *brane* or a *coassociative submanifold* [19], i.e. submanifolds in M where the restriction of Ω is zero and have the largest possible dimension. Branes are the analog of Lagrangian submanifolds in symplectic geometry.

Definition 4 *Suppose that C is a four dimensional submanifold of an almost G_2-manifold M. We call C a* ***coassociative submanifold*** *if*

$$\Omega|_C = 0 \textit{ and } \dim C = 4.$$

For example when $M = X \times S^1$ with X a Calabi-Yau threefold, $H \times S^1$ (resp. $C \times \{p\}$) is a coassociative submanifold in M if and only if H (resp. C) is a special Lagrangian submanifold with phase $\pi/2$ (resp. complex surface) in X. In [19] J.H. Lee and the first author showed that the isotropic knot space $\hat{\mathcal{K}}_{S^1}X$ of X admits a natural holomorphic symplectic structure. Moreover $\hat{\mathcal{K}}_{S^1}H$ (resp. $\hat{\mathcal{K}}_{S^1}C$) is a complex Lagrangian submanifold in $\hat{\mathcal{K}}_{S^1}X$ with respect to the complex structure J (resp. K).

Constructing special Lagrangian submanifolds with zero phase in X with boundaries lying on H (resp. C) corresponds to the Dirichlet (resp. Neumann) free boundary value problem for minimizing volume among Lagrangian submanifolds as studied by Schoen and others. For a general G_2-manifold M, the natural free boundary value for an instanton is a coassociative submanifold. Similar to the intersection theory of Lagrangian submanifolds in symplectic manifolds. We propose to study the following problem: Count the number of instantons in G_2-manifolds bounding two coassociative submanifolds.

The product of a coassociative submanifold with a two dimensional plane inside the eleven dimension spacetime $M \times \mathbb{R}^{3,1}$ is called a *D5-brane* in M-theory. Counting the number of M2-branes between two D5-branes has also been studied in the physics literatures.

In general this is a very difficult problem. For instance, counting S^1-invariant instantons in $M = X \times S^1$ is the open Gromov-Witten invariants. However when the two coassociative submanifolds C and C' are close to each other, we can relate the number of instantons between them to the Seiberg-Witten invariant of C.

3 Relationships to Seiberg-Witten invariants

To determine the number of instantons between nearby coassociative submanifolds, we first recall the deformation theory of coassociative submanifolds C inside any G_2-manifold M, as developed by McLean [22]. Given any normal vector $n \in N_{C/M}$, the interior product $\iota_n \Omega$ is naturally a self-dual two form on C because of $\Omega|_C = 0$. This gives a natural identification,

$$\begin{aligned} N_{C/M} &\overset{\cong}{\to} \Lambda^2_+(C) \\ n &\to \eta_0 = \iota_n \Omega. \end{aligned}$$

Furthermore infinitesimal deformations of coassociative submanifolds are parametrized by self-dual harmonic two forms $\eta_0 \in H^2_+(C)$, and they are always unobstructed. Notice that the zero set of η_0 is the intersection of C with a infinitesimally near coassociative submanifold, that is

$$\{\eta_0 = 0\} = \lim_{t \to 0} (C \cap C_t),$$

where $C = C_0$ and $\eta_0 = dC_t/dt|_{t=0}$.

Since

$$\eta_0 \wedge \eta_0 = \eta_0 \wedge * \eta_0 = |\eta_0|^2 * 1,$$

η_0 defines a natural symplectic structure on $C^{reg} := C\backslash\{\eta_0 = 0\}$. If we normalize η_0,

$$\eta = \eta_0 / |\eta_0|,$$

then the equation

$$\eta(u,v) = g(Ju,v)$$

defines a Hermitian almost complex structure on C^{reg}.

The next proposition says that when two coassociative submanifolds C and C' come together, then the limit of instantons bounding them will be a holomorphic curve Σ in C^{reg} with boundary $C \cap C'$.

Proposition 5 *Suppose that C_t is an one parameter family of coassociative submanifolds in a G_2-manifold M. Suppose that A_t is a smooth family of instantons in M bounding $C_0 \cup C_t$ for nonzero t and*

$$\lim_{t\to 0} A_t \cap C_0 = \Sigma$$

exists in C^1-topology. Then Σ is a J-holomorphic curve in C_0 .

Proof. For simplicity we assume that $n = dC_t/dt|_{t=0}$ is nowhere vanishing. Let us denote the boundary component of A_t in C_0 as Σ_t and the unit normal vector field for Σ_t in A_t as $_t$. Note that n_t is perpendicular to C_0. This is because A_t being preserved by the vector cross product implies that

$$n_t = u \times v,$$

for some tangent vectors u and v in Σ_t, therefore given any tangent vector w along C_0, we have

$$g(n_t, w) = g(u \times v, w) = \Omega(u, v, w) = 0.$$

The last equality follows from C_0 being coassociative and $\Sigma_t \subset C_0$. Using this and the fact that A_t bounds $C_0 \cup C_t$ with $\lim_{t\to 0} C_t = C_0$, i.e. n_t is *pointing towards* C_t, we obtain

$$\iota_{n_0}\Omega = \eta_0 \text{ where } n_0 := \lim_{t\to 0} n_t$$

Therefore $\Sigma = \lim_{t\to 0} \Sigma_t$ is a holomorphic curve in C_0 with respect to the almost complex structure J defined by $\eta(u,v) = g(Ju,v)$. ■

The reverse of the above proposition should also hold true. The Lagrangian analog of it is proven by Fukaya and Oh in [7]. On the other hand, by the celebrated work of Taubes, we expect that the number of such open holomorphic curves in C_0 equals to the Seiberg-Witten invariant of C_0. We conjecture the following statement.

Proposition 6 *Let M be a G_2-manifold. Suppose that*

$$\psi : C \times [0,1] \longrightarrow M$$

is a smooth map such that for each $t \in [0,1]$, $\psi_t(\cdot) := \psi(\cdot, t)$ *is a smooth immersion of* C *into* M *as a coassociative submanifolds, and*

$$\phi_t : \Sigma \times [0,t] \longrightarrow M$$

is a smooth family of instantons in M *such that for* $t > 0$, $\operatorname{Im} \phi_t$ *is associative and*

$$\phi_t(\Sigma \times \{0\}) \subset C_0 := \psi(C \times \{0\}) \text{ and } \phi_t(\Sigma \times \{t\}) \subset C_t := \psi(C \times \{t\}).$$

Then Σ *is a* J*-holomorphic curve in* C_0 .

Proof. For simplicity we assume that $n = dC_t/dt|_{t=0}$ is nowhere vanishing. Let us denote the boundary component of $A_t = \operatorname{Im} \phi_t$ in C_0 as Σ_t, i.e. $\Sigma_t := \phi_t(\Sigma \times \{0\})$, and the unit normal vector field for Σ_t in A_t as w_t. Note that w_t is *perpendicular to* C_0. This is because A_t being preserved by the vector cross product implies that

$$w_t = u \times v,$$

for some tangent vectors u and v in Σ_t, therefore given any tangent vector w along C_0, we have

$$g(w_t, w) = g(u \times v, w) = \Omega(u, v, w) = 0.$$

The last equality follows from C_0 being coassociative and $\Sigma_t \subset C_0$. Using this and the fact that A_t *bounds* $C_0 \cup C_t$ with $\lim_{t \to 0} C_t = C_0$, we conclude that w_t is *pointing towards* C_t. To be precise, we have along Σ

$$\lim_{t \to 0} n_t = \left(\left. \frac{dC_t}{dt} \right|_{t=0} \right) \Bigg|_{\Sigma} \in \Gamma\left(\Sigma, N_{C_0/M}\right).$$

Therefore $\Sigma = \lim_{t \to 0} \Sigma_t$ is a holomorphic curve in C_0 with respect to the almost complex structure J defined by $\eta_0(u, v) = g(Ju, v)$. ■

The reverse of the above proposition should also hold true. The Lagrangian analog of it is proven by Fukaya and Oh in [7]. On the other hand, by the celebrated work of Taubes, we expect that the number of such open holomorphic curves in C_0 equals to the Seiberg-Witten invariant of C_0. We conjecture the following statement.

Conjecture: Suppose that C and C' are nearby coassociative submanifolds in a G_2-manifold M. Then the number of instantons in M with small volume and with boundary lying on $C \cup C'$ is given by the Seiberg-Witten invariants of C.

In the next section we will discuss the case when C and C' do not intersect. The basic ideas are (i) the limit of such instantons is a holomorphic curve with respect to the (degenerated) symplectic form η on C coming from its deformations as coassociative submanifolds and this process can be reversed; (ii) the number of holomorphic curves in the four manifold C should be related to the Seiberg-Witten invariant of C by the work of Taubes ([26], [27]).

Suppose that η is a self-dual two form on C with constant length $\sqrt{2}$, in particular it is a (non-degenerate) symplectic form, and Σ is a holomorphic curve in C, possibly disconnected. If Σ is **regular** in the sense that the linearized operator $\bar{\partial}$ has trivial cokernel [25], then Taubes showed that the perturbed Seiberg-Witten equations,

$$F_a^+ = \tau(\psi \otimes \psi^*) - r\sqrt{-1}\eta,$$
$$D_{A(a)}\psi = 0,$$

have solutions for all sufficient large r. Here a is a connection on the complex line bundle E over C whose first Chern class equals the Poincaré dual of Σ, $PD[\Sigma]$, ψ is a section of the twisted spinor bundle $S_+ = E \oplus (K^{-1} \otimes E)$ and $D_{A(a)}$ is the twisted Dirac operator. The number of such solutions is the Seiberg-Witten invariant $SW_C(\Sigma)$ of C. Furthermore the converse is also true, thus Taubes established an equivalence between Seiberg-Witten theory and Gromov-Witten theory for symplectic four manifolds. This result has far reaching applications in four dimensional symplectic geometry.

For a general four manifold C with nonzero $b^+(C)$, using a generic metric, any self-dual two form η on C defines a degenerate symplectic form on C, i.e. η is a symplectic form on the complement of $\{\eta = 0\}$, which is a finite union of circles (see [9][16]). Therefore, one might expect to have a relationship between the Seiberg-Witten of C and the number of holomorphic curves with boundaries $\{\eta = 0\}$ in C. Part of this Taubes' program has been verified in [26], [27].

4 Proof of the main theorem

Suppose that η is a nowhere vanishing self-dual harmonic two form on a coassociative submanifold C in a G_2-manifold M. For any holomorphic curve Σ in C, we want to construct an instanton in M bounding C and C', where C' is a small deformation of the coassociative submanifold C along the normal direction η. Notice that C and C' do not intersect. We will construct such an instanton using a perturbation argument which requires a lower bound on the first eigenvalue for the appropriate elliptic operator. Recall that the deformation of an instanton is governed by a twisted Dirac operator. We will reinterpret it as a complexified version of the Cauchy-Riemann operator.

4.1 Deformation of instantons

To construct an instanton A in M from a holomorphic curve Σ in C, we need to perturb an almost instanton A' to a honest one using a quantitative version of the implicit function theorem. Let us first recall the deformation theory of instantons A ([13] and [19]) in a Riemannian manifold M with a parallel (or closed) r-fold vector cross product

$$\times : \Lambda^r T_M \to T_M.$$

In our situation, we have $r = 2$. By taking the wedge product with T_M we obtain a homomorphism τ,

$$\tau : \Lambda^{r+1} T_M \to \Lambda^2 T_M \cong \Lambda^2 T_M^*,$$

where the last isomorphism is induced from the Riemannian metric. As a matter of fact, the image of τ lies inside the subbundle $\mathfrak{g}_M^{\perp}$ which is the orthogonal complement of $\mathfrak{g}_M \subset \mathfrak{so}(T_M) \cong \Lambda^2 T_M^*$, the bundle infinitesimal isometries of T_M preserving $\times$. That is,

$$\tau \in \Omega^{r+1}\left(M, \mathfrak{g}_M^{\perp}\right).$$

Lemma 7 *([13], [19]) An $r+1$ dimensional submanifold $A \subset M$ is an instanton, i.e. preserved by $\times$, if and only if*

$$\tau|_A = 0 \in \Omega^{r+1}\left(A, \mathfrak{g}_M^{\perp}\right).$$

This lemma is important in describing deformations of an instanton. Namely it shows that the normal bundle to an instanton A is a twisted spinor bundle over A and infinitesimal deformations of A are parametrized by twisted harmonic spinors.

In our present situation, M is a G_2-manifold. Using the interior product with Ω, we can identify $\mathfrak{g}_M^{\perp}$ with the tangent bundle T_M and we can also characterize $\tau \in \Omega^3(M, T_M)$ by the following formula,

$$(*\Omega)(u, v, w, z) = g\left(\tau(u, v, w), z\right).$$

Therefore $A \subset M$ is an instanton if and only if $*_A(\tau|_A) = 0 \in T_M|_A$. As a matter of fact, if A is already close to be an instanton, then we only need the normal components of $*_A(\tau|_A)$ to vanish.

Proposition 8 *There is a positive constant δ such that for any three dimensional linear subspace A in $M \cong \operatorname{Im}\mathbb{O}$ with $|\tau|_A| < \delta$, A is an instanton if and only if $*_A(\tau|_A) \in T_A$.*

Proof. McLean [22] observed that if A_t is a family of linear subspaces in $M \cong \mathbb{R}^7$ with A_0 an instanton, then

$$*_{A_t}\left(\frac{d\tau|_{A_t}}{dt}\right)\Bigg|_{t=0} \in N_{A_0/M} \subset T_M|_{A_0}.$$

Explicitly, if we denote the standard base for $\mathbb{R}^7$ as e_i's, e.g. $e_1 \times e_2 = e_3$, then we can assume that A is spanned by e_1, e_2 and $\tilde{e}_3 = e_3 + \sum_{i=4}^{7} t_i e_i$ for some small t_i's because the natural action of G_2 on the Grassmannian $Gr(2,7)$ is transitive. Then an easy computation(c.f. equation (5.4) in [**?**]) shows that the normal component of $*(\tau|_A)$ in $N_{A/M}$ is given by

$$*(\tau|_A)^{\perp} = -t_5 (e_4)^{\perp} + t_4 (e_5)^{\perp} + t_7 (e_6)^{\perp} - t_6 (e_7)^{\perp},$$

where $(\cdot)^{\perp}$ denote the projection to $N_{A/M}$.When t_j's are all zero, we have $(e_j)^{\perp} = e_j$ for $4 \leq j \leq 7$. In particular, they are linearly independent when t_j's are small. In that case, $*(\tau|_A)^{\perp} = 0$ will actually imply that $t_j = 0$ for all j, i.e. A is an instanton in M. Hence the proposition. ■

This proposition will be needed later when we perturb an almost instanton to an honest one. We also need to identify the normal bundle $N_{A/M}$ to an instanton A with a twisted spinor bundle over A as follow [22]: We denote P the $SO(4)$-frame bundle of $N_{A/M}$. Using the identification

$$SO(4) = Sp(1)\,Sp(1) \to SO(\mathbb{H}),$$

$$(p,q) \cdot y = py\bar{q},$$

the tangent bundle to A can be identified as an associated bundle to P for the representation $SO(4) \to SO(\operatorname{Im} H)$, $(p,q) \cdot y = qy\bar{q}$. As a result the spinor bundle $\mathbb{S}$ of A is associated to the representation $SO(4) \to SO(\mathbb{H})$ given by $(p,q) \cdot y = y\bar{q}$. Hence we obtain

$$N_{A/M} \cong \mathbb{S} \otimes_{\mathbb{H}} E,$$

where E is the associated bundle to P for the representation $SO(4) \to SO(\mathbb{H})$ given by $(p,q) \cdot y = py$.

4.2 Complexified Cauchy-Riemann equation

Recall that the normal bundle to any instanton A is a twisted spinor bundle $\mathbb{S}\otimes_{\mathbb{H}}E$, or simply $\mathbb{S}$, over A. Let $\mathcal{D}$ be the Dirac operator on A. If $V := V^a \frac{\partial}{\partial \omega^a}$ is a normal vector field to A and we write the covariant differentiation of V as $\nabla(V) := V_i^a \frac{\partial}{\partial \omega^a} \otimes \omega^i$, then by viewing V as a twisted spinor or a quaternion valued function on A,

$$V = V^4 + \mathbf{i}V^5 + \mathbf{j}V^6 + \mathbf{k}V^7,$$

we have,

$$\begin{aligned} \mathcal{D}V = &-\left(V_1^5 + V_2^6 + V_3^7\right) + \mathbf{i}\left(V_1^4 + V_3^6 - V_2^7\right) \\ &+ \mathbf{j}\left(V_2^4 - V_3^5 + V_1^7\right) + \mathbf{k}\left(V_3^4 + V_2^5 - V_1^6\right), \end{aligned}$$

where $\mathcal{D} := \nabla_1 \mathbf{i} + \nabla_2 \mathbf{j} + \nabla_3 \mathbf{k}$.

Let us first consider a simplified model, suppose that $\mathtt{A}$ is a product Riemannian three manifold $[0,\varepsilon] \times \Sigma$ with coordinates (x_1, z) where $z = x_2 + ix_3$. Let e_1 be the unit tangent vector field on $\mathtt{A}$ normal to Σ, namely along the x_1-direction. We have

$$\mathcal{D} = e_1 \cdot \frac{\partial}{\partial x_1} + \bar{\partial},$$

where $\bar{\partial}$ is the Dolbeault operator on the Riemann surface Σ for the holomorphic line bundle $\mathbb{S}^+$ and $\mathbb{S}^-$.

The Clifford multiplication of e_1 on $\mathbb{S}$ satisfies $e_1^2 = -1$ and therefore we have an eigenspace decomposition $\mathbb{S} := \mathbb{S}^+ \oplus \mathbb{S}^-$ corresponding to eigenvalues $\pm i$.

If we write $V = (u, v)$ with $u = V^4 + \mathbf{i}V^5 \in \mathbb{S}^+$ and $v = V^6 + \mathbf{i}V^7 \in \mathbb{S}^-$, then we have

$$
\begin{aligned}
\mathcal{D}V &= \left(\frac{\partial u}{\partial x_1}\mathbf{i} - \partial_z v\right) + \left(-\frac{\partial v}{\partial x_1}\mathbf{i} + \bar{\partial}_z u\right) \cdot \mathbf{j} \\
&= \left(\left(\frac{\partial u}{\partial x_1} + \mathbf{i}\partial_z v\right) + \left(\frac{\partial v}{\partial x_1} + \mathbf{i}\bar{\partial}_z u\right) \cdot \mathbf{j}\right) \cdot \mathbf{i} \\
&= \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \left(\frac{\partial}{\partial x_1} + \begin{bmatrix} 0 & \mathbf{i}\partial_z \\ \mathbf{i}\bar{\partial}_z & 0 \end{bmatrix}\right) \begin{bmatrix} u \\ v \end{bmatrix},
\end{aligned}
$$

where

$$
\bar{\partial}_z := \frac{\partial}{\partial x_2} + \mathbf{i}\frac{\partial}{\partial x_3} \text{ and } \partial_z := \frac{\partial}{\partial x_2} - \mathbf{i}\frac{\partial}{\partial x_3}.
$$

We will also denote $\mathbf{i}\partial_z$ and $\mathbf{i}\bar{\partial}_z$ by ∂^+ and ∂^- respectively. They are Dirac operators on Σ and they satisfy

$$
\partial^+ = \left(\partial^-\right)^*.
$$

This implies that the Dirac equation $\mathcal{D}V = 0$ is equivalent to the following complexified Cauchy-Riemann equations,

$$
\begin{aligned}
\bar{\partial}_z u &= \frac{\partial v}{\partial x_1}\mathbf{i}, \\
\partial_z v &= \frac{\partial u}{\partial x_1}\mathbf{i}.
\end{aligned}
$$

4.3 Eigenvalue estimates

In this subsection we first give a quantitative estimate of the eigenvalue of the linearized operator for the simplified model $A_\varepsilon = [0, \varepsilon] \times \Sigma$ with product metric $g_{\mathtt{A}_\varepsilon} = dx_1^2 + g_\Sigma$. Then we use the conformal property of the Dirac operator to obtain a corresponding result for any warped product metric on A_ε.

We introduce the following function spaces for spinors $V = (u, v)$ over $\mathtt{A}_\varepsilon$.

Definition 9 *Let $\mathbb{S}$ be the spinor bundle over $(\mathtt{A}_\varepsilon, g_{\mathtt{A}_\varepsilon})$ and V be a smooth section of $\mathbb{S}$,*

1. *We define the norm*

$$
\|V\|_{L^{m,p}(\mathtt{A}_\varepsilon, \mathbb{S})} := \left(\sum_{\alpha+\beta \le m} \int_0^\varepsilon \int_\Sigma \left|(\nabla_x)^\alpha (\nabla_\Sigma)^\beta V\right|^p dx d\Sigma\right)^{1/p}
$$

and

$$\|V\|_{C^m(\mathtt{A}_\varepsilon,\mathbb{S})} := \sum_{\alpha+\beta\leq m} \sup \left| (\nabla_x)^\alpha (\nabla_\Sigma)^\beta V \right|$$

where the covariant derivatives and L^p-norm are all with respect to $g_{\mathtt{A}_\varepsilon}$. Consequently, we have an ε independent constant C so that for any smooth section V,

$$\|V\|_{L^p(\mathtt{A}_\varepsilon,\mathbb{S})} \leq C \|V\|_{L^{m,q}(\mathtt{A}_\varepsilon,\mathbb{S})}, \text{ for } p \leq \frac{3q}{3-mq}$$
$$\|V\|_{C^m(\mathtt{A}_\varepsilon,\mathbb{S})} \leq C \|V\|_{L^{l,p}(\mathtt{A}_\varepsilon,\mathbb{S})}, \text{ for } p \geq \frac{3}{l-m}$$

as long as $\varepsilon \in [1/2, 3/2]$.

2. *We define the function spaces*

$$L^{m,p}(\mathtt{A}_\varepsilon,\mathbb{S}) := \left\{ V = (u,v) \in \Gamma(\mathtt{A}_\varepsilon,\mathbb{S}) \mid \|V\|_{L^{m,p}(\mathtt{A}_\varepsilon,\mathbb{S})} < +\infty \right\}$$

and $L_-^{m,p}(\mathtt{A}_\varepsilon,\mathbb{S})$ (resp.$L_+^{m,p}(\mathtt{A}_\varepsilon,\mathbb{S})$) be the closure (with respect to the norm $\|\cdot\|_{L^{m,p}(\mathtt{A}_\varepsilon)}$) of the subspace of smooth sections $V = (u,v) \in \Gamma(\mathtt{A}_\varepsilon,\mathbb{S})$ such that $v \in C_0^\infty(\mathtt{A}_\varepsilon \backslash \partial \mathtt{A}_\varepsilon)$ (resp. $u \in C_0^\infty(\mathtt{A}_\varepsilon \backslash \partial \mathtt{A}_\varepsilon)$). Let us also introduce the space

$$C^m(\mathtt{A}_\varepsilon,\mathbb{S}) := \left\{ V = (u,v) \in \Gamma(\mathtt{A}_\varepsilon,\mathbb{S}) \mid \|V\|_{C^m(\mathtt{A}_\varepsilon,\mathbb{S})} < +\infty \right\}$$
$$C_-^m(\mathtt{A}_\varepsilon,\mathbb{S}) := \left\{ V = (u,v) \in \Gamma(\mathtt{A}_\varepsilon,\mathbb{S}) \mid \|V\|_{C^m(\mathtt{A}_\varepsilon,\mathbb{S})} < +\infty, v|_{\partial \mathtt{A}_\varepsilon} = 0 \right\}.$$

It is known (c.f. [5] Theorem 21.5) that the Dirac operators

$$\mathcal{D}_\pm := \mathcal{D}|_{L_\pm^{1,2}} : L_\pm^{1,2}(\mathtt{A}_\varepsilon,\mathbb{S}) \to L^2(\mathtt{A}_\varepsilon,\mathbb{S})$$

give well-posed local elliptic boundary problems and their formal adjoint operators are $\mathcal{D}_\pm^* = \mathcal{D}_\mp$.

The following theorem compare the first eigenvalue for Dirac operator on the Riemann surface Σ with the one on the product three manifold $\mathtt{A}_\varepsilon$.

Theorem 10 *Suppose λ_{∂^+} (resp. λ_{∂^-}) is the first eigenvalue of $\Delta_\Sigma = \partial^-\partial^+$ (resp. $\Delta_\Sigma = \partial^+\partial^-$) acting on the space $L^{1,2}(\Sigma,\mathbb{S}^+)$(resp.$L^{1,2}(\Sigma,\mathbb{S}^-)$) and let*

$$\lambda_{\mathcal{D}} := \inf_{V \in L_-^{1,2}(\mathtt{A}_\varepsilon)} \frac{\|\mathcal{D}V\|_{L^2(\mathtt{A}_\varepsilon)}^2}{\|V\|_{L^2(\mathtt{A}_\varepsilon)}^2}.$$

Then

$$\lambda_{\mathcal{D}} \geq \min\left\{ \lambda_{\partial^+}, \frac{2}{\varepsilon^2} \right\}.$$

Proof. For any $V = (u, v) \in L^{1,2}_{-}(\mathtt{A}_\varepsilon)$, we have

$$\langle \mathcal{D}V, \mathcal{D}V\rangle_{L^2} = \int_{[0,\varepsilon]\times\Sigma} \left|\frac{\partial V}{\partial x_1}\right|^2 + 2\left\langle \frac{\partial V}{\partial x_1}, \begin{bmatrix} 0 & \partial^+ \\ \partial^- & 0 \end{bmatrix} V\right\rangle + \left\|\partial^- v\right\|^2 + \left\|\partial^+ u\right\|^2.$$

Using the formula $\partial^- = (\partial^+)^*$, we have

$$\begin{aligned}
&\int_{[0,\varepsilon]\times\Sigma} \left\langle \frac{\partial V}{\partial x_1}, \begin{bmatrix} 0 & \partial^+ \\ \partial^- & 0 \end{bmatrix} V\right\rangle \\
&= \int_{[0,\varepsilon]\times\Sigma} \left\langle \frac{\partial u}{\partial x_1}, \partial^+ v\right\rangle + \left\langle \frac{\partial v}{\partial x_1}, \partial^- u\right\rangle \\
&= \int_{[0,\varepsilon]\times\Sigma} \left\langle \partial^-\left(\frac{\partial u}{\partial x_1}\right), v\right\rangle - \left\langle v, \partial^-\left(\frac{\partial u}{\partial x_1}\right)\right\rangle + \int_{\{\varepsilon\}\times\Sigma} \langle v, \partial^- u\rangle - \int_{\{0\}\times\Sigma} \langle v, \partial^- u\rangle \\
&= 0.
\end{aligned}$$

because $v|_{\partial\mathtt{A}} = 0$

In order to estimate $\int_{\mathtt{A}} |V_{x_1}|^2$, we notice that, for any fixed point $p \in \Sigma$, $v|_{[0,\varepsilon]\times\{p\}}$ can be treated as a function over the interval $[0, \varepsilon]$ and we compute

$$\begin{aligned}
\int_0^\varepsilon v^2 dx_1 &= \int_0^\varepsilon \left(\int_0^{x_1} \frac{\partial v}{\partial x_1}(t)\, dt\right)^2 dx_1 \\
&\le \int_0^\varepsilon \left(\int_0^{x_1} ds\right)\left(\int_0^{x_1} \left|\frac{\partial v}{\partial x_1}(t)\right|^2 dt\right) dx_1 \\
&\le \int_0^\varepsilon x_1 dx_1 \int_0^\varepsilon \left|\frac{\partial v}{\partial x_1}(t)\right|^2 dt \\
&= \frac{\varepsilon^2}{2}\int_0^\varepsilon \left|\frac{\partial v}{\partial x_1}(t)\right|^2 dt.
\end{aligned}$$

Put all these together, we have

$$\begin{aligned}
&\langle \mathcal{D}V(x_1), \mathcal{D}V(x_1)\rangle \\
&= \int_{[0,\varepsilon]\times\Sigma} \left(\|u_{x_1}\|^2 + \|v_{x_1}\|^2 + \left\|\partial^- v\right\|^2 + \left\|\partial^+ u\right\|^2\right) \\
&\ge \int_0^\varepsilon \int_\Sigma \left\|\partial^+ u\right\|^2 + \int_\Sigma \int_0^\varepsilon \|v_{x_1}\|^2 \\
&\ge \lambda_{\partial^+} \int_0^\varepsilon \int_\Sigma \|u\|^2 + \frac{2}{\varepsilon^2}\int_\Sigma \int_0^\varepsilon \|v\|^2 \\
&\ge \min\left\{\lambda_{\partial^+}, 2/\varepsilon^2\right\}\left(\int_0^\varepsilon \int_\Sigma \|u\|^2 + \int_\Sigma \int_0^\varepsilon \|v\|^2\right) \\
&= \min\left\{\lambda_{\partial^+}, 2/\varepsilon^2\right\}\|V(x_1)\|^2_{L^2(\mathtt{A}_\varepsilon, \mathbb{S})}.
\end{aligned}$$

Hence the result. ■

For our later application to the perturbation arguments, we need the eigenvalue estimate for a warped product metric $g_{\mathtt{A}_\varepsilon,h} = h(x)\,dx_1^2 + g_\Sigma$. on $\mathtt{A}_\varepsilon$. Note that such a metric is always conformally equivalent to a product metric. We recall that if $\mathcal{D}_g$ is the Dirac operator on a Riemannian spin manifold $(\mathtt{A}, g)$ with metric g then the conformal change of the metric $g \to hg$ by any positive function $h \in C^\infty(\mathtt{A})$ will lead to the change of Dirac operator as the following

$$\mathcal{D}_{hg} = h^{-\frac{n+1}{4}} \circ \mathcal{D}_g \circ h^{\frac{n-1}{4}},$$

where n is the dimension of $\mathtt{A}$. If we compare the Rayleigh quotient we find

$$\frac{1}{K} \inf_{V\in\mathbb{S}} \frac{\int_X \|\mathcal{D}_{hg}V\|_{hg}^2}{\int_X \|V\|_{hg}} \le \inf_{V\in\mathbb{S}} \frac{\int_X \|\mathcal{D}_g V\|_g^2}{\int_X \|V\|_g} \le K \inf_{V\in\mathbb{S}} \frac{\int_X \|\mathcal{D}_{hg}V\|_{hg}^2}{\int_X \|V\|_{hg}},$$

where $K > 0$ is a constant depending only on $\min_{x\in\mathtt{A}} h(x)$ and $\max_{x\in\mathtt{A}} h(x)$. In particular, this implies

$$\frac{1}{K}\lambda_{\mathcal{D}_g} \le \lambda_{\mathcal{D}_{hg}} \le K\lambda_{\mathcal{D}_g}.$$

This allows us to extend the above theorem to any product three manifold $\mathtt{A}_\varepsilon = [0,\varepsilon] \times \Sigma$ with a *warped* product metric,

$$g_{\mathtt{A}_\varepsilon,h} = h(x)\,dx_1^2 + g_\Sigma.$$

where g_Σ is a metric on Σ and h is a smooth positive function on Σ. This is because $g_{\mathtt{A}_\varepsilon,h}$ is conformally equivalent to a product metric $dx_1^2 + h^{-1}g_\Sigma$ with conformal factor $h(x)$.

Suppose $V = (u,v) \in L^{1,2}_-(\mathtt{A}_\varepsilon, \mathbb{S})$ and $W = (f,g) \in L^2(\mathtt{A}_\varepsilon, \mathbb{S})$ and $\mathcal{D}$ is the Dirac operator with respect to the metric $g_{\mathtt{A}_\varepsilon,h} = h(x)\,dx_1^2 + g_\Sigma$ then

$$\begin{aligned}
&\int_0^\varepsilon \int_\Sigma \langle \mathcal{D}V, W\rangle \\
&= \int_0^\varepsilon \int_\Sigma \left\langle \mathbf{i}\left(h^{1/2}u_{x_1} + \partial^+ v\right), f\right\rangle - \left\langle \mathbf{i}\left(h^{1/2}v_{x_1} + \partial^- u\right), g\right\rangle \\
&= \mathbf{i}\int_0^\varepsilon \int_\Sigma \left(-\left\langle u, h^{1/2}f_{x_1}\right\rangle + \langle v, \partial^- f\rangle - \langle u, \partial^+ g\rangle + \left\langle v, h^{1/2}g_{x_1}\right\rangle\right) + \mathbf{i}\int_\Sigma \left(\left\langle u, h^{1/2}f\right\rangle|_0^\varepsilon - \left\langle v, h^{1/2}g\right\rangle|_0^\varepsilon\right) \\
&= \mathbf{i}\int_0^\varepsilon \int_\Sigma \left\langle u, -h^{1/2}f_{x_1} - \partial^+ g\right\rangle + \left\langle v, h^{1/2}g_{x_1} + \partial^- f\right\rangle + \mathbf{i}\int_\Sigma h^{1/2}\left(\langle u, f\rangle|_0^\varepsilon - \langle v, g\rangle|_0^\varepsilon\right) \\
&= \int_0^\varepsilon \int_\Sigma \left\langle u, \mathbf{i}\left(h^{1/2}f_{x_1} + \partial^+ g\right)\right\rangle - \left\langle v, \mathbf{i}\left(h^{1/2}g_{x_1} + \partial^- f\right)\right\rangle + \mathbf{i}\int_\Sigma h^{1/2}\left(\langle u, f\rangle|_0^\varepsilon - \langle v, g\rangle|_0^\varepsilon\right) \\
&= \int_0^\varepsilon \int_\Sigma \langle V, \mathcal{D}W\rangle + \mathbf{i}\int_\Sigma h^{1/2}\left(\langle u, f\rangle|_0^\varepsilon - \langle v, g\rangle|_0^\varepsilon\right)
\end{aligned}$$

When $V \in L^{1,2}_-(\mathtt{A}_\varepsilon, \mathbb{S})$ and $W \in L^{1,2}_+(\mathtt{A}_\varepsilon, \mathbb{S})$, the above boundary terms are zero and we have

$$\int_0^\varepsilon \int_\Sigma \langle \mathcal{D}V, W\rangle = \int_0^\varepsilon \int_\Sigma \langle V, \mathcal{D}W\rangle.$$

This implies that $\mathcal{D}$ is self adjoint operator from $L^{1,2}_{\pm}(\mathtt{A}_\varepsilon, \mathbb{S})$ to $L^{1,2}_{\mp}(\mathtt{A}_\varepsilon, \mathbb{S})$ in the sense of [5]. Since $L^{1,2}_{+}(\mathtt{A}_\varepsilon, \mathbb{S})$ is dense in $L^2(\mathtt{A}_\varepsilon, \mathbb{S})$, this implies that $\mathcal{D} : L^{1,2}_{-}(\mathtt{A}_\varepsilon, \mathbb{S}) \to L^2(\mathtt{A}_\varepsilon, \mathbb{S})$ is surjective if and only if $\ker \mathcal{D}|_{L^{1,2}_{+}(\mathtt{A}_\varepsilon, \mathbb{S})} = 0 \subset L^{1,2}_{+}(\mathtt{A}_\varepsilon, \mathbb{S})$, which is equivalent to $\ker \partial^- = 0$ by the above arguments. Hence we have obtained the following result.

Theorem 11 *Suppose that the first eigenvalue for $\partial^+\partial^-$ and $\partial^-\partial^+$ are both strictly positive. Then*

$$\mathcal{D} : L^{1,2}_{-}(\mathtt{A}_\varepsilon, \mathbb{S}) \to L^2(\mathtt{A}_\varepsilon, \mathbb{S})$$

is one-to-one and onto.

4.4 Estimates for the linearized problem

In this section we will develop the necessary linear theory for the Dirac equation

$$\mathcal{D}V = W \text{ on } \mathtt{A}_\varepsilon = \Sigma \times [0, \varepsilon]$$

with a warped product metric $g_{\mathtt{A}_\varepsilon, h} := h(x)\, dx_1^2 + g_\Sigma$. The key issue is to obtain a priori estimates for V with explicit dependence of ε, as ε goes to zero. When ε is away from zero, say $\varepsilon \in [1/2, 3/2]$, we have ε-free Schauder estimates. For ε small, we overcome the difficulty coming from ε by choosing an appropriate integer k so that $k\varepsilon \in [1/2, 3/2]$ and we extend any solution $V = (u, v)$ on $\mathtt{A}_\varepsilon$ to $\mathtt{A}_{k\varepsilon}$ in weak sense by reflection suitably. However much care will be needed to obtain the C^α-estimate, because after the reflection of W across the boundary of $\mathtt{A}_\varepsilon$, it will no longer be continuous in general. This problem will be resolved in the case (ii) part of the proof of the following theorem. To make the exposition more transparent we will assume that $h \equiv 1$, and it is clear from the proof below that the argument works equally well for any $h(x) \in C^\infty(\Sigma)$.

Theorem 12 *For any $0 < \alpha < 1$ and $p > 3$ there is a positive constant $C = C(\alpha, p, \lambda)$ independent of ε such that for any $V \in C^\infty_-(\mathtt{A}_\varepsilon, \mathbb{S})$ and $W \in C^\infty(\mathtt{A}_\varepsilon, \mathbb{S})$ satisfying*

$$\mathcal{D}V = W \text{ on } \mathtt{A}_\varepsilon = \Sigma \times [0, \varepsilon].$$

We have

$$C \|V\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon, \mathbb{S})} \le \varepsilon^{-\left(\frac{3}{p} + \alpha\right)} \|W\|_{C^\alpha(\mathtt{A}_\varepsilon, \mathbb{S})}.$$

Proof. We write $V = (u, v) \in C^\infty_-(\mathtt{A}_\varepsilon, \mathbb{S})$, $W = (w_1, w_2) \in C^\infty(\mathtt{A}_\varepsilon, \mathbb{S})$ and the metric on $\mathtt{A}_\varepsilon$ as

$$g_{\mathtt{A}_\varepsilon, h} := h(x)\, dx_1^2 + g_\Sigma$$

so the equation may be written as

$$\begin{cases} h^{1/2} u_{x_1} + \partial^+ v &= w_1 \\ h^{1/2} v_{x_1} + \partial^- u &= w_2 \end{cases} \quad \text{with } v|_{\partial \mathtt{A}_\varepsilon} = 0.$$

In the following we assume $h \equiv 1$. Let us fix an integer $k > 0$ such that $k\varepsilon \in [1/2, 3/2]$ and divide the estimates into two cases:

Case (i): Suppose that $w_1 = 0$, then we will have along the boundary $\partial \mathtt{A}_\varepsilon$, $u_{x_1} = 0$ since $v = 0$. We extend v from $\mathtt{A}_\varepsilon$ to $\mathtt{A}_{k\varepsilon}$ by odd reflection along the walls $\Sigma \times \{j\varepsilon\}$ with $1 \leq j \leq k-1$. Similarly we consider an even extension of u to $\mathtt{A}_{k\varepsilon}$. That is,

$$v(x,z) = \begin{cases} -v\left((2j+2)\varepsilon - x\right) & \text{for } x \in [(2j+1)\varepsilon, (2j+2)\varepsilon] \\ v(x - 2j\varepsilon) & \text{for } x \in [2j\varepsilon, (2j+1)\varepsilon] \end{cases}$$

$$u(x,z) = \begin{cases} u\left((2j+2)\varepsilon - x\right) & \text{for } x \in [(2j+1)\varepsilon, (2j+2)\varepsilon] \\ u(x - 2j\varepsilon) & \text{for } x \in [2j\varepsilon, (2j+1)\varepsilon] \end{cases}$$

This will induce an even extension of w_2 so that the Dirac equation

$$\mathcal{D}V = W$$

is satisfied in the weak sense on $\mathtt{A}_{k\varepsilon}$. By differentiating both sides of the equation $v_{x_1} + \partial^- u = w_2$ with respect to x_1, we obtain an equation which is equivalent to the Dirichlet problem of the second order elliptic equation

$$v_{x_1x_1} - \partial^-\partial^+ v = \frac{\partial w_2}{\partial x_1} \text{ and } v|_{\partial \mathtt{A}_{k\varepsilon}} = 0$$

Note that $\partial^-\partial^+$ is a positive operator. Since the C^α-norm is preserved under the odd extension, Schauder estimate and L^p-estimate for the second order elliptic equation would then imply that there are constants $C(\alpha)$ and $\tilde{C}(p)$ independent of ε such that

$$\begin{aligned} & \|w_2\|_{C^\alpha(\mathtt{A}_\varepsilon, \mathbb{S}^-)} + \|V\|_{C^0(\mathtt{A}_\varepsilon, \mathbb{S})} \\ & = \|w_2\|_{C^\alpha(\mathtt{A}_{k\varepsilon}, \mathbb{S}^-)} + \|V\|_{C^0(\mathtt{A}_{k\varepsilon}, \mathbb{S})} \\ & \geq C(\alpha) \|V\|_{C^{1,\alpha}(\mathtt{A}_{k\varepsilon}, \mathbb{S})} \\ & = C(\alpha) \|V\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon, \mathbb{S})} \end{aligned}$$

and

$$\|w_2\|_{L^p(\mathtt{A}_{k\varepsilon}, \mathbb{S}^-)} + \|V\|_{L^p(\mathtt{A}_{k\varepsilon}, \mathbb{S})} \geq \tilde{C}(p) \|V\|_{L^{1,p}_-(\mathtt{A}_{k\varepsilon}, \mathbb{S})}$$

since $k\varepsilon \in [1/2, 3/2]$.

Case (ii): suppose that $w_2 = 0$ and $w_1 \in C_0^\alpha(\mathtt{A}_\varepsilon, \mathbb{S}^+)$. Since $v = 0$, this implies that if we consider the odd extension of v and the even extension of u to $\mathtt{A}_{k\varepsilon}$, as in the previous case, then they induce an odd extension of w_1 so that the equation

$$\begin{cases} u_{x_1} + \partial^+ v &= w_1 \\ v_{x_1} + \partial^- u &= 0 \end{cases}$$

is satisfied in the weak sense on $\mathtt{A}_{k\varepsilon}$. Notice that w_1 does not vanish on $\partial \mathtt{A}_\varepsilon$ in general, so after the odd extension, w_1 is no longer continuous but still we have

$w_1 \in L^p(\mathtt{A}_{k\varepsilon}, \mathbb{S})$ for $\forall p$. The L^p-estimate for the second order elliptic equation then implies that there is a constant $\tilde{C}(p)$ independent of ε such that

$$\|w_1\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S}^-)} + \|V\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})} \geq \tilde{C}(p)\|V\|_{L^{1,p}_-(\mathtt{A}_{k\varepsilon},\mathbb{S})}. \tag{1}$$

Differentiate the second equation with respect to x_1, we obtain that the equation is equivalent to the Dirichlet problem of the second order elliptic equation

$$v_{x_1x_1} - \partial^-\partial^+ v = -\partial^- w_1 \text{ and } v|_{\partial \mathtt{A}_{k\varepsilon}} = 0.$$

The Schauder estimate for second order elliptic equation (c.f. [10] section 4.4 and 6.4) implies that there is a constant $C(\alpha)$ independent of ε such that

$$\varepsilon\|w_1\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S}^+)} + \|v\|_{C^0_-(\mathtt{A}_\varepsilon,\mathbb{S})} \geq C(\alpha)\varepsilon^{1+\alpha}\|v\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon,\mathbb{S})}. \tag{2}$$

It follows from (1), the eigenvalue estimate of the previous subsection

$$\|W\|_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})} \geq C(\lambda)\|V\|_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})} \text{ for } V \in C^\infty_-(\mathtt{A}_\varepsilon,\mathbb{S})$$

and the interpolation inequality

$$\|V\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})} \leq \delta\|V\|_{L^{1,p}(\mathtt{A}_{k\varepsilon},\mathbb{S})} + \frac{1}{\delta}\|V\|_{L^2(\mathtt{A}_{k\varepsilon},\mathbb{S})} \text{ with } \delta = \tilde{C}(p)/2$$

that

$$\begin{aligned}
\tilde{C}(p)\|V\|_{L^{1,p}_-(\mathtt{A}_{k\varepsilon},\mathbb{S})} &\leq \|W\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})} + \frac{1}{\delta}\|V\|_{L^2(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\
&\leq \|W\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})} + \frac{1}{C(\lambda,p)}\|W\|_{L^2(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\
&\leq \|W\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})}\left(1 + \frac{1}{C(\lambda,p)}\right).
\end{aligned}$$

So

$$\|V\|_{L^{1,p}_-(\mathtt{A}_{k\varepsilon},\mathbb{S})} \leq \tilde{C}(p)^{-1}\left(1 + \frac{1}{\delta C(\lambda)}\right)\|W\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})}$$

in particular, there are constant C independent of ε such that

$$\begin{aligned}
\|V\|_{C^{1-n/p}_-(\mathtt{A}_\varepsilon,\mathbb{S})} &= \|V\|_{C^{1-n/p}_-(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\
&\leq C\|V\|_{L^{1,p}_-(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\
&\leq \tilde{C}(p)^{-1}\left(1 + \frac{1}{C(\lambda,p)}\right)\|W\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\
&\leq C(p,\lambda)\|W\|_{C^0(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\
&= C(p,\lambda)\|W\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})}.
\end{aligned}$$

By combining this with the fact that $v|_{\partial \mathtt{A}_\varepsilon =0}$, we have

$$\|v\|_{C^0_-(\mathtt{A}_\varepsilon,\mathbb{S})} \leq \|v\|_{C^{1-n/p}_-(\mathtt{A}_\varepsilon,\mathbb{S})} \varepsilon^{1-n/p} \leq C(p,\lambda)\|w_1\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})} \varepsilon^{1-\frac{n}{p}}.$$

Plug these into (2) we obtain

$$C(\alpha)\|v\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon,\mathbb{S})} \leq \varepsilon^{-\alpha}\|w_1\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S}^+)} + C(p,\lambda)\|w_1\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})} \varepsilon^{-\left(\frac{n}{p}+\alpha\right)}.$$

We can also obtain a similar estimate for u because $u = (\partial^-)^{-1} v_{x_1}$ and $(\partial^-)^{-1}$ is independent on ε. Thus we have

$$C(\alpha)\|V\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon,\mathbb{S})} \leq \varepsilon^{-\alpha}\|w_1\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S}^+)} + C(p,\lambda)\|w_1\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})} \varepsilon^{-\left(\frac{n}{p}+\alpha\right)}.$$

By combining cases (i) and (ii) together and let $n=3$, we obtain

$$C(\alpha,p,\lambda)\|V\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon,\mathbb{S})} \leq \varepsilon^{-\left(\frac{3}{p}+\alpha\right)}\|W\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})}.$$

Hence the result. ■

Corollary 13 *Let $W \in C^\infty(\mathtt{A}_\varepsilon,\mathbb{S})$ and suppose $V \in C^\infty_-(\mathtt{A}_\varepsilon,\mathbb{S})$ solves*

$$\mathcal{D}V = W \text{ on } \mathtt{A}_\varepsilon = \Sigma\times[0,\varepsilon] \text{ with } h \in C^\infty(\Sigma).$$

Then for any $0<\alpha<1, p>3$ there is a positive constant $C = C(\alpha,p,\lambda,h)$ independent of ε such that

$$C\|V\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon,\mathbb{S})} \leq \varepsilon^{-\left(\frac{3}{p}+\alpha\right)}\|W\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})}.$$

4.5 Perturbation arguments

Let $C_0 \subset M$ be a coassociative submanifold. Suppose that n is a normal vector field on C_0 such that its corresponding self-dual two form, $\eta_0 = \iota_n\Omega \in \wedge^2_+(C_0)$ is harmonic with respect to the induced metric. So η_0 is actually a symplectic form on the complement of the zero set $Z(\eta_0)$ of η_0 in C_0. Furthermore

$$J_n(u) := |n|^{-1} n\times u$$

defines an almost complex structure J_n on the $C\backslash Z(\eta_0)$. Since deformations of coassociative submanifolds are unobstructed, we may assume that there is an one parameter family of coassociative submanifolds $\varphi : [0,\varepsilon]\times C_0 \longrightarrow M$ which corresponds to integrating out the normal vector field n, that is

$$\left.\frac{\partial\varphi}{\partial t}\right|_{t=0} = n \in \Gamma\left(C_0, N_{C_0/M}\right) \text{ and } C_t := \varphi(\{t\}\times C_0)$$

In the remaining part of this article we assume that η_0 is nowhere vanishing on C_0, that is (C_0,η_0) is a symplectic four manifold. We are going to establish a correspondence between the regular J_n-holomorphic curves Σ in C_0 and the existence of instantons with coassociative boundary conditions.

Given such a $\Sigma \subset C_0$, we obtain a family $\Sigma_t \subset C_t$ whose total space A'_ε is close to be associative and the induced metric on A'_ε is close to be a warped product metric for small ε. We want to perturb A'_ε to become an honest associative submanifold in M. In order to apply the implicit function theorem to obtain the desired perturbation for A'_ε, we need the estimates for the linearized problem to behave well as ε approach zero. Such an estimate was established in the previous section.

To prove this result, we will construct a map

$$F_\varepsilon : C^{m,\alpha}_{-}(\mathtt{A}_\varepsilon, \mathbb{S}) \to C^{m-1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})$$

such that the solution to the equation $F_\varepsilon(V) = 0$ will give rise to an associative submanifold with boundary lying on $C_0 \cup C_\varepsilon$. Let us briefly describe the seven steps construction of F_ε here: We construct a three dimensional submanifold $A'_\varepsilon \subset M$ by flowing Σ along with C_t and an identification between the normal bundle of $A'_\varepsilon \subset M$ with the spinor bundle $\mathbb{S} \to \mathtt{A}_\varepsilon$, this makes the linear theory developed in the previous subsection applicable. Then we need to define an exponential map on $C^{m,\alpha}_{-}(\mathtt{A}_\varepsilon, \mathbb{S})$ carefully so that $\exp V$ always satisfies the coassociative boundary condition. To do that, we need to deal with those normal directions to $\mathtt{A}_\varepsilon$ inside $\mathcal{C} = \varphi([0,\varepsilon] \times C_0)$ and perpendicular to $\mathcal{C}$ separately.

Now we are ready to describe our construction:

1. For ε small, let $\mathtt{C} := [0,\varepsilon] \times C_0$ and $\mathcal{C} := \varphi(\mathtt{C})$ is diffeomorphic to $\mathtt{C}$, and all coassociative submanifolds C_t's are mutually disjoint and $\varphi(t,\cdot)$ is an embedding for $\forall t \in [0,\varepsilon]$.

2. Let $\Sigma \subset C_0$ be a J_n-holomorphic curve, we denote

$$\mathtt{A}_\varepsilon := [0,\varepsilon] \times \Sigma \text{ and } A'_\varepsilon := \varphi(\mathtt{A}_\varepsilon)$$

then A'_ε is close to be associative in the sense that $\left|\tau|_{A'_\varepsilon}\right| \leq K\varepsilon$ for some constant K depending on the geometry of the family $\{C_t\}$ and M for small ε. Notice that if we identify $N_{\Sigma/C} \otimes \mathbb{C}$ with $\mathbb{S}^+ \otimes \mathbb{C}$ the complexified positive spinor bundle over Σ, then $N_{\mathcal{C}/M} \otimes \mathbb{C}|_\Sigma = \mathbb{S}^- \otimes \mathbb{C} = K_\Sigma \otimes \left(N^{\mathbb{C}}_{\Sigma/C}\right)^*$, where K_Σ is the canonical line bundle of Σ. This follows from the fact that M is a Riemannian manifold with G_2-holonomy, then at every point $x \in C_0$ we may canonically identify T_xM with $T_xC_0 + \wedge^2_+ T^*_x C_0$. For $\forall x \in \Sigma \subset C_0$, we may choose $\{e_i\}_{i=1}^4$ to be an orthonormal basis of T_xC_0 such that

$$T^{\mathbb{C}}_x \Sigma = \operatorname{span}_{\mathbb{C}} \{e_1 - ie_2\} \text{ and } N^{\mathbb{C}}_{\Sigma/C}|_x = \operatorname{span}_{\mathbb{C}} \{e_3 - ie_4\}$$

then

$$\begin{aligned}
& N^{\mathbb{C}}_{\mathcal{C}/M}|_{x\in\Sigma} \\
& = \operatorname{span}_{\mathbb{C}}\left\{(e_1^*\wedge e_3^* - e_2^*\wedge e_4^*) + i\,(e_1^*\wedge e_4^* + e_2^*\wedge e_3^*)\right\} \\
& = \operatorname{span}_{\mathbb{C}}\left\{(e_1^* + ie_2^*)\wedge(e_3^* + ie_4^*)\right\} \\
& = K_\Sigma \otimes \left(N^{\mathbb{C}}_{\Sigma/C}\right)^* .
\end{aligned}$$

In particular,

$$H^0\left(\Sigma, N^{\mathbb{C}}_{\mathcal{C}/M}|_\Sigma\right) = H^0\left(\Sigma, K_\Sigma\otimes\left(N^{\mathbb{C}}_{\Sigma/C}\right)^*\right) = H^1\left(\Sigma, N^{\mathbb{C}}_{\Sigma/C}\right)^*$$

since the dimension for the Seiberg-Witten moduli is 0. This implies that $\dim H^0\left(\Sigma, N^{\mathbb{C}}_{\Sigma/C}\right) = \dim H^1\left(\Sigma, N^{\mathbb{C}}_{\Sigma/C}\right) = 0$, by our assumption that Σ is regular. Moreover by Theorem 11, we have the linear operator

$$\mathcal{D} : L^{1,2}_{-}\left(\mathtt{A}_\varepsilon, \mathbb{S}\right) \to L^2\left(\mathtt{A}_\varepsilon, \mathbb{S}\right)$$

is one-to-one and onto.

3. Let
$$g_{\mathtt{A}_\varepsilon,h} := h\,(x)\,dt^2 + g_\Sigma$$
be the *warped* product metric on $[0,\varepsilon]\times\Sigma$ with g_Σ being the induced metric on Σ and $h\,(x)$ is the squared length of $n = dC_t/dt|_{t=0}$ restricted to Σ. Then
$$(1-K\varepsilon)\,g_{\mathtt{A}_\varepsilon,h} \le \varphi^* g_M \le (1+K\varepsilon)\,g_{\mathtt{A}_\varepsilon,h}$$
for some constant K depending on the geometry of family $\{C_t\}_{0\le t\le\varepsilon}$, η_0 and M.

4. Let $g_{\mathtt{C}} := dt^2 \oplus g|_{C_0}$ be the product metric on $\mathcal{C}$ and $\exp^{\mathtt{C}}$ is the exponential map associated to the metric $g_{\mathtt{C}}$.

5. Using the metric g_M on M we have an orthogonal decomposition $\varphi^* N_{A'_\varepsilon/M} = \varphi^* N_{A'_\varepsilon/\mathcal{C}} \oplus \varphi^* N_{\mathcal{C}/M}$. We define a vector bundle $\mathbb{S}$ on $\mathtt{A}_\varepsilon$ as the pullback of the bundle $\varphi^* N_{A'_\varepsilon/M}|_{\{0\}\times\Sigma}$ by the projection map $\mathtt{A}_\varepsilon \xrightarrow{\pi} \Sigma$. Thus we obtain a Cartesian product,
$$\begin{array}{ccc}
\mathbb{S} & \longrightarrow & \varphi^* N_{A'_\varepsilon/M}|_{\{0\}\times\Sigma} \\
\downarrow & & \downarrow \\
\mathtt{A}_\varepsilon & \xrightarrow{\pi} & \Sigma
\end{array}$$
and orthogonal decomposition
$$\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^- = \pi^*\left(\varphi^* N_{A'_\varepsilon/\mathcal{C}}|_{\{0\}\times\Sigma}\right) + \pi^*\left(\varphi^* N_{\mathcal{C}/M}|_{\{0\}\times\Sigma}\right).$$

6. Let $N_{\mathtt{A}_\varepsilon/\mathtt{C}}$ be the normal bundle of $\mathtt{A}_\varepsilon \subset \mathtt{C}$ with respect to the metric $g_{\mathtt{C}}$. For ε small, we define a bundle isomorphism

$$\begin{array}{ccc} \mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^- & \overset{I=\left(I^+,I^-\right)}{\longrightarrow} & N_{\mathtt{A}_\varepsilon/\mathtt{C}} \oplus \varphi^* N_{\mathcal{C}/M} \\ \downarrow & & \downarrow \\ \mathtt{A}_\varepsilon & \overset{id}{\longrightarrow} & \mathtt{A}_\varepsilon \end{array}.$$

Fix a $0 \le t \le \varepsilon$, for $u \in \mathbb{S}^+|_{\{t\}\times\Sigma\subset\mathtt{A}_\varepsilon}$ and $v \in \mathbb{S}^-|_{\{t\}\times\Sigma\subset\mathtt{A}_\varepsilon}$, we may treat u as a section of $\varphi^* N_{A'_\varepsilon/\mathcal{C}}|_{\{0\}\times\Sigma}$ and v as a section of $\varphi^* N_{\mathcal{C}/M}|_{\{0\}\times\Sigma}$. Then $I^+(u) \in N_{\mathtt{A}_\varepsilon/\mathtt{C}}|_{\{t\}\times\Sigma}$ is obtained by parallel transport along $[0,t]\times\{x\} \subset \mathtt{C}$ and then orthogonally project to $N_{\mathtt{A}_\varepsilon/\mathtt{C}}$ with respect to the metric $g_{\mathtt{C}}$, and $I^-(v)$ is obtained by parallel transport along $[0,t]\times\{x\} \subset \mathtt{C}$ and orthogonally project to $\varphi^* N_{\mathcal{C}/M}$ with respect to the pull back metric $\varphi^* g$.

7. We introduce

$$\begin{array}{ccc} \widetilde{\exp} : \Gamma(\mathtt{A}_\varepsilon, \mathbb{S}) & \longrightarrow & Map(\mathtt{A}_\varepsilon, M) \\ V = (u,v) & & \exp^M_{\exp^{\mathtt{C}} I^+(u)} \left(T_{\exp^{\mathtt{C}} I^+(u)} I^-(v)\right) \end{array}$$

where $\exp^M$ is the exponential map with respect to the metric g on M, $\exp^{\mathtt{C}}$ is the exponential map with respect to the metric $g_{\mathtt{C}}$, and

$$T_{\exp^{\mathtt{C}} I^+(u)} : N_{\mathcal{C}/M}|_{r_u(0,x)} \longrightarrow N_{\mathcal{C}/M}|_{r_u(1,x)}$$

is the parallel transport with respect to the metric g on M along the curve $\gamma_u(s,x) := \exp^{\mathtt{C}}_x sI^+(u) \subset \mathtt{C}$, for $x \in \mathtt{A}_\varepsilon \subset \mathtt{C}$. We will denote the image of $\widetilde{\exp} V$ by $\mathtt{A}_\varepsilon(V) \subset M$. It follows from our construction that for any $V \in C^m_-(\mathtt{A}_\varepsilon, \mathbb{S})$, $\partial\mathtt{A}_\varepsilon(V) \subset C_0 \cup C_\varepsilon$ and $\widetilde{\exp}(0,0) = \varphi$.

Next we define the nonlinear map F_ε with the important property that elements in $F_\varepsilon^{-1}(0)$ with small norm correspond to associative submanifolds in M near A'_ε for small ε.

$$\begin{array}{c} F_\varepsilon : C^{m,\alpha}_-(\mathtt{A}_\varepsilon, \mathbb{S}) \to C^{m-1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S}) \\ F_\varepsilon(V) = \tilde{T}_V \left(*_{\mathtt{A}_\varepsilon(V)} \left(\widetilde{\exp} V\right)^* \tau\right)^\perp \end{array}$$

where $V = (u,v)$ and the map $\tilde{T}_V : \left(\widetilde{\exp} V\right)^* N_{\mathtt{A}_\varepsilon(V)} \to \mathbb{S}$ is obtained as following

- Parallel transport with respect to the metric g on M along $\gamma^M(s) := \exp^M_{\exp^{\mathtt{C}} I^+(u)} s\left(T_{\exp^{\mathtt{C}} I^+(u)} I^-(v)\right)$ with $s \in [0,1]$.
- Identifying it as a section of $N_{\mathtt{A}_\varepsilon/\mathtt{C}} \oplus \varphi^* N_{\mathcal{C}/M}$, which in turn can be identified as a section of $\mathbb{S}$ via

$$N_{\mathtt{A}_\varepsilon/\mathtt{C}} \oplus \varphi^* N_{\mathcal{C}/M} \overset{I^{-1}}{\longrightarrow} \mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-.$$

defined in the step 6. In particular, $\tilde{T}_0$ is I^{-1}.

Proposition 14 *For any $0 < \alpha < 1, p > 3$ and $R > 0$, there is a positive constant $C = C(\alpha, p, \lambda, R)$ such that for any sufficiently small $\varepsilon > 0$ we have*

$$\|(DF_\varepsilon(V) - DF_\varepsilon(W))\,\delta V\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})} \le C\,\|\delta V\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon,\mathbb{S})}$$

and

$$\|DF_\varepsilon(0)\,\delta V\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})} \ge C\varepsilon^{\left(\frac{3}{p}+\alpha\right)}\,\|\delta V\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon,\mathbb{S})}$$

for any $V \in C^{1,\alpha}_-(\mathtt{A}_\varepsilon,\mathbb{S})$, $W \in C^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})$ with $\|V\|_{C^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})}, \|W\|_{C^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})} \le R$ and for any $\delta V \in C^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})$.

Proof. First, we notice that

$$\begin{aligned} DF_\varepsilon(V)(\delta V) &= \left\{D\tilde{T}_V(\delta V) + \tilde{T}_{(V)}\left(\operatorname{div}_{\mathtt{A}_\varepsilon(V)}(\delta V)\right)\right\}\left(*_{\mathtt{A}_\varepsilon(V)}\left(\widetilde{\exp}V\right)^*\tau\right) \\ &\quad + \tilde{T}_V\left(*_{\mathtt{A}_\varepsilon(V)}\left(D\widetilde{\exp}V\right)^*\tau\right) \end{aligned}$$

and

$$\begin{aligned} & D\widetilde{\exp}_{(u,v)}(\delta u, \delta v) \\ &= \left.\frac{d}{ds}\right|_{s=0} \exp^M_{\exp^{\mathsf{c}} I^+(u+s\delta u)}\left(T_{\exp^{\mathsf{c}} I^+(u+s\delta u)} I^-(v + s\delta v)\right) \\ &= \Big\{\left(D_1 \exp^M\right)_{\exp^{\mathsf{c}} I^+(u)}\left(T_{\exp^{\mathsf{c}} I^+(u)} I^-(v)\right) \\ &+ \left(D_2 \exp^M\right)_{\exp^{\mathsf{c}} I^+(u)}\left(T_{\exp^{\mathsf{c}} I^+(u)} I^-(v)\right) D_1 T_{\exp^{\mathsf{c}} I^+(u)}\Big\}\, D\exp^{\mathsf{C}} I^+(\delta u) \\ &+ \left(D_2 \exp^M\right)_{\exp^{\mathsf{c}} I^+(u)}\left(T_{\exp^{\mathsf{c}} I^+(u)} I^-(v)\right) T_{\exp^{\mathsf{c}} I^+(u)} I^-(\delta v), \end{aligned}$$

so there are smooth functions $G_i, i = 1, 2, 3$ depend on $\mathcal{C}$ and M such that

$$DF_\varepsilon(V)\,\delta V = G_1(V, \nabla V)\,\delta V + G_2(V, \nabla V)\,\nabla\delta u + G_3(V, \nabla V)\,\nabla\delta v.$$

Since

$$\begin{aligned} & G_i(V, \nabla V) - G_i(W, \nabla W) \\ &= (G_i(V, \nabla V) - G_i(V, \nabla W)) + (G_i(V, \nabla W) - G_i(W, \nabla W)), \end{aligned}$$

there is a constant $C(R)$ only depend on the geometry of $\mathcal{C}$ and M but independent of ε such that for $\|V\|_{C^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})}, \|W\|_{C^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})} \le R$, we have

$$\|(DF_\varepsilon(V) - DF_\varepsilon(W))\,\delta V\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})} \le C(R)\,\|V - W\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon,\mathbb{S})}\,\|\delta V\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon,\mathbb{S})}.$$

Second, by setting $V = 0$, we have

$$DF_\varepsilon(0)\,\delta V = \left\{D\tilde{T}_{(0)}(\delta V) + \left(\operatorname{div}_{\mathtt{A}_\varepsilon(0)}(\delta V)\right)\right\} *_{\mathtt{A}_\varepsilon} \varphi^*\tau + *_{\mathtt{A}_\varepsilon}\left(D\widetilde{\exp}|_0\right)^*\tau$$

and

$$\begin{aligned}
&D\widetilde{\exp}_{(0,0)}(\delta u, \delta v) \\
&= \left.\frac{d}{ds}\right|_{s=0} \exp^{M}_{\exp^{\mathtt{C}} I^{+}(s\delta u)}\left(T_{\exp^{\mathtt{C}} I^{+}(s\delta u)} I^{-}(s\delta v)\right) \\
&= D\left(\exp^{\mathtt{C}} \circ I^{+}\right)(\delta u) + d\exp^{M}_{\mathtt{A}_\varepsilon} \circ dI^{-}(\delta v) \\
&= dI^{+}(\delta u) + dI^{-}(\delta v) \in \Gamma\left(\mathtt{A}_\varepsilon, N_{\mathtt{A}_\varepsilon/\mathtt{C}} \oplus \varphi^* N_{\mathcal{C}/M}\right)
\end{aligned}$$

with

$$\begin{aligned}
&dI^{+}(\delta u)|_{\{0\}\times\Sigma} + dI^{-}(\delta v)|_{\{0\}\times\Sigma} \\
&= \delta u + \delta v \in \Gamma\left(\{0\}\times\Sigma, \left(N_{\mathtt{A}_\varepsilon/\mathtt{C}} \oplus \varphi^* N_{\mathcal{C}/M}\right)|_{\{0\}\times\Sigma} = \mathbb{S}\right).
\end{aligned}$$

Now we have

$$\begin{aligned}
\varphi^*\tau &= \tau\left(d\varphi(\cdot), d\varphi(\cdot), d\varphi(\cdot)\right)(x_1, z) \\
&= \tau\left(d\varphi(\cdot), d\varphi(\cdot), d\varphi(\cdot)\right)(0, z) + x_1 E
\end{aligned}$$

with

$$|E(x_1, z; w_1, w_2)| \leq C\left(|x_1| + |w_1| + |w_2|\right).$$

for some constant C only depends on $\mathcal{C}$ and M but independent of ε.Since $DF_\varepsilon(0) = \mathcal{D}_{\mathtt{A}_\varepsilon} + x_1 E\left(x_1, z; \nabla_\Sigma, \partial_{x_1}\right)$ with (x_1, z) being the coordinates of $\mathtt{A}_\varepsilon$, this implies that

$$\left\|\left(DF_\varepsilon(0) - \mathcal{D}_{\mathtt{A}_\varepsilon}\right)\delta V\right\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})} \leq C\varepsilon \left\|\delta V\right\|_{C^{1,\alpha}_{-}(\mathtt{A}_\varepsilon,\mathbb{S})}$$

It follow from Corollary 13 that

$$C(\alpha, p, \lambda, h)\left\|\delta V\right\|_{C^{1,\alpha}_{-}(\mathtt{A}_\varepsilon,\mathbb{S})} \leq \varepsilon^{-\left(\frac{3}{p}+\alpha\right)}\left\|\mathcal{D}_{\mathtt{A}_\varepsilon}\delta V\right\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})}.$$

therefore for ε small we have

$$\begin{aligned}
&\left\|DF_\varepsilon(0)\delta V\right\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})} \\
&\geq \left\|\mathcal{D}_{\mathtt{A}_\varepsilon}\delta V\right\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})} - \left\|\left(DF_\varepsilon(0) - \mathcal{D}_{\mathtt{A}_\varepsilon}\right)\delta V\right\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})} \\
&\geq C(\alpha, p, \lambda, h)\left(\varepsilon^{-\left(\frac{3}{p}+\alpha\right)} + \varepsilon\right)\left\|\delta V\right\|_{C^{1,\alpha}_{-}(\mathtt{A}_\varepsilon,\mathbb{S})} \\
&\geq C(\alpha, p, \lambda, h)\,\varepsilon^{-\left(\frac{3}{p}+\alpha\right)}\left\|\delta V\right\|_{C^{1,\alpha}_{-}(\mathtt{A}_\varepsilon,\mathbb{S})},
\end{aligned}$$

hence the proposition. ■

To find the zeros of F_ε, we are going to apply the following quantitative version of the implicit function theorem (c.f. Theorem 15.6 [6]).

Theorem 15 *Let X and Y be Banach space and $F: B_r(x_0) \subset X \to Y$ a C^1-map, such that*

1. $(DF(x_0))^{-1}$ *is a bounded linear operator with* $\left|(DF(x_0))^{-1} F(x_0)\right| \leq \alpha$ *and* $\left|(DF(x_0))^{-1}\right| \leq \beta$;
2. $|DF(x_1) - DF(x_2)| \leq \kappa |x_1 - x_2|$ *for all* $x_1, x_2 \in B_r(x_0)$;
3. $2\kappa\alpha\beta < 1$ *and* $2\alpha < r$.

   Then F *has a unique zero in* $B_{2\alpha}(x_0)$.

To apply above theorem, we define the map

$$\tilde{F}_\varepsilon := \varepsilon^{-\left(\frac{3}{p}+\alpha\right)} F_\varepsilon : C^{1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S}) \cap C_0^\infty(\mathtt{A}_\varepsilon, \mathbb{S}) \quad \longrightarrow \quad C^{0,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})$$

then for any $\delta V \in C^{1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S}) \cap C_0^\infty(\mathtt{A}_\varepsilon, \mathbb{S})$,

$$\left\| D\tilde{F}_\varepsilon(0)\, \delta V \right\|_{C^{0,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})} \geq C(\alpha, p) \left\| \delta V \right\|_{C_-^{1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})}$$

and for $\|V\|_{C^{1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})}, \|W\|_{C^{1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})} \leq R$ we have

$$\left\| \left(D\tilde{F}_\varepsilon(V) - D\tilde{F}_\varepsilon(W) \right) \delta V \right\|_{C^\alpha(\mathtt{A}_\varepsilon, \mathbb{S})} \leq C\varepsilon^{-\left(\frac{3}{p}+\alpha\right)} \|\delta V\|_{C_-^{1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})}$$

Now

$$\left\| \tilde{F}_\varepsilon(u, v) \right\|_{C^{0,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})} \leq C\varepsilon^{-\left(\frac{3}{p}+\alpha\right)} \varepsilon = C\varepsilon^{1-\left(\frac{3}{p}+\alpha\right)}$$

When we choose $\alpha = 1/4$ and $p > 12$ then we have

$$\begin{aligned} & \left\| D\tilde{F}_\varepsilon(0) \right\| \left\| \tilde{F}_\varepsilon(u, v) \right\|_{C^{0,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})} \varepsilon^{-\left(\frac{3}{p}+\alpha\right)} \\ & \sim \varepsilon^{1-2\left(\frac{3}{p}+\alpha\right)} \\ & = \varepsilon^{1/2} \longrightarrow 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

Theorem 15 then implies that there is a unique $\|V_\varepsilon\|_{C^{1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})} \leq 2\varepsilon^{1-2\left(\frac{3}{p}+\alpha\right)}$ solves $F_\varepsilon(V_\varepsilon) = 0$.

Claim 16 *For* ε *small* $\mathtt{A}_\varepsilon(V_\varepsilon) := \widetilde{\exp} V_\varepsilon$ *is an instanton.*

Proof. First we notice that by our construction the tangent space $T\mathtt{A}_\varepsilon$ is already ε-away from being associative. The estimate $\|V_\varepsilon\|_{C^{1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})} \leq 2\varepsilon^{1-2\left(\frac{3}{p}+\alpha\right)}$ then implies that the tangent space of $T\mathtt{A}_\varepsilon(V_\varepsilon)$ is also ε-close to be associative and it also implies that the map $\tilde{T}_{V_\varepsilon}$ defined in step 7 is an isomorphism for small ε. So Proposition 8 implies that the tangent space $T\mathtt{A}_\varepsilon(V_\varepsilon)$ is indeed associative. Now standard elliptic regularity implies that V_ε is actually smooth, thus $\mathtt{A}_\varepsilon(V_\varepsilon) = \widetilde{\exp} V_\varepsilon$ is an instanton in M. ■

Finally, we obtain our main result

Theorem 17 *Suppose that M is a G_2-manifold and C_t is an one parameter family of coassociative submanifolds in M. Suppose that the self-dual two form $\eta = dC_t/dt|_{t=0} \in \Omega_+^2(C)$ is nonvanishing, then it defines an almost complex structure J on C_0.*

For any regular J-holomorphic curve Σ in C_0, there is an instanton A_ε in M which is diffeomorphic to $[0,1] \times \Sigma$ and $\partial A_\varepsilon \subset C_0 \cup C_\varepsilon$, for all sufficiently small positive ε.

In particular, by combining Taubes' result on GW=SW [25][26][27] with the above theorem we obtain the following existence result.

Corollary 18 *Suppose that C is a coassociative submanifold in a G_2-manifold M with non-trivial Seiberg-Witten invariants. Given any symplectic form on C, we write C_t's the corresponding coassociative deformations of C in M. Then there is an instanton A_t in M with boundaries lying on $C_0 \cup C_t$ for each sufficiently small t.*

Lastly we expect that any instanton A in M bounding $C_0 \cup C_t$ and with small volume must arise in the above manner. Namely we need to prove a ε-regularity result for instantons.

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