MODULI OF BUNDLES OVER RATIONAL SURFACES AND ELLIPTIC CURVES I: SIMPLY LACED CASES

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Abstract. It is well-known that del Pezzo surfaces of degree $9-n$ one-to-one correspond to flat E_n bundles over an elliptic curve. In this paper, we construct ADE bundles over a broader class of rational surfaces which are called ADE surfaces, and extend the above correspondence to all flat G bundles over an elliptic curve, where G is any simply laced, simple, compact and simply-connected Lie group. In the sequel, we will construct G bundles for non-simply laced Lie group G over these rational surfaces, and extend the above correspondence to non-simply laced cases.

Introduction

Let S be a smooth rational surfaces. If the anti-canonical line bundle $-K_S$ is ample, then S is called a *del Pezzo surface*. It is well-known that a del Pezzo surface can be classified as a blow-up of $\mathbb{CP}^2$ at $n (n \leq 8)$ points in general position or $\mathbb{CP}^1 \times \mathbb{CP}^1$. When these blown-up points are in *almost general position*, such a surface is called a *generalized del Pezzo surface*, according to Demazure [6]. It is also well-known that the sub-lattice $K_S^\perp$ of $Pic(S)$ is a root lattice of type E_n. For more results on (generalized) del Pezzo surfaces one can see [6] and [20]. Thus there is a natural Lie algebra bundle of type E_n over S. By restriction to a fixed smooth anti-canonical curve Σ, one obtains a flat E_n bundle over Σ. Moreover, Donagi [7] [8] and Friedman-Morgan-Witten [11] [12] prove that the moduli space of del Pezzo surfaces with fixed anti-canonical curve Σ can be identified with the moduli space of flat E_n bundles over the elliptic curve Σ.

In this paper, we will extend this correspondence to all compact, simple, simply laced and simply connected Lie groups and to a broader class of rational surfaces, which are called ADE surfaces. This paper can be regarded as the final and revised version of the preprint [17], although we still refer to [17] for some of the proofs, in order to avoid repetition. Next we sketch the contents briefly.

In Section 1, we first analyze the structure of the Picard lattice of a rational surface which is a blow-up of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or the Hirzebruch surface $\mathbb{F}_1$ at some points. We shall see that there is a sub-lattice of the Picard lattice which is a root lattice of ADE-type.

Next we generalize the definition of del Pezzo surfaces to that of ADE surfaces, where an E_n surface is just a del Pezzo surface of degree $9-n$. Roughly speaking, an ADE surface S is a rational surface with a smooth rational curve C on S such that the sub-lattice $\langle K_S, C\rangle^\perp$ of $Pic(S)$ is an irreducible root lattice (see

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Definition 7). The condition in Definition 7 implies that $C^2 = -1, 0$ or 1, and that the sub-lattice $\langle K_S, C\rangle^\perp$ is a root lattice of type E_n, D_n, or A_n respectively (Proposition 8). Therefore such a surface is called a rational surface of E_n-type, D_n-type, or A_n-type accordingly.

Note that the definition of an E_n surface implies that after blowing down the (-1) curve C, the anti-canonical line bundle $-K$ will be ample. So the resulting surface is just a del Pezzo surface. Thus the definition of ADE surfaces naturally generalizes that of del Pezzo surfaces.

After this, we prove that an ADE surface is nothing but a blow-up of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at some points in general position. This gives us an explicit construction for any ADE surface.

In Section 2, we construct Lie algebra bundles of ADE-type, and their natural representation bundles over those surfaces discussed in Section 1. By a Lie algebra bundle over a surface S, we mean a vector bundle which has a fiberwise Lie algebra structure, and this structure is compatible with any trivialization. Similarly, by a representation bundle, we mean a vector bundle which is a fiberwise representation of a Lie algebra bundle, and this fiberwise representation is compatible with any trivialization.

More precisely, let S be an ADE surface. Since the sub-lattice $\langle K_S, C\rangle^\perp$ of $Pic(S)$ is a root lattice, we can explicitly construct a natural Lie algebra bundle of corresponding type over S, using the root system of the root lattice $\langle K_S, C\rangle^\perp$. Using the lines and rulings on S, we can also construct natural fundamental representation bundles over S.

In Section 3, we relate the above Lie algebra bundles of ADE-type over ADE rational surfaces to flat G bundles over an elliptic curve Σ, where G is a compact Lie group of corresponding type. If an ADE rational surface S contains a fixed smooth elliptic curve Σ as an anti-canonical curve, then by restriction, one obtains flat ADE-bundles over Σ. We can prove this restriction identifies the moduli space of flat ADE bundles over Σ and the moduli space of the pairs $(S, \Sigma \in |-K_S|)$ with extra structure ζ_G which is called a *G-configuration* (Definition 18). One of the main results in this paper is the following theorem.

Theorem 1. *Let Σ be a fixed elliptic curve, and let G be a simple, compact, simply laced and simply connected Lie group. Denote $\mathcal{S}(\Sigma, G)$ the moduli space of the pairs (S, Σ), where S is an ADE rational surface with $\Sigma \in |-K_S|$. Denote $\mathcal{M}_\Sigma^G$ the moduli space of flat G-bundles over Σ. Then by restriction, we have*

(i) $\mathcal{S}(\Sigma, G)$ can be embedded into $\mathcal{M}_\Sigma^G$ as an open dense subset.

(ii) There exists a natural and explicit compactification for $\mathcal{S}(\Sigma, G)$, denoted by $\overline{\mathcal{S}(\Sigma, G)}$, such that this embedding can be extended to an isomorphism from $\overline{\mathcal{S}(\Sigma, G)}$ onto $\mathcal{M}_\Sigma^G$.

(iii) Any surface corresponding to a boundary point in $\overline{\mathcal{S}(\Sigma, G)} \backslash \mathcal{S}(\Sigma, G)$ is equipped with a G-configuration, and on such a surface, any smooth rational curve has a self-intersection number at least -2. Furthermore, in E_n case, all (-2) curves form chains of ADE-type, and the anti-canonical model of such a surface admits at worst ADE-singularities.

Physically, when $G = E_n$ is a simple subgroup of $E_8 \times E_8$, these G bundles are related to the duality between F-theory and string theory. Among other things, this duality predicts the moduli of flat E_n bundles over a fixed elliptic curve Σ can be identified with the moduli of del Pezzo surfaces with fixed anti-canonical curve Σ. For details, one can consult [7] [8] [11] and [12].

Notation 2. In this paper, we will fix some notations from Lie theory. Let G be a compact, simple and simply-connected Lie group. We denote

$r(G)$: the rank of G;
$R(G)$: the root system;
$R_c(G)$: the coroot system;
$W(G)$: the Weyl group;
$\Lambda(G)$: the root lattice;
$\Lambda_c(G)$: the coroot lattice;
$\Lambda_w(G)$: the weight lattice;
$T(G)$: a maximal torus;
$ad(G)$: the adjoint group of G, i.e. $G/C(G)$ where $C(G)$ is the center of G;
$\Delta(G)$: a simple root system of G.

When there is no confusion, we just ignore the letter G.

1. Rational surfaces of ADE-type

Before defining what ADE surfaces are, we first give their explicit constructions.

1.1. First consider the E_n case, that is, the case of del Pezzo surfaces. We start with a complex projective plane $\mathbb{P}^2$ and n points $x_1, \cdots, x_n$ on $\mathbb{P}^2$ with $n \leq 8$. Note that $x_2, \cdots, x_n$ may be *infinitely near* points. For example, we say that x_2 is *infinitely near* x_1 if x_2 lies on the exceptional curve obtained by blowing up x_1. Blowing up $\mathbb{P}^2$ at these points in turn, we obtain a rational surface, denoted $X_n(x_1, \cdots, x_n)$ or X_n for brevity.

These points are said to be *in general position* if they satisfy the following conditions:

(i) They are distinct points;
(ii) No three of them are collinear;
(iii) No six of them lie on a common conic curve;
(iv) No cubics pass through 8 points with one of them a double point.

The following result is well-known (see [6] and [20]).

Lemma 3. *Let $x_i \in \mathbb{P}^2, i = 1, \cdots, n, n \leq 8$. Then the following conditions are equivalent:*

(i) These points are in general position.

(ii) The self-intersection number of any rational curve on X_n is bigger than or equal to -1.

(iii) The anti-canonical class $-K_{X_n}$ is ample. □

A surface X_n is called a *del Pezzo surface* if it satisfies one of the above equivalent conditions.

We say that $x_i \in \mathbb{P}^2, i = 1, \cdots, n$ with $n \leq 8$ are in *almost general position* if any smooth rational curve on X_n has a self-intersection number at least -2, and such a surface is called a *generalized del Pezzo surface* (see [6]).

Let h be the class of lines in $\mathbb{P}^2$ and l_i be the exceptional divisor corresponding to the blow-up at $x_i \in \mathbb{P}^2, i = 1, \cdots, n$. Denote $Pic(X_n)$ the Picard group of X_n, which is isomorphic to $H^2(X_n,\mathbb{Z})$. Then $Pic(X_n)$ is a lattice with basis $h, l_1, \cdots, l_n$, of signature $(1,n)$. Let $K = -3h + l_1 + \cdots + l_n$ be the canonical class. We extend the definition of the (real) Lie algebras $E_n, n = 6,7,8$ to all n with $0 \le n \le 8$ by setting $E_0 = 0, E_1 = \mathbb{R}, E_2 = A_1 \times \mathbb{R}, E_3 = A_1 \times A_2, E_4 = A_4$ and $E_5 = D_5$.

Denote

$$\begin{aligned} P_n &= \{x \in H^2(X_n,\mathbb{Z}) \mid x \cdot K = 0\}, \\ R_n &= \{x \in H^2(X_n,\mathbb{Z}) \mid x \cdot K = 0,\ x^2 = -2\} \subset P_n, \\ I_n &= \{x \in H^2(X_n,\mathbb{Z}) \mid x^2 = -1 = x \cdot K\}, \text{ and} \\ C_n &= \{\zeta_n = (e_1, \cdots, e_n) \mid e_i \in I_n,\ e_i \cdot e_j = 0,\ i \neq j\}. \end{aligned}$$

An element of I_n is called an *exceptional divisor*, and an element $\zeta_n \in C_n$ is called an *exceptional system* (of divisors) (see [6] and [20]).

Lemma 4. *(i) R_n is a root system of type E_n with a system of simple roots $\alpha_1 = l_1 - l_2,\ \alpha_2 = l_2 - l_3,\ \alpha_3 = h - l_1 - l_2 - l_3,\ \alpha_4 = l_3 - l_4,\ \cdots,\ \alpha_n = l_{n-1} - l_n$. Its root lattice is just P_n, and its weight lattice is $Q_n = H^2(X_n,\mathbb{Z})/\mathbb{Z}K$. Let $l \in I_n$, then $R_n \cap l^\perp$ is a root system of type E_{n-1}, and $P_n \cap l^\perp$ is its root lattice.*

(ii) The Weyl group $W(E_n)$ acts on C_n simply transitively.

Proof. (i) For the proof that R_n is a root system of type E_n with given simple roots, see Manin's book [20]. $H^2(X_n,\mathbb{Z})$ is a lattice with $\mathbb{Z}$-basis $h, l_1, \cdots, l_n$. Obviously, $\{e_0 = l_1, e_1 = \alpha_1, \cdots, e_n = \alpha_n\}$ forms another $\mathbb{Z}$-basis. Take any $x \in P_n \subset H^2(X_n,\mathbb{Z})$. Let $x = \sum a_i \cdot e_i$. Then $x \cdot K = 0$ implies $a_0 = 0$. So P_n is the root lattice of R_n.

The natural pairing $P_n \otimes H^2(X_n,\mathbb{Z}) \to \mathbb{Z}$ induces a perfect pairing

$$P_n \otimes (H^2(X_n,\mathbb{Z})/\mathbb{Z}K) \to \mathbb{Z}.$$

So the weight lattice is just $H^2(X_n,\mathbb{Z})/\mathbb{Z}K$.

For the last assertion, we can assume $l = l_8$, then it is true obviously.

(ii) See [20]. □

The Dynkin diagram is the following

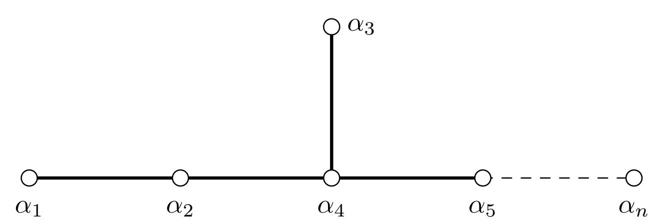

Figure 1. The root system E_n.

1.2. Next we consider the D_n case. Let $Y = \mathbb{F}_1$ be a *Hirzebruch surface*, and fix the ruling f and the section s, where $s^2 = -1$. In fact $\mathbb{F}_1$ is the blow-up of $\mathbb{P}^2$ at one point x_0. Thus $f = h - l_0,\ s = l_0$ where h is the class of lines on $\mathbb{P}^2$ and l_0 is the exceptional curve. Blowing up Y at n points $x_1, \cdots, x_n$ we obtain Y_n. The

Picard group of Y_n is $H^2(Y_n,\mathbb{Z})$, which is a lattice with basis $s, f, l_1, \cdots, l_n$. The canonical class $K = -(2s + 3f - \sum_{i=1}^{n} l_i)$.

Denote

$$\begin{aligned} P_n &= \{x \in H^2(Y_n,\mathbb{Z}) \mid x \cdot K = 0 = x \cdot f\}, \\ R_n &= \{x \in H^2(Y_n,\mathbb{Z}) \mid x \cdot K = 0 = x \cdot f,\ x^2 = -2\}, \\ I_n &= \{x \in H^2(Y_n,\mathbb{Z}) \mid x^2 = -1 = x \cdot K,\ x \cdot f = 0\}, \\ C_n &= \{\zeta_n = (e_1, \cdots, e_n) \mid e_i \in I_n, e_i \cdot e_j = 0, i \neq j, \\ &\qquad \sum e_i \cdot s \equiv 0 \ mod\ 2\}. \end{aligned}$$

Similarly as before, an element $\zeta_n \in C_n$ is called an *exceptional system* (of divisors).

Lemma 5. *(i) R_n is a root system of type D_n with a system of simple roots $\alpha_1 = f - l_1 - l_2, \alpha_2 = l_1 - l_2, \cdots, \alpha_n = l_{n-1} - l_n$. Its root lattice is just P_n and its weight lattice is $Q_n = H^2(Y_n,\mathbb{Z})/\mathbb{Z}\langle f, K\rangle$.*

(ii) The Weyl group $W(D_n)$ acts on C_n simply transitively.

Proof. (i) $H^2(Y_n,\mathbb{Z})$ is a lattice with $\mathbb{Z}$-basis $s, f, l_i, i = 1, \cdots, n$. Let $x = as + bf + \sum c_i l_i \in R_n$ where $a, b, c_i \in \mathbb{Z}$. Then we have a system of linear equations

$$\begin{cases} x^2 = -2, \\ x \cdot K = 0 = x \cdot f. \end{cases}$$

Solving this, we obtain

$$\begin{cases} a = 0, \\ \sum c_i^2 = 2, \\ 2b = -\sum c_i. \end{cases}$$

So, $x = \pm(l_i - l_j), i \neq j$ or $x = \pm(f - l_i - l_j), i \neq j$. That is $R_n = \{\pm(l_i - l_j), \pm(f - l_i - l_j) | \ i \neq j\}$. This implies that R_n is a root system of D_n-type with indicated simple roots.

Obviously, $\{e_1 = s, e_2 = l_1, e_{i+2} = \alpha_i, i = 1, \cdots, n\}$ forms another $\mathbb{Z}$-basis. Take any $x \in P_n \subset H^2(Y_n,\mathbb{Z})$. Let $x = \sum a_i \cdot e_i$. Then $x \cdot K = 0 = x \cdot f$ implies $a_1 = a_2 = 0$. So P_n is the root lattice of R_n.

The natural pairing $P_n \otimes H^2(Y_n,\mathbb{Z}) \to \mathbb{Z}$ has kernel $\mathbb{Z}\langle f, -2s + \sum l_i\rangle = \mathbb{Z}\langle f, K\rangle$. So the pairing induces a perfect pairing $P_n \otimes (H^2(Y_n,\mathbb{Z})/\mathbb{Z}\langle f, K\rangle) \to \mathbb{Z}$. Hence the weight lattice is just $H^2(Y_n,\mathbb{Z})/\mathbb{Z}\langle f, K\rangle$.

(ii) A simple computation shows that

$$I_n = \{l_i, f - l_i | i = 1, \cdots, n\}.$$

Thus all the elements of C_n are of the form $\zeta_n = (u_1, \cdots, u_n)$ where the number of u_i's, such that $u_i = f - l_k$ for some k, is even. Then by the structure of $W(D_n)$, the result is clear. □

The Dynkin diagram is the following

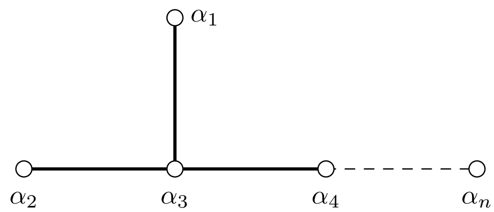

Figure 2. The root system D_n.

1.3. In the following we consider the A_{n-1} case. For this, let Z_n be just the same as Y_n.

Denote

$$\begin{array}{rcl} P_{n-1} & = & \{x \in H^2(Z_n, \mathbb{Z}) \mid x \cdot K = x \cdot f = x \cdot s = 0\}, \\ R_{n-1} & = & \{x \in H^2(Z_n, \mathbb{Z}) \mid x \cdot K = x \cdot f = x \cdot s = 0,\ x^2 = -2\}, \\ I_{n-1} & = & \{x \in H^2(Z_n, \mathbb{Z}) \mid x^2 = -1 = x \cdot K,\ x \cdot f = 0 = x \cdot s\}, \\ C_{n-1} & = & \{\zeta_n = (e_1, \cdots, e_n) \mid e_i \in I_{n-1}, e_i \cdot e_j = 0, i \neq j\}. \end{array}$$

As before, an element of $\zeta_n \in C_{n-1}$ is called an *exceptional system* (of divisors).

Lemma 6. *(i) R_{n-1} is a root system of type A_{n-1} with a system of simple roots $\alpha_1 = l_1 - l_2, \cdots, \alpha_{n-1} = l_{n-1} - l_n$. Its root lattice is just P_{n-1} and its weight lattice is $H^2(Z_n, \mathbb{Z})/\mathbb{Z}\langle f, s, K\rangle$.*

(ii) The Weyl group $W(A_{n-1})$ acts on C_{n-1} simply transitively. In fact, $W(A_{n-1})$ acts as the permutation group of $l_1, \cdots, l_n$.

(iii) Let e be a (-1) curve which does not meet s. Then there exist i, j with $i \neq j$ such that $e = s + f - l_i - l_j$, and when $n \geq 4$, $\langle K, s, f, e\rangle^\perp$ is a reducible root lattice of type $A_1 \times A_{n-3}$; when $n = 3$, $\langle K, s, f, e\rangle^\perp$ is not a root lattice; when $n = 2$, $\langle K, s, f, e\rangle^\perp$ is the same as P_1, which is of type A_1.

(iv) Let $e_i, 1 \leq i \leq k,\ k \geq 2$ be (-1) curves such that $s, e_i, 1 \leq i \leq k$ are disjoint pairwise. Then when $k \neq 3$, $\langle K, s, f, e_i, 1 \leq i \leq k\rangle^\perp$ is not a root lattice. When $k = 3$, (a) if $e_1 = s + f - l_{i_2} - l_{i_3},\ e_2 = s + f - l_{i_1} - l_{i_3},\ e_3 = s + f - l_{i_1} - l_{i_2}$ then $\langle K, s, f, e_1, e_2, e_3\rangle^\perp$ is a root lattice of A-type; (b) otherwise, $\langle K, s, f, e_1, e_2, e_3\rangle^\perp$ is not a root lattice.

Proof. (i) $H^2(Z_n, \mathbb{Z})$ is a lattice with $\mathbb{Z}$-basis $s, f, l_i, i = 1, \cdots, n$. A simple computation shows that

$$R_{n-1} = \{l_i - l_j \mid i \neq j\}.$$

Then it is obviously a root system of type A_{n-1} with given simple roots.

Obviously, $\{e_1 = s, e_2 = f, e_3 = l_1, e_{i+3} = \alpha_i, i = 1, \cdots, n\}$ forms another $\mathbb{Z}$-basis. Take any $x \in P_{n-1} \subset H^2(Z_n, \mathbb{Z})$. Let $x = \sum a_i \cdot e_i$. Then $x \cdot K = x \cdot f = x \cdot s = 0$ implies $a_1 = a_2 = a_3 = 0$. So P_{n-1} is the root lattice of R_{n-1}.

The natural pairing

$$P_{n-1} \otimes H^2(Z_n, \mathbb{Z}) \to \mathbb{Z}$$

has a kernel

$$\mathbb{Z}\langle f, s, \sum l_i\rangle = \mathbb{Z}\langle f, s, K\rangle.$$

So the pairing induces a perfect pairing

$$P_{n-1} \otimes (H^2(Z_n, \mathbb{Z})/\mathbb{Z}\langle f, s, K\rangle) \to \mathbb{Z}.$$

Hence the weight lattice is just $H^2(Z_n,\mathbb{Z})/\mathbb{Z}\langle f,s,K\rangle$.

(ii) In fact $I_{n-1}=\{l_1,\cdots,l_n\}$. So an element of C_{n-1} is just a permutation of $l_1,\cdots,l_n$.

(iii) Let $e=as+bf+\sum c_il_i$, then e is a (-1) curve and $e\cdot s=0$ imply that e must be of the form $s+f-l_i-l_j, i\neq j$. Without loss of generality, we can assume that $e=s+f-l_1-l_2$. Then the result follows from a simple computation.

(iv) First let $k=2$. From the proof of (iii), we know both e_1 and e_2 are the form $s+f-l_i-l_j, i\neq j$. Since $e_1\cdot e_2=0$, we can assume $e_1=s+f-l_1-l_2$ and $e_2=s+f-l_1-l_3$. Then the result follows easily. For $k=3$, if $e_1=s+f-l_{i_2}-l_{i_3},\ e_2=s+f-l_{i_1}-l_{i_3},\ e_3=s+f-l_{i_1}-l_{i_2}$ then $\langle K,s,f,e_1,e_2,e_3\rangle^{\perp}=\langle K,s,f,l_{i_1},l_{i_2},l_{i_3}\rangle^{\perp}$. We can assume $l_{i_1}=l_1,l_{i_2}=l_2,l_{i_3}=l_3$. Then $\langle K,s,f,l_1,l_2,l_3\rangle^{\perp}$ is a root lattice of A-type. Other cases are similar. $\square$

The Dynkin diagram is the following

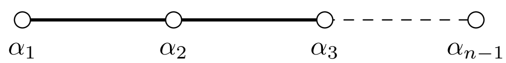

Figure 3. The root system A_{n-1}.

Note that Lemma 5 and Lemma 6 (i) (ii) are still true if we replace $\mathbb{F}_1$ by any *Hirzebruch surface* $\mathbb{F}_k(k\geq 0)$.

1.4. Now we show that in a suitable sense, the converse of the above lemmas is also true. As promised in the introduction, we will see that the following definition generalizes that of *del Pezzo surfaces*.

Definition 7. *Let (S,C) be a pair consisting of a smooth rational surface S and a smooth rational curve $C\subset S$ with $C^2\neq 4$. The pair (S,C) is called of ADE-type (or an ADE surface) if it satisfies the following two conditions:*

(i) Any (smooth) rational curve on S has a self-intersection number at least -1;

(ii) The sub-lattice $\langle K_S,C\rangle^{\perp}$ of $Pic(S)$ is an irreducible root lattice of rank equal to $rank(Pic(S))-2$.

The following proposition shows that such surfaces can be classified into three types.

Proposition 8. *Let (S,C) be a rational surface of ADE-type. Let $n=rank(Pic(S))-2$. Then $C^2\in\{-1,0,1\}$ and*

(i) when $C^2=-1$, $\langle K_S,C\rangle^{\perp}$ is of E_n-type, where $4\leq n\leq 8$;

(ii) when $C^2=0$, $\langle K_S,C\rangle^{\perp}$ is of D_n-type, where $n\geq 3$;

(iii) when $C^2=1$, $\langle K_S,C\rangle^{\perp}$ is of A_n-type.

Proof. By the first condition in Definition 7, $C^2\geq -1$. Therefore there are the following four cases.

Firstly, suppose $C^2=-1$. Then we can contract C to obtain a smooth surface $\widetilde{S}$. Let $\pi:S\to\widetilde{S}$ be the blow-down. Then the projection

$$Pic(S)=Pic(\widetilde{S})\oplus\mathbb{Z}\langle C\rangle\to Pic(\widetilde{S})$$

induces an isomorphism $\langle K_S, C\rangle^\perp \cong \langle K_{\widetilde{S}}\rangle^\perp$. But the latter is an irreducible root system if and only if $\widetilde{S}$ is a blow-up of $\mathbb{CP}^2$ at $n(4 \le n \le 8)$ points. At this time $\langle K_{\widetilde{S}}\rangle^\perp$ is a root system of E_n-type. Thus S is a blow-up of $\mathbb{CP}^2$ at $n+1(4 \le n \le 8)$ points.

Secondly, suppose $C^2 = 0$. Then by Riemann-Roch theorem, the linear system $|C|$ defines a ruling over $\mathbb{P}^1$ with fiber C. Contract all (-1) curves in fiber, we obtain a relatively minimal model (not unique), which is $\mathbb{P}^1 \times \mathbb{P}^1$ or the Hirzebruch surface $\mathbb{F}_1$. So, S is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at n points. And the lattice $\langle K_S, C\rangle^\perp$ must be of D_n-type by Lemma 5.

Thirdly, suppose $C^2 = 1$. Then blow up one point $p_0 \in C$, we obtain $\widetilde{S}$ which is a ruling over $\mathbb{P}^1$ with fiber $\widetilde{C} = C - E$ and section E where E is the exceptional curve associated to this blow-up. Contracting all (-1) curves in fiber which do not intersect with E, we will obtain $\mathbb{F}_1$. Thus $\widetilde{S}$ is a blow-up of $\mathbb{F}_1$ at n points. And we have $\langle K_S, C\rangle^\perp \cong \langle K_{\widetilde{S}}, \widetilde{C}, E\rangle^\perp$. Therefore the lattice is a root lattice of A_n-type by Lemma 6.

Finally, suppose $C^2 \ge 2$. Note that since we assume $C^2 \ne 4$, the situation of Lemma 6 (iv) (a) can not happen. So we only need to discuss the case where $C^2 = 2$, because the discussion on general cases is similar. Blowing up S at two points $p, q \in C, p \ne q$, we obtain $\widetilde{S}$ with exceptional curves E_p, E_q. Let $\widetilde{C} = C - E_p - E_q$ be the strict transform of C, then $|\widetilde{C}|$ defines a ruling with fiber $\widetilde{C}$ and section $s = E_p$ (fixed). Similarly as before, contracting all (-1) curves E in fiber which satisfy $E \cdot \widetilde{C} = 0 = E \cdot s$, we will obtain $\mathbb{F}_1$. Then $\widetilde{S}$ can be considered as a blow-up of $\mathbb{F}_1$ at n points. Note that $\langle K_S, C\rangle^\perp \cong \langle K_{\widetilde{S}}, \widetilde{C}, s, E_q\rangle^\perp$. We know that $\langle K_{\widetilde{S}}, \widetilde{C}, s\rangle^\perp$ is a root lattice of A_n-type from Lemma 6. Then the result follows also from Lemma 6. □

Remark 9. We extend the definition of E_n surfaces to all n with $0 \le n \le 8$, by defining $E_n(n \le 3)$ surfaces to be del Pezzo surfaces of degree $9 - n$.

Corollary 10. *On an ADE surface, any exceptional divisor perpendicular to C is represented by an irreducible curve. Therefore, any exceptional system consists of exceptional curves.*

Proof. In E_n case, the result follows from Proposition 8 and Lemma 3. In D_n and A_n cases, according to Proposition 8, the result is obvious. □

In the following we generalize the definition for $n \le 8$ points being *in general position* to any $n \ge 0$. Denote $S = \mathbb{P}^2$ (or $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$). Denote $S_n(x_1, \cdots, x_n)$ (or S_n for brevity) the blow-up of S at n points $x_1, \cdots, x_n$. We say that $x_1, \cdots, x_n$ are *in general position* if any smooth rational curve on S_n has a self-intersection number at least -1. And we say that $x_1, \cdots, x_n$ are *in almost general position* if any smooth rational curve on S_n has a self-intersection number at least -2.

Corollary 11. *Let (S, C) be an ADE surface.*

(i) In E_n case, blowing down the (-1) curve C, we obtain a del Pezzo surface of degree $9 - n$.

(ii) In D_n case, S is just a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at n points in general position with C as the natural ruling.

(iii) In A_n case, let $\widetilde{S}$ be the blow-up of S at a point on C, with the exceptional curve E, then $\widetilde{S}$ is a blow-up of $\mathbb{F}_1$ at $n+1$ points, and the strict transform $\widetilde{C}$ of C defines a ruling with E as the section of $\mathbb{F}_1$. □

2. Lie algebra bundles over rational surfaces of ADE-type and their representation bundles

When G is of ADE-type, to each ADE surface S, we can construct a natural $\mathcal{G} = Lie(G)$ bundle and natural fundamental representation bundles over S, which are determined by the lines (or exceptional divisors in general) and rulings on S.

Definition 12. *By a Lie algebra $\mathcal{G} = Lie(G)$ bundle, we mean a vector bundle which fiberwise carries a Lie algebra structure of $\mathcal{G}$-type, and this Lie algebra structure is compatible with trivialization of this bundle. By a representation bundle of a $\mathcal{G}$ bundle, we mean a vector bundle $\mathcal{V}$ which fiberwise is a representation of $\mathcal{G}$, and the action of $\mathcal{G}$ on $\mathcal{V}$ is compatible with trivialization of them.*

We describe these bundles in the following, and give the detailed arguments just in E_n case.

2.1. E_n bundles over E_n surfaces. Let (S, C) be an E_n surface. Recall that $S = X_{n+1}(x_1, \cdots, x_{n+1})$ where C be the exceptional divisor associated to the blow-up at x_{n+1}. Denote $\widetilde{S} = X_n(x_1, \cdots, x_n)$. Since $\langle K_S, C\rangle^\perp \cong K_{\widetilde{S}}^\perp$, we can just consider the surface $\widetilde{S} = X_n(x_1, \cdots, x_n)$.

Since we have a root system of E_n-type attached to X_n, we can construct a Lie algebra bundle over X_n as follows:

$$\mathscr{E}_n = \mathcal{O}^{\oplus n} \bigoplus_{D\in R_n} \mathcal{O}(D).$$

The fiberwise Lie algebra structure of $\mathscr{E}_n$ is defined as the following. Fix the system of simple roots of R_n as

$$\Delta(E_n) = \{\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, \alpha_3 = h - l_1 - l_2 - l_3, \cdots, \alpha_n = l_{n-1} - l_n\},$$

and take a trivialization of $\mathscr{E}_n$. Then over a trivializing open subset U, $\mathscr{E}_n|_U \cong U \times (\mathbb{C}^{\oplus n} \bigoplus_{\alpha\in R_n} \mathbb{C}_\alpha)$. Take a Chevalley basis $\{x_\alpha^U, \alpha \in R_n; h_i, 1 \le i \le n\}$ for $\mathscr{E}_n|_U$ and define the Lie algebra structure by the following four relations, namely, Serre's relations on Chevalley basis (see [14], p147):

(a) $[h_i h_j] = 0, 1 \le i, j \le n$.
(b) $[h_i x_\alpha^U] = \langle\alpha, \alpha_i\rangle x_\alpha^U, 1 \le i \le n, \alpha \in R_n$.
(c) $[x_\alpha^U x_{-\alpha}^U] = h_\alpha$ is a $\mathbb{Z}$-linear combination of $h_1, \cdots, h_n$.
(d) If α, β are independent roots, and $\beta - r\alpha, \cdots, \beta + q\alpha$ are the α-string through β, then $[x_\alpha^U x_\beta^U] = 0$ if $q = 0$, while $[x_\alpha^U x_\beta^U] = \pm(r+1)x_{\alpha+\beta}^U$ if $\alpha + \beta \in R_n$.

Note that $h_i, 1 \le i \le n$ are independent of any trivialization, so the relation (a) is always invariant under different trivializations. If $\mathscr{E}_n|_V \cong V \times (\mathbb{C}^{\oplus n} \bigoplus_{\alpha\in R_n})$ is another trivialization, and f_α^{UV} is the transition function for the line bundle $\mathcal{O}(\alpha)(\alpha \in R_n)$, that is, $x_\alpha^U = f_\alpha^{UV} x_\alpha^V$, then the relation (b) is

$$[h_i (f_\alpha^{UV} x_\alpha^V)] = \langle\alpha, \alpha_i\rangle f_\alpha^{UV} x_\alpha^V,$$

that is,

$$[h_i x_\alpha^V] = \langle \alpha, \alpha_i \rangle x_\alpha^V.$$

So (b) is also invariant. (c) is also invariant since $(f_\alpha^{UV})^{-1}$ is the transition function for $\mathcal{O}(-\alpha)(\alpha \in R_n)$. Finally, (d) is invariant since $f_\alpha^{UV} f_\beta^{UV}$ is the transition function for $\mathcal{O}(\alpha+\beta)(\alpha, \beta \in R_n)$.

Therefore, the Lie algebra structure is compatible with the trivialization. Hence it is well-defined. In other words, we can construct globally a Lie algebra bundle over a surface once we are given a root system consisting of divisors on this surface.

The following relations are intricate. One is the relation between I_n (the set of all exceptional divisors) and the fundamental representation associated to the highest weight λ_n which is dual to the simple root α_n (see Figure 1). Another one is the relation between the set of rulings and the fundamental representation associated to the highest weight λ_1 which is dual to the simple root α_1 (Figure 1). We explain the relations in the following.

Let $\mathbb{L}_n$ be the fundamental representation with the highest weight λ_n. Then we have:

n	1	2	3	4	5	6	7	8
$dim\ \mathbb{L}_n$	1	3	6	10	16	27	56	248
$\lvert I_n \rvert$	1	3	6	10	16	27	56	240

Denotes Ru_n the set of all rulings on X_n. Let $\mathbb{R}_n$ be the fundamental representation with the highest weight λ_1. Then we have:

n	1	2	3	4	5	6	7	8
$dim\ \mathbb{R}_n$	1	2	3	5	10	27	133	3875
$\lvert Ru_n \rvert$	1	2	3	5	10	27	126	2120

Inspired by these, we can construct a fundamental representation bundle $\mathscr{L}_n$ (respectively $\mathscr{R}_n$) using the exceptional divisors (respectively the rulings) on X_n as follows.

$$\begin{aligned} \mathscr{L}_n &= \bigoplus_{l \in I_n} \mathcal{O}(l) \text{ when } n \leq 7, \\ \mathscr{L}_8 &= \bigoplus_{l \in I_8} \mathcal{O}(l) \oplus \mathcal{O}(-K)^{\oplus 8}. \end{aligned}$$

Respectively,

$$\begin{aligned} \mathscr{R}_n &= \bigoplus_{R \in Ru_n} \mathcal{O}(R) \text{ when } n \leq 6, \\ \mathscr{R}_7 &= \bigoplus_{R \in Ru_7} \mathcal{O}(R) \oplus \mathcal{O}(-K)^{\oplus 7}. \end{aligned}$$

The fiberwise action is defined naturally, which is in fact compatible with any trivialization.

For example we consider the bundle $\mathscr{L}_n$ and suppose $n \leq 7$. Take U, V as before, and suppose they also trivialize $\mathscr{L}_n$, that is $\mathscr{L}_n|_U \cong U \times (\bigoplus_{l \in I_n} \mathbb{C}_l)$ and $\mathscr{L}_n|_V \cong V \times (\bigoplus_{l \in I_n} \mathbb{C}_l)$. Take e_l^U (resp. $e_l^V = g^{VU} e_l^U$) to be the basis of $\mathbb{C}_l$ over U (resp. V). Then define $x_\alpha^U . e_l^U$ to be equal to $e_{l'}^U$ if $l' = \alpha + l \in I_n$ and be equal to 0 otherwise. And define $h_\alpha . e_l^U = (\alpha \cdot l) e_l^U$.

Note that the situation here is slightly different from some standard usage, for example [3] [14], since the self-intersection number of an element of R_n or I_n is negative. But this does not matter if we take the simple root system to be $\{-\alpha_1, \cdots, -\alpha_n\}$, and take the pairing to be $(x, y) := -(x \cdot y)$. Firstly since $\lambda_n(-\alpha_i) = (-\alpha_i, l_n) = \alpha_i \cdot l_n = \delta_{in}$, we have $\lambda_n \cong (\cdot, l_n)$. Secondly the action is irreducible since the Weyl group acts on I_n transitively. Lastly $e_{l_n}^U$ is the maximal vector of weight λ_n. Therefore this fiberwise action does define the highest weight module with the highest weight λ_n (see [14]).

Obviously, this fiberwise Lie algebra action is compatible with the trivialization.

For $\mathscr{L}_8$, note that the bijection $I_8 \to R_8$ given by $l \mapsto l + K$ induces an isomorphism

$$\mathscr{E}_8 \cong \mathscr{L}_8 \otimes \mathcal{O}(K).$$

This implies $\mathscr{L}_8$ is just the adjoint representation bundle.

Similarly, $\mathscr{R}_n$ is the fundamental representation bundle with the highest weight $\lambda_1 \cong (\cdot, h - l_1)$ and the maximal vector $e_{h-l_1}^U$, where the simple root system and the pairing are defined as above. We also have that $\mathscr{R}_7 \otimes \mathcal{O}(K) \cong \mathscr{E}_7$ is the adjoint representation bundle.

Example 13. Let us look at the $sl(2)$ sub-bundle

$$\mathcal{O} \oplus \mathcal{O}(\alpha) \oplus (-\alpha),$$

where $\alpha = l_1 - l_2$. Then the bundle $\mathcal{O}(l_1) \oplus \mathcal{O}(l_2)$ is the standard representation bundle. And the line bundle $\mathcal{O}(h - l_1 - l_2)$ is a trivial representation.

In fact, the Lie algebra bundle $\mathscr{E}_n$ is uniquely determined by its representation bundles $\mathscr{L}_n$ and $\mathscr{R}_n$, according to [1]. Concretely (see [17] for more details),

(i) $\mathscr{E}_4$ is the automorphism bundle of $\mathscr{R}_4$ preserving $\wedge^5 \mathscr{R}_4 \cong \mathcal{O}(-2K)$.

(ii) $\mathscr{E}_5$ is the automorphism bundle of $\mathscr{R}_5$ preserving $q_5 : \ \mathscr{R}_5 \otimes \mathscr{R}_5 \to \mathcal{O}(-K)$, where q_5 is defined by $\mathcal{O}(R') \otimes \mathcal{O}(R'') \to \mathcal{O}(-K)$ if $R' + R'' = -K$, and 0 otherwise.

(iii) $\mathscr{E}_6$ is the automorphism bundle of $\mathscr{R}_6$ and $\mathscr{L}_6$ preserving

$$\begin{cases} c_6 : \ \mathscr{L}_6 \otimes \mathscr{L}_6 \to \mathscr{R}_6, \ and \\ c_6^* : \ \mathscr{R}_6 \otimes \mathscr{R}_6 \to \mathscr{L}_6 \otimes \mathcal{O}(-K), \end{cases}$$

where c_6 is defined by the map $(l_i, \ l_j) \mapsto 2h - \sum_{k \neq i,j} l_k$ and c_6^* is defined by the map $(h - l_i, h - l_j) \mapsto h - l_i - l_j$.

(iv) $\mathscr{E}_7$ is the automorphism bundle of $\mathscr{L}_7$ preserving

$$f_7 : \ \mathscr{L}_7 \otimes \mathscr{L}_7 \otimes \mathscr{L}_7 \otimes \mathscr{L}_7 \to \mathcal{O}(-2K),$$

where f_7 is defined by the map $(C_1, C_2, C_3, C_4) \mapsto -2K$ if $C_1 + C_2 + C_3 + C_4 = -2K$, 0 otherwise.

(v) $\mathscr{E}_8$ is the automorphism bundle of $\mathscr{L}_8$ preserving

$$\mathscr{L}_8 \wedge \mathscr{L}_8 \to \mathscr{L}_8 \otimes \mathcal{O}(-K).$$

For X_6, the bijection $Ru_6 \to I_6$ defined by $R \mapsto -(R+K)$ induces an isomorphism $\mathscr{R}_6 \cong \mathscr{L}_6^* \otimes \mathcal{O}(-K)$, which is consistent with the duality between $\mathbb{L}_6$ and $\mathbb{R}_6$ for the Lie group E_6.

2.2. D_n **bundles over rational ruled surfaces.** Let (S, C) be a D_n surface. By Proposition 8, S dominates $\mathbb{F}_1$ or $\mathbb{F}_0(=\mathbb{P}^1 \times \mathbb{P}^1)$ with ruling C. We can suppose that S dominates $\mathbb{F}_1$ since for another case the arguments is the same. Thus $S = Y_n(x_1, \cdots, x_n)$ is the blow-up of $\mathbb{F}_1$ at n points $x_i, i = 1, \cdots, n$, where for any i, x_i does not lie on the section s.

Since R_n is a root system of type D_n, the Lie algebra bundle can be constructed as follows.

$$\mathscr{D}_n = \mathcal{O}^{\oplus n} \bigoplus_{D \in R_n} \mathcal{O}(D).$$

Recall that in D_n case,

$$\begin{aligned} I_n &= \{C |\ C^2 = C \cdot K = -1, C \cdot f = 0\} \\ &= \{l_i, f - l_i \mid i = 1, \cdots, n\}. \end{aligned}$$

The fundamental representation with the highest weight λ_n, where λ_n is the fundamental weight corresponding to $\alpha_n = l_{n-1} - l_n$, is

$$\mathscr{W}_n = \bigoplus_{C \in I_n} \mathcal{O}(C).$$

In fact, $\mathscr{W}_n$ is the standard representation bundle of $\mathscr{D}_n$.

Note that there are n singular fibers, and each singular fiber is of the form $l_i + l_i'$ where $l_i' = f - l_i, i = 1, \cdots, n$. The relation

$$\mathcal{O}(l_i) \otimes \mathcal{O}(l_i') = \mathcal{O}(f)$$

implies we can define a non-degenerated fiberwise quadratic form

$$q_n : \mathscr{W}_n \otimes \mathscr{W}_n \to \mathcal{O}(f).$$

The two spinor bundles are defined as

$$\mathcal{S}_n^+ = \bigoplus_{S^2 = S\cdot K = -1, S \cdot f = 1} \mathcal{O}(S) \text{ and } \mathcal{S}_n^- = \bigoplus_{T^2 = -2, T \cdot K = 0, T \cdot f = 1} \mathcal{O}(T).$$

Moreover, there are all kinds of structures on these representation bundles, for example, the Clifford multiplication:

$$\mathcal{S}_n^+ \otimes \mathscr{W}_n^* \to \mathcal{S}_n^- \text{ and } \mathcal{S}_n^- \otimes \mathscr{W}_n \to \mathcal{S}_n^+.$$

When $n = 2m-1$ is odd, we have isomorphism

$$(\mathcal{S}_n^+)^* \otimes \mathcal{O}_{Y_n}((m-4)f - K) \cong \mathcal{S}_n^-.$$

When $n = 2m$ is even, we have isomorphisms

$$\begin{aligned} (\mathcal{S}_n^+)^* \otimes \mathcal{O}_{Y_n}((m-3)f - K) &\cong \mathcal{S}_n^+, \\ (\mathcal{S}_n^-)^* \otimes \mathcal{O}_{Y_n}((m-4)f - K) &\cong \mathcal{S}_n^-. \end{aligned}$$

For more details, see [17].

2.3. A_{n-1} bundles and their representation bundles. Let S be an A_{n-1} surface. By Proposition 8, we can assume that $S = Z_n(x_1, \cdots, x_n)$ be the blow-up of $\mathbb{F}_1$ at n points $x_i, i = 1, \cdots, n$, where for any i, x_i does not lie on the section s. Recall that

$$\begin{aligned} R_{n-1} &= \{l_i - l_j |\ i \neq j\} \text{ and} \\ I_{n-1} &= \{l_1, \cdots, l_n\}. \end{aligned}$$

Since R_{n-1} is a root system of A_{n-1}-type, the Lie algebra bundle can be constructed as

$$\mathscr{A}_{n-1} = \mathcal{O}^{\oplus n-1} \bigoplus_{D \in R_{n-1}} \mathcal{O}(D).$$

And the standard representation bundle is

$$\mathcal{V}_{n-1} = \bigoplus_{C \in I_{n-1}} \mathcal{O}(C) = \bigoplus_{i=1}^{n} \mathcal{O}(l_i).$$

The k^{th} fundamental representation bundle is just

$$\wedge^k(\mathcal{V}_{n-1}) \cong \bigoplus_{i_1 < \cdots < i_k} \mathcal{O}(l_{i_1} + \cdots + l_{i_k}).$$

We also have $\mathscr{A}_{n-1} = \mathcal{E}nd_0(\mathcal{V}_{n-1})$.

We summarize the content of this section as the following form.

Conclusion 14. *For every ADE surface S, there is a natural Lie algebra bundle of corresponding ADE-type over S. Furthermore, we can construct two natural fundamental representation bundles over S, using lines and rulings on S. Moreover, the Lie algebra bundle can be considered as the automorphism (Lie algebra) bundle of these fundamental representation bundles preserving natural structures.* □

3. Flat G bundles over elliptic curves

In this section we review some well-known results about flat G bundles over elliptic curves.

Let Σ be an elliptic curve with identity element 0. The fundamental group $\pi_1(\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$. Let G be a compact, simple and simply connected Lie group of rank r with root system R, coroot system R_c, Weyl group $W(G)$, root lattice Λ, coroot lattice Λ_c and maximal torus T. The dual lattice $\Lambda_c^\vee$ of Λ_c is the weight lattice. We denote the moduli space of flat G-bundles over Σ by $\mathcal{M}_\Sigma^G$. It is well-known that we have the following isomorphisms.

$$\begin{aligned} \mathcal{M}_\Sigma^G &\cong Hom(\pi_1(\Sigma), G)/ad(G) \\ &\cong Hom(\pi_1(\Sigma), T)/W \\ &\cong T \times T/W \\ &\cong \Sigma \otimes_{\mathbb{Z}} \Lambda_c/W. \end{aligned}$$

The second isomorphism is because of Borel's theorem [5] which says that a commuting pair of elements in G can be diagonalized simultaneously. The last isomorphism comes from

$$Hom(\pi_1(\Sigma), T) = Hom(\pi_1(\Sigma), U(1) \otimes_{\mathbb{Z}} \Lambda_c) \cong Hom(\pi_1(\Sigma), U(1)) \otimes_{\mathbb{Z}} \Lambda_c$$

and

$$Hom(\pi_1(\Sigma), U(1)) \cong Pic^0(\Sigma) \cong \Sigma.$$

A famous theorem [18][19] of Looijenga's says that

$$\Sigma \otimes_{\mathbb{Z}} \Lambda_c / W \cong \mathbb{WP}^r_{s_0=1, s_1, \cdots, s_r},$$

where the latter is the weighted projective space with weights s_i's, and $s_1, \cdots, s_r$ are the coefficients of the highest coroot of R_c.

One element of $Hom(\Lambda, \Sigma)/W$ can only determine a flat $ad(G) = G/C(G)$ bundle in general. For the adjoint group $ad(G)$, the moduli space of flat $ad(G)$ bundles $\mathcal{M}_\Sigma^{ad(G)}$ contains $Hom(\Lambda, \Sigma)/W$ as a connected component (see [11]). On the other hand, we have the following short exact sequences:

$$0 \to \Lambda \to \Lambda_c^\vee \to \Gamma \to 0$$

and

$$0 \to Hom(\Gamma, \Sigma) \to Hom(\Lambda_c^\vee, \Sigma) \to Hom(\Lambda, \Sigma) \to 0.$$

Here Γ is a finite abelian group. The second sequence is exact since Σ is a divisible abelian group. It follows that $Hom(\Lambda, \Sigma)$ and $\Sigma \otimes_{\mathbb{Z}} \Lambda_c$ are isogenous as abelian varieties. Let d be the exponent of the finite group Γ. If we fix a d^{th} root of $\Sigma \cong Jac(\Sigma)$ then we can extend uniquely a homomorphism $f_0 \in Hom(\Lambda, \Sigma)$ to a homomorphism $f \in Hom(\Lambda_c^\vee, \Sigma)$. Hence when we fix a d^{th} root of Σ, we obtain the following isomorphism

$$\mathcal{M}_\Sigma^G \cong Hom(\Lambda, \Sigma)/W.$$

When G is of ADE-type, the root lattice and the coroot lattice coincide, hence the weight lattice is just the dual lattice of the root lattice.

Remark 15. We have constructed ADE (Lie algebra) bundles over ADE rational surfaces. For such a bundle, taking the compact form of its automorphism bundle, we obtain the adjoint Lie group bundle P. When the surface S has a smooth anti-canonical curve Σ, restricting P to Σ (fixing the identity element $0 \in E$), we shall obtain a flat $ad(G)$ bundle of ADE-type over Σ. We can also first restrict the Lie algebra bundles to Σ, and then take the compact form. We still obtain the same flat $ad(G)$ bundle over Σ. To obtain a simple Lie group E_n (resp. D_n), we need to assume that $4 \leq n \leq 8$ (resp. $n \geq 3$).

4. Flat G bundles over elliptic curves and rational surfaces: simply laced cases

From this section on, we fix our ADE surface S to be the rational surface $X_n(x_1, \cdots, x_n)$, $Y_n(x_1, \cdots, x_n)$, or $Z_n(x_1, \cdots, x_n)$. For X_n, we assume $n \leq 8$.

In last section, we saw that once we are given a root system of type E_n (respectively $D_n,\ A_{n-1}$) in the Picard lattice of X_n (respectively $Y_n,\ Z_n$), we can construct a Lie algebra bundle of that type and its natural fundamental representation bundles over this surface. Furthermore, we can construct an adjoint compact Lie group bundle over this surface. To obtain the corresponding Lie group bundle over the fixed elliptic curve Σ by restriction, we need to assume that Σ is an anti-canonical curve of our rational surfaces. That is, we first embed Σ into $\mathbb{P}^2$ as an anti-canonical curve, using the projective embedding ϕ determined by the linear system $|3(0)|$ where (0) is the divisor of the identity element of Σ, and assume that all these blown up points $x_i \in \Sigma$ for $i = 1, \cdots, n$, and that $0, x_1, \cdots, x_n$ are in general position. Moreover, we blow up $\mathbb{P}^2$ at 0 to obtain the embedding of Σ into $\mathbb{F}_1$ as an anti-canonical curve, and take the exceptional curve l_0 as the section s for the ruled surface $\mathbb{F}_1$.

Convention 16. *In Z_n case, it is well-known that in order to obtain a flat $SU(n)$-bundle over Σ we need one more assumption:*

$$\sum x_i = 0 \text{ in } \Sigma.$$

We explain how the moduli space $\mathcal{M}_\Sigma^G$ is related to the moduli space of rational surfaces of the above types. Denote $\mathcal{S}(\Sigma, G)$ the moduli space of the pairs (S, Σ), where S is an ADE rational surface of type the same as that of G and $\Sigma \in |-K_S|$.

Proposition 17. *There exists a well-defined map*

$$\phi: \ \mathcal{S}(\Sigma, G) \to Hom(\Lambda, \Sigma)/W,$$

where Λ is the lattice P_n or P_{n-1} defined in Section 1.

Proof. First we consider the case where $S = X_n$ is a Del Pezzo surface, that is, all blown up points are in general position. Suppose we are given the pair $(X_n, \Sigma \in |-K_{X_n}|\ )$. For each element $y \in P_n$, y stands for a holomorphic line bundle over S. Restricting y to Σ, we obtain a holomorphic line bundle over Σ, denoted by $\mathcal{L}_y$. The degree of $\mathcal{L}_y$ is

$$deg(\mathcal{L}_y) = y \cdot (-K) = 0.$$

So $\mathcal{L}_y$ is an element of the Jacobian of Σ, which is canonically isomorphic to Σ since the identity element of Σ is given. Thus we obtain a map from P_n to $\Sigma : y \mapsto \mathcal{L}_y$, which is obviously a homomorphism of abelian groups. But for one pair (X_n, Σ), we can have different choices of simple roots in order to identify P_n with the root lattice of E_n, and all choices are only differed by the action of the Weyl group $W(E_n)$. So finally we obtain a well-defined map from the moduli space $\mathcal{S}(\Sigma, E_n)$ of such pairs (X_n, Σ) to the projective variety $Hom(P_n, \Sigma)/W(E_n)$.

The other two cases are similar. Roughly speaking, given a pair (Y_n, Σ) (resp. (Z_n, Σ)), we obtain an element in

$$Hom(P_n, \Sigma)/W(D_n) \text{ (resp. } Hom(P_{n-1}, \Sigma)/W(A_{n-1})). \qquad \square$$

In fact we can prove a theorem of Torelli type for the above correspondings. Roughly speaking, the moduli space of the pairs (S, Σ) is isomorphic to

$$Hom(\Lambda, \Sigma)/W,$$

where Λ is our root lattice.

Definition 18. *Let $S = X_n$, Y_n, or Z_n. An exceptional system $\zeta_n = (e_1, \cdots, e_n) \in C_n$ on X_n (resp. Y_n, Z_n) is called a G-configuration for $G = E_n$ (resp. D_n, A_{n-1}) if e_n is a (-1) curve, and after blowing down e_n, e_{n-1} is a (-1) curve. And this process can be proceeded successively until after blowing down e_1, we obtain $\mathbb{P}^2$ (resp. $\mathbb{F}_1$) for $G = E_n$ (resp. D_n and A_{n-1}). Denote ζ_G a G-configuration. When S is equipped with a G-configuration ζ_G, and S has Σ as an anti-canonical curve, we call S a rational surface with G-configuration and denote it by a pair (S, G).*

Equivalently, a G-configuration ζ_{E_n} (resp. ζ_{D_n} or $\zeta_{A_{n-1}}$) on $S = X_n$ (resp. Y_n, Z_n), means that S could be considered as the blow-up of $\mathbb{P}^2$ (resp. $\mathbb{F}_1$, $\mathbb{F}_1$) at n (maybe not distinct) points $y_1, \cdots, y_n \in S$ successively, such that $e_1, \cdots, e_n$ are the corresponding exceptional divisors.

Lemma 19. *Let S be a surface with G-configuration. Then any smooth rational curve on S has a self-intersection number at least -2. Furthermore, in E_n case, all these (-2) curves form chains of ADE-type.*

Proof. Let L be a smooth rational curve on S. Then $L \cdot \Sigma \geq 0$. By adjoint formula, we have $-2 = L^2 + L \cdot K_S$. Since Σ is linearly equivalent to $-K_S$, we have $L^2 \geq -2$. For the last assertion, see [6]. □

On an ADE surface, by Corollary 10, any exceptional system is an ADE-configuration. Thus, we can restate the result of Lemma 4 (ii), Lemma 5 (ii) and Lemma 6 (ii) as follows.

Proposition 20. *For an ADE surface, $W(G)$ acts on the set of all G-configurations simply transitively.* □

This proposition implies that a G-configuration determines exactly an isomorphism from P_n (or P_{n-1} for A_{n-1}) to the corresponding root lattice $\Lambda(G)$.

An A_{n-1}-configuration on Z_n is illustrated in the following figure

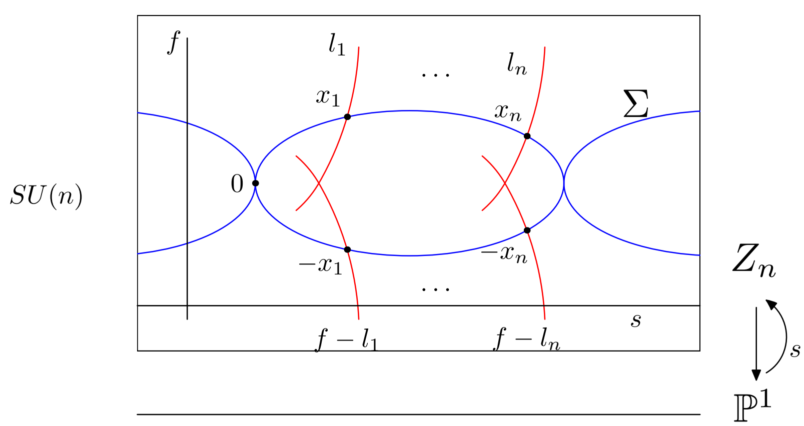

Figure 4. A surface with an A_{n-1}-configuration $(l_1, \cdots, l_n)$.

A D_n-configuration on Y_n is illustrated in the following figure

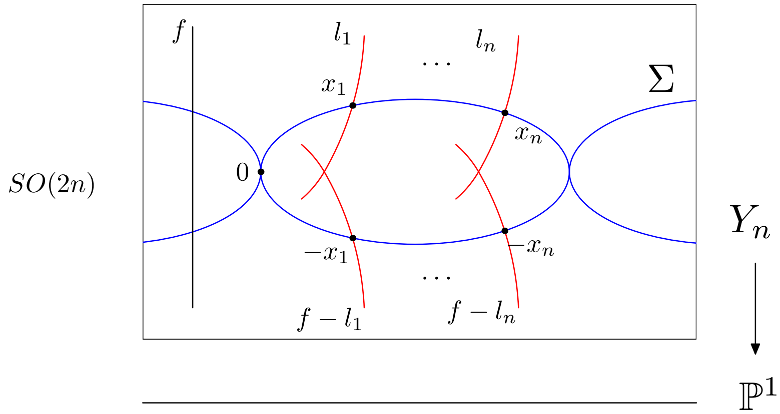

Figure 5. A surface with a D_n-configuration $(l_1, \cdots, l_n)$.

And an E_n-configuration on X_n is illustrated in the following figure

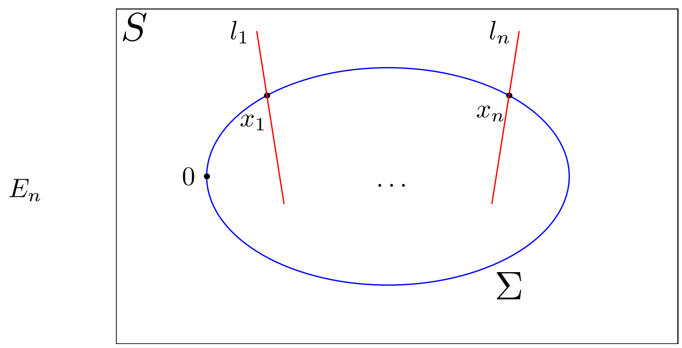

Figure 6. A surface with an E_n-configuration $(l_1, \cdots, l_n)$,

Recall the definition $\zeta_{D_n} = (e_1, \cdots, e_n)$ where $e_i \cdot K_{Y_n} = -1,\ e_i \cdot f = 0,\ e_i \cdot e_j = \delta_{ij}$ and $\sum e_i \cdot s \equiv 0 \ mod\ 2$. Next we explain geometrically why we need to assume that $\sum e_i \cdot s \equiv 0 \ mod\ 2$.

Definition 21. *Let $C \subset \mathbb{P}^2$ be a curve of degree d. A point $P \in C$ is called a ordinary k-fold point of C if P is a k-fold singular point and C has k distinct tangent directions at P.*

Lemma 22. *Let C be a plane curve of degree d with an ordinary $(d-1)$-fold point P. Then*

(i) P is the only singular point of C.

(ii) The normalization of C is a smooth rational curve.

(iii) Fix a point $P \in \mathbb{P}^2$. Then the variety of all plane curves of degree d with P as an ordinary $(d-1)$-fold point is of dimension $2d$.

(iv) Given P and other $2d$ generic points, there exists a unique curve $C \subset \mathbb{P}^2$ of degree d, such that C has P as an ordinary $(d-1)$-fold point and passes through these $2d$ generic points.

Proof. (i) Apply Bezout's theorem. (ii) Apply the genus formula. (iii) Let $[x, y, z]$ be the homogenous coordinates of $\mathbb{P}^2$, and $P = [1, 0, 0]$. Then C is defined by the polynomial

$$f(x, y, z) = g(y, z) + \prod_{i=1}^{d-1}(a_i y - b_i z)x,$$

where $deg(g) = d$. Therefore, the dimension is $2d$. □

Proposition 23. *Let Σ be embedded into $\mathbb{F}_1$ (with section s) as a smooth anti-canonical curve and $x_1, \cdots, x_n$ are distinct points of Σ. Blowing up $\mathbb{F}_1$ at x_i's we obtain Y_n with corresponding exceptional curves $l_i, i = 1, \cdots, n$.*

(i) When $n = 2k$, if $x_1, \cdots, x_n$ are in general position, then after contracting $f-l_1, \cdots, f-l_n$, we still obtain the surface $\mathbb{F}_1$. In other words, we obtain the same surface Y_{2k} by blowing up either $\{x_1, \cdots, x_n\}$, or $\{-x_1, \cdots, -x_n\}$.

(ii) When $n = 2k+1$, if $x_1, \cdots, x_n$ are in general position, then after contracting $f - l_1, \cdots, f - l_n$, the resulting surface is $\mathbb{P}^1 \times \mathbb{P}^1$, but not $\mathbb{F}_1$.

Proof. Let C be a negative rational curve in Y_n which doesn't intersect $f - l_i, i = 1, \cdots, n$. Then C satisfies the following equations

$$\begin{cases} C \cdot C = -m, m > 0; \\ C \cdot K = m - 2; \\ C \cdot (f - l_i) = 0, i = 1, \cdots, n. \end{cases}$$

Since C is a rational curve and $\Sigma \in |-K|$, $C \cdot (-K) \geq 0$. So $m \leq 2$. Then $m = 1$ or 2. Considering $\mathbb{F}_1$ as the blow-up of $\mathbb{P}^2$ at $0 \in \Sigma$ with exceptional curve s, we can assume $C = a \cdot h - b \cdot s - \sum c_i \cdot l_i, a \geq 0, b \geq 0, c_i \geq 0$. Solving the system of equations, we obtain

$$\begin{cases} m = 1 \text{ or } 2, \\ b = a - 1, \\ c_i = 1, i = 1, \cdots, n, \\ a = (n - 1 + m)/2. \end{cases}$$

For $m = 1$, $n = 2a$ is even. The class

$$C = ah - (a-1)s - \sum_{i=1}^{n=2a} l_i = af + s - \sum_{i=1}^{2a} l_i.$$

This means that all of the points $0, x_1, \cdots, x_n$ lie on the curve $\pi(C)$, where $\pi : Y_n \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ successively at $0, x_1, \cdots, x_n$. There exists exactly one such curve C for generic $x_1, \cdots, x_n$, and it is smooth, by Lemma 22. Hence, after contracting $f - l_1, \cdots, f - l_{2a}$, we still obtain $\mathbb{F}_1$.

For $m = 2$, $n = 2a + 1$ is odd. The class

$$C = ah - (a-1)s - \sum_{i=1}^{n=2a+1} l_i = af + s - \sum_{i=1}^{2a+1} l_i.$$

This means that all of the points $0, x_1, \cdots, x_n$ lie on the curve $\pi(C)$, where $\pi : Y_n \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ successively at $0, x_1, \cdots, x_n$. There exists no such curves for generic $x_1, \cdots, x_n$, by Lemma 22. Hence, after contracting $f - l_1, \cdots, f - l_{2a+1}$, no rational curves with negative self-intersection number can survive. Therefore the resulting surface is $\mathbb{P}^1 \times \mathbb{P}^1$, but not $\mathbb{F}_1$. □

Example 24. Blowing up $\mathbb{F}_1$ at 2 points x_1, x_2 we obtain Y_2. Contracting $f-l_1$ and $f-l_2$, or contracting l_1 and l_2, we always obtain the surface $\mathbb{F}_1$. But contracting $f-l_1$ and l_2, we just obtain the surface $\mathbb{P}^1 \times \mathbb{P}^1$, not $\mathbb{F}_1$!

Remark 25. (i) Lemma 22 has a corresponding version for $\mathbb{P}^1 \times \mathbb{P}^1$.

(ii) A G-configuration $\zeta_G = (e_1, \cdots, e_n)$ for $S = X_n$ (resp. Y_n, Z_n) just means that after blowing down $e_n, e_{n-1}, \cdots, e_1$ successively, we still obtain $\mathbb{P}^2$ (resp. $\mathbb{F}_1$, $\mathbb{F}_1$).

Let S be an ADE surface equipped with a G-configuration ζ_G. we denote the moduli space of the pairs (S, Σ) by $\mathcal{S}(\Sigma, G)$, where two pairs (S, Σ) and $(S^{'}, \Sigma^{'})$ are equivalent if and only if there is an isomorphism π from S to S' such that $\pi|_\Sigma$ is also an isomorphism from Σ to Σ'.

We show that $\mathcal{S}(\Sigma, G)$ is isomorphic to an open dense subset U of the variety $Hom(\Lambda, \Sigma)/W$. In fact, for any element $\theta \in (Hom(\Lambda, \Sigma)/W) \backslash U$, the boundary component, we can find possibly non-equivalent pairs (S, Σ) such that θ comes from the restriction. Thus, we can complete $\mathcal{S}(\Sigma, G)$ by adding these pairs and identifying them as one point. Denote the completion by $\overline{\mathcal{S}(\Sigma, G)}$. Then we can identify $\overline{\mathcal{S}(\Sigma, G)}$ with the projective variety $Hom(\Lambda, \Sigma)/W$. This provides a natural compactification for the moduli space $\mathcal{S}(\Sigma, G)$.

More precisely, let $S = X_n$ (respectively, Y_n, Z_n) be an ADE surface and Λ be the root lattice of E_n (respectively, D_n, A_{n-1}) with corresponding Weyl group W. And we fix a 3^{rd} (respectively, 2^{nd}, n^{th}) root of Σ in E_n (respectively, D_n, A_{n-1}) case. Then we have

Theorem 26. *(i) There is an injective map ϕ from the moduli space $\mathcal{S}(\Sigma, G)$ onto an open dense subset of $Hom(\Lambda, \Sigma)/W$.*

(ii) ϕ can be extended to a bijective map from the completion $\overline{\mathcal{S}(\Sigma, G)}$ onto $Hom(\Lambda, \Sigma)/W$.

(iii) Moreover, the completion is obtained by including all rational surfaces with G-configurations to $\mathcal{S}(\Sigma, G)$. Any smooth rational curve on a surface corresponding to a boundary point has a self-intersection number at least -2, and in E_n case these (-2) curves form chains of ADE-type.

Proof. First we suppose $S = X_n$. We have constructed the map ϕ in Proposition 17. We prove the injectivity. Fix a G-configuration $\zeta_G = (l_1, \cdots, l_n)$ on X_n, and a simple root system $\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, \alpha_3 = h - l_1 - l_2 - l_3, \alpha_4 = l_3 - l_4, \cdots, \alpha_n = l_{n-1} - l_n$. Blowing down $l_n, l_{n-1}, \cdots, l_1$ successively, we obtain $\mathbb{P}^2$ with Σ as an anti-canonical curve. For all $i = 1, \cdots, n$, let $x_i \in X_n$ be the unique intersection points of l_i and Σ. Then X_n can be considered as a blow-up of $\mathbb{P}^2$ at these n points $x_i \in \Sigma, i = 1, \cdots, n$ with exceptional curves $l_i, i = 1, \cdots, n$.

According to previous arguments, we have a homomorphism $g \in Hom(\Lambda, \Sigma)$. Let $g(\alpha_i) = p_i \in \Sigma$, then we have the following equations by the group law of Σ as an abelian group

$$\begin{cases} x_1 - x_2 = p_1, \\ x_2 - x_3 = p_2, \\ -x_1 - x_2 - x_3 = p_3, \\ x_{k-1} - x_k = p_k, k = 4, \cdots, n. \end{cases}$$

The determinant of the coefficient matrix of this system of linear equations is ± 3. So it has unique solution (if we fix a 3^{rd} root of Σ). That is, x_i's are uniquely determined by g up to Weyl group actions. The Weyl group actions just lead to choices of other G-configurations. By Proposition 20, this doesn't change the pair (X_n, Σ). Hence, ϕ is injective. These points x_i's are not ″in general position″ if and only if p_i's will satisfy some (finitely many) equations. That means the image of ϕ must be open dense in $Hom(\Lambda, \Sigma)/W$. The extendability of ϕ is also because of the existence and uniqueness of the solution of the above equations.

For the cases of Y_n and Z_n, the arguments is similar. It is easy to see that the map ϕ is well defined in both cases. For Y_n, the system of linear equations is

$$\begin{cases} -x_1 - x_2 = p_1, \\ x_{k-1} - x_k = p_k, k = 2, \cdots, n. \end{cases}$$

The determinant is ± 2. So the solution is uniquely determined (if we fix a 2^{nd} root of Σ). The remained arguments is just like the first case. At last, for the case of Z_n, the system of equations is

$$\begin{cases} \sum x_i = 0, \\ x_{k-1} - x_k = p_{k-1}, k = 2, \cdots, n. \end{cases}$$

The determinant is $\pm n$. Then the solution is uniquely determined (if we fix an n^{th} root of Σ). The remaining arguments are just the same as that in the E_n case. These prove (i) and (ii).

As for (iii), the result follows from Lemma 19. □

In ADE case, when the finite group $\Lambda_c^\vee/\Lambda$ (the fundamental group of the Lie group $ad(G)$) is non-trivial, and of exponent d, a homomorphism ϕ_0 from Λ to Σ would determine only an $ad(G)$ bundle. For a given pair (S, Σ), suppose we are given a distinguished d^{th} root of $\Sigma \cong Jac(\Sigma)$. Then there is a homomorphism ϕ from the weight lattice to Σ, which extending and determined uniquely by ϕ_0. And ϕ will determine a G bundle on Σ. Thus we can still identify the moduli space of pairs and the moduli space of flat G bundles. Precisely, the construction is as following. In E_n case, we fix a $d^{th}(d = 9-n)$ root of Σ and take the d^{th} root $\mathcal{N}_0$ of the line bundle $\mathcal{O}_\Sigma(-K)$ and define the homomorphism by $\mu \in Hom(\Lambda_c^\vee, \Sigma) : y \mapsto L_y \otimes \mathcal{N}_0^{y \cdot K}$. In D_n case, when n is even, we fix a 2^{nd} root of Σ, and take the 2^{nd} root $\mathcal{N}_0$ of the line bundle $\mathcal{O}_\Sigma(f)$ and a 2^{nd} root $\mathcal{N}_1$ of the line bundle $\mathcal{O}_\Sigma(K + (4 - n/2)f)$, and define the homomorphism by $\mu \in Hom(\Lambda_c^\vee, \Sigma) : y \mapsto L_y \otimes \mathcal{N}_0^{y \cdot K} \otimes \mathcal{N}_1^{y \cdot f}$; when n is odd, we fix a 4^{th} root of Σ, and take the 2^{nd} root $\mathcal{N}_0$ of the line bundle $\mathcal{O}_\Sigma(f)$ and a 4^{th} root $\mathcal{N}_1$ of the line bundle $\mathcal{O}_\Sigma(2K + (8-n)f)$, and define the homomorphism by $\mu \in Hom(\Lambda_c^\vee, \Sigma) : y \mapsto L_y \otimes \mathcal{N}_0^{y \cdot K} \otimes \mathcal{N}_1^{y \cdot f}$. In A_{n-1} case, we fix an n^{th} root of $Jac(\Sigma) = \Sigma$ and take the n^{th} root $\mathcal{N}_0$ of the line bundle $\mathcal{O}_\Sigma(K + 2s + 2f)$ and define the homomorphism by $\mu \in Hom(\Lambda_c^\vee, \Sigma) : y \mapsto L_y \otimes \mathcal{N}_0^{y(K+2f+2s)} \otimes \mathcal{O}(-s)^{y \cdot f} \otimes \mathcal{O}(-f)^{y \cdot s}$. To apply Theorem 26, we need to fix an e^{th} root of Σ where $e = LCM(3, d)$. Then we obtain the following identification.

Theorem 27. *Under the above construction, we have a bijection*

$$\overline{\mathcal{S}(\Sigma, G)} \xrightarrow{\sim} Hom(\Lambda_c^\vee, \Sigma)/W \cong (\Lambda_c \otimes \Sigma)/W \cong \mathcal{M}_\Sigma^G.$$

Proof. We just prove it for D_n (n is odd) case. For other cases, the proof is similar. Since the exponent of Γ is $d = 4$, we fix a 4^{th} root of $Jac(\Sigma)$. Since the degree of $\mathcal{O}_\Sigma(f)$ is $f \cdot (-K) = 2$, there is a point $0 \in \Sigma$ determined uniquely by the 4^{th} root of $Jac(\Sigma)$, such that $\mathcal{O}_\Sigma(f) = \mathcal{O}_\Sigma(2(0)) = \mathcal{O}_\Sigma((0))^2$. Take $\mathcal{N}_0 = \mathcal{O}_\Sigma((0))$. Since the degree of the bundle $\mathcal{O}_\Sigma(2K+(8-n)f)$ is $2(n-8)+2(8-n)=0$, it is an element of $Jac(\Sigma)\cong\Sigma$. Of course $\mathcal{O}_\Sigma(2K+(8-n)f)$ is 4-divisible. The degree of $\mu(y)$ is 0, so μ is a homomorphism from $\Lambda_c^\vee$ to $Jac(\Sigma)\cong\Sigma$ induced by the homomorphism $\mu_0(y)=L_y$. And μ is uniquely determined by μ_0. □

Remark 28. [23][11][12]. The moduli space of flat A_n bundles over Σ is exactly the ordinary projective space $\mathbb{CP}^n$. This can be described as follows: a flat $SU(n+1)$ bundle is determined uniquely by $n+1$ points on Σ with sum equal to 0, up to isomorphism. And $n+1$ points on Σ with sum equal to 0 are determined uniquely by a global section $H^0(\Sigma,\mathcal{O}_\Sigma(n(0)))$ up to scalar, where (0) is the divisor of the identity element 0. So the moduli space of flat $SU(n+1)$ bundles is isomorphic to $\mathbb{P}(H^0(\Sigma,\mathcal{O}_\Sigma((n+1)P)))=\mathbb{P}^n$. From this we see that the moduli space of pairs $(S,\ \Sigma)$ is just the ordinary complex projective space $\mathbb{P}^n$.

Example 29. Let us look at what the pre-image of a trivial G-bundle is. For example, in E_8 case, the trivial bundle means the element $0\in Hom(\Lambda(E_8),\Sigma)/W(G)$. By the above correspondence, all $x_i=0$ in Σ. This means that we can blow up $\mathbb{P}^2$ at the identity element 0 (an inflection point) eight times to obtain the surface represented by this pre-image, which is a boundary point in the moduli space $\overline{\mathcal{S}(\Sigma,G)}$. Blowing up once more, we obtain an elliptic fibration with a singular fiber of $\widetilde{E_8}$-type [4].

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