HARD LEFSCHETZ ACTIONS IN RIEMANNIAN GEOMETRY WITH SPECIAL HOLONOMY

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Abstract. It is known that the hard Lefschetz action, together with Kähler identities for Kähler (resp. hyperkähler) manifolds, determines a $\mathfrak{su}(1,1)_{sup}$ (resp. $\mathfrak{sp}(1,1)_{sup}$) Lie superalgebra action on differential forms. In this paper, we explain the geometric origin of this action, and we also generalize it to manifolds with other holonomy groups.

For semi-flat Calabi-Yau (resp. hyperkähler) manifolds, these symmetries can be enlarged to a $\mathfrak{so}(2,2)_{sup}$ (resp. $\mathfrak{su}(2,2)_{sup}$) action.

1. Introduction

Lefschetz's work (see e.g. [1]) related the topology of a complex projective manifold M with its hyperplane section. In modern terminology, this implies the cohomology group of M admits a natural $\mathfrak{sl}(2,\mathbb{R})$ action. This is the celebrated hard Lefschetz theorem. Hodge (see e.g. [5]) reinterpreted this action on the level of differential forms $\Omega^\bullet(M)$ which commutes with Laplacian operator. Thus the hard Lefschetz theorem follows from the Hodge theorem. Furthermore if we consider the vector space $\mathbb{C}^2 \oplus \mathbb{R}$ spanned by $\partial, \bar{\partial}, \partial^*, \bar{\partial}^*$ and Δ, then all Kähler identities, for instances $[L, \partial^*] = i\bar{\partial}$ and $\Delta = 2\Delta_{\bar{\partial}}$, can be combined with the hard Lefschetz action to give a Lie superalgebra action of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{C}^2 \oplus \mathbb{R}$ on $\Omega^\bullet(M)$.

There is an analogous theorem for hyperkähler manifolds M, namely there is a Lie superalgebra action of $\mathfrak{so}(4,1) \oplus \mathbb{C}^4 \oplus \mathbb{R}$ on $\Omega^\bullet(M)$. The $\mathfrak{so}(4,1)$ part of this action on $H^*(M)$, by zeroth order operators, was discovered by Verbitsky in [13]. Following a suggestion of Witten, Figueroa-O'Farrill, Köhl and Spence [4] gave a physical interpretation of all these actions in terms of supersymmetric algebra in sigma models. It was further studied by Cao and Zhou in [3].

The followings are two natural questions which will be answered in this paper: (1) What is the geometric origin of these Lie superalgebra actions on the spaces of differential forms on Kähler manifolds (i.e. $U(n)$ holonomy) and hyperkähler manifolds (i.e. $Sp(n)$ holonomy)? (2) Are there analogous hard Lefschetz type results for manifolds with other holonomy groups, for example quaternionic-Kähler manifolds, G_2-manifolds and $Spin(7)$-manifolds?

In [9] the first author revisited the Berger classification of holonomy groups of Riemannian manifolds which are not locally symmetric spaces. Given any normed algebra $\mathbb{K}$, which must be one of $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, we defined the notion of $\mathbb{K}$-manifolds. Their holonomy groups are precisely $O(n), U(n), Sp(n)Sp(1)$ and $Spin(7)$ respectively. If they are also $\mathbb{K}$-oriented, then their holonomy groups reduce to $SO(n), SU(n), Sp(n)$ and G_2 respectively.

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Note that $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1)$ and $\mathfrak{so}(4,1) \cong \mathfrak{sp}(1,1)$. For any normed algebra $\mathbb{K}$, we could define analogously a Lie algebra $\mathfrak{su}_{\mathbb{K}}(1,1)$ and a Lie superalgebra $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup} = \mathfrak{su}_{\mathbb{K}}(1,1) \oplus \mathbb{K}^{1,1} \oplus \mathbb{R}$. On any $\mathbb{K}$-manifold M, we will construct a natural Lie superalgebra bundle $E^{\mathfrak{su}}$ with fiber $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$. To relate this to the hard Lefschetz action, we use the fact that differential forms on M can be regarded as spinors for the direct sum $T \oplus T^*$ of the tangent and cotangent bundles of M, which admits a tautological quadratic form of type (m,m). Roughly speaking, we have the following bundle,

$$\mathbb{K}^{n,n} \to T \oplus T^* \to M.$$

Using the Clifford algebra for $\mathbb{K}^{n,n}$ and the Dirac operator, we construct differential operators of order zero, one and two on $\Omega^\bullet(M)$. For example, the second order operator is simply the Laplacian operator Δ. We will show that all these operators together with their commutating relations, which in case of Kähler manifolds are the hard Lefschetz action and Kähler identities, generate a Lie superalgebra $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ action. We have

Theorem 1.1. *Let M be an oriented Riemannian manifold. Suppose M is a $\mathbb{K}$-manifold with $\mathbb{K}$ a normed algebra, i.e. $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. Then there is a Lie superalgebra bundle $E^{\mathfrak{su}}$ over M with fiber $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$:*

$$\mathfrak{su}_{\mathbb{K}}(1,1) \oplus \mathbb{K}^{1,1} \oplus \mathbb{R} \to E^{\mathfrak{su}} \to M.$$

When $\mathbb{K}$ is associative, i.e. $\mathbb{K} \neq \mathbb{O}$, each section of $E^{\mathfrak{su}} \to M$ determines a differential operator of order at most two on differential forms on M. Thus, we have

$$\Psi : \Gamma(M, E^{\mathfrak{su}}) \to \mathrm{Diff}(\textstyle\bigwedge^\bullet T^*, \bigwedge^\bullet T^*).$$

Furthermore, composing Ψ with the symbol map gives a Lie superalgebra homomorphism

$$\sigma \circ \Psi : \Gamma(M, E^{\mathfrak{su}}) \to \mathrm{Symb}(\textstyle\bigwedge^\bullet T^*, \bigwedge^\bullet T^*).$$

We call this the *super hard Lefschetz action* for $\mathbb{K}$-manifolds.

When $E^{\mathfrak{su}}$ is trivial, we can take constant sections of $E^{\mathfrak{su}}$ and obtain a Lie superalgebra action of $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ on $\Omega^\bullet(M)$. This happens when the holonomy group of M is inside $SO(n), U(n)$ or $Sp(n)$. When M is compact, the $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ action on $\Omega^\bullet(M)$ descends to the cohomology $H^*(M)$ by Hodge theory, for which only $\mathfrak{su}_{\mathbb{K}}(1,1)$ acts non-trivially on $H^*(M)$. Our results apply equally well for every normed algebra. However, it is more involved to describe precisely the algebraic relations for the super hard Lefschetz action for $\mathbb{O}$-manifolds due to the non-associative nature of $\mathbb{O}$ (see Theorem 3.15 for details).

For Calabi-Yau manifolds M, the "mirror" of the hard Lefschetz action should give us another $\mathfrak{sl}(2,\mathbb{R})$-action, at least in the semi-flat limit. They combine to form a $\mathfrak{so}(2,2)$-action on differential forms on M [10]. We can adapt our method easily to this case and obtain an enlarged super hard Lefschetz action for semi-flat Calabi-Yau and hyperkähler manifolds. For $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, we write $\mathbb{K}' = \mathbb{R}$ or $\mathbb{C}$ respectively, and we have

Theorem 1.2. *Suppose that M is a semi-flat $\mathbb{K}$-manifold with $\mathbb{K}$ being $\mathbb{C}$ or $\mathbb{H}$. Then there is a natural $\mathfrak{su}_{\mathbb{K}'}(2,2)_{sup}$ action, extending the super hard Lefschetz $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ action, on the space of differential forms on M via differential operators of order at most two.*

This paper is organized as follows. In section 2, we construct the Lie superalgebra $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$-bundle $E^{\mathfrak{su}}$ over a $\mathbb{K}$-manifold and introduce a (Lie superalgebra) bundle morphism ι. In section 3, we construct differential operators via spin actions, apply them to $\mathbb{K}$-manifolds and prove our main theorems. In section 4, we obtain $\mathfrak{so}(2,2)_{sup}$ (resp. $\mathfrak{su}(2,2)_{sup}$) action on differential forms on semi-flat Calabi-Yau (resp. hyperkähler) manifolds. Finally in the appendix, we interpret the differential operators we constructed in terms of the usual ones.

2. Lie superalgebra bundles over $\mathbb{K}$-manifolds

In this section, we first introduce the notion of a $\mathbb{K}$-manifold in terms of its holonomy group $G_{\mathbb{K}}$. We then introduce a Lie superalgebra $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ and construct a $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$-bundle $E^{\mathfrak{su}}$ over any $\mathbb{K}$-manifold. Finally, we show that there exists another Lie superalgebra bundle E over any $\mathbb{K}$-manifold, and we introduce a natural bundle morphism $\iota : E^{\mathfrak{su}} \to E$.

2.1. $G_{\mathbb{K}}$ and $\mathbb{K}$-manifolds.

A normed algebra $\mathbb{K}$ is a finite dimensional real algebra with unit 1 and a norm $\|\cdot\|$ satisfying $\|a \cdot b\| = \|a\| \cdot \|b\|$ for any $a, b \in \mathbb{K}$. It is a classical fact that $\mathbb{K}$ is exactly (isomorphic to) one of the following four algebras: the real $\mathbb{R}$, the complex $\mathbb{C}$, the quaternion $\mathbb{H}$ and the octonion $\mathbb{O}$.

For $m = n \cdot \dim_{\mathbb{R}} \mathbb{K}$, where $n = 1$ if $\mathbb{K} = \mathbb{O}$, we can identify $V = \mathbb{R}^m$ with $\mathbb{K}^n$. The standard metric on V gives an inner product on $\mathbb{K}^n$ satisfying $g(x \cdot \alpha, y \cdot \alpha) = g(x, y)\|\alpha\|^2$ for any $x, y \in V$ and $\alpha \in \mathbb{K}$.

Definition 2.1. *A twisted isomorphism ϕ of V is a $\mathbb{R}$-isometry ϕ of V such that there exists $\theta \in SO(\mathbb{K})$ with the property $\phi(x\alpha) = \phi(x)\theta(\alpha)$ for any $x \in V$ and any $\alpha \in \mathbb{K}$. ϕ is called special if it preserves the "$\mathbb{K}$-orientation" in terms of "$\lambda_{\mathbb{K}}(\phi)$" as defined in* [9].

We denote by $G_{\mathbb{K}}(n)$ (resp. $H_{\mathbb{K}}(n)$) the group of (resp. special) twisted isomorphisms of V.

Definition 2.2. *A Riemannian manifold (M, g) is called a (resp. special) $\mathbb{K}$-manifold, if the holonomy group of its Levi-Civita connection is a subgroup of $G_{\mathbb{K}}(n)$ (resp. $H_{\mathbb{K}}(n)$) with $m = \dim M = n \cdot \dim \mathbb{K}$.*

From the viewpoint of normed algebras, (non-locally symmetric) Riemannian manifolds with various holonomy groups are classified as follows [9].

$\mathbb{K}$	$G_{\mathbb{K}}(n)$ ($\mathbb{K}$-manifolds)	$H_{\mathbb{K}}(n)$ (Special $\mathbb{K}$-manifolds)
$\mathbb{R}$	$O(n)$ (Riemannian manifolds)	$SO(n)$ (Oriented Riemannian manifolds)
$\mathbb{C}$	$U(n)$ (Kähler manifolds)	$SU(n)$ (Calabi-Yau manifolds)
$\mathbb{H}$	$Sp(n)Sp(1)$ (Quaternionic-Kähler manifolds)	$Sp(n)$ (Hyperkähler manifolds)
$\mathbb{O}$	$Spin(7)$ (Spin(7)-manifolds)	G_2 (G_2-manifolds)

In this paper, we denote $G_{\mathbb{K}}(n)$ (resp. $H_{\mathbb{K}}(n)$) by $G_{\mathbb{K}}$ (resp. $H_{\mathbb{K}}$) whenever the dimension is well understood.

2.2. **$\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$-bundles over $\mathbb{K}$-manifolds.** Let $\mathbb{K}$ be a normed algebra, and $\mathrm{Mat}(2,\mathbb{K})$ be 2×2 matrices with entries in $\mathbb{K}$.

2.2.1. $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$. Each matrix $A\in\mathrm{Mat}(2,\mathbb{K})$ induces a real endomorphism $\phi_A:\mathbb{K}^2\to\mathbb{K}^2; u=(u_1,u_2)\mapsto\phi_A(u)=uA^\star$, where $A^\star\triangleq(\overline{A_{ij}})^T$. Denote $(\mathbb{K}^2,\check{q})$ by $\mathbb{K}^{1,1}$, where $\check{q}$ is the quadratic form of type $(\dim_{\mathbb{R}}\mathbb{K},\dim_{\mathbb{R}}\mathbb{K})$ defined by $\check{q}(u,v)\triangleq\mathrm{Re}\big(\frac{1}{2}(u_1\bar{v}_2+u_2\bar{v}_1)\big)$ for any $u,v\in\mathbb{K}^2$.

Since $\mathbb{H}$ is non-commutative and $\mathbb{O}$ is the worst for its non-associativity, it is a little tricky to define $\mathfrak{sl}(2,\mathbb{K})$ uniformly. Following [2], we define $\mathfrak{sl}(2,\mathbb{K})$ to be the real Lie algebra of operators on $\mathbb{K}^2$ generated by $\{\phi_A\mid A_{11}+A_{22}=0,\ A=\big(A_{ij}\big)\in\mathrm{Mat}(2,\mathbb{K})\}$. And we use the following notations:

$$\mathfrak{su}_{\mathbb{K}}(1,1)\triangleq\{\phi\in\mathfrak{sl}(2,\mathbb{K})\mid\check{q}(\phi(u),v)+\check{q}(u,\phi(v))=0,\forall u,v\in\mathbb{K}^2\};$$
$$\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}\triangleq\mathfrak{su}_{\mathbb{K}}(1,1)\oplus\mathbb{K}^{1,1}\oplus\mathbb{R}.$$

In fact, $\mathfrak{sl}(2,\mathbb{K})$ and $\mathfrak{su}_{\mathbb{K}}(1,1)$ are isomorphic to classical Lie algebras below (see the appendix for more details).

$\mathbb{K}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathfrak{sl}(2,\mathbb{K})$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$\mathfrak{su}_{\mathbb{K}}(1,1)$	$\mathfrak{so}(1,1)$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(4,1)$	$\mathfrak{so}(8,1)$

Furthermore, $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ is naturally a Lie superalgebra because of the following remark.

Remark 2.3. *Let Q be a quadratic form on a real vector space W, and let $\mathfrak{a}$ be a Lie subalgebra of $\mathfrak{so}(W,Q)$. Then $\mathfrak{a}\oplus W\oplus\mathbb{R}$ is naturally a Lie superalgebra with the following super Lie bracket: $\forall\phi,\psi\in\mathfrak{a},\forall u,v\in W,\forall a,b\in\mathbb{R}$,*

$$[\phi+a,\psi+b]=\phi\psi-\psi\phi,\quad [u,v]=-2Q(u,v),\quad [\phi+a,u]=\phi(u).$$

2.2.2. $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$*-bundles.* Let (M,g) be a $\mathbb{K}$-manifold. Since $\mathrm{Hol}(g)\subset G_{\mathbb{K}}$, its frame bundle can be reduced to a principal $G_{\mathbb{K}}$-bundle $P_{G_{\mathbb{K}}}$

By Definition 2.1, there exists a unique $\theta\in SO(\mathbb{K})$ associated to $\phi\in G_{\mathbb{K}}$. In fact, it induces an action Φ of $G_{\mathbb{K}}$ on $\mathbb{K}^{1,1}$ by $\phi\cdot u\triangleq(\theta(u_1),\theta(u_2))$ for any $u\in\mathbb{K}^{1,1}$. It is easy to show that $\Phi(G_{\mathbb{K}})\subset SO(\mathbb{K}^2,\check{q})$ and that $Ad\circ\Phi$ preserves the Lie subalgebra $\mathfrak{su}_{\mathbb{K}}(1,1)\subset\mathfrak{so}(\mathbb{K}^2,\check{q})$. Therefore, Φ induces an action $Ad\circ\Phi$ of $G_{\mathbb{K}}$ on $\mathfrak{su}_{\mathbb{K}}(1,1)$. We take the trivial action of $G_{\mathbb{K}}$ on $\mathbb{R}$, and simply denote by Φ all these actions. Hence, there exist the following associated bundles over the $\mathbb{K}$-manifold M:

$$E_0^{\mathfrak{su}}\triangleq P_{G_{\mathbb{K}}}\times_{\Phi}\mathfrak{su}_{\mathbb{K}}(1,1),\quad E_1^{\mathfrak{su}}\triangleq P_{G_{\mathbb{K}}}\times_{\Phi}\mathbb{K}^{1,1},\quad E_2^{\mathfrak{su}}\triangleq P_{G_{\mathbb{K}}}\times_{\Phi}\mathbb{R}.$$

Note that Φ preserves the super Lie bracket of $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$, we have

Proposition 2.4. *There exists a Lie superalgebra bundle $E^{\mathfrak{su}}=E_0^{\mathfrak{su}}\oplus E_1^{\mathfrak{su}}\oplus E_2^{\mathfrak{su}}$ over any $\mathbb{K}$-manifold M with fiber $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$.*

Example 2.5. *The action of $G_{\mathbb{K}}$ (resp. $H_{\mathbb{K}}$) on $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ is trivial, if and only if $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ (resp. $\mathbb{K}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$). Therefore, $E^{\mathfrak{su}}$ is trivial, if and only if $Hol(g)\subset O(n),U(n)$ or $Sp(n)$.*

2.3. **Lie superalgebra bundle morphisms over $\mathbb{K}$-manifolds.**

2.3.1. *$\mathcal{L}$-bundles over $\mathbb{K}$-manifolds.* Let g be an inner product on a real vector space V, and let Q be the natural quadratic form on $W = V \oplus V^*$ given by

$$Q(X+\xi, Y+\eta) \triangleq \frac{\eta(X)+\xi(Y)}{2}$$

for any $X, Y \in V$ and any $\xi, \eta \in V^*$. It induces a quadratic form $\hat{Q}$ on $\mathrm{Hom}(V^*, W) \cong V \otimes W$ given by $\hat{Q}(v_1 \otimes w_1, v_2 \otimes w_2) \triangleq g(v_1, v_2)Q(w_1, w_2)$. Note that the induced action of $\mathfrak{so}(W, Q)$ on $V \otimes W$ preserves $\hat{Q}$. Hence, it follows from Remark 2.3 that

$$\mathcal{L} \triangleq \mathfrak{so}(W, Q) \oplus \mathrm{Hom}(V^*, W) \oplus \mathbb{R}$$

is naturally a Lie superalgebra.

Let (M, g) be a $\mathbb{K}$-manifold of real dimension m. The natural action of $G_{\mathbb{K}} \subset O(m)$ on $V = \mathbb{R}^m$ induces actions on $\mathfrak{so}(W, Q)$, $\mathrm{Hom}(V^*, W)$ and $\mathbb{R}$ respectively in the standard way, which we also denote by Φ. Hence, there exist the following associated vector bundles over M:

$$E_0 \triangleq P_{G_{\mathbb{K}}} \times_{\Phi} \mathfrak{so}(W, Q), \quad E_1 \triangleq P_{G_{\mathbb{K}}} \times_{\Phi} \mathrm{Hom}(V^*, W), \quad E_2 \triangleq P_{G_{\mathbb{K}}} \times_{\Phi} \mathbb{R}.$$

In fact, $E_0 = \bigwedge^2(T \oplus T^*)$, $E_1 = \mathrm{Hom}(T^*, T \oplus T^*)$ and E_2 is a trivial line bundle. From the above discussion, we have the following proposition.

Proposition 2.6. *There exists a natural Lie superalgebra bundle $E = E_0 \oplus E_1 \oplus E_2$ over any $\mathbb{K}$-manifold M with fiber $\mathcal{L}$.*

2.3.2. *Lie superalgebra bundle morphisms.* Let (M, g) be a $\mathbb{K}$-manifold of real dimension m. Note that $TM = P_{G_{\mathbb{K}}} \times_{\Phi} V$, where $V = \mathbb{R}^m$ is identified with $\mathbb{K}^n$.

There is a natural monomorphism of Lie algebras

$$\iota : \mathfrak{su}_{\mathbb{K}}(1,1) \hookrightarrow \mathfrak{so}(W, Q)$$

defined as follows. If $\mathbb{K}$ is associative, $\iota(L)$ is given by the following procedure

$$\iota(L) : V \oplus V^* \xrightarrow{\psi_1} \mathbb{K}^n \oplus \mathbb{K}^n \xrightarrow{\psi_2} \mathbb{K}^n \otimes_{\mathbb{K}} \mathbb{K}^{1,1} \xrightarrow{\mathrm{Id} \otimes L} \mathbb{K}^n \otimes_{\mathbb{K}} \mathbb{K}^{1,1} \xrightarrow{(\psi_2 \circ \psi_1)^{-1}} V \oplus V^*,$$

where $\psi_1(\mathbf{x}, \xi) = (\mathbf{x}_{\mathbb{K}}, \xi_{\mathbb{K}})$ is the natural identification of $V \oplus V^*$ with $\mathbb{K}^n \oplus \mathbb{K}^n$ and $\psi_2(\mathbf{x}_{\mathbb{K}}, \xi_{\mathbb{K}}) = \mathbf{x}_{\mathbb{K}} \otimes (1,0) + \xi_{\mathbb{K}} \otimes (0,1)$ is the natural isomorphism. If $\mathbb{K}$ is not associative, in which case we note that $\mathbb{K} = \mathbb{O}$ and $V \cong \mathbb{O}$, then $\iota(L)$ is given by the following procedure $\iota(L) : V \oplus V^* \xrightarrow{\psi_1} \mathbb{O} \oplus \mathbb{O} \xrightarrow{L} \mathbb{O} \oplus \mathbb{O} \xrightarrow{\psi_1^{-1}} V \oplus V^*$.

There is also a natural inclusion $\iota : \mathbb{K}^{1,1} \to \mathrm{Hom}(V^*, W); u = (u_1, u_2) \mapsto \iota(u)$ as defined by the following procedure $\iota(u) : V^* \xrightarrow{\psi_1} \mathbb{K}^n \xrightarrow{\psi_u} \mathbb{K}^n \oplus \mathbb{K}^n \xrightarrow{\psi_1^{-1}} V \oplus V^*$, where $\psi_u(\xi_{\mathbb{K}}) = (\xi_{\mathbb{K}} u_1, \xi_{\mathbb{K}} u_2)$. Together with the map $\iota : \mathbb{R} \to \mathbb{R}$ given by $\iota(a) \triangleq ma$, we obtain a map

$$\iota : \mathfrak{su}_{\mathbb{K}}(1,1)_{sup} \to \mathcal{L}.$$

Note that the action of $G_{\mathbb{K}}$ on $\mathcal{L}$ via the inclusion into $O(m)$ is standard. Then it is straightforward to get the following lemma, the proof of which we omit.

Lemma 2.7. (1) *$G_{\mathbb{K}}$ preserves the subspace $\iota(\mathfrak{su}_{\mathbb{K}}(1,1)_{sup})$ of $\mathcal{L}$.*

(2) *If $\mathbb{K}$ is associative, $\iota : \mathfrak{su}_{\mathbb{K}}(1,1)_{sup} \hookrightarrow \mathcal{L}$ is an injective morphism of Lie superalgebras. If $\mathbb{K} = \mathbb{O}$, $\iota(\mathfrak{su}_{\mathbb{K}}(1,1)) \cdot \iota(\mathbb{O}^{1,1}) = \mathrm{Hom}(V^*, V \oplus V^*)$.*

Thus, there is an action of $G_{\mathbb{K}}$ on $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ by viewing it as the subspace $\iota(\mathfrak{su}_{\mathbb{K}}(1,1)_{sup})$ of $\mathcal{L}$. In fact, this action is exactly the same as the $G_{\mathbb{K}}$ action as introduced in section 2.2.2. Since all the actions come out in the standard way, we denote all of them by the same notation Φ. Consequently, we have an induced vector bundle embedding

$$\iota : E^{\mathfrak{su}} \hookrightarrow E.$$

Following from Lemma 2.7, we have

Proposition 2.8. *Let M be a $\mathbb{K}$-manifold. If $\mathbb{K}$ is associative, then $\iota : E^{\mathfrak{su}} \hookrightarrow E$ is an injective Lie superalgebra bundle morphism.*

For a bundle B over M, we denote the space of sections as $\Gamma(M, B)$, or simply $\Gamma(B)$. We denote by $\check{q}$ the bilinear form on $\Gamma(E_1^{\mathfrak{su}})$ induced from the quadratic form $\check{q}$ on $\mathbb{K}^{1,1}$. And we denote by ι the induced inclusion $\Gamma(E^{\mathfrak{su}}) \to \Gamma(E)$ from $\iota : \mathfrak{su}_{\mathbb{K}}(1,1)_{sup} \to \mathcal{L}$.

3. Lie superalgebra bundle action on forms

In this section, we construct differential operators of order zero, one and two on differential forms on a $\mathbb{K}$-manifold M, and compute (some of) their supercommutators. Using these, we proceed to obtain the main result of this paper, namely there is a natural Lie superalgebra homomorphism $\sigma \circ \Psi : \Gamma(E^{\mathfrak{su}}) \longrightarrow \mathrm{Symb}(\bigwedge^{\bullet} V^*, \bigwedge^{\bullet} V^*)$, when $\mathbb{K}$ is associative (i.e. $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$).

3.1. **Spin action on $\bigwedge^{\bullet} V^*$.** Let V be a real vector space. The vector space $W = V \oplus V^*$ has a natural quadratic form Q and a natural spin structure [6]. The spinor representation S of $Spin\,(W, Q)$ can be naturally identified with $\bigwedge^{\bullet} V^*$ using the following linear action of W on $\bigwedge^{\bullet} V^*$:

$$(X + \xi) \cdot \varphi \triangleq \xi \wedge \varphi - i_X(\varphi), \text{ where } X \in V, \xi \in V^* \text{ and } \varphi \in \textstyle\bigwedge^{\bullet} V^*.$$

Recall that $Spin\,(W, Q)$ is a double cover of $SO\,(W, Q)$ and the induced isomorphism on the Lie algebra level is given by (c.f. [8]):

$$\begin{aligned} \mathrm{ad} \quad : \quad & \mathfrak{spin}\,(W, Q) \xrightarrow{\cong} \mathfrak{so}\,(W, Q)\,; \\ & x \mapsto \mathrm{ad}(x), \text{ where } \mathrm{ad}(x) : W \to W\,; \mathrm{ad}(x)(w) = xw - wx. \end{aligned}$$

Thus given a metric on V, we can identify $\mathfrak{so}\,(V)$ with a Lie subalgebra of $\mathfrak{spin}(W, Q)$ via $\mathrm{ad}^{-1} \circ \psi_4$, where ψ_4 is the diagonal embedding of $\mathfrak{so}(V)$ into $\mathfrak{so}(W, Q)$. Using this identification, one can show that this spin action of $\mathfrak{spin}\,(W, Q)$ on $S = \bigwedge^{\bullet} V^*$ restricts to the usual action of $\mathfrak{so}\,(V)$ on $\bigwedge^{\bullet} V^*$. Globally over a manifold, $\bigwedge^{\bullet} T^*$ can be identified as a spinor bundle of $T \oplus T^*$ [6].

3.2. 0^{th} **order operators.** Let (M, g) be a $\mathbb{K}$-manifold. From now on, we always assume that M is orientable (in the usual sense). Then $\mathrm{Hol}(g) \subset G_{\mathbb{K}}^{\circ}$, where $G_{\mathbb{K}}^{\circ}$ is the connected component of $G_{\mathbb{K}}$. We note that $G_{\mathbb{K}}^{\circ} = G_{\mathbb{K}}$ if $\mathbb{K} \neq \mathbb{R}$.

Let $\mathcal{S} = \bigwedge^{\bullet} T^*$. Denote by $\mathrm{Diff}_k(\mathcal{S}, \mathcal{S})$ the space of differential operators of order k on $\Gamma(\mathcal{S}) = \Omega^{\bullet}(M)$, and put $\mathrm{Diff}(\mathcal{S}, \mathcal{S}) = \bigoplus_{k=0}^{\infty} \mathrm{Diff}_k(\mathcal{S}, \mathcal{S})$. In particular, $\mathrm{Diff}_0(\mathcal{S}, \mathcal{S}) = \Gamma(\mathrm{End}(\mathcal{S}))$. With the natural isomorphism ad and the spinor representation as mentioned in section 3.1, together with the natural inclusion $\iota : \Gamma(E_0^{\mathfrak{su}}) \hookrightarrow \Gamma(E_0)$, we obtain the following natural maps.

Definition 3.1. *Define $\Psi : \Gamma(E_0) \to \mathrm{Diff}_0(\mathcal{S}, \mathcal{S})$ by $\Psi(x) = \rho_x$, where ρ_x is defined as follows: $(\rho_x \varphi)(p) = \mathrm{ad}^{-1}(x(p)) \cdot \varphi(p)$ for any $\varphi \in \Gamma(\mathcal{S})$ and any $p \in M$.*

Definition 3.2. *Define* $\Psi_\iota : \Gamma(E_0^{\mathfrak{su}}) \to \mathrm{Diff}_0(\mathcal{S}, \mathcal{S})$ *by* $\Psi_\iota(x) = \rho_{\iota(x)}$. *We simply denote* Ψ_ι *(resp.* $\rho_{\iota(x)}$*) by* Ψ *(resp.* ρ_x*).*

In [11], the second author has studied the cases $\mathrm{Hol}(g) \subset SO(n), U(n)$ and $Sp(n)$. We will restate the results in the appendix.

Let $V = \mathbb{R}^m$ and $S = \bigwedge^\bullet V^*$. Since $\mathrm{Hol}(g) \subset G_{\mathbb{K}}^\circ$, the frame bundle of M can be reduced to a $G_{\mathbb{K}}^\circ$-bundle P such that $\mathcal{S} = P \times_\Phi S$. Note that there is a canonical bijection between $\Gamma(\mathcal{S})$ and the $G_{\mathbb{K}}^\circ$-invariant sections $\Gamma(P, S)^{G_{\mathbb{K}}^\circ}$ [7]. In order to obtain an operator on $\Gamma(\mathcal{S})$, it is enough to construct an operator on $\Gamma(P, S)^{G_{\mathbb{K}}^\circ}$.

Example 3.3. *Let* $\{f_j\}_{j=1}^m$ *be the standard basis of* V, *and* $\{f^j\}$ *be the dual basis. Then* $\mathrm{ad}^{-1}(\psi_4(\mathfrak{so}(m))) = Span\{e_{i+m}e_{j+m} - e_ie_j \mid 1 \leq i < j \leq m\} \subset \mathfrak{spin}(W, Q)$, *where* $e_j = f^j + f_j, e_{j+m} = f^j - f_j, j = 1, \cdots, m$.

Note that $\nu = e_1 \cdots e_m \in Cl(W, Q)$ *and that* $(e_{i+m}e_{j+m} - e_ie_j)\nu = \nu(e_{i+m}e_{j+m} - e_ie_j)$ *for any* $1 \leq i < j \leq m$. *As mentioned in section 3.1, the spin action of* $\mathrm{ad}^{-1}(\psi_4(\mathfrak{so}(m)))$ *equals the usual action of* $\mathfrak{so}(m)$ *on* S. *Hence, the natural action* $\mathbb{R}\nu \to \mathrm{End}(S)$ *commutes with the standard action of the connected compact group* $G_{\mathbb{K}}^\circ$ *on* S. *Hence,* ν *provides an operator on* $\Gamma(P, S)^{G_{\mathbb{K}}^\circ}$, *and therefore it induces a global operator* ρ_ν *of order zero. In fact,* $\rho_\nu|_{\Omega^r(M)} = (-1)^{mr + \frac{r(r-1)}{2}} \star|_{\Omega^r(M)}$.

3.3. 1^{st} **order operators.** Recall that $\Gamma(E_1) = \Gamma\big(\mathrm{Hom}(T^*, T \oplus T^*)\big)$, and that the Levi-Civita connection ∇ of M is a $G_{\mathbb{K}}^\circ$-connection. With the help of ∇, we obtain the following natural map.

Definition 3.4. *Define* $\Psi : \Gamma(E_1) \to \mathrm{Diff}_1(\mathcal{S}, \mathcal{S})$ *by* $\Psi(u) = D_u$, *where* D_u *is the first order operator given by composition of the following maps*

$$D_u : \ \Gamma(\mathcal{S}) \xrightarrow{\nabla} \Gamma(T^* \otimes \mathcal{S}) \xrightarrow{u} \Gamma((T \oplus T^*) \otimes \mathcal{S}) \xrightarrow{\textit{Clifford product}} \Gamma(\mathcal{S}).$$

By the natural identification of $Cl = Cl(T \oplus T^*, Q)$ with $\bigwedge^\bullet(T \oplus T^*)$, D_u can also act on $\Gamma(Cl)$ through a similar procedure:

$$D_u : \ \Gamma(Cl) \xrightarrow{\nabla} \Gamma(T^* \otimes Cl) \xrightarrow{u} \Gamma((T \oplus T^*) \otimes Cl) \xrightarrow{\text{Clifford product}} \Gamma(Cl).$$

In particular, for any $x \in \Gamma(E_0) = \Gamma(\bigwedge^2(T \oplus T^*))$, $D_u x$ is meaningful, where we regard x as a section in $\Gamma(Cl)$ via ad^{-1}. Note that $\nabla_X(s \cdot \varphi) = (\nabla_X s) \cdot \varphi + s \cdot \nabla_X \varphi$, for any $X \in \Gamma(TM)$, any $s \in \Gamma(Cl)$ and any $\varphi \in \Gamma(\mathcal{S})$. We have

Proposition 3.5. *For any* $x \in \Gamma(E_0)$ *and any* $u \in \Gamma(E_1)$,

$$\rho_x \circ D_u - D_u \circ \rho_x = D_{x \cdot u} - D_u x.$$

Proof: It is sufficient to prove it locally. Let U be a coordinate chart with local coordinate $(y_1, \cdots, y_m)$. Denote $\nabla_{\frac{\partial}{\partial y_j}}$ by ∇_j. For any $\varphi \in \Gamma(U, \mathcal{S})$, we have

$(\rho_x \circ D_u)\varphi = \mathrm{ad}^{-1}(x) \cdot \sum_{j=1}^m u(dy^j) \cdot \nabla_j \varphi$, and we have

$(D_u \circ \rho_x)\varphi = \sum\limits_{j=1}^m u(dy^j) \cdot \nabla_j(\mathrm{ad}^{-1}(x) \cdot \varphi) = \sum\limits_{j=1}^m u(dy^j) \cdot \big((\nabla_j \mathrm{ad}^{-1}(x)) \cdot \varphi + \mathrm{ad}^{-1}(x) \cdot \nabla_j \varphi\big)$.

Hence, $\rho_x \circ D_u \varphi - D_u \circ \rho_x \varphi$

$$= \Big(\sum_{j=1}^m (\mathrm{ad}^{-1}(x) \cdot u(dy^j) - u(dy^j) \cdot \mathrm{ad}^{-1}(x)) \cdot \nabla_j \varphi \Big) - \sum_{j=1}^m u(dy^j) \cdot \nabla_j \mathrm{ad}^{-1}(x) \cdot \varphi$$

$$
\begin{aligned}
&=\Big(\sum_{j=1}^{m}(\mathrm{ad}(\mathrm{ad}^{-1}(x))\cdot u(dy^j))\cdot\nabla_j\varphi\Big)-(D_u x)\varphi\\
&=\Big(\sum_{j=1}^{m}(x\cdot u)(dy^j)\nabla_j\varphi\Big)-(D_u x)\varphi\\
&=D_{x\cdot u}\varphi-(D_u x)\varphi.
\end{aligned}
$$

Hence, $\rho_x\circ D_u - D_u\circ\rho_x = D_{x\cdot u} - D_u x$. □

Because of the inclusion $\iota:\Gamma(E_1^{\mathfrak{su}})\to\Gamma(E_1)$, we have the following natural map.

Definition 3.6. *Define $\Psi_\iota:\Gamma(E_1^{\mathfrak{su}})\to\mathrm{Diff}_1(\mathcal{S},\mathcal{S})$ by $\Psi_\iota(u)=D_{\iota(u)}$. We simply denote Ψ_ι (resp. $D_{\iota(u)}$) by Ψ (resp. D_u).*

Note that the action of $G^\circ_{\mathbb{K}}$ on $\mathbb{R}^{1,1}=\mathbb{R}\epsilon_1\oplus\mathbb{R}\epsilon_2$ is always trivial, where $\epsilon_1=(1,0),\epsilon_2=(0,1)\in\mathbb{K}^{1,1}$. Hence, $E_1^{\mathfrak{su}}$ has a trivial subbundle $M\times\mathbb{R}^{1,1}$. Therefore the constant section ϵ_j induces a first order operator $D_{\epsilon_j}, j=1,2$. Moreover, it follows from the observation $\nu\cdot(\iota(\epsilon_2)(f^k))\cdot\nu^{-1}=(-1)^{m-1}\iota(\epsilon_1)(f^k)$ and the construction of ρ_ν as in Example 3.3 that $D_{\epsilon_1}=(-1)^{m-1}\rho_\nu D_{\epsilon_2}\rho_\nu^{-1}$.

Because of the use of the Levi-Civita connection, we have $D_{\epsilon_2}=\sum_j dy^j\wedge\nabla_{\frac{\partial}{\partial y_j}}=d$ and $D_{\epsilon_1}=\sum_j -i_{\frac{\partial}{\partial y_j}}\circ\nabla_{\frac{\partial}{\partial y_j}}=d^*$ (c.f. [8]). In particular, ${D_{\epsilon_1}}^2={D_{\epsilon_2}}^2=0$. However, we would rather make the assumption "${D_{\epsilon_2}}^2=0$" in Proposition 3.8, for possible application to other cases.

3.4. 2^{nd} **order operators.** For any linear operators a,b,c on $\Gamma(\mathcal{S})$, we define $\{a,b\}\triangleq ab+ba$ and $[a,b]\triangleq ab-ba$. Clearly, $[a,\{b,c\}]=\{[a,b],c\}+\{b,[a,c]\}$.

Define $\triangle=\{D_{\epsilon_1},D_{\epsilon_2}\}$. Then we have

Proposition 3.7. *For any $u,v\in\Gamma(E_1^{\mathfrak{su}})$, $\{D_u,D_v\}-2\check{q}(u,v)\triangle$ is a first order differential operator.*

We will give a proof by computing the symbols in the appendix. At the moment, we would like to give an extension of $\mathfrak{su}_{\mathbb{K}}(1,1)$. We define $\mathfrak{u}_{\mathbb{K}}(1,1)$ to be $\mathfrak{su}_{\mathbb{K}}(1,1)$ itself if $\mathbb{K}\neq\mathbb{C}$, and let $\mathfrak{u}_{\mathbb{C}}(1,1)\triangleq\mathfrak{su}_{\mathbb{C}}(1,1)\oplus\mathbb{R}\phi_A$, where $\phi_A\in\mathfrak{so}(\mathbb{K}^2,\check{q})$ is as defined in section 2.2.1 with $A=\sqrt{-1}\cdot\mathrm{I}_2$, the product of $\sqrt{-1}$ and the identity matrix $\mathrm{I}_2\in\mathrm{Mat}(2,\mathbb{C})$. Then we have $\mathfrak{u}_{\mathbb{K}}(1,1)=\Big\{\begin{pmatrix}\beta_1&\beta_2\\ \beta_3&-\bar{\beta}_1\end{pmatrix}\Bigm|\beta_1\in\mathbb{K},\beta_2,\beta_3\in\mathrm{Im}\mathbb{K}\Big\}$, if $\mathbb{K}$ is associative. Furthermore, all the statements after section 2.2.1 that involve $\mathfrak{su}_{\mathbb{K}}(1,1)$ still hold true if we replace $\mathfrak{su}_{\mathbb{K}}(1,1)$ with $\mathfrak{u}_{\mathbb{K}}(1,1)$. With this observation, we can provide another proof for the most relevant case as below.

Proposition 3.8. *Suppose $E^{\mathfrak{su}}$ is trivial and ${D_{\epsilon_2}}^2=0$. Then for any constant sections $u,v\in\Gamma(E_1^{\mathfrak{su}})=\Gamma(M\times\mathbb{K}^{1,1})$,*

$$\{D_u,D_v\}=2\check{q}(u,v)\triangle.$$

Remark 3.9. *It follows from Example 2.5 that $E^{\mathfrak{su}}$ is trivial only if $\mathbb{K}$ is associative.*

Proof of Proposition 3.8: Because of the decomposition $\mathbb{K}^{1,1}=\mathbb{R}\epsilon_1\oplus\mathrm{Im}\mathbb{K}\epsilon_1\oplus\mathbb{R}\epsilon_2\oplus\mathrm{Im}\mathbb{K}\epsilon_2$, we can write any $u,v\in\mathbb{K}^{1,1}$ as $u=u_{1\mathrm{r}}+u_{1\mathrm{i}}+u_{2\mathrm{r}}+u_{2\mathrm{i}}$ and $v=v_{1\mathrm{r}}+v_{1\mathrm{i}}+v_{2\mathrm{r}}+v_{2\mathrm{i}}$.

Case $u, v \in \mathbb{R}\epsilon_1 \oplus \mathbb{R}\epsilon_2$:

As mentioned in section 3.3, $D_{\epsilon_1} = (-1)^{m-1}\rho_\nu D_{\epsilon_2}\rho_\nu^{-1}$. Hence, it follows from ${D_{\epsilon_2}}^2 = 0$ that ${D_{\epsilon_1}}^2 = 0$. Note that $\check{q}(\epsilon_1, \epsilon_2) = \frac{1}{2}$ and $\check{q}(\epsilon_1, \epsilon_1) = \check{q}(\epsilon_2, \epsilon_2) = 0$. For the constant sections $u, v \in \mathbb{R}\epsilon_1 \oplus \mathbb{R}\epsilon_2$, there exist $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that

$$\begin{aligned}\{D_u, D_v\} &= \{D_{a_1\epsilon_1+a_2\epsilon_2}, D_{b_1\epsilon_1+b_2\epsilon_2}\}\\ &= \{a_1D_{\epsilon_1} + a_2D_{\epsilon_2}, b_1D_{\epsilon_1} + b_2D_{\epsilon_2}\}\\ &= (a_2b_1 + a_1b_2)\triangle\\ &= 2\check{q}(u, v)\triangle.\end{aligned}$$

Case $u \in \mathrm{Im}\mathbb{K}\epsilon_1, v \in \mathbb{R}\epsilon_1 \oplus \mathrm{Im}\mathbb{K}\epsilon_1 \oplus \mathbb{R}\epsilon_2$:

Note that for any constant sections $x \in \Gamma(E_0^{\mathfrak{su}})$ and $w \in \Gamma(E_1^{\mathfrak{su}})$, $D_w x = 0$ and $\iota(x)\cdot\iota(\epsilon_\ell) = \iota(x\cdot\epsilon_\ell)$, $\ell = 1, 2$. Since $\check{q}(\epsilon_\ell, \epsilon_\ell) = 0$, we have

$$\begin{aligned}0 = [\rho_x, 0] &= [\rho_x, \{D_{\epsilon_\ell}, D_{\epsilon_\ell}\}]\\ &= \{[\rho_x, D_{\epsilon_\ell}], D_{\epsilon_\ell}\} + \{D_{\epsilon_\ell}, [\rho_x, D_{\epsilon_\ell}]\}\\ &= 2\{D_{x\cdot\epsilon_\ell}, D_{\epsilon_\ell}\} \qquad \text{(by Proposition 3.5).}\end{aligned}$$

Again note that $0 = \check{q}(x\cdot\epsilon_\ell, \epsilon_\ell) + \check{q}(\epsilon_\ell, x\cdot\epsilon_\ell) = 2\check{q}(x\cdot\epsilon_\ell, \epsilon_\ell)$. Hence,

$$\{D_{x\cdot\epsilon_\ell}, D_{\epsilon_\ell}\} = 0 = 2\check{q}(x\cdot\epsilon_\ell, \epsilon_\ell)\triangle.$$

Note that $u = c\epsilon_1$ for some $c \in \mathrm{Im}\mathbb{K}$, $x \triangleq \begin{pmatrix} 0 & -c \\ 0 & 0 \end{pmatrix}$ and $y \triangleq \begin{pmatrix} -c & 0 \\ 0 & -c \end{pmatrix}$ are constant sections in $\Gamma(E_0^{\mathfrak{su}})$ such that $x\cdot\epsilon_2 = u$ and $y\cdot\epsilon_1 = u$. Take $b_1, b_2 \in \mathbb{R}$ such that $v = b_1\epsilon_1 + v_{1\mathrm{i}} + b_2\epsilon_2$ where $v_{1\mathrm{i}} \in \mathrm{Im}\mathbb{K}\epsilon_1$. Then we have

$\{D_u, D_{b_1\epsilon_1}\} = b_1\{D_{y\cdot\epsilon_1}, D_{\epsilon_1}\} = 0$ and $\{D_u, D_{b_2\epsilon_2}\} = b_2\{D_{x\cdot\epsilon_2}, D_{\epsilon_2}\} = 0$.

Note that $\{D_{\epsilon_2}, D_{v_{1\mathrm{i}}}\} = \{D_{v_{1\mathrm{i}}}, D_{\epsilon_2}\} = 0$ and that $\iota(x)\cdot\iota(v_{1\mathrm{i}}) = 0$, we have

$$\begin{aligned}\{D_u, D_{v_{1\mathrm{i}}}\} = \{D_{x\cdot\epsilon_2}, D_{v_{1\mathrm{i}}}\} &= \{[\rho_x, D_{\epsilon_2}], D_{v_{1\mathrm{i}}}\}\\ &= [\rho_x, \{D_{\epsilon_2}, D_{v_{1\mathrm{i}}}\}] - \{D_{\epsilon_2}, [\rho_x, D_{v_{1\mathrm{i}}}]\}\\ &= [\rho_x, 0] - \{D_{\epsilon_2}, D_{\iota(x)\cdot\iota(v_{1\mathrm{i}})}\}\\ &= 0 - \{D_{\epsilon_2}, 0\} = 0\end{aligned}$$

Since $u \in \mathrm{Im}\mathbb{K}\epsilon_1$ and $v \in \mathbb{R}\epsilon_1 \oplus \mathrm{Im}\mathbb{K}\epsilon_1 \oplus \mathbb{R}\epsilon_2$, $\check{q}(u, v) = 0$. Therefore we have,

$$\{D_u, D_v\} = \{D_u, D_{b_1\epsilon_1}\} + \{D_u, D_{v_{1\mathrm{i}}}\} + \{D_u, D_{b_2\epsilon_2}\} = 0 = 2\check{q}(u, v)\triangle.$$

Case $u \in \mathrm{Im}\mathbb{K}\epsilon_2, v \in \mathbb{R}\epsilon_1 \oplus \mathbb{R}\epsilon_2 \oplus \mathrm{Im}\mathbb{K}\epsilon_2$:

$u = c\epsilon_2$ for some $c \in \mathrm{Im}\mathbb{K}$. Define $x \triangleq \begin{pmatrix} 0 & 0 \\ -c & 0 \end{pmatrix}$ and $y \triangleq \begin{pmatrix} -c & 0 \\ 0 & -c \end{pmatrix}$, and use the same method as above, we can show the formula $\{D_u, D_v\} = 0 = 2\check{q}(u, v)\triangle$.

Case $u \in \mathrm{Im}\mathbb{K}\epsilon_1, v \in \mathrm{Im}\mathbb{K}\epsilon_2$:

Take $c_1, c_2 \in \mathrm{Im}\mathbb{K}$ such that $u = c_1\epsilon_1$ and $v = c_2\epsilon_2$. Then we have the constant sections $x \triangleq \begin{pmatrix} 0 & -c_1 \\ 0 & 0 \end{pmatrix}$ and $y \triangleq \begin{pmatrix} 0 & 0 \\ -c_2 & 0 \end{pmatrix}$ in $\Gamma(E_0^{\mathfrak{su}})$ such that $x\cdot\epsilon_2 = u$ and $y\cdot\epsilon_1 = v$. Note that $[\rho_x, \rho_y] = \rho_{[x,y]}$ and $x\cdot\epsilon_1 = 0$, we have

$$\begin{aligned}\{D_u, D_v\} = \{D_{x\cdot\epsilon_2}, D_v\} &= \{[\rho_x, D_{\epsilon_2}], D_v\}\\ &= [\rho_x, \{D_{\epsilon_2}, D_v\}] - \{D_{\epsilon_2}, [\rho_x, D_v]\}\end{aligned}$$

$$
\begin{aligned}
&= -\{D_{\epsilon_2}, [\rho_x, [\rho_y, D_{\epsilon_1}]]\} \\
&= -\{D_{\epsilon_2}, [[\rho_x, \rho_y], D_{\epsilon_1}] + [\rho_y, [\rho_x, D_{\epsilon_1}]]\} \\
&= -\{D_{\epsilon_2}, D_{[x,y]\cdot\epsilon_1} + 0\} \\
&= -2\check{q}(\epsilon_2, [x,y]\cdot\epsilon_1)\triangle \\
&= 2\check{q}(x\cdot\epsilon_2, y\cdot\epsilon_1)\triangle \\
&= 2\check{q}(u,v)\triangle
\end{aligned}
$$

Since the product $\{\cdot,\cdot\}$ is symmetric, the formula $\{D_u, D_v\} = 2\check{q}(u,v)\triangle$ also holds true for the remaining cases. Hence, we have completed the proof. □

3.5. **Main results.** Both $\Gamma(E^{\mathfrak{su}})$ and $\Gamma(E)$ have induced Lie superalgebra structures. From Lemma 2.7, $\iota : \Gamma(E^{\mathfrak{su}}) \to \Gamma(E)$ is no longer a Lie superalgebra morphism when $\mathbb{K} = \mathbb{O}$. However, $\iota(\Gamma(E_0^{\mathfrak{su}})) \oplus \big(\iota(\Gamma(E_0^{\mathfrak{su}}))\cdot\iota(\Gamma(E_1^{\mathfrak{su}}))\big) \oplus \iota(\Gamma(E_2^{\mathfrak{su}}))$ is always a super Lie subalgebra of $\Gamma(E)$.

For any differential operator D of order k, its symbol $\sigma_k(D)$ is an element in $\mathrm{Symb}_k(\mathcal{S},\mathcal{S}) = \Gamma\big(M, \mathrm{Sym}^k T^* \otimes \mathrm{Hom}(\mathcal{S},\mathcal{S})\big)$ [14]. This symbol map fits the following exact sequence

$$0 \longrightarrow \mathrm{Diff}_{k-1}(\mathcal{S},\mathcal{S}) \xrightarrow{j} \mathrm{Diff}_k(\mathcal{S},\mathcal{S}) \xrightarrow{\sigma_k} \mathrm{Symb}_k(\mathcal{S},\mathcal{S}),$$

where j is the natural inclusion. Furthermore, $\mathrm{Symb}(\mathcal{S},\mathcal{S}) = \bigoplus_{k=0}^{\infty} \mathrm{Symb}_k(\mathcal{S},\mathcal{S})$ has a natural Lie superalgebra structure such that

$$\sigma : \mathrm{Diff}(\mathcal{S},\mathcal{S}) \longrightarrow \mathrm{Symb}(\mathcal{S},\mathcal{S})$$

is a Lie superalgebra homomorphism.

Recall that for any section (x,u) in $\Gamma(E_0)\oplus\Gamma(E_1)$ (resp. $\Gamma(E_0^{\mathfrak{su}})\oplus\Gamma(E_1^{\mathfrak{su}})$), we have constructed the associated differential operator (of order zero and one) (ρ_x, D_u) (resp. $(\rho_{\iota(x)}, D_{\iota(u)})$). Note that both $E_2^{\mathfrak{su}}$ and E_2 are trivial line bundles, any smooth section f of E_2 (resp. $E_2^{\mathfrak{su}}$) is a smooth function on M. Then we obtain the following natural maps.

Definition 3.10. *Define* $\Psi : \Gamma(E) \to \bigoplus_{k=0}^{2} \mathrm{Diff}_k(\mathcal{S},\mathcal{S}) \subset \mathrm{Diff}(\mathcal{S},\mathcal{S})$ *by* $\Psi(x,u,f) = (\rho_x, D_u, -\frac{1}{\dim M} f\triangle)$ *for any* $(x,u,f) \in \Gamma(E_0)\oplus\Gamma(E_1)\oplus\Gamma(E_2) = \Gamma(E)$.

Definition 3.11. *Define* $\Psi_\iota : \Gamma(E^{\mathfrak{su}}) \to \bigoplus_{k=0}^{2} \mathrm{Diff}_k(\mathcal{S},\mathcal{S}) \subset \mathrm{Diff}(\mathcal{S},\mathcal{S})$ *by* $\Psi_\iota(x,u,f) = (\rho_{\iota(x)}, D_{\iota(u)}, -f\triangle)$ *for any* $(x,u,f) \in \Gamma(E_0^{\mathfrak{su}})\oplus\Gamma(E_1^{\mathfrak{su}})\oplus\Gamma(E_2^{\mathfrak{su}}) = \Gamma(E^{\mathfrak{su}})$. *We simply denote* Ψ_ι *by* Ψ.

Theorem 3.12. *Let* M *be a Riemannian manifold with its holonomy group inside* $SO(n), U(n)$ *or* $Sp(n)$. *Then* $\Omega^\bullet(M)$ *admits a* $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ *action with* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$ *respectively.*

Remark 3.13. *The* $\mathbb{R}$ *part of the* $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ *action consists of* $\mathbb{R}\triangle$, *where* $\triangle = \{D_{\epsilon_1}, D_{\epsilon_2}\}$ *is the Laplacian operator* Δ *since* $D_{\epsilon_1} = d^*$ *and* $D_{\epsilon_2} = d$ *as mentioned in section 3.3. Since the* $\mathbb{R}$ *part is the center of* $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$, *the* $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ *action on* $\Omega^\bullet(M)$ *descends to the cohomology* $H^*(M)$ *by Hodge theory, if* M *is compact.*

Proof of Theorem 3.12: It follows from Example 2.5 that $E^{\mathfrak{su}}$ is trivial. Identify constant sections of $\Gamma(E^{\mathfrak{su}})$ with $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ naturally. We need to show that $\Psi : \mathfrak{su}_{\mathbb{K}}(1,1)_{sup} \to \mathrm{Diff}(\mathcal{S},\mathcal{S})$ is an injective Lie superalgebra homomorphism.

It follows from the construction of the operators of order zero that

$$\Psi([x,y]) = \rho_{[x,y]} = [\rho_x, \rho_y] = [\Psi(x), \Psi(y)], \quad \text{for any } x, y \in \mathfrak{su}_{\mathbb{K}}(1,1).$$

For any $x \in \mathfrak{su}_{\mathbb{K}}(1,1)$ and $u \in \mathbb{K}^{1,1}$, $D_u x = 0$; it follows from Proposition 3.5 that

$$\Psi([x,u]) = D_{x\cdot u} = [\rho_x, D_u] = [\Psi(x), \Psi(u)].$$

By Proposition 3.8, we have for any $u, v \in \mathbb{K}^{1,1}$ that

$$\Psi([u,v]) = \Psi(-2\check{q}(u,v)) = 2\check{q}(u,v)\triangle = \{D_u, D_v\} = \{\Psi(u), \Psi(v)\}.$$

It remains to show that

$$[\rho_x + D_u, \triangle] = 0, \quad \text{for any } x \in \mathfrak{su}_{\mathbb{K}}(1,1) \text{ and any } u \in \mathbb{K}^{1,1}.$$

In fact,

$$[\rho_x, \triangle] = \{[\rho_x, D_{\epsilon_1}], D_{\epsilon_2}\} + \{D_{\epsilon_1}, [\rho_x, D_{\epsilon_2}]\} = 2\check{q}(x\cdot\epsilon_1, \epsilon_2)\triangle + 2\check{q}(\epsilon_1, x\cdot\epsilon_2)\triangle = 0$$

Take the decomposition $u = u_{1\mathrm{r}} + u_{1\mathrm{i}} + u_{2\mathrm{r}} + u_{2\mathrm{i}}$. Note that ${D_{\epsilon_2}}^2 = 0$, it is obvious that $[D_{u_{2\mathrm{r}}}, \triangle] = 0$. We can take $x \in \mathfrak{su}_{\mathbb{K}}(1,1)$ such that $x \cdot \epsilon_2 = u_{1\mathrm{i}}$ (as we did in the proof of Proposition 3.8). Hence,

$$[D_{u_{1\mathrm{i}}}, \triangle] = [[\rho_x, D_{\epsilon_2}], \triangle] = [\rho_x, [D_{\epsilon_2}, \triangle]] - [D_{\epsilon_2}, [\rho_x, \triangle]] = [\rho_x, 0] - [D_{\epsilon_2}, 0] = 0.$$

Similarly, we have $[D_{u_{1\mathrm{r}}+u_{2\mathrm{i}}}, \triangle] = 0$. Hence,

$$\Psi([x+u, c]) = \Psi(0) = 0 = [\rho_x + D_u, -c\triangle] = [\Psi(x+u), \Psi(c)].$$

Clearly, Ψ is injective; and $\Psi(\mathfrak{su}_{\mathbb{K}}(1,1)_{sup})$, consisting of differential operators, acts on $\Omega^\bullet(M)$. Hence, $\Omega^\bullet(M)$ admits a $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ action. □

Note that the decomposition $\mathbb{K}^{1,1} = \mathbb{R}\epsilon_1 \oplus \mathrm{Im}\mathbb{K}\epsilon_1 \oplus \mathbb{R}\epsilon_2 \oplus \mathrm{Im}\mathbb{K}\epsilon_2$, induces a bundle decomposition $E^{\mathfrak{su}} = E^{\mathfrak{su}}_{1\mathrm{r}} \oplus E^{\mathfrak{su}}_{1\mathrm{i}} \oplus E^{\mathfrak{su}}_{2\mathrm{r}} \oplus E^{\mathfrak{su}}_{2\mathrm{i}}$ for any normed algebra $\mathbb{K}$. Therefore for any $u \in \Gamma(E^{\mathfrak{su}})$, we can write it as $u = u_{1\mathrm{r}} + u_{1\mathrm{i}} + u_{2\mathrm{r}} + u_{2\mathrm{i}}$. Using the same arguments as in the proof of Theorem 3.12, together with Proposition 3.5 and Proposition 3.7, we have the following theorems.

Theorem 3.14. *Let M be an oriented Riemannian manifold. Suppose M is a $\mathbb{K}$-manifold with $\mathbb{K}$ being an associative normed algebra. Then*

$$\sigma \circ \Psi : \Gamma(E^{\mathfrak{su}}) \longrightarrow \mathrm{Symb}(\mathcal{S}, \mathcal{S})$$

is a Lie superalgebra monomorphism.

Theorem 3.15. *Let M be an oriented Riemannian manifold. Suppose M is a $\mathbb{K}$-manifold with $\mathbb{K}$ a normed algebra. Then*

$$\sigma \circ \Psi : \iota(\Gamma(E^{\mathfrak{su}}_0)) \oplus \big(\iota(\Gamma(E^{\mathfrak{su}}_0)) \cdot \iota(\Gamma(E^{\mathfrak{su}}_1))\big) \oplus \iota(\Gamma(E^{\mathfrak{su}}_2)) \longrightarrow \mathrm{Symb}(\mathcal{S}, \mathcal{S})$$

is a Lie superalgebra monomorphism.

4. $\mathfrak{su}_{\mathbb{K}'}(2,2)_{sup}$-action for Semi-flat Calabi-Yau and hyperkäher manifolds

Mirror symmetry is a highly nontrivial duality transformation for Calabi-Yau manifolds (i.e. special $\mathbb{C}$-manifolds) and hyperkähler manifolds (i.e. special $\mathbb{H}$-manifolds). From the SYZ proposal [12], mirror Calabi-Yau manifolds should admit special Lagrangian fibrations, which becomes semi-flat in the large complex structure limit. In particular, the hard Lefschetz action should have a mirror version, as it was discussed in [10]. We conjecture that this mirror hard Lefschetz action should be closely related to the Schmid SL_2-orbit theorem for the large complex structure degeneration.

Putting both $\mathfrak{su}(1,1)$ actions together, we have a $\mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1) = \mathfrak{so}(2,2)$ action on differential forms on semi-flat Calabi-Yau manifolds. We are going to explain this enlarged (super) hard Lefschetz action below. In this article, we use the following definition of semi-flatness.

Definition 4.1. *A $\mathbb{K}$-manifold is called semi-flat if its holonomy group can be reduced from $G_{\mathbb{K}}$ to $G^{\circ}_{\mathbb{K}'}$, the connected component of $G_{\mathbb{K}'}$. Here $\mathbb{K}'$ means $\mathbb{R}$ or $\mathbb{C}$ when $\mathbb{K}$ equals $\mathbb{C}$ or $\mathbb{H}$ respectively.*

In particular, the tangent bundle $\mathbb{K}^n \to T \to M$ of a semi-flat $\mathbb{K}$-manifold M is the complexification of another bundle $(\mathbb{K}')^n \to T' \to M$.

Recall when $V \cong \mathbb{K}^n$ then $V \oplus V^*$ with the canonical quadratic form Q identifies it with $\mathbb{K}^n \otimes_{\mathbb{K}} \mathbb{K}^{1,1}$. Thus $\mathfrak{su}_{\mathbb{K}}(1,1)$ acts on $V \oplus V^*$ and its spinor representation $S = \bigwedge^{\bullet} V^*$. Now $V \cong V' \otimes_{\mathbb{R}} \mathbb{C}$ with $V' \cong (\mathbb{K}')^n$. By same reasonings, we have

$$V \oplus V^* \cong (\mathbb{K}')^n \otimes_{\mathbb{K}'} (\mathbb{K}')^{2,2}$$

Thus we obtain a $\mathfrak{su}_{\mathbb{K}'}(2,2)$ action on $(V \oplus V^*, Q)$, and therefore also on its spinor representation $S = \bigwedge^{\bullet} V^*$. Furthermore this action commutes with the natural $\mathfrak{u}_{\mathbb{K}'}(n)$ action. Therefore, we obtain a $\mathfrak{su}_{\mathbb{K}'}(2,2)$ action on the space of differential forms on a semi-flat $\mathbb{K}$-manifold M. One can check directly that for semi-flat Calabi-Yau manifolds, this $\mathfrak{so}(2,2) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ action corresponds to the hard Lefschetz action and its mirror action as defined in [10] (see also [3]).

To see these Lie algebras concretely, we note that $\mathfrak{su}_{\mathbb{K}'}(2,2) \cong \mathfrak{so}(\dim \mathbb{K}, 2)$.

$$\begin{aligned} \mathfrak{su}_{\mathbb{C}}(1,1) &= \mathfrak{so}(2,1) \subset \mathfrak{su}_{\mathbb{R}}(2,2) = \mathfrak{so}(2,2) \\ \mathfrak{su}_{\mathbb{H}}(1,1) &= \mathfrak{so}(4,1) \subset \mathfrak{su}_{\mathbb{C}}(2,2) = \mathfrak{so}(4,2). \end{aligned}$$

Clearly $\mathfrak{su}_{\mathbb{K}'}(2,2)_{sup} = \mathfrak{su}_{\mathbb{K}'}(2,2) \oplus (\mathbb{K}')^{2,2} \oplus \mathbb{R}$ is naturally a Lie superalgebra, which includes $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ as a super Lie subalgebra. Thus, the real vector space $(\mathbb{K}')^{2,2} \oplus \mathbb{R} = \mathbb{K}^{1,1} \oplus \mathbb{R}$ acts on $\Omega^{\bullet}(M)$ via differential operators of order one and two. Together with the $\mathfrak{su}_{\mathbb{K}'}(2,2)$ action, which extends the $\mathfrak{su}_{\mathbb{K}}(1,1)$ action, it gives a Lie superalgebra $\mathfrak{su}_{\mathbb{K}'}(2,2)_{sup}$ action on $\Omega^{\bullet}(M)$.

In conclusion, we have obtained the following result for semi-flat Calabi-Yau and hyperkähler manifolds.

Theorem 4.2. *Suppose that M is a semi-flat $\mathbb{K}$-manifold with $\mathbb{K}$ being $\mathbb{C}$ or $\mathbb{H}$, then there is a natural $\mathfrak{su}_{\mathbb{K}'}(2,2)_{sup}$ action, extending the super hard Lefschetz $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ action, on the space of differential forms on M via differential operators of order at most two.*

5. Appendix

5.1. $\mathfrak{su}_{\mathbb{K}}(1,1) \cong \mathfrak{so}(\dim \mathbb{K}, 1)$**.** There is a natural isomorphism $\tau_* : \mathfrak{sl}(2,\mathbb{K}) \xrightarrow{\cong} \mathfrak{so}(\mathbb{R}^{1,1} \oplus \mathbb{K})$. One can refer to [2] for the geometric meaning of the isomorphism. Furthermore, we have

$$\tau_*|_{\mathfrak{su}_{\mathbb{K}}(1,1)} : \mathfrak{su}_{\mathbb{K}}(1,1) \xrightarrow{\cong} \mathfrak{so}(\mathbb{R}^{1,1} \oplus \mathrm{Im}\mathbb{K}) \cong \mathfrak{so}(\dim_{\mathbb{R}} \mathbb{K}, 1).$$

We will write down it more explicitly for the case $\mathbb{K}$ is associative. Identify $\mathbb{R}^{1,1} \oplus \mathbb{K}$ with $\mathfrak{h}_2(\mathbb{K})$, the hermitian 2×2 matrices with entries in $\mathbb{K}$, via the map $(\alpha, \beta, x) \mapsto \begin{pmatrix} \alpha+\beta & x \\ \bar{x} & \alpha-\beta \end{pmatrix}$, where $\alpha, \beta \in \mathbb{R}$, $x \in \mathbb{K}$. Then there is a double cover given by $\tau : SL(2,\mathbb{K}) \longrightarrow SO^+(\mathbb{R}^{1,1} \oplus \mathbb{K})$; $A \mapsto \tau_A$, where

$$\begin{aligned} \tau_A : \quad & \mathbb{R}^{1,1} \oplus \mathbb{K} \longrightarrow \mathbb{R}^{1,1} \oplus \mathbb{K}; \\ & \begin{pmatrix} \alpha+\beta & x \\ \bar{x} & \alpha-\beta \end{pmatrix} \mapsto A \begin{pmatrix} \alpha+\beta & x \\ \bar{x} & \alpha-\beta \end{pmatrix} A^\star. \end{aligned}$$

Therefore, it induces an isomorphism τ_* of Lie algebras.

For the associative normed algebra $\mathbb{K}$, the natural inclusion $\mathbb{R}^{1,1} \oplus \mathrm{Im}\mathbb{K} \hookrightarrow \mathbb{R}^{1,1} \oplus \mathrm{Im}\mathbb{H} \hookrightarrow \mathbb{R}^{1,1} \oplus \mathbb{H}$, induces an embedding of $\mathfrak{so}(\mathbb{R}^{1,1} \oplus \mathrm{Im}\mathbb{K})$ into $\mathfrak{so}(1, 1+\dim_{\mathbb{R}}\mathbb{H})$ naturally. Therefore we only write down $\tau_*(\mathfrak{su}_{\mathbb{H}}(1,1))$ explicitly. Let E_{ij} be the matrix with 1 in the (i,j)th entry and 0 elsewhere. Take the basis of $\mathfrak{su}_{\mathbb{H}}(1,1)$ as in section 5.3, then we have

$$\begin{aligned} \tau_*(L_s) &= E_{1(3+s)} + E_{(3+s)1} - E_{2(3+s)} + E_{(3+s)2}, \; s = 1,2,3; \\ \tau_*(\Lambda_s) &= E_{1(3+s)} + E_{(3+s)1} + E_{2(3+s)} - E_{(3+s)2}, \; s = 1,2,3; \\ \tau_*(K_1) &= 2(E_{65} - E_{56}), \qquad \tau_*(K_2) = 2(E_{46} - E_{64}); \\ \tau_*(K_3) &= 2(E_{54} - E_{45}), \qquad \tau_*(H) = 2(E_{12} + E_{21}). \end{aligned}$$

5.2. **Proof of Proposition 3.7.** We use the notation of k-symbol σ_k as in [14] for differential operators.

For any $p \in M$, let $(y_1, \cdots, y_m)$ be a normal coordinate system around p. Then for any $u \in \Gamma(E_1^{\mathfrak{su}})$, $D_u = \sum_{j=1}^m \iota(u)(dy^j) \cdot \nabla_{\frac{\partial}{\partial y_j}}$.

For any $\xi \in T_p^*M$ and any $\varphi \in \bigwedge^\bullet T_p^*M$, take $g \in \Omega^0(M)$ and $s \in \Gamma(\mathcal{S})$ such that $dg(p) = \xi$ (i.e. $\sum_j \frac{\partial g}{\partial y_j}(p) dy^j = \xi$) and $s(p) = \varphi$, then we have $\sigma_k(D_u)(p,\xi)\varphi = 0$ for any $k \geq 2$, and

$$\sigma_1(D_u)(p,\xi)\varphi = \Big(D_u\big(\frac{i}{1!}(g - g(p))s\big)\Big)(p) = \sum_{j=1}^m \iota(u)(dy^j) \cdot \frac{\partial g}{\partial y_j}(p)\varphi.$$

In particular, $\sigma_1(D_{\epsilon_2})(p,\xi)\varphi = dy^j \cdot \frac{\partial g}{\partial y_j}(p)\varphi$ and $\sigma_1(D_{\epsilon_1})(p,\xi)\varphi = \frac{\partial}{\partial y_j} \cdot \frac{\partial g}{\partial y_j}(p)\varphi$. Hence,

$$\sigma_2(\triangle)(p,\xi)\varphi = \sum_{j,k} \frac{\partial g}{\partial y_j}(p) \frac{\partial g}{\partial y_k}(p)(dy^j \cdot \frac{\partial}{\partial y_k} + \frac{\partial}{\partial y_k} \cdot dy^j) \cdot \varphi = -\sum_{j=1}^m \big(\frac{\partial g}{\partial y_j}(p)\big)^2 \varphi.$$

On the other hand,

$$\sigma_2(\{D_u, D_v\})(p,\xi)\varphi$$

$$=\sum_{j,k}\iota(u)(dy^j)\cdot\frac{\partial g}{\partial y_j}(p)\iota(v)(dy^k)\cdot\frac{\partial g}{\partial y_k}(p)\varphi+\sum_{j,k}\iota(v)(dy^j)\cdot\frac{\partial g}{\partial y_j}(p)\iota(u)(dy^k)\cdot\frac{\partial g}{\partial y_k}(p)\varphi$$
$$=\sum_{j,k}\frac{\partial g}{\partial y_j}(p)\frac{\partial g}{\partial y_k}(p)\big(\iota(u)(dy^j)\cdot\iota(v)(dy^k)+\iota(v)(dy^k)\cdot\iota(u)(dy^j)\big)\cdot\varphi$$
$$=\sum_{j,k}-\frac{\partial g}{\partial y_j}(p)\frac{\partial g}{\partial y_k}(p)\cdot 2Q\big(\iota(u)(dy^j),\iota(v)(dy^k)\big)\cdot\varphi$$
$$=\sum_{j,k}-2\frac{\partial g}{\partial y_j}(p)\frac{\partial g}{\partial y_k}(p)\cdot\check{q}(u,v)\delta_{jk}\cdot\varphi$$
$$=-\,2\check{q}(u,v)\sum_{j=1}^{m}\big(\frac{\partial g}{\partial y_j}(p)\big)^2\cdot\varphi.$$

Hence, $\sigma_2(\{D_u,D_v\})(p,\xi)\varphi=\sigma_2(2\check{q}(u,v)\triangle)(p,\xi)\varphi$.
Therefore, $\sigma_2(\{D_u,D_v\}-2\check{q}(u,v)\triangle)=0$. Since D_u and D_v are of order one, $\{D_u,D_v\}-2\check{q}(u,v)\triangle$ is of order at most two. Therefore, $\{D_u,D_v\}-2\check{q}(u,v)\triangle$ is a first order operator. □

5.3. **Identifying $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ with the usual hard Lefschetz actions.** We can reinterpret those operators in the $\mathfrak{su}_{\mathbb{K}}(1,1)_{sup}$ action in Theorem 3.12 as follows.

Case $\mathbb{K}=\mathbb{R}$:
In this case, $G^{\circ}_{\mathbb{K}}=SO(n)$ and M is an oriented Riemannian manifold. Furthermore we have $\mathfrak{su}_{\mathbb{R}}(1,1)_{sup}=\mathbb{R}h\oplus\mathbb{R}^{1,1}\oplus\mathbb{R}$, where $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, so that

$$\rho_h|_{\Omega^p(M)}=(\frac{m}{2}-p)\mathrm{Id},\quad D_{\epsilon_2}=d,\quad D_{\epsilon_1}=d^*,\quad \Psi(1)=-\Delta.$$

Case $\mathbb{K}=\mathbb{C}$:
In this case, $G^{\circ}_{\mathbb{K}}=G_{\mathbb{K}}=U(n)$ and M is a Kähler manifold with Kähler form ω. Furthermore we have $\mathfrak{su}_{\mathbb{C}}(1,1)_{sup}=\mathfrak{su}_{\mathbb{C}}(1,1)\oplus\mathbb{C}^{1,1}\oplus\mathbb{R}$ with $\mathfrak{su}_{\mathbb{C}}(1,1)=\mathrm{Span}_{\mathbb{R}}\{L,\Lambda,H\}$, where

$$L=\begin{pmatrix}0&0\\-\sqrt{-1}&0\end{pmatrix},\ \Lambda=\begin{pmatrix}0&\sqrt{-1}\\0&0\end{pmatrix},\ H=\begin{pmatrix}-1&0\\0&1\end{pmatrix}.$$

And we have $\rho_L=\omega\wedge$, $\rho_\Lambda=\rho_L^*$ and $\rho_H=[\rho_L,\rho_\Lambda]$, which are exactly those defining the hard Lefschetz action on Kähler manifolds [5] (see also [11] for more details). On complex valued differential forms, we have

$$D_{\epsilon_2}=d=\partial+\bar\partial,\qquad D_{\sqrt{-1}\epsilon_2}=\sqrt{-1}(\bar\partial-\partial),$$
$$D_{\epsilon_1}=d^*=\partial^*+\bar\partial^*,\qquad D_{\sqrt{-1}\epsilon_1}=\sqrt{-1}(\bar\partial^*-\partial^*),$$
$$\{D_{\epsilon_1},D_{\epsilon_2}\}=\{D_{\sqrt{-1}\,\epsilon_1},D_{\sqrt{-1}\,\epsilon_2}\}=\Delta=-\Psi(1).$$

Case $\mathbb{K}=\mathbb{H}$:
In this case, $G^{\circ}_{\mathbb{K}}=G_{\mathbb{K}}=Sp(n)$ and M is a hyperkähler manifold. Furthermore, $\mathfrak{su}_{\mathbb{H}}(1,1)=\left\{\begin{pmatrix}\beta_1&\beta_2\\\beta_3&-\bar\beta_1\end{pmatrix}\ \middle|\ \beta_1\in\mathbb{H},\beta_2,\beta_3\in\mathrm{Im}\mathbb{H}\right\}$ is ten dimensional, and is

spanned by $\{L_s, \Lambda_s, K_s, H \mid s = 1, 2, 3\}$, where

$$L_s = \begin{pmatrix} 0 & 0 \\ -J_s & 0 \end{pmatrix}, \Lambda_s = \begin{pmatrix} 0 & J_s \\ 0 & 0 \end{pmatrix}, K_s = \begin{pmatrix} J_s & 0 \\ 0 & J_s \end{pmatrix}, H = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $J_1^2 = J_2^2 = J_3^2 = J_1J_2J_3 = -1$.

ρ_{L_s} is exactly the same as the operator "$\omega_s \wedge$" where ω_s is the Kähler form with respect to the complex structure J_s, and ρ_{Λ_s} is the adjoint operator of ρ_{L_s} for each s. Moreover, for each $s \in \{1, 2, 3\}$,

$$\begin{array}{ll} D_{\epsilon_2} = d = \partial_s + \bar{\partial}_s, & D_{J_s\epsilon_2} = \sqrt{-1}(\bar{\partial}_s - \partial_s), \\ D_{\epsilon_1} = d^* = \partial_s^* + \bar{\partial}_s^*, & D_{J_s\epsilon_1} = \sqrt{-1}(\bar{\partial}_s^* - \partial_s^*), \\ \{D_{\epsilon_1}, D_{\epsilon_2}\} = \{D_{J_s\epsilon_1}, D_{J_s\epsilon_2}\} = \Delta = -\Psi(1). & \end{array}$$

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