

# Counting Elliptic Curves in K3 Surfaces

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## Abstract

We compute the genus  $g = 1$  family GW-invariants of K3 surfaces for non-primitive classes. These calculations verify Göttsche-Yau-Zaslow formula for non-primitive classes with index two. Our approach is to use the genus two topological recursion formula and the symplectic sum formula to establish relationships among various generating functions.

The number of elliptic curves in K3 surfaces  $X$  representing a homology class  $A \in H_2(X, \mathbb{Z})$  and pass through one generic point depends only on the self-intersection number  $A \cdot A = 2d - 2$  and the index<sup>1</sup>  $r$  of the class  $A$  ([BL1], [BL2]). We denote it as  $N_1(d, r)$ . The conjectural formula of Göttsche [G] for elliptic curves, which generalize the Yau-Zaslow formula [YZ], assert that the generating function for those numbers  $N_1(d, r)$ 's for any given index  $r$  is given by

$$\sum_{d \geq 0} N_1(d, r) t^d = t G_2'(t) \prod_{l \geq 1} \left( \frac{1}{1 - t^l} \right)^{24} \quad (0.1)$$

where  $G_2(t)$  is the Eisenstein series of weight 2, i.e.

$$G_2(t) = \sum_{d \geq 0} \sigma(d) t^d \quad \text{where} \quad \sigma(d) = \sum_{k|d} k, \quad d \geq 1 \quad \text{and} \quad \sigma(0) = -\frac{1}{24}.$$

In particular  $N_1(d, r)$  should be independent of the index of the homology class. In [BL1, L2], this formula was verified for primitive classes by using modified Gromov-Witten invariants of K3 surfaces. In this article, we verify the formula (0.1) for index two classes by computing the family GW-invariants defined in [L1]. Our main theorem is the following result.

**Theorem 0.1** *Let  $X$  be a K3 surface and  $A/2 \in H_2(X; \mathbb{Z})$  be a primitive class. Then, the genus  $g = 1$  family GW-invariant of  $X$  for the class  $A$  is given by*

$$GW_{A,1}^{\mathcal{H}} = GW_{B,1}^{\mathcal{H}} + 2GW_{A/2,1}^{\mathcal{H}} \quad (0.2)$$

where  $B$  is any primitive class with  $B^2 = A^2$ .

<sup>1</sup>The index of  $A$  is the largest positive integer  $r$  such that  $r^{-1}A$  is integral. An index one class is called primitive.

To explain the equivalence between the above theorem and the Göttsche-Yau-Zaslow formula, we first notice that the *family Gromov-Witten* invariant counts the number of *J-holomorphic maps* for any complex structure  $J$  in the twistor family of the K3 surface  $X$ . As explained in [BL1], there is a unique  $J$  in the twistor family which supports holomorphic curves representing  $A$  and this justifies the use of our family invariant. An important issue is the distinction between holomorphic maps to  $X$  and holomorphic curves in  $X$ . This is because a multiple curve in  $X$  can be the image of different holomorphic maps. This issue does not arise when the homology class  $A/2$  they represent is primitive and therefore  $N_1(d', 1) = GW_{A/2,1}^{\mathcal{H}}$ , where  $(A/2)^2 = 2d' - 2$ . The number of genus one holomorphic maps covering a fix elliptic curve with degree  $r$  equals the partition function  $p(r)$ , for instance  $p(2) = 1 + 2 = 3$ . Each primitive elliptic curve in  $X$  contributes 3 to  $GW_{A,1}^{\mathcal{H}}$ , but it only contributes 1 to  $N_1(d, 2)$ . Therefore the above theorem is equivalent to,

$$N_1(d, 2) = N_1(d, 1).$$

Together with the validation of the formula for primitive classes, this implies the Göttsche-Yau-Zaslow formula for the number of elliptic curves in K3 surfaces representing index two homology classes.

The organization of this paper is as follows: The construction of family GW-invariants is briefly described in section 1. This section also contain a family version of the composition law. In section 2, using the composition law and the genus  $g = 2$  TRR (Topological Recursion Relation) formula [Ge] we establish the  $g = 2$  TRR formula for family GW-invariants. In section 3, we prove Theorem 0.1 by combining that TRR formula with the symplectic sum formulas of [LL].

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## 1 Composition Law for Family GW-Invariants

This section briefly describes family GW-invariants defined in [L1]. Let  $X$  be a Kähler surface with a Kähler structure  $(\omega, J, g)$ . For each 2-form  $\alpha$  in the linear space

$$\mathcal{H} = \text{Re}(H^{2,0} \oplus H^{0,2})$$

we define an endormorphism  $K_\alpha$  of  $TX$  by the equation  $\langle u, K_\alpha v \rangle = \alpha(u, v)$ . Since  $Id + JK_\alpha$  is invertible,

$$J_\alpha = (Id + JK_\alpha)^{-1} J (Id + JK_\alpha)$$

is an almost complex structure on  $X$ .

Denote by  $\overline{\mathcal{F}}_{g,k}(X, A)$  the space of all stable maps  $f : (C, j) \rightarrow X$  of genus  $g$  with  $k$ -marked points which represent the homology class  $A$ . For each such map, collapsing unstable components

of the domain determines a point in the Deligne-Mumford space  $\overline{\mathcal{M}}_{g,k}$  and evaluation of marked points determines a point in  $X^k$ . Thus we have a map

$$\overline{\mathcal{F}}_{g,k}(X, A) \xrightarrow{st \times ev} \overline{\mathcal{M}}_{g,k} \times X^k \quad (1.3)$$

where  $st$  and  $ev$  denote the stabilization map and the evaluation map, respectively. On the other hand, there is a *generalized orbifold bundle*  $E$  over  $\overline{\mathcal{F}}_{g,k}(X, A) \times \mathcal{H}$  whose fiber over  $((f, j), \alpha)$  is  $\Omega_{j, J_\alpha}^{0,1}(f^*TX)$ . This bundle has a section  $\Phi$  defined by

$$\Phi(f, j, J_\alpha) = df + J_\alpha df j.$$

When  $X$  is a K3 surface and  $A \neq 0$ , the moduli space  $\Phi^{-1}(0) = \overline{\mathcal{M}}_{g,k}^{\mathcal{H}}(X, A)$  is compact. By the same manner as in the theory of the ordinary GW-invariants [LT], this section then gives rise to a well-defined rational homology class

$$GW_{g,k}^{\mathcal{H}}(X, A) \in H_{2r}(\overline{\mathcal{F}}_{g,k}(X, A); \mathbb{Q}) \quad \text{where } r = g + k.$$

The family GW invariants of  $(X, J)$  are defined by

$$GW_{g,k}^{\mathcal{H}}(X, A)(\beta; \alpha) = GW_{g,k}^{\mathcal{H}}(X, A) \cap (st^*(\beta^*) \cup ev^*(\alpha^*))$$

where  $\beta^*$  and  $\alpha^*$  are Poincaré dual of  $\beta \in H_*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q})$  and  $\alpha \in H_*(X^k; \mathbb{Q})$ , respectively.

In [L1] the first author proved that the above family GW-invariants of K3 surfaces are same as the invariants defined by Bryan and Leung [BL1] using the twistor family. In particular, they are independent of complex structures and for any two classes  $A, B$  of the same index with  $A^2 = B^2$ , we have

$$GW_{g,k}^{\mathcal{H}}(X, A) = GW_{g,k}^{\mathcal{H}}(X, B). \quad (1.4)$$

We will often denote the above family GW-invariants simply as  $GW_{A,g}^{\mathcal{H}}$ .

The family GW-invariants have a property analogous to the composition law of ordinary GW-invariants (cf. [RT]). Consider a node of a stable curve  $C$  in the Deligne-Mumford space  $\overline{\mathcal{M}}_{g,k}$ . When the node is separating, the normalization of  $C$  has two components. The genus and the number of marked points decompose as  $g = g_1 + g_2$  and  $k = k_1 + k_2$  and there is a natural map

$$\sigma : \overline{\mathcal{M}}_{g_1, k_1+1} \times \overline{\mathcal{M}}_{g_2, k_2+1} \rightarrow \overline{\mathcal{M}}_{g,k}.$$

defined by identifying  $(k_1 + 1)$ -th marked points of the first component to the first marked point of the second component. We denote by  $PD(\sigma)$  the Poincaré dual of the image of this map  $\sigma$ . For non-separating node, there is another natural map

$$\theta : \overline{\mathcal{M}}_{g-1, k+2} \rightarrow \overline{\mathcal{M}}_{g,k}$$

defined by identifying the last two marked points. We also write  $PD(\theta)$  for the Poincaré dual of the image of this map  $\theta$ .

Recall that the ordinary GW-invariants of K3 surfaces are all zero except for trivial homology class.

**Proposition 1.1** ([L1]) *Let  $\{H_a\}$  be a base of  $H_*(X; \mathbb{Z})$  and  $\{H^a\}$  be its dual base with respect to the intersection form.*

(a) *Given any decomposition  $g = g_1 + g_2$  and  $k = k_1 + k_2$ , we have*

$$\begin{aligned} & GW_{A,g}^{\mathcal{H}}(PD(\sigma); \alpha_1, \dots, \alpha_k) \\ &= \sum_a GW_{A,g_1}^{\mathcal{H}}(\alpha_1, \dots, \alpha_{k_1}, H_a) GW_{0,g_2}(H^a, \alpha_{k_1+1}, \dots, \alpha_k) \\ &+ \sum_a GW_{0,g_1}(\alpha_1, \dots, \alpha_{k_1}, H_a) GW_{A,g_2}^{\mathcal{H}}(H^a, \alpha_{k_1+1}, \dots, \alpha_k) \end{aligned}$$

where  $GW_{0,g_1}$  and  $GW_{0,g_2}$  denotes the ordinary GW invariants of K3 surfaces.

(b)  $GW_{A,g}^{\mathcal{H}}(PD(\theta); \alpha_1, \dots, \alpha_k) = \sum_a GW_{A,g-1}^{\mathcal{H}}(\alpha_1, \dots, \alpha_k, H_a, H^a)$ .

## 2 Topological Recursion Relations

The first composition law, Proposition 1.1 a, relates family invariants and ordinary invariants (for trivial homology class) of K3 surfaces. In this section, we first recall the ordinary GW-invariants of a closed symplectic 4-manifold for the trivial homology class. As in [L2], we then combine the composition law with the genus  $g = 2$  TRR formula [Ge] to establish a family version of  $g = 2$  TRR formula.

Let  $\tau_i$  be the first Chern class of the line bundle  $\mathcal{L}_i \rightarrow \overline{\mathcal{M}}_{g,k}^{\mathcal{H}}(X, A)$  whose geometric fiber at the point  $(C; x_1, \dots, x_k, f, \alpha)$  is  $T_{x_i}^*C$ .

**Lemma 2.1** *Let  $X$  be a closed symplectic 4-manifold. Its Gromov-Witten invariants satisfy the following properties:*

- (a)  $GW_{0,0}(\alpha_1, \dots, \alpha_k) = 0$  unless  $k = 3$  and  $\sum \deg \alpha_i = 4$ . In that case,  $GW_{0,0}(\alpha_1, \alpha_2, \alpha_3) = \int_X \alpha_1^* \alpha_2^* \alpha_3^*$ ,
- (b)  $GW_{0,1}(\alpha_1, \dots, \alpha_k) = 0$  unless  $k = 1$ . In that case  $GW_{0,1}(\alpha) = -\frac{1}{24} c_1(\alpha)$ ,
- (c)  $GW_{0,2}(\tau(\alpha_1), \alpha_2) = 0$ . When  $c_1(T_X) = 0$ , we also have  $GW_{0,2}(\tau(\alpha)) = 0$ .

**Proof.** (a) and (b) directly follows from Proposition 1.4.1 of [KM]. On the other hand, the formula (7) of [KM] says that

$$GW_{0,2}(\tau(\alpha_1), \alpha_2) = (\alpha_1 \cdot \alpha_2) \int_{\overline{\mathcal{M}}_{2,2}} \lambda_2^2 \psi_1,$$

and

$$GW_{0,2}(\tau(\alpha)) = -c_1(\alpha) \int_{\overline{\mathcal{M}}_{2,1}} \lambda_1 \lambda_2 \psi_1$$

where  $\lambda_i = c_i(E)$  is the Chern class of the Hodge bundle  $E$  and  $\psi_i = c_1(L_i)$  is the first Chern class of the line bundle  $L_i \rightarrow \overline{\mathcal{M}}_{g,k}$  whose geometric fiber at the point  $(C; x_1, \dots, x_k)$  is  $T_{x_i}^*C$ . The first invariant in (c) is zero since  $\lambda_2^2 = 0$  (cf. [M]), while the second one vanishes when  $c_1(T_X) = 0$ .  $\square$

Let  $E(2) \rightarrow \mathbb{P}^1$  be an elliptic K3 surface with a section of self intersection number  $-2$ . Denote by  $s$  and  $f$  the section class and the fiber class, respectively. We will also denote by  $(S, F)$  either  $(2s, f)$  or  $(s - 3f, 2f)$ .

**Proposition 2.2** *The family Gromov-Witten invariants of an elliptic K3 surface  $E(2)$  satisfy the following formula,*

$$GW_{S+dF,2}^{\mathcal{H}}(\tau_1(F), \tau_2(F)) = -\frac{2}{3} GW_{S+dF,1}^{\mathcal{H}}(pt) + \frac{(d-2)^2}{9} GW_{S+dF,0}^{\mathcal{H}}.$$

**Proof.** In the same manner as for ordinary GW-invariants, combining Proposition 1.1 with the formula (5) of [Ge] yields an expression for the family invariants. Let  $\{H_a\}$  and  $\{H^a\}$  be bases of  $H^*(E(2); \mathbb{Z})$  which are dual by the intersection form. We then have

$$\begin{aligned} & GW_{S+dF,2}^{\mathcal{H}}(\tau_1(F), \tau_2(F)) \\ &= \sum_a 2 \left( GW_{S+dF,2}^{\mathcal{H}}(\tau_1(F), H_a) GW_{0,0}(H^a, F) + GW_{0,2}(\tau_1(F), H_a) GW_{S+dF,0}^{\mathcal{H}}(H^a, F) \right) \\ & - \sum_{a,b} GW_{S+dF,0}^{\mathcal{H}}(F, H_a) GW_{0,0}(F, H_b) GW_{0,2}(H^a, H^b) \\ & - \sum_{a,b} GW_{0,0}(F, H_a) GW_{S+dF,0}^{\mathcal{H}}(F, H_b) GW_{0,2}(H^a, H^b) \\ & - \sum_{a,b} GW_{0,0}(F, H_a) GW_{0,0}(F, H_b) GW_{S+dF,2}^{\mathcal{H}}(H^a, H^b) \\ & + \sum_a 3 \left( GW_{S+dF,0}^{\mathcal{H}}(F, F, H_a) GW_{0,2}(\tau(H^a)) + GW_{0,0}(F, F, H_a) GW_{S+dF,2}^{\mathcal{H}}(\tau(H^a)) \right) \\ & - \sum_{a,b} 3 GW_{S+dF,0}^{\mathcal{H}}(F, F, H_a) GW_{0,0}(H^a, H_b) GW_{0,2}(H^b) \\ & - \sum_{a,b} 3 GW_{0,0}(F, F, H_a) GW_{S+dF,0}^{\mathcal{H}}(H^a, H_b) GW_{0,2}(H^b) \\ & - \sum_{a,b} 3 GW_{0,0}(F, F, H_a) GW_{0,0}(H^a, H_b) GW_{S+dF,2}^{\mathcal{H}}(H^b) \\ & + \sum_{a,b} \frac{13}{10} GW_{S+dF,0}^{\mathcal{H}}(F, F, H_a, H_b) GW_{0,1}(H^a) GW_{0,1}(H^b) \\ & + \sum_{a,b} \frac{13}{10} GW_{0,0}(F, F, H_a, H_b) GW_{S+dF,1}^{\mathcal{H}}(H^a) GW_{0,1}(H^b) \\ & + \sum_{a,b} \frac{13}{10} GW_{0,0}(F, F, H_a, H_b) GW_{0,1}(H^a) GW_{S+dF,1}^{\mathcal{H}}(H^b) \end{aligned}$$

$$\begin{aligned}
& + \sum_{a,b} \frac{8}{5} GW_{S+dF,1}^{\mathcal{H}}(F, H_a) GW_{0,0}(H^a, F, H_b) GW_{0,1}(H^b) \\
& + \sum_{a,b} \frac{8}{5} GW_{0,1}(F, H_a) GW_{S+dF,0}^{\mathcal{H}}(H^a, F, H_b) GW_{0,1}(H^b) \\
& + \sum_{a,b} \frac{8}{5} GW_{0,1}(F, H_a) GW_{0,0}(H^a, F, H_b) GW_{S+dF,1}^{\mathcal{H}}(H^b) \\
& - \sum_{a,b} \frac{4}{5} GW_{S+dF,0}^{\mathcal{H}}(F, F, H_a) GW_{0,1}(H^a, H_b) GW_{0,1}(H^b) \\
& - \sum_{a,b} \frac{4}{5} GW_{0,0}(F, F, H_a) GW_{S+dF,1}^{\mathcal{H}}(H^a, H_b) GW_{0,1}(H^b) \\
& - \sum_{a,b} \frac{4}{5} GW_{0,0}(F, F, H_a) GW_{0,1}(H^a, H_b) GW_{S+dF,1}^{\mathcal{H}}(H^b) \\
& + \sum_{a,b} \frac{23}{240} \left( GW_{S+dF,0}^{\mathcal{H}}(F, F, H_a, H^a, H_b) GW_{0,1}(H_b) + GW_{0,0}(F, F, H_a, H^a, H_b) GW_{S+dF,1}^{\mathcal{H}}(H_b) \right) \\
& + \sum_{a,b} \frac{2}{48} \left( GW_{S+dF,0}^{\mathcal{H}}(F, H_a, H^a, H_b) GW_{0,1}(H_b, F) + GW_{0,0}(F, H_a, H^a, H_b) GW_{S+dF,1}^{\mathcal{H}}(H_b, F) \right) \\
& - \sum_{a,b} \frac{1}{80} \left( GW_{S+dF,1}^{\mathcal{H}}(F, F, H_a) GW_{0,0}(H^a, H_b, H^b) + GW_{0,1}(F, F, H_a) GW_{S+dF,0}^{\mathcal{H}}(H^a, H_b, H^b) \right) \\
& + \sum_{a,b} \frac{7}{30} \left( GW_{S+dF,0}^{\mathcal{H}}(F, F, H_a, H_b) GW_{0,1}(H^a, H^b) + GW_{0,0}(F, F, H_a, H_b) GW_{S+dF,1}^{\mathcal{H}}(H^a, H^b) \right) \\
& + \sum_{a,b} \frac{2}{30} \left( GW_{S+dF,0}^{\mathcal{H}}(F, H_a, H_b) GW_{0,1}(H^a, H^b, F) + GW_{0,0}(F, H_a, H_b) GW_{S+dF,1}^{\mathcal{H}}(H^a, H^b, F) \right) \\
& - \sum_{a,b} \frac{1}{30} \left( GW_{S+dF,0}^{\mathcal{H}}(F, F, H_a) GW_{0,1}(H^a, H_b, H^b) + GW_{0,0}(F, F, H_a) GW_{S+dF,1}^{\mathcal{H}}(H^a, H_b, H^b) \right) \\
& + \sum_{a,b} \frac{1}{576} GW_{S+dF,0}^{\mathcal{H}}(F, F, H_a, H^a, H_b, H^b) \tag{2.5}
\end{aligned}$$

(cf. (17) of [Li]). Using the vanishing results in Lemma 2.1, one can simplify the right hand side of (2.5) to have

$$\begin{aligned}
GW_{S+dF,2}^{\mathcal{H}}(\tau_1(F), \tau_2(F)) & = \sum_{a,b} \left( -\frac{1}{80} \right) GW_{S+dF,1}^{\mathcal{H}}(F, F, H_a) GW_{0,0}(H^a, H_b, H^b) \\
& + \sum_{a,b} \frac{1}{15} GW_{S+dF,1}^{\mathcal{H}}(F, H_a, H_b) GW_{0,0}(F, H^a, H^b) \\
& + \sum_{a,b} \frac{1}{576} GW_{S+dF,0}^{\mathcal{H}}(F, F, H_a, H^a, H_b, H^b). \tag{2.6}
\end{aligned}$$

This can be further simplified by using Lemma 2.1 a. The right-hand side of (2.6) becomes

$$-\frac{2}{3} GW_{S+dF,1}^{\mathcal{H}}(pt) + \sum_{a,b} \frac{1}{576} GW_{S+dF,0}^{\mathcal{H}}(F, F, H_a, H^a, H_b, H^b). \quad (2.7)$$

On the other hand, genus  $g = 0$  invariants with point constraints vanish by dimensional reasons. This observation, combined with  $\sum_a (H_a \cdot A)(A \cdot H^a) = A^2$ , shows that

$$\sum_{a,b} \frac{1}{576} GW_{S+dF,0}^{\mathcal{H}}(F, F, H_a, H^a, H_b, H^b) = \frac{4}{576} (4d - 8)^2 GW_{S+dF,0}^{\mathcal{H}}. \quad (2.8)$$

Then, the proposition follows directly from (2.6), (2.7) and (2.8).  $\square$

### 3 Proof of Theorem 0.1

Our goal is to compute the genus  $g = 1$  family GW-invariants of K3 surfaces for classes  $A$  of index 2. By (1.4), it suffices to compute the family GW-invariants of  $E(2)$  for the classes  $2(s + df)$ . We introduce four generating functions by the following formulas

$$\begin{aligned} M_g(t) &= \sum GW_{2s+df,g}^{\mathcal{H}}(pt^g) t^d, \\ P_g(t) &= \sum GW_{(s-3f)+d(2f),g}^{\mathcal{H}}(pt^g) t^d, \\ N_g(t) &= \sum GW_{s+df,g}^{\mathcal{H}}(pt^g) t^d, \\ H_g(\cdot)(t) &= \sum GW_{S+dF,g}^{\mathcal{H}}(\cdot) t^d. \end{aligned}$$

Notice that the coefficients of the even terms of  $M_1(t)$  give the invariants  $GW_{A,1}^{\mathcal{H}}$  for all index two classes  $A$ . Therefore our main theorem 0.1 is equivalent to the following proposition by restricting only to even terms.

**Proposition 3.1** *The above generating functions satisfy the following relation,*

$$M_1(t) = P_1(t) + 2N_1(t^2).$$

**Proof.** Since  $(d-2)^2 = d(d-1) - 3d + 4$ , it follows from Proposition 2.2 that

$$\begin{aligned} &GW_{S+dF,2}^{\mathcal{H}}(\tau_1(F), \tau_2(F)) \\ &= -\frac{2}{3} GW_{S+dF,1}^{\mathcal{H}}(pt) + \frac{1}{9} d(d-1) GW_{S+dF,0}^{\mathcal{H}} - \frac{1}{3} d GW_{S+dF,0}^{\mathcal{H}} + \frac{4}{9} GW_{S+dF,0}^{\mathcal{H}}. \end{aligned} \quad (3.9)$$

Then, by combining (3.9) and the definition of  $H_g(\cdot)(t)$ , we obtain

$$H_2(\tau_1(F), \tau_2(F))(t) = \frac{1}{9} H_0''(t) - \frac{1}{3} t H_0'(t) + \frac{4}{9} H_0(t) - \frac{2}{3} H_1(t). \quad (3.10)$$

In [LL], we used the symplectic sum formula of [IP] to obtain

$$\begin{aligned} & H_2(\tau_1(F), \tau_2(F))(t) - 2H_1(t) \\ &= \frac{20}{3} G_2(t) t H'_0(t) - \left( 64 G_2^2(t) + \frac{40}{3} G_2(t) - 8t G'_2(t) \right) H_0(t). \end{aligned} \quad (3.11)$$

The equations (3.11) and (3.10) then yield

$$\begin{aligned} & \frac{1}{9} t^2 H''_0(t) - \frac{1}{3} t H'_0(t) + \frac{4}{9} H_0(t) - \frac{8}{3} H_1(t) \\ &= \frac{20}{3} G_2(t) t H'_0(t) - \left( 64 G_2^2(t) + \frac{40}{3} G_2(t) - 8t G'_2(t) \right) H_0(t). \end{aligned} \quad (3.12)$$

On the other hand, we have

$$\frac{1}{8} N_0(t^2) = M_0(t) - P_0(t), \quad (3.13)$$

$$t \frac{d}{dt} N_0(t) = 24 G_2(t) N_0(t) + N_0(t), \quad (3.14)$$

$$N_1(t) = \left( t \frac{d}{dt} G_2(t) \right) N_0(t), \quad (3.15)$$

(cf. [LL, BL1, L2]). It then follows from (3.13) and (3.14) that

$$\frac{d}{dt} (M_0(t) - P_0(t)) = (48 G_2(t^2) + 2) (M_0(t) - P_0(t)). \quad (3.16)$$

Recalling  $(S, F)$  is either  $(2s, f)$  or  $(s - 3f, 2f)$ , we combine (3.12) and (3.16) to obtain

$$M_1(t) - P_1(t) = (4t^2 G'_2(t^2) + 3F(t)) (M_0(t) - P_0(t)) \quad (3.17)$$

where  $F(t) = 32 G_2^2(t^2) - 40 G_2(t^2) G_2(t) + 8 G_2^2(t) - t G'_2(t)$ . Then, we have

$$M_1(t) - P_1(t) = 16 t^2 G'_2(t^2) (M_0(t) - P_0(t)) = 2 t^2 G'_2(t^2) N_0(t^2) = 2 N_1(t^2)$$

where the first equality follows from (3.17) and the fact  $F(t) = 4t^2 G'_2(t^2)$  [LL], the second equality follows from (3.13), and the last equality follows from (3.15).  $\square$

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