Hyper-Lagrangian Submanifolds of Hyperkähler Manifolds and Mean Curvature Flow

By *Naichung Conan Leung and Tom Y. H. Wan*

ABSTRACT.   *In this article, we define a new class of middle dimensional submanifolds of a Hyperkähler manifold which contains the class of complex Lagrangian submanifolds, and show that this larger class is invariant under the mean curvature flow. Along the flow, the complex phase map satisfies the generalized harmonic map heat equation. It is also related to the mean curvature vector via a first order differential equation. Moreover, we proved a result on nonexistence of Type I singularity.*

A hyperkähler manifold M is a Riemanninan manifold with holonomy $\mathrm{Sp}(n)$. It admits a complex structure J and a holomorphic symplectic form $\Omega_J \in \Omega^{2,0}(M)$ because $\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{Sp}(2n, \mathbb{C})$. In fact it admits a 2-sphere family of such structures, called the twistor family $\mathbb{S}^2$. Hyperkähler geometry arises naturally in many moduli problems, see, for example, [1, 7] and it is also intimately related to physical theory with $N = 4$ supersymmetries, see, for example, [6]. A submanifold L in M is called a complex Lagrangian submanifold if it is Lagrangian with respect to Ω_J for some $J \in \mathbb{S}^2$. Since Ω_J is a complex form, Lagrangian respect to Ω_J implies two independent conditions. This is more restrictive than the usual meaning of being Lagrangian in a symplectic manifold. Complex Lagrangian submanifold with respect to Ω_J is always a J-complex submanifold of M. In particular, it is calibrated by the Kähler form ω_J and an absolute minimal submanifold in M. The geometry of such submanifolds are studied in [8, 9], for instance.

Since we have a 2-sphere family of holomorphic symplectic structures Ω_J with $J \in \mathbb{S}^2$ on M, we could relax the assumption to the tangent spaces

$$T_x L \subset T_x M$$

to be a complex Lagrangian subspace with respect to $\Omega_{J(x)}$ with varying $J(x) \in \mathbb{S}^2$. Such an L is called a *hyper-Lagrangian submanifold* of M and the above map $x \to J(x)$ is called the *complex phase map*

$$J : L \to \mathbb{S}^2 \, .$$

It is not difficult to see that this notion is defined for a larger class of manifolds, namely Quaternionic-Kähler manifolds, i.e., Riemannian manifolds with holonomy $\mathrm{Sp}(n)\mathrm{Sp}(1)$. For example, $\mathbb{CP}^n \subset \mathbb{HP}^n$ and any surface in an oriented four manifold are examples of hyper-Lagrangian submanifolds.

In this article we study the geometry of hyper-Lagrangian submanifolds in a hyperkähler manifold. We prove the formula relating the mean curvature vector H of L and the complex phase map J,

$$\partial J = \frac{i}{2} H \lrcorner \Omega_J \, . \tag{1.1}$$

In particular, if L is minimal and the cohomology class $[J] \in H^2(L, \mathbb{Z})$ vanishes, then L is a complex Lagrangian submanifold of M. This class $[J]$ is called the *complex Maslov index* of L.

We show that the mean curvature flow of a hyper-Lagrangian submanifold L in M preserves the hyper-Lagrangian condition. To prove this, we need to couple the mean curvature flow of L with a harmonic map flow for J and derive appropriate estimates in order to use a maximum principle argument as in [11].

We also show that if the image of J lies in a hemisphere of $\mathbb{S}^2$, then this remains so under the mean curvature flow. Furthermore, Type I singularity will not occur in this situation.

When the hyperkähler manifold is of dimension four, or more generally a Kähler Einstein four manifold, these results were obtained by Jingyi Chen [2] and Mutao Wang [13] independently. In this case, the above hemisphere condition is equivalent to the surface being a symplectic submanifold in the Kähler Einstein four manifold.

Another well-behaved class of middle dimensional submanifolds under mean curvature flow are Lagrangian submanifolds in a Calabi-Yau manifold[2, 11, 12, 13]. The results there are completely analogous to those we obtain in this article. This is not a coincidence as our results are the quaternion version of their complex geometric statements. The comparisons of geometry defined over different normed algebras were studied by the first author in [10]. The following table compares the corresponding notions used in this article:

Complex $\mathbb{C}$	Quaternion $\mathbb{H}$
Lagrangian Submanifold	Hyper-Lagrangian Submanifold
$\cap$	$\cap$
Kähler Manifold	Quaternionic-Kähler Manifold
Special Lagrangian Submanifold	Complex Lagrangian Submanifold
$\cap$	$\cap$
Calabi-Yau Manifold	Hyperkähler Manifold

Remark that given any hyper-Lagrangian submanifold L in a hyperkähler manifold M, suppose that $\mathrm{Re}\left(\Omega_{J(x)}\right)$ is constant for all $x \in L$. Then L is also a Lagrangian submanifold in M,

regarded as a Calabi-Yau manifold via a natural inclusion $\mathrm{Sp}\,(n) \subset \mathrm{SU}\,(2n)$. In this circumstance, our results for hyper-Lagrangian submanifolds in hyperkähler manifolds are indeed the same as the corresponding results for Lagrangian submanifolds in Calabi-Yau manifolds.

Our article will be organized as follows. We will first prove some formulae which are needed in the rest of the article in Section 2. In Section 3, we study the mean curvature flow and show that the hyper-Lagrangian condition is preserved. After this, we study the compatibility condition for a hyper-Lagrangian submanifold and prove the formula (1.1) in Section 4. Finally, we give a brief discussion on the regularity of mean curvature flow of a hyper-Lagrangian submanifold in Section 5.

2. Terminologies and basic computations

Let $(M^{4n}, \overline{g})$ be a hyperkähler manifold, $L^{2n} \subset M$ be a submanifold of middle dimension, and $F : L \rightarrow M$ be the inclusion. Let J_α, $\alpha = 1, 2, 3$ be parallel complex structures such that $J_1 J_2 = J_3 = -J_2 J_1$. Then any $J = \sum_{\alpha=1}^{3} a_\alpha J_\alpha$ with constant $a = (a_1, a_2, a_3) \in \mathbb{S}^2$ is also a parallel complex structure. Let $\mathcal{J}$ be the set of all $J \in \Gamma(F^{-1}(\mathrm{End}(TM)))$ such that

$$J(p) = \sum_{\alpha=1}^{3} a_\alpha(p) J_\alpha \quad \text{with} \quad (a_1(p), a_2(p), a_3(p)) \in \mathbb{S}^2 \ .$$

Note that $\mathcal{J}$ can be identified with the set of smooth maps $a : L \rightarrow \mathbb{S}^2$.

For any $J \in \mathcal{J}$ we follow the definitions in [11] to define

(1)   the 2-form
$$\omega(J)(\cdot, \cdot) = \overline{g}(J\cdot, \cdot) \ ,$$

(2)   an operator $N(J) : T_pL \rightarrow (T_pL)^\perp$ by
$$N(J)v := (Jv)^\perp, and$$

(3)   a tensor on L:
$$h(J)(u, v, w) := -\overline{g}\big(N(J)(u), \overline{\nabla}_v w\big) = \overline{g}\big(\overline{\nabla}_v(N(J)(u)), w\big) \ .$$

In order to simplify notations, we will write ω, N and h for $\omega(J)$, $N(J)$, and $h(J)$, respectively, if the dependence on J is clear. And we will reserve the notations ω_α, N_α, and h_α for $\omega(J_\alpha)$, $N(J_\alpha)$, and $h(J_\alpha)$, respectively, once a choice of $\{J_\alpha\}_{\alpha=1}^{3}$ is fixed. It is then obvious that $\omega = \sum a_\alpha \omega(J_\alpha)$, $N = \sum a_\alpha N(J_\alpha)$, and $h = \sum a_\alpha h(J_\alpha)$; and the construction is independent on the choice of the orthogonal set of parallel complex structures J_α, $\alpha = 1, 2, 3$.

Let x^i, $i = 1, \ldots, 2n$, and y^κ, $\kappa = 1, \ldots, 4n$ be local coordinates on L and M, respectively. Let $e_i = dF(\partial_{x_i})$. Then $\{e_i\}$ is a basis for TL in $F^{-1}TM$. And we will denote $h_{ijk} = h(J)_{ijk} = h(J)(e_i, e_j, e_k)$ for $J \in \mathcal{J}$ and $h_{\alpha,ijk} = h_\alpha(e_i, e_j, e_k)$. Similarly, we will write $\omega_{\alpha,ij} = \omega_\alpha(e_i, e_j)$.

Suppose that $N(J)$ is an isomorphism in a coordinate neighborhood. Then $\eta(J)_{ij} := \overline{g}(N(J)(e_i), N(J)(e_j))$ is invertible and we will denote the inverse by $\eta(J)^{ij}$. As before, we will write η_{ij} and η^{ij} for the purpose of simplifying notations.

Most of the equations used in [11] are still valid for general $J \in \mathcal{J}$. In particular, we have

$$N(e_i) = J e_i - \sum_{j=1}^{2n} \omega_i^j e_j \, , \tag{2.1}$$

$$\eta_{ij} = g_{ij} + \omega_i^k \omega_{kj} \, . \tag{2.2}$$

$$\overline{\nabla}_{e_k} e_j = \Gamma_{kj}^n e_n - \eta^{mn} h_{mkj} N(e_n) \, . \tag{2.3}$$

Note that only quantities in the second term on the right-hand side of the Equation (2.3) depends on J. We also have

$$\begin{aligned}
\nabla_l h_{ikj} - \nabla_k h_{ilj} &= \overline{R}_{ijkl} + \eta^{mn} \omega_n^s \left(h_{mlj} h_{ski} - h_{mkj} h_{sli} \right) \\
&\quad + \eta^{mn} \omega_i^s \left(h_{mkj} h_{nls} - h_{mlj} h_{nks} \right) ,
\end{aligned} \tag{2.4}$$

where $\overline{R}_{ijkl} = \overline{g}(\overline{R}(e_k, e_l)e_j, N(e_i))$, and

$$R_{ijkl} = \overline{R}_{ijkl} + \eta^{mn} \left(h_{mik} h_{njl} - h_{mil} h_{njk} \right) \, . \tag{2.5}$$

However, when ω is no longer parallel or closed, other equations have to be modified.

Proposition 2.1. *For any orthogonal set of parallel complex structures $\{J_\alpha\}_{\alpha=1}^3$ and any $J = \sum_{\alpha=1}^3 a_\alpha J_\alpha$, the corresponding $h = h(J)$ and $\omega = \omega(J)$ satisfy*

$$h_{kij} = h_{ikj} + \nabla_j \omega_{ik} - \sum_{\alpha=1}^3 (\nabla_j a_\alpha) \omega_{\alpha,ik} \, .$$

Proof. The proof is straightforward by the corresponding proposition of [11]. As each J_α is parallel,

$$\begin{aligned}
h_{kij} &= \sum_\alpha a_\alpha h_{\alpha,kij} \\
&= \sum_\alpha a_\alpha (h_{\alpha,ikj} + \nabla_j \omega_{\alpha,ik}) \\
&= h_{ikj} + \sum_\alpha a_\alpha \nabla_j \omega_{\alpha,ik} \\
&= h_{ikj} + \nabla_j \omega_{ik} - \sum_\alpha (\nabla_j a_\alpha) \omega_{\alpha,ik} \, . \qquad \square
\end{aligned}$$

Proposition 2.2. *For any $J \in \mathcal{J}$, the corresponding 2-form $\omega = \omega(J)$ satisfies*

$$\nabla_k \nabla_j \omega_{li} - \nabla_l \nabla_j \omega_{ki} = \nabla_j \nabla_i \omega_{lk} + R_{ilj}^s \omega_{ks} + R_{ijk}^s \omega_{ls} - R_{jkl}^s \omega_{si} + \nabla_j (d\omega)_{lik} \, .$$

Proof. The proof is exactly the same as in [11]. The only modification is that ω is not closed, so the last term does not vanish in general. $\qquad \square$

Proposition 2.3. *For any orthogonal set of parallel complex structures $\{J_\alpha\}_{\alpha=1}^3$ and any $J =$*

$\sum_{\alpha=1}^{3} a_\alpha J_\alpha$, we have

$$
\begin{aligned}
\nabla_l h_{kij} - \nabla_k h_{lij} &= \overline{R}_{ijkl} + \nabla_j \nabla_i \omega_{lk} + \omega_i^s \overline{R}_{sjkl} + \omega_k^s R_{silj} + \omega_l^s R_{sijk} \\
&\quad + \eta^{mn} \omega_n^s \left(h_{mlj} h_{ski} - h_{mkj} h_{sli} \right) \\
&\quad + \sum_\alpha (\nabla_i \nabla_j a_\alpha) \omega_{\alpha,kl} \\
&\quad + \sum_\alpha (\nabla_i a_\alpha \nabla_j \omega_{\alpha,kl} + \nabla_j a_\alpha \nabla_i \omega_{\alpha,kl}) \\
&\quad + \sum_\alpha (\nabla_l a_\alpha \nabla_j \omega_{\alpha,ik} + \nabla_k a_\alpha \nabla_j \omega_{\alpha,li}) \ .
\end{aligned}
$$

Proof.   By Proposition 2.1,

$$
\begin{aligned}
\nabla_l h_{kij} - \nabla_k h_{lij} &= \nabla_l \left(h_{ikj} + \nabla_j \omega_{ik} - \sum_\alpha (\nabla_j a_\alpha) \omega_{\alpha,ik} \right) \\
&\quad - \nabla_k \left(h_{ilj} + \nabla_j \omega_{il} - \sum_\alpha (\nabla_j a_\alpha) \omega_{\alpha,il} \right) \\
&= \left(\nabla_l h_{ikj} - \nabla_k h_{ilj} \right) + \left(\nabla_k \nabla_j \omega_{li} - \nabla_l \nabla_j \omega_{ki} \right) \\
&\quad + \sum_\alpha \left[\nabla_k \left((\nabla_j a_\alpha) \omega_{\alpha,il} \right) - \nabla_l \left((\nabla_j a_\alpha) \omega_{\alpha,ik} \right) \right] \ .
\end{aligned}
$$

The first two terms can be handled using Proposition 2.2 and Equation (2.4) as in [11] and we obtain

$$
\begin{aligned}
\nabla_l h_{kij} - \nabla_k h_{lij} &= \overline{R}_{ijkl} + \nabla_j \nabla_i \omega_{lk} + \omega_i^s \overline{R}_{sjkl} + \omega_k^s R_{silj} + \omega_l^s R_{sijk} \\
&\quad + \eta^{mn} (h_{mlj} h_{ski} - h_{mkj} h_{sli}) \\
&\quad + \nabla_j (d\omega)_{lik} + \sum_\alpha \left[\nabla_k \left((\nabla_j a_\alpha) \omega_{\alpha,il} \right) - \nabla_l \left((\nabla_j a_\alpha) \omega_{\alpha,ik} \right) \right] \ . \ (2.6)
\end{aligned}
$$

Since ω_α are closed for all $\alpha = 1, 2, 3$, we have

$$
\begin{aligned}
\nabla_j (d\omega)_{lik} &= \sum_\alpha \nabla_j (da_\alpha \wedge \omega_\alpha)_{lik} \\
&= \sum_\alpha \nabla_j \left[(\nabla_l a_\alpha) \omega_{\alpha,ik} - (\nabla_i a_\alpha) \omega_{\alpha,lk} + (\nabla_k a_\alpha) \omega_{\alpha,li} \right] \\
&= \sum_\alpha \left[(\nabla_j \nabla_l a_\alpha) \omega_{\alpha,ik} + (\nabla_j \nabla_i a_\alpha) \omega_{\alpha,kl} + (\nabla_j \nabla_k a_\alpha) \omega_{\alpha,li} \right] \\
&\quad + \sum_\alpha \left[(\nabla_l a_\alpha) \nabla_j \omega_{\alpha,ik} + (\nabla_i a_\alpha) \nabla_j \omega_{\alpha,kl} + (\nabla_k a_\alpha) \nabla_j \omega_{\alpha,li} \right] \ .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\sum_\alpha \left[\nabla_k \left((\nabla_j a_\alpha) \omega_{\alpha,il} \right) - \nabla_l \left((\nabla_j a_\alpha) \omega_{\alpha,ik} \right) \right] \\
&= \sum_\alpha \left[(\nabla_k \nabla_j a_\alpha) \omega_{\alpha,il} + \nabla_j a_\alpha \nabla_k \omega_{\alpha,il} \right] \\
&\quad - \sum_\alpha \left[(\nabla_l \nabla_j a_\alpha) \omega_{\alpha,ik} + \nabla_j a_\alpha \nabla_l \omega_{\alpha,ik} \right] \\
&= \sum_\alpha \left[(\nabla_k \nabla_j a_\alpha) \omega_{\alpha,il} - (\nabla_l \nabla_j a_\alpha) \omega_{\alpha,ik} + \nabla_j a_\alpha \nabla_i \omega_{\alpha,kl} \right] \ .
\end{aligned}
$$

Therefore, the extra terms in the last line of Equation (2.6) can be written as

$$\nabla_j (d\omega)_{lik} + \sum_\alpha \left[\nabla_k \left((\nabla_j a_\alpha) \omega_{\alpha,il} \right) - \nabla_l \left((\nabla_j a_\alpha) \omega_{\alpha,ik} \right) \right]$$

$$= \sum_\alpha (\nabla_j \nabla_i a_\alpha) \omega_{\alpha,kl} + \sum_\alpha (\nabla_i a_\alpha \nabla_j \omega_{\alpha,kl} + \nabla_j a_\alpha \nabla_i \omega_{\alpha,kl})$$

$$+ \sum_\alpha (\nabla_l a_\alpha \nabla_j \omega_{\alpha,ik} + \nabla_k a_\alpha \nabla_j \omega_{\alpha,il}) ,$$

which is the desired result. $\square$

3. Mean curvature flow

In this section, we are going to prove that hyper-Lagrangian submanifolds are preserved under mean curvature flow. Our proof is similar to that of Smoczyk [11]. However, the defining almost complex structure J of the hyper-Lagrangian submanifold and the 2-forms here corresponding the symplectic form are no longer parallel or closed. This introduces a lot of extra terms and a need of an estimate of the covariant derivative of the corresponding symplectic form. Moreover, unlike the situation in [11], the harmonic map heat flow equation is not just a consequence of the mean curvature flow, it will also be used to obtain the necessary estimate.

3.1. Mean curvature flow

As in [11], the assumption that N is an isomorphism ensures that the mean curvature vector of L can be written as

$$H = -\eta^{mn} g^{kl} h_{mkl} N(e_n) .$$

From now on, the corresponding *mean curvature form* with respect to J is defined by

$$\sigma_H := H \lrcorner \omega$$

which can be written in term of coordinates as follow

$$\sigma_H = H_i \, dx^i := g^{kl} h_{ikl} \, dx^i .$$

Similarly, for $\alpha = 1, 2, 3$, we write

$$\sigma_{H_\alpha} := H \lrcorner \omega_\alpha = H_{\alpha,i} \, dx^i .$$

In terms of the coefficients of σ_H, the mean curvature vector can be written as

$$H = -\eta^{mn} H_m N(e_n) .$$

Note that all the above depend on the $J \in \mathcal{J}$ except the mean curvature vector.

Now, we assume that $F : L \to M$ is deformed under the mean curvature flow

$$\frac{d}{dt} F_t = H, \quad F_0 = F ,$$

which can be written as

$$\frac{d}{dt} F_t = -\eta^{mn} H_m N(e_n), \quad F_0 = F$$

for any $J \in \mathcal{J}$. Suppose that the mean curvature flow exists for $t \in [0, T)$ and the metric at time t is g_t. Let $a(t, p) : [0, T) \times L \to \mathbb{S}^2$ be a smooth mapping and $J(t) = \sum_\alpha a_\alpha(t, p) J_\alpha$ be the corresponding deformation of the tensor J along the mean curvature flow. We are going to calculate the deformation of the corresponding 2- form $\omega(t)$ on L_t.

Proposition 3.1. *Suppose that $J(t) = \sum_\alpha a_\alpha(t, p) J_\alpha$ is a deformation of $J = J(0)$ in $\mathcal{J}$ along the mean curvature flow. Then the corresponding 2-form $\omega(t) = \omega(J(t))$ satisfies*

$$
\begin{aligned}
\frac{d}{dt}\omega_{ij} = \ & \overline{R_{\underline{s}}}^{\,s}{}_{ji} + \Delta\omega_{ij} \\
& + \omega_k^s \overline{R}_s^{\,k}{}_{ji} + \omega_j^s R_s^{\,k}{}_{ik} + \omega_i^s \overline{R}_s^{\,k}{}_{kj} \\
& + \eta^{mn}\omega_n^s\left(h_{mi}{}^k h_{sjk} - h_{mj}{}^k h_{sik}\right) \\
& + \sum_\alpha\left[(\partial_t a_\alpha - \Delta a_\alpha)\omega_{\alpha,ij} - 2\nabla_k a_\alpha \nabla_k \omega_{\alpha,ij}\right] \\
& + \sum_\alpha\left(h_{\alpha,kkj}\nabla_i a_\alpha - h_{\alpha,kki}\nabla_j a_\alpha\right) \ .
\end{aligned}
$$

Proof. Since ω_α are parallel, we have for $\alpha = 1, 2, 3$ that

$$
\frac{\partial}{\partial t}\omega_{\alpha,ij} = \left(d\sigma_{H_\alpha}\right)_{ij} + (H \lrcorner \, d\omega_\alpha)_{ij} = \left(d\sigma_{H_\alpha}\right)_{ij} \ .
$$

Hence,

$$
\begin{aligned}
\frac{d}{dt}\omega_{ij} &= \sum_\alpha\left[\frac{\partial a_\alpha}{\partial t}\omega_{\alpha,ij} + a_\alpha\left(d\sigma_{H_\alpha}\right)_{ij}\right] \\
&= \sum_\alpha\left[\frac{\partial a_\alpha}{\partial t}\omega_{\alpha,ij} + \left(d\left(a_\alpha\sigma_{H_\alpha}\right)\right)_{ij} - \left(da_\alpha \wedge \sigma_{H_\alpha}\right)_{ij}\right] \\
&= \sum_\alpha\left[\frac{\partial a_\alpha}{\partial t}\omega_{\alpha,ij} - \left(da_\alpha \wedge \sigma_{H_\alpha}\right)_{ij}\right] + (d\sigma_H)_{ij} \ .
\end{aligned}
$$

Using Proposition 2.3, we have

$$
\begin{aligned}
(d\sigma_H)_{ij} = \ & \overline{R_{\underline{s}}}^{\,s}{}_{ji} + \Delta\omega_{ij} \\
& + \omega_k^s \overline{R}_s^{\,k}{}_{ji} + \omega_j^s R_s^{\,k}{}_{ik} + \omega_i^s \overline{R}_s^{\,k}{}_{kj} \\
& + \eta^{mn}\omega_n^s\left(h_{mi}{}^k h_{sjk} - h_{mj}{}^k h_{sik}\right) \\
& + \sum_\alpha\left[(\Delta a_\alpha)\omega_{\alpha,ji} + 2\nabla_k a_\alpha \nabla_k \omega_{\alpha,ji}\right] \\
& + \sum_\alpha\left(\nabla_i a_\alpha \nabla_k \omega_{\alpha,kj} - \nabla_j a_\alpha \nabla_k \omega_{\alpha,ki}\right) \ .
\end{aligned}
$$

Therefore, together with

$$
\sum_\alpha (da_\alpha \wedge \sigma_{H_\alpha})_{ij} = \sum_\alpha\left[(\nabla_i a_\alpha)H_{\alpha,j} - (\nabla_j a_\alpha)H_{\alpha,i}\right] ,
$$

we have

$$
\begin{aligned}
\frac{d}{dt}\omega_{ij} \;=\;& \overline{R}_{\underline{s}}{}^{s}{}_{ji} + \Delta\omega_{ij} \\[4pt]
& + \omega_k^s \overline{R}_s{}^{k}{}_{ji} + \omega_j^s R_s{}^{k}{}_{ik} + \omega_i^s \overline{R}_s{}^{k}{}_{kj} \\[4pt]
& + \eta^{mn}\omega_n^s\left(h_{mi}{}^{k}h_{sjk} - h_{mj}{}^{k}h_{sik}\right) \\[4pt]
& + \sum_\alpha \left[(\partial_t a_\alpha - \Delta a_\alpha)\omega_{\alpha,ij} - 2\nabla_k a_\alpha \nabla_k \omega_{\alpha,ij}\right] \\[4pt]
& + \sum_\alpha \left[\nabla_i a_\alpha(\nabla_k\omega_{\alpha,kj} - H_{\alpha,j}) - \nabla_j a_\alpha(\nabla_k\omega_{\alpha,ki} - H_{\alpha,i})\right] .
\end{aligned}
$$

Finally, by Proposition 2.1, for each $\alpha = 1, 2, 3$, we have

$$
H_{\alpha,i} = h_{\alpha,ikk} = h_{\alpha,kik} + \nabla_k \omega_{\alpha,ki} .
$$

Putting this in the above and note that $h_{\alpha,kik} = h_{\alpha,kki}$, we have proved the proposition. $\qquad\square$

Lemma 3.2. *Let $a : L \to \mathbb{S}^2$ be the map corresponding to some $J \in \mathcal{J}$ with respect to an orthogonal set of complex structures $\{J_\alpha\}_{\alpha=1}^3$. Then we have*

$$
\sum_\alpha (\nabla_i a_\alpha) h_{\alpha,kkj} = \eta^{ls} h_{lkj} \sum_\alpha \left[(a \times \nabla_i a)_\alpha \omega_{\alpha,ks} + \omega_s^n(\nabla_i a_\alpha)\omega_{\alpha,kn}\right] ,
$$

where $a \times \nabla_i a$ is the cross product of $a = (a_\alpha)$ and $\nabla_i a = (\nabla_i a_\alpha)$ by regarding them as vectors in 3-space.

Proof. By definition

$$
\begin{aligned}
h_{\alpha,kkj} \;=\;& -\overline{g}\left(N_\alpha e_k, \overline{\nabla}_{e_k} e_j\right) \\[4pt]
=\;& -\overline{g}\left((J_\alpha e_k)^\perp, \overline{\nabla}_{e_k} e_j\right) \\[4pt]
=\;& -\overline{g}\left(J_\alpha e_k, (\overline{\nabla}_{e_k} e_j)^\perp\right) \\[4pt]
=\;& -\overline{g}\left(J_\alpha e_k, \eta^{ls} h_{lkj} N(e_s)\right) \\[4pt]
=\;& -\eta^{ls} h_{lkj} \overline{g}\left(J_\alpha e_k, J e_s - \omega_s^n e_n\right) \\[4pt]
=\;& \eta^{ls} h_{lkj}\left[\omega_{\alpha,kn}\omega_s^n - \overline{g}(J_\alpha e_k, J e_s)\right] \\[4pt]
=\;& \eta^{ls} h_{lkj}\left[\omega_{\alpha,kn}\omega_s^n - \sum_\beta a_\beta \overline{g}(J_\alpha e_k, J_\beta e_s)\right] .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_\alpha (\nabla_i a_\alpha) h_{\alpha,kkj} \;=\;& \eta^{ls} h_{lkj} \sum_\alpha (\nabla_i a_\alpha)\omega_{\alpha,kn}\omega_s^n \\[4pt]
& - \eta^{ls} h_{lkj} \sum_{\alpha,\beta} (a_\beta \nabla_i a_\alpha)\overline{g}(J_\alpha e_k, J_\beta e_s)
\end{aligned}
$$

$$
\begin{aligned}
= \ & \eta^{ls}h_{lkj}\sum_{\alpha}(\nabla_i a_\alpha)\omega_{\alpha,kn}\omega_s^n \\
& - \eta^{ls}h_{lkj}\sum_{\alpha}(a_\alpha\nabla_i a_\alpha)\bar{g}(J_\alpha e_k, J_\alpha e_s) \\
& + \eta^{ls}h_{lkj}\sum_{\beta<\alpha}(a_\alpha\nabla_i a_\beta - a_\beta\nabla_i a_\alpha)\bar{g}(J_\alpha J_\beta e_k, e_s) \\
= \ & \eta^{ls}h_{lkj}\sum_{\alpha}(\nabla_i a_\alpha)\omega_{\alpha,kn}\omega_s^n \\
& - \eta^{ls}h_{lkj}g_{ks}\sum_{\alpha}(a_\alpha\nabla_i a_\alpha) \\
& + \eta^{ls}h_{lkj}\sum_{\beta<\alpha}(a_\alpha\nabla_i a_\beta - a_\beta\nabla_i a_\alpha)\bar{g}(J_\alpha J_\beta e_k, e_s) \ .
\end{aligned}
$$

Using the fact that $\sum_\alpha a_\alpha^2 = 1$, we have $\sum_\alpha a_\alpha\nabla_i a_\alpha = 0$ and hence

$$
\begin{aligned}
\sum_\alpha(\nabla_i a_\alpha)h_{\alpha,kkj} \ = \ & \eta^{ls}h_{lkj}\sum_{\alpha}(\nabla_i a_\alpha)\omega_{\alpha,kn}\omega_s^n \\
& + \eta^{ls}h_{lkj}\sum_{\beta<\alpha}(a_\alpha\nabla_i a_\beta - a_\beta\nabla_i a_\alpha)\bar{g}(J_\alpha J_\beta e_k, e_s) \ .
\end{aligned}
$$

This is the desired result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.3. *Let $\{J_\alpha\}_{\alpha=1}^3$ be an orthogonal set of complex structures of a hyperkähler manifold $(M^{4n}, \bar{g})$ such that $N_2 = N(J_2)$ and $N_3 = N(J_3)$ are isomorphisms at each point of a submanifold $L^{2n} \subset M$. Then*

$$
\begin{aligned}
\nabla_i\omega_{1,kl} \ = \ & -\eta_3{}^{mn}h_{3,mik}\omega_{2,ln} + \eta_3{}^{mn}h_{3,mik}\omega_{1,lj}\omega_{3,n}^j \\
& - \eta_2{}^{mn}h_{2,mil}\omega_{3,kn} - \eta_2{}^{mn}h_{2,mil}\omega_{1,kj}\omega_{2,n}^j \ .
\end{aligned}
$$

Moreover, if L is compact, then there is a constant $C > 0$ such that

$$
|\nabla\omega_1| \le C\sqrt{|\omega_2|^2 + |\omega_3|^2} \ .
$$

Proof. Let e_i be an orthonormal basis of TL in a neighborhood of a point p. Then by definition,

$$
\nabla_i\omega_{1,kl} = (\nabla_{e_i}\omega_1)(e_k, e_l) = e_i(\omega_1(e_k, e_l)) - \omega_1(\nabla_{e_i}e_k, e_l) - \omega_1(e_k, \nabla_{e_i}e_l) \ .
$$

Since $\bar{\omega}_1$ is parallel, we have

$$
\begin{aligned}
\nabla_i\omega_{1,kl} \ = \ & \bar{\omega}_1(\bar{\nabla}_{e_i}e_k, e_l) + \bar{\omega}_1(e_k, \bar{\nabla}_{e_i}e_l) - \bar{\omega}_1(\nabla_{e_i}e_k, e_l) - \bar{\omega}_1(e_k, \nabla_{e_i}e_l) \\
= \ & \bar{\omega}_1\big((\bar{\nabla}_{e_i}e_k)^\perp, e_l\big) + \bar{\omega}_1\big(e_k, (\bar{\nabla}_{e_i}e_l)^\perp\big) \ .
\end{aligned}
$$

Using $J_1 = J_2 J_3$,

$$
\begin{aligned}
\nabla_i\omega_{1,kl} \ = \ & \bar{g}\big(J_2 J_3(\bar{\nabla}_{e_i}e_k)^\perp, e_l\big) + \bar{g}\big(J_2 J_3 e_k, (\bar{\nabla}_{e_i}e_l)^\perp\big) \\
= \ & -\bar{g}\big(J_2 e_l, J_3(\bar{\nabla}_{e_i}e_k)^\perp\big) - \bar{g}\big(J_3 e_k, J_2(\bar{\nabla}_{e_i}e_l)^\perp\big) \ .
\end{aligned}
$$

Note that

$$\left(\overline{\nabla}_{e_i} e_k\right)^{\perp} = -\eta_2^{mn} h_{2,mik} N_2(e_n) = -\eta_3^{mn} h_{3,mik} N_3(e_n) \,,$$

we have

$$
\begin{aligned}
\nabla_i \omega_{1,kl} &= \overline{g}\left(J_2 e_l, J_3\left(\eta_3^{mn} h_{3,mik} N_3(e_n)\right)\right) + \overline{g}\left(J_3 e_k, J_2\left(\eta_2^{mn} h_{2,mil} N_2(e_n)\right)\right) \\
&= \eta_3^{mn} h_{3,mik} \overline{g}(J_2 e_l, J_3 N_3(e_n)) + \eta_2^{mn} h_{2,mil} \overline{g}(J_3 e_k, J_2 N_2(e_n)) \,.
\end{aligned}
$$

Then by the definition of the operator N, we have

$$J_2 N_2(e_n) = J_2\left(J_2 e_n - \omega_{2,n}^j e_j\right) = -e_n - \omega_{2,n}^j J_2 e_j \,,$$

and similarly for J_3. Putting this into the above, we conclude that

$$
\begin{aligned}
\nabla_i \omega_{1,kl} &= \eta_3^{mn} h_{3,mik} \left[\overline{g}(J_2 e_l, -e_n) - \omega_{3,n}^j \overline{g}(J_2 e_l, J_3 e_j)\right] \\
&\quad + \eta_2^{mn} h_{2,mil} \left[\overline{g}(J_3 e_k, -e_n) - \omega_{2,n}^j \overline{g}(J_3 e_k, J_2 e_j)\right] \\
&= -\eta_3^{mn} h_{3,mik} \omega_{2,ln} - \eta_3^{mn} h_{3,mik} \omega_{1,lj} \omega_{3,n}^j \\
&\quad - \eta_2^{mn} h_{2,mil} \omega_{3,kn} - \eta_2^{mn} h_{2,mil} \omega_{1,kj} \omega_{2,n}^j \,.
\end{aligned}
$$

This completes the proof of the first part of the lemma and the last part of the lemma follows immediately. □

3.2. Main theorems and harmonic map heat flow

Theorem 3.4. *Let $(M^{4n}, \overline{g})$ be a hyperkähler manifold. Let $L_t \subset M^{4n}$, $t \in [0, T)$ for some $T > 0$, be a family of middle dimensional submanifolds given by the mean curvature flow. Suppose that for each point $(t, p) \in L_t$, there is a parallel complex structure J of M such that, for all parallel complex structure K of M orthogonal to J,*

$$\overline{g}(K\cdot, \cdot)|_{T_p L_t} = 0 \,,$$

and that J is smooth in (t, p). Then the defining map $J = J(t, x)$ from $[0, T) \times L_0$ to $\mathbb{S}^2$, the space of parallel complex structures of M, satisfies the harmonic map heat flow equations with variable metric

$$\partial_t J = \triangle_t J \,,$$

where $\triangle_t J$ is the tension field of J with respect to the induced metric g_t on L_t.

Proof. Let $t_o \in [0, T)$ and p be a point in L_{t_o}. Then $J_1 = J(t_o, p)$ is a fixed parallel complex structure of M. We may complete this to an orthogonal set $\{J_\alpha\}_{\alpha=1}^3$, i.e., $J_1 J_2 = J_3 = -J_2 J_1$. With respect to this basis, we can use standard spherical coordinates on $\mathbb{S}^2$ to represent J in a neighborhood of $(t_o, p) \in [0, T) \times L_0$ by

$$J = \cos\theta \sin\varphi J_1 + \sin\theta \sin\varphi J_2 + \cos\varphi J_3 \,,$$

where θ and φ are functions of (t, x) in a neighborhood of (t_0, p). Then it is easy to see that

$$K = -\sin\theta J_1 + \cos\theta J_2 \,,$$

and

$$JK = -\cos\theta \cos\varphi J_1 - \sin\theta \cos\varphi J_2 + \sin\varphi J_3$$

are complex structures orthogonal to J. Let's denote these three combinations by $J = \sum_\alpha a_\alpha J_\alpha$, $K = \sum_\alpha b_\alpha J_\alpha$, $JK = \sum_\alpha c_\alpha J_\alpha$ and write $a = (a_1, a_2, a_3)$ and so on. Then the map J is represented by

$$a(t, x) = (\cos\theta \sin\varphi, \ \sin\theta \sin\varphi, \ \cos\varphi) \,.$$

By a straightforward calculation, we have

$$a(t_o, p) = (1, 0, 0), \quad b(t_o, p) = (0, 1, 0), \quad c(t_o, p) = (0, 0, 1) \,,$$

$$\nabla b(t_o, p) = (-\nabla\theta|_{(t_o, p)}, 0, 0), \quad \nabla c(t_o, p) = (\nabla\varphi|_{(t_o, p)}, 0, 0) \,,$$

$$\partial_t b(t_o, p) = (-\partial_t\theta|_{(t_o, p)}, 0, 0), \quad \partial_t c(t_o, p) = (\partial_t\varphi|_{(t_o, p)}, 0, 0) \,,$$

$$b \times \nabla b|_{(t_o, p)} = (0, 0, \nabla\theta|_{(t_o, p)}), \quad c \times \nabla c|_{(t_o, p)} = (0, \nabla\varphi|_{(t_o, p)}, 0) \,,$$

$$\triangle b(t_o, p) = \left(-\triangle\theta, -|\nabla\theta|^2, 0\right)\big|_{(t_o, p)} \,,$$

and

$$\triangle c(t_o, p) = \left(\triangle\varphi, 2\nabla\theta \cdot \nabla\varphi, -|\nabla\varphi|^2\right)\big|_{(t_o, p)} \,.$$

In the above, ∇ and $\triangle$ denote the gradient and the Laplacian with respect to the induced metric on L_{t_o}; $b \times \nabla b$ and $c \times \nabla c$ the cross products by regarding b, ∇b, c, and ∇c as 3-vectors.

By assumption, both $\overline{g}(K\cdot, \cdot)$ and $\overline{g}(JK\cdot, \cdot)$ vanish on $T_x L_t$ for any (t, x) in a neighborhood of (t_o, p). In particular, $\omega_2 = \omega_3 = 0$ at (t_o, p). Then by applying Proposition 3.1 to the 2-form $\overline{g}(K\cdot, \cdot)$, we have at the point (t_o, p),

$$\begin{aligned}
0 &= [(\partial_t - \triangle)b_1]\omega_{1,ij} - 2\nabla_k b_1 \nabla_k \omega_{1,ij} \\
&\quad + \sum_\alpha (h_{\alpha, kkj}\nabla_i b_\alpha - h_{\alpha, kki}\nabla_j b_\alpha) \,.
\end{aligned}$$

Note that we have used the fact that $\sum_s \overline{R}^s_{\underline{s}ji} = 0$ whenever $\overline{g}(K\cdot, \cdot) \equiv 0$ by the flatness of the Ricci curvature, see [11]. On the other hand, Lemmas 3.3 and 3.2 imply

$$\nabla\omega_{1,ij} = 0 \,,$$

and

$$\sum_\alpha h_{\alpha, kkj}\nabla_i b_\alpha = 0 \,.$$

Hence, we conclude that

$$0 = [(\partial_t - \triangle_t)b_1]\omega_{1,ij} \,.$$

Since $\omega_2 = \omega_3 = 0$, we have $\omega_1 \neq 0$. Therefore,

$$(\partial_t - \triangle_t)b_1 = 0 \,,$$

which is equivalent to

$$(\partial_t - \triangle_t)\theta = 0 \quad \text{at } (t_o, p) \,.$$

Similarly, by applying Proposition 3.1, and Lemmas 3.3 and 3.2 to the symplectic form $\overline{g}(JK\cdot, \cdot)$, we have

$$(\partial_t - \triangle_t)\varphi = 0 \quad \text{at } (t_o, p) \,.$$

All together, we have proved that J satisfies

$$(\partial_t - \triangle_t)J = 0$$

as a mapping from $[0, T) \times L_0$ to $\mathbb{S}^2$. $\qquad\square$

Note that we have used the same notation to denote the Laplacian of functions and the tension field of a map. However, as we have seen, no confusion will be created.

Lemma 3.5. *Suppose that $(M^{4n}, \overline{g})$ is a hyperkähler manifold and $L_t \subset M^{4n}$, $t \in [0, T)$ for some $T > 0$, is a family of compact middle dimensional submanifolds given by the mean curvature flow. Let $J = J(t, x) \in \mathcal{J}$ be a harmonic map heat flow along the mean curvature flow L_t, i.e., $(\partial_t - \triangle_t)J = 0$; and let $K = K(t, p)$ be the parallel complex structure of M corresponding to $J(t, p)$ constructed in the proof of Theorem 3.4. If for each point $(t, p) \in L_t$, $N(K), N(JK) : T_p L_t \to (T_p L_t)^\perp$ are isomorphisms, then there is a constant $c > 0$ such that*

$$\frac{\partial}{\partial t}|\Omega|^2 \le \triangle_t |\Omega|^2 + c|\Omega|^2 ,$$

where $\Omega(\cdot, \cdot) = \overline{g}(K\cdot, \cdot) + i\overline{g}(JK\cdot, \cdot)$.

Proof.   Lets fix a point (t_0, p) and use the same notations and conventions as in the proof of the Theorem 3.4. Then as in [11], we use the flatness of the Ricci curvature, Proposition 3.1, and Lemma 3.2 to deduce that, there is a constant $A_1 > 0$ such that

$$\begin{aligned}
\frac{\partial}{\partial t}|\omega(K)|^2 &\le& A_1|\omega(K)|^2 + 2\langle \omega(K), \triangle\omega(K)\rangle \\
&& + 2\sum_\alpha [(\partial_t b_\alpha - \triangle b_\alpha)\langle\omega_\alpha, \omega(K)\rangle - 2\nabla_k b_\alpha \langle\nabla_k\omega_\alpha, \omega(K)\rangle] \\
&& + 2\eta(K)^{ls} h(K)_{lkj} \sum_\alpha (b \times \nabla_i b)_\alpha \langle\omega_\alpha, \omega(K)\rangle ,
\end{aligned}$$

where $b \times \nabla_i$ is the cross product by regarding b and ∇b as 3-vectors.

Evaluating at (t_0, p), we have

$$\begin{aligned}
(\partial_t - \triangle)|\omega(K)|^2 &\le& A_2|\omega_2|^2 + 2|\nabla\theta|^2|\omega_2|^2 \\
&& + 4\nabla_k\theta\langle\nabla_k\omega_1, \omega_2\rangle + 2\eta_2^{ls} h_{2,lkj}\nabla_i\theta\langle\omega_3, \omega_2\rangle ,
\end{aligned}$$

for some constant $A_2 > 0$. Since L_t are compact, we can apply the last part of Lemma 3.3 to conclude that there is a constant $A_3 > 0$ such that

$$(\partial_t - \triangle)|\omega(K)|^2 \le A_3\big(|\omega_2|^2 + |\omega_3|^2\big) .$$

Similarly, we have at the point (t_0, p),

$$\begin{aligned}
(\partial_t - \triangle)|\omega(JK)|^2 &\le& B_2|\omega_3|^2 - 4\nabla\theta \cdot \nabla\varphi\langle\omega_2, \omega_3\rangle \\
&& + 2|\nabla\varphi|^2|\omega_3|^2 - 4\nabla_k\varphi\langle\nabla_k\omega_1, \omega_3\rangle \\
&& + 2\eta_3^{ls} h_{3,lkj}\nabla_i\varphi\langle\omega_2, \omega_3\rangle ,
\end{aligned}$$

for some constant $B_2 > 0$. Hence, there exists a constant $B_3 > 0$ such that

$$(\partial_t - \triangle)|\omega(JK)|^2 \le B_3\big(|\omega_2|^2 + |\omega_3|^2\big) .$$

Combining these two inequalities, we have shown that

$$(\partial_t - \triangle)|\Omega|^2 \le c|\Omega|^2$$

with $c = A_3 + B_3$. $\square$

Next, we prove the main theorem of this section, namely, the mean curvature flow preserves the hyper-Lagrangian condition.

Theorem 3.6. *Let $(M^{4n}, \overline{g})$ be a hyperkähler manifold. Let $L_t \subset M^{4n}$, $t \in [0, T)$ for some $T > 0$, be a family of middle dimensional submanifolds given by the mean curvature flow. Suppose that $L_0 \subset M$ is hyper-Lagrangian. Then L_t is hyper-Lagrangian for all $t \in [0, T)$.*

Proof. Consider the initial value problem for the generalized harmonic map heat flow to $\mathbb{S}^2$ along the mean curvature flow:

$$\begin{cases} (\partial_t - \triangle_t)\Theta = 0, & \text{on} \quad (0, T) \times L_0 \\ \Theta(0, x) = J(x) \,. \end{cases}$$

The short time existence implies that there is a $T_1 \in (0, T)$ such that solution $\Theta(t, x) = J(t, x)$ exists for $t \in [0, T_1)$. Recall that if L_0 is hyper-Lagrangian, then for each point $(0, p) \in L_0$, there is a parallel complex structure J of M such that, for all parallel complex structure K of M orthogonal to J,

$$\overline{g}(K\cdot, \cdot)|_{T_p L_0} = 0 \,.$$

So we can pick $K(t, x)$ satisfying the condition in the Lemma 3.5 and conclude that the corresponding complex 2-form Ω satisfies

$$\begin{cases} (\partial_t - \triangle)|\Omega|^2 \le c|\Omega|^2, & \text{on } (0, T_1) \times L_0 \\ |\Omega|^2(0, p) \equiv 0 \,. \end{cases}$$

Therefore maximum principle implies that $|\Omega|^2 \equiv 0$, by the assumption that L_0 is hyper-Lagrangian. This proves the result for a short time.

Now using the same notation as in the proof of the theorem, we see that

$$\overline{g}(K\cdot, \cdot) \equiv \overline{g}(JK\cdot, \cdot) \equiv 0$$

along L_t. Therefore at any point $(t, p) \in [0, T_1) \times L_0$ and in the basis and coordinates chosen as in the proof of the theorem, we have

$$\nabla(b_\alpha \omega_\alpha) = \nabla(c_\alpha \omega_\alpha) = 0 \,.$$

This implies, at the point (t, p),

$$\nabla b_1 \omega_1 + \nabla \omega_2 = \nabla c_1 \omega_1 + \nabla \omega_3 = 0 \,.$$

So

$$|\nabla \theta| \le |\nabla \omega_2| \quad \text{and} \quad |\nabla \varphi| \le |\nabla \omega_3| \,.$$

Noting that $\nabla_j \omega_{\alpha, ik} = h_{\alpha, kij} - h_{\alpha, ikj}$ and $|h_{\alpha, kij}| \le |\overline{\nabla}_{e_i} e_j|$, we conclude that $|\nabla \theta|$ and $|\nabla \varphi|$ are bounded in terms of the geometry of mean curvature flow L_t as long as it exists. Hence, if $T_1 < T$, we can find subsequence of $J(t)$ converges to some element J in $\mathcal{J}$ as $t \to T_1$. It is clear that J enjoys the same property on L_{T_1} as $J(t)$ on L_t, for $t < T_1$. This implies that L_{T_1} is also hyper-Lagrangian.

In conclusion, we have proved that the set of $t \in [0, T)$ with hyper-Lagrangian L_t is both open and closed in $[0, T)$. This, of course, implies that L_t is hyper-Lagrangian for all $t \in [0, T)$.

$\square$

4. Hyper-Lagrangian submanifolds

Let (L^{2n}, g) be a hyper-Lagrangian submanifold in a hyperkähler manifold $(M^{4n}, \bar{g})$ with respect to some $J(x)$ in $\Lambda^2_{+,x}$ and $K(x) : L \to \Lambda^2_{+,x}$ be a smooth map such that $K(x)$ is orthogonal to $J(x)$ for each $x \in L$. Then by the hyper-Lagrangian condition, we have

$$\omega_K|_L = 0, \quad \forall x \in L .$$

Suppose that $\{e_i\}_1^{2n}$ is an orthonormal frame field for TL around a point $x = p$ satisfying

$$Je_{2\nu-1} = e_{2\nu} \quad \text{for} \quad \nu = 1, \dots, n ,$$

then $\{f_i = Ke_i\}_{i=1}^{2n}$ is an orthonormal frame field for the normal bundle satisfying

$$Jf_{2\nu-1} = -f_{2\nu}, \quad \text{for} \quad \nu = 1, \dots, n .$$

For the frame $\{e_i, f_i\}$, we denote

$$\begin{cases} e'_\nu = \frac{1}{2}(e_{2\nu-1} - ie_{2\nu}), & e''_\nu = \frac{1}{2}(e_{2\nu-1} + ie_{2\nu}) , \\ f'_\nu = \frac{1}{2}(f_{2\nu-1} + if_{2\nu}), & f''_\nu = \frac{1}{2}(f_{2\nu-1} - if_{2\nu}) . \end{cases}$$

Then it is obvious that

$$\begin{cases} Je'_\nu = ie'_\nu, & Je''_\nu = -ie''_\nu , \\ Jf'_\nu = if'_\nu, & Jf''_\nu = -if''_\nu , \end{cases}$$

$$\begin{cases} Ke'_\nu = f''_\nu, & Ke''_\nu = f'_\nu , \\ Kf'_\nu = -e''_\nu, & Kf''_\nu = -e'_\nu . \end{cases}$$

The frame fields $\{e'_\nu, f'_\mu\}$ defined in the above will be referred as the *canonical frame adapted to* (J, K) *around the point* p.

Correspondingly, we set

$$\begin{cases} \phi_\nu = \zeta_{2\nu-1} + i\zeta_{2\nu}, & \overline{\phi}_\nu = \zeta_{2\nu-1} - i\zeta_{2\nu}, \\ \psi_\nu = \varphi_{2\nu-1} - i\varphi_{2\nu}, & \overline{\psi}_\nu = \varphi_{2\nu-1} + i\varphi_{2\nu} , \end{cases}$$

for the frame $\{\zeta_1, \dots, \zeta_{2n}, \varphi_1, \dots, \varphi_{2n}\}$ dual to $\{e_1, \dots, e_{2n}, f_1 \dots, f_{2n}\}$. Then the J-holomorphic symplectic form is given by

$$\Omega_J = \omega_{JK} - i\omega_K = -i \sum_{\nu=1}^n \phi_\nu \wedge \psi_\nu ,$$

and the structural equations implies

$$\begin{cases} d\zeta_i & = \sum_{j=1}^{2n} \zeta_{ij} \wedge \zeta_j - \sum_{j=1}^{2n} \Psi_{ji} \wedge \varphi_j \\ d\varphi_i & = \sum_{j=1}^{2n} \Psi_{ij} \wedge \zeta_j + \sum_{j=1}^{2n} \varphi_{ij} \wedge \varphi_j , \end{cases}$$

for some 1-forms ζ_{ij}, Ψ_{ij}, and φ_i satisfying

$$\zeta_{ij} + \zeta_{ji} = 0 \quad \text{and} \quad \varphi_{ij} + \varphi_{ji} = 0 .$$

Along the submanifold L, there exist h_{ijk}, symmetric in j, k, such that

$$\Psi_{ij} = \sum_{k=1}^{2n} h_{ijk}\zeta_k \,.$$

It is clear that if we choose a normal coordinates around the point p, then the h_{ijk} defined here coincides with the $h(K)_{ijk}$ defined in the previous section at the point p. Finally, the mean curvature vector is of course given by

$$H = -\sum_{i=1}^{2n}\left(\sum_{k=1}^{2n} h_{ikk}\right) f_i \,.$$

From the above, the covariant derivatives of the orthonormal frames are given by

$$\begin{cases} \bar{\nabla}e_i & = \sum_{j=1}^{2n} \zeta_{ij} \otimes e_j - \sum_{j=1}^{2n} \Psi_{ji} \otimes f_j \\ \bar{\nabla}f_i & = \sum_{j=1}^{2n} \Psi_{ij} \otimes e_j + \sum_{j=1}^{2n} \varphi_{ij} \otimes f_j \,, \end{cases}$$

where ζ_{ij}, Ψ_{ij}, and φ_{ij} are 1-forms. If we set

$$\begin{cases} \mathcal{A}_{\mu\nu} & = \frac{1}{2}\left[(\zeta_{2\mu-1,2\nu-1} + \zeta_{2\mu,2\nu}) - i\,(\zeta_{2\mu,2\nu-1} - \zeta_{2\mu-1,2\nu})\right] \,, \\ \mathcal{B}_{\mu\nu} & = \frac{1}{2}\left[(\zeta_{2\mu-1,2\nu-1} - \zeta_{2\mu,2\nu}) - i\,(\zeta_{2\mu,2\nu-1} + \zeta_{2\mu-1,2\nu})\right] \,, \\ \mathcal{C}_{\mu\nu} & = \frac{1}{2}\left[(\Psi_{2\mu-1,2\nu-1} - \Psi_{2\mu,2\nu}) - i\,(\Psi_{2\mu,2\nu-1} + \Psi_{2\mu-1,2\nu})\right] \,, \\ \mathcal{D}_{\mu\nu} & = \frac{1}{2}\left[(\Psi_{2\mu-1,2\nu-1} + \Psi_{2\mu,2\nu}) - i\,(\Psi_{2\mu-1,2\nu} - \Psi_{2\mu,2\nu-1})\right] \,, \\ \mathcal{E}_{\mu\nu} & = \frac{1}{2}\left[(\varphi_{2\mu-1,2\nu-1} + \varphi_{2\mu,2\nu}) + i\,(\varphi_{2\mu,2\nu-1} - \varphi_{2\mu-1,2\nu})\right] \,, \\ \mathcal{F}_{\mu\nu} & = \frac{1}{2}\left[(\varphi_{2\mu-1,2\nu-1} - \varphi_{2\mu,2\nu}) + i\,(\varphi_{2\mu,2\nu-1} + \varphi_{2\mu-1,2\nu})\right] \,, \end{cases}$$

for $\nu, \mu = 1, \dots, n$, then straightforward calculation using $\zeta_{ij} + \zeta_{ji} = 0$ and $\varphi_{ab} + \varphi_{ba} = 0$ shows for any $\mu, \nu = 1, \dots, n$,

$$\begin{cases} \bar{\nabla}e'_\nu & = \sum_\mu \left(\mathcal{A}_{\nu\mu} \otimes e'_\mu + \mathcal{B}_{\nu\mu} \otimes e''_\mu\right) - \sum_\mu \left(\mathcal{C}_{\mu\nu} \otimes f'_\mu + \mathcal{D}_{\mu\nu} \otimes f''_\mu\right) \,, \\ \bar{\nabla}f'_\nu & = \sum_\mu \left(\overline{\mathcal{C}_{\nu\mu}} \otimes e'_\mu + \mathcal{D}_{\nu\mu} \otimes e''_\mu\right) + \sum_\mu \left(\mathcal{E}_{\nu\mu} \otimes f'_\mu + \mathcal{F}_{\nu\mu} \otimes f''_\mu\right) \end{cases} \tag{4.1}$$

and

$$\begin{cases} \mathcal{A}_{\mu\nu} + \overline{\mathcal{A}}_{\mu\nu} & = \mathcal{B}_{\nu\mu} + \mathcal{B}_{\mu\nu} = 0 \,, \\ \mathcal{E}_{\mu\nu} + \overline{\mathcal{E}}_{\nu\mu} & = \mathcal{F}_{\mu\nu} + \mathcal{F}_{\nu\mu} = 0 \,. \end{cases} \tag{4.2}$$

In order to prove the compatibility condition, for any fixed $p \in L$, we set $J_1 = J(p)$, $J_2 = K(p)$, and $J_3 = J_1 J_2$. Then $\{J_\alpha\}_{\alpha=1}^{3}$ is an orthogonal set of parallel complex structures and they will be called the *canonical basis adapted to* (J, K) *at* p. At the point p, we have

$$J_1 = \sum_{\nu=1}^{n} (e_{2\nu-1} \wedge e_{2\nu} - f_{2\nu-1} \wedge f_{2\nu})$$

$$J_2 = \sum_{\nu=1}^{n} (e_{2\nu-1} \wedge f_{2\nu-1} + e_{2\nu} \wedge f_{2\nu})$$

$$J_3 = \sum_{\nu=1}^{n} (f_{2\nu} \wedge e_{2\nu-1} - f_{2\nu-1} \wedge e_{2\nu})$$

in the canonical identification between 2-forms and 2-vectors.

Now we can state the following.

Theorem 4.1.   *Let L^{2n} be a hyper-Lagrangian in a hyperkähler manifold M^{4n} with defining map J. Let $K : L \to \mathbb{S}^2$ be any smooth map orthogonal to J at every point of L. Then for any fixed point $p \in L$, the coefficient 1-forms of the structural equations for the canonical frame adapted to (J, K) around p satisfy the following conditions at the point p:*

$$\mathcal{A}_{\mu\nu} + \overline{\mathcal{A}}_{\mu\nu} = 0 , \tag{4.3}$$

$$\mathcal{B}_{\nu\mu} = 0, \tag{4.4}$$

$$\mathcal{C}_{\nu\mu} = \mathcal{C}_{\mu\nu} , \tag{4.5}$$

$$\mathcal{D}_{\nu\mu} = \frac{1}{2}\delta_{\nu\mu}\,(da_3 + i\,da_2) , \tag{4.6}$$

$$\mathcal{E}_{\mu\nu} + \overline{\mathcal{E}}_{\nu\mu} = 0 , \tag{4.7}$$

$$\mathcal{F}_{\nu\mu} = 0 , \tag{4.8}$$

$$\mathcal{E}_{\nu\mu} = \overline{\mathcal{A}_{\nu\mu}} + i\delta_{\nu\mu}\,db_3 , \tag{4.9}$$

where a_α and b_α are, respectively, the components of J and K with respect to the canonical basis adapted to (J, K) at p.

Moreover, the differential of J at p as a map into $\mathbb{S}^2$ is given by

$$dJ_p = 2\,(\,\mathrm{Im}\mathcal{D}_{\nu\nu}\,J_2 + \mathrm{Re}\mathcal{D}_{\nu\nu}\,J_3) \quad \text{for any} \quad \nu = 1, \dots, n .$$

Proof.   We note that (4.3) and (4.7) are just (4.2). To prove the other conditions, we differentiate $Je'_\nu = ie'_\nu$ to obtain

$$(\nabla J)e'_\nu + J(\nabla e'_\nu) = i\nabla e'_\nu$$

for all $\nu = 1, \dots, n$. Let J_α be the canonical basis adapted to (J, K) at p, and a_α, b_α be the coefficients of $J = \sum_\alpha a_\alpha J_\alpha$ and $K = \sum_\alpha b_\alpha J_\alpha$, respectively. Then by the hyper-Lagrangian condition, ∇J at the point p is a linear combination of $J_2 = K(p)$ and J_3. Therefore, we have the following.

$$\nabla J = da_2 J_2 + da_3 J_3$$

at the point p. Putting this and the structural equations in the above equation, we obtain at the point p,

$$(da_2 - i\,da_3)\,f''_\nu = 2i\mathcal{B}_{\nu\mu}e''_\mu - 2i\mathcal{D}_{\nu\mu}f''_\mu ,$$

which implies the Equations (4.6) and (4.4).

Similarly, applying the same argument to $Jf'_\nu = if'_\nu$ and $Ke''_\nu = f'_\nu$, we obtain (4.6) again and all other Equations (4.8), (4.5), (4.9) together with the identity $da_2 + db_1 = 0$ at p. The last identity is automatically satisfied by the definition of a_α and b_α.

The final part is an obvious conclusion of (4.6) and the proof is completed. $\square$

The theorem indicates that hyper-Lagrangian is a strong condition which imposes a lot of restrictions on the structural equations. As a consequence, we have the following.

Corollary 4.2.   *Let L^{2n} be a hyper-Lagrangian in a hyperkähler manifold M^{4n} and for any $p \in L$, $J(p)$ be the almost complex structure at p such that $\Omega_J(p)|_{T_pL} = 0$. Then J induces an integrable Kähler structure on L with holomorphic normal bundle.*

Proof.   From Theorem 4.1, we have

$$\begin{cases} \nabla e'_\nu & = (\bar{\nabla} e'_\nu)^T = \sum_\mu \mathcal{A}_{\mu\nu} \otimes e'_\mu \, , \\ \nabla^\perp f'_\nu & = (\bar{\nabla} f'_\nu)^\perp = \sum_\mu \mathcal{E}_{\nu\mu} \otimes f'_\mu \, . \end{cases}$$

The conclusions follow immediately.   □

In the neighborhood of the point p, it is more convenient to use the following local representation of the map J

$$\Theta(x) = \frac{a_1(x) + i a_2(x)}{1 - a_3(x)}$$

via stereographic projection. Then the theorem implies that at the point p,

$$\mathcal{D}_{\nu\mu} = \frac{1}{2} \delta_{\nu\mu} \, d\Theta \, . \tag{4.10}$$

Using Θ, the conditions on the second fundamental form can be written as in the following.

Corollary 4.3.   *Let L^{2n} be a hyper-Lagrangian in a hyperkähler manifold M^{4n} with defining map J. Let $K : L \to \mathbb{S}^2$ be any smooth map orthogonal to J at every point of L. Then for any fixed point $p \in L$, $\Psi_{2\mu-1,2\nu-1}$, $\Psi_{2\mu,2\nu}$, $\Psi_{2\mu-1,2\nu}$, $\Psi_{2\mu,2\nu-1}$ are symmetric in μ, ν; and*

$$\begin{cases} \Psi_{2\mu-1,2\nu-1} + \Psi_{2\mu,2\nu} & = \frac{1}{2}\delta_{\mu\nu} \, \text{Re} \, (d\Theta) \\ \Psi_{2\mu-1,2\nu} - \Psi_{2\mu,2\nu-1} & = -\frac{1}{2}\delta_{\mu\nu} \, \text{Im} \, (d\Theta) \, , \end{cases}$$

where $\{\Psi_{ij}\}$ is the second fundamental form in the canonical frame adapted to (J, K) at p.

Proof.   By definition, we have

$$\begin{aligned} \Psi_{2\mu-1,2\nu-1} & = \text{Re} \left(\mathcal{C}_{\mu\nu} + \mathcal{D}_{\mu\nu} \right) \, , \\ \Psi_{2\mu,2\nu} & = \text{Re} \left(-\mathcal{C}_{\mu\nu} + \mathcal{D}_{\mu\nu} \right) \, , \\ \Psi_{2\mu-1,2\nu} & = \text{Im} \left(-\mathcal{C}_{\mu\nu} - \mathcal{D}_{\mu\nu} \right) \, , \\ \Psi_{2\mu,2\nu-1} & = \text{Im} \left(-\mathcal{C}_{\mu\nu} + \mathcal{D}_{\mu\nu} \right) \, . \end{aligned}$$

Therefore, Equation (4.5) and (4.6) imply that all four 1-forms listed are symmetric in μ, ν. And the two equations follow immediately by adding and subtracting corresponding equations.   □

Corollary 4.4.   *Let L^{2n} be a hyper-Lagrangian in a hyperkähler manifold M^{4n} with defining map J. Let $K : L \to \mathbb{S}^2$ be any smooth map orthogonal to J at every point of L. Then for any fixed point $p \in L$, the second fundamental form $\{h_{ijk}\}$ in the canonical frame adapted to (J, K) at p satisfies*

$$h_{ijk} = h_{jik} \quad \text{for} \quad \{i, j\} \neq \{2\mu - 1, 2\mu\} \quad \text{for any} \quad \mu,$$

and

$$h_{2\mu-1,2\mu,k} = h_{2\mu,2\mu-1,k} - \, \text{Im} \, (d\Theta(e_k)) \, .$$

Moreover, for any μ, ν,

$$h_{2\mu-1,2\nu-1,k} + h_{2\mu,2\nu,k} = \delta_{\mu\nu} \, \text{Re} \, (d\Theta(e_k)) \, .$$

Proof.   It is immediately from the symmetries of Ψ that $h_{ijk} = h_{jik}$ if $i + j$ is even. If $i + j$ is odd, then we consider $h_{2\mu-1,2\nu,k} - h_{2\nu,2\mu-1,k}$. By the symmetries of $\Psi_{2\mu-1,2\nu}$ and equation for $\Psi_{2\mu-1,2\nu} - \Psi_{2\mu,2\nu-1}$ we have

$$h_{2\mu-1,2\nu,k} - h_{2\nu,2\mu-1,k} = h_{2\mu-1,2\nu,k} - h_{2\mu,2\nu-1,k} = -\frac{1}{2}\delta_{\mu\nu}\mathrm{Im}\,(d\Theta(e_k))\ .$$

So we conclude that $h_{2\mu-1,2\nu,k} = h_{2\mu,2\nu-1,k}$ if $\mu \neq \nu$. Otherwise, we have

$$h_{2\mu-1,2\mu,k} = h_{2\mu,2\mu-1,k} - \mathrm{Im}\,(d\Theta(e_k))\ .$$

This completes the proof of the first part. The second part follows immediately from the corresponding equation for Ψ. $\qquad\square$

Note that the formula for commuting i, j in h_{ijk} in this corollary can also be obtained from Proposition 2.1.

It is interesting to compare the result in this corollary with the complete symmetry of all three indices of the second fundamental form of a special Lagrangian submanifold in a Calabi-Yau manifold. In our case, the second fundamental form of a hyper-Lagrangian does not have the complete symmetry in all three indices in general. However, the corollary shows that the second fundamental form has a lot of symmetry which is sufficient to implies the first-order equation on Θ relating $\partial\Theta$ to the mean curvature vector H and the holomorphic symplectic form Ω_J.

If we denote $d\Theta_p(e_k) = P_k - iQ_k$, then

$$P_k = h_{11k} + h_{22k} = h_{33k} + h_{44k} = \cdots = h_{2n-1,2n-1,k} + h_{2n,2n,k}$$

and

$$Q_k = h_{12k} - h_{21k} = h_{34k} - h_{43k} = \cdots = h_{2n-1,2n,k} - h_{2n,2n-1,k}$$

for all $k = 1, \ldots, 2n$. And we have

$$d\Theta_p = \sum_{k=1}^{2n}(P_k - iQ_k)\zeta_k\ .$$

In terms of the forms ϕ_ν and $\overline{\phi}_\nu$, this is given by

$$\begin{aligned}
d\Theta_p \quad = \quad & \frac{1}{2}\sum_\nu\left[(P_{2\nu-1} - Q_{2\nu}) - i(P_{2\nu} + Q_{2\nu-1})\right]\phi_\nu \\
& + \frac{1}{2}\sum_\nu\left[(P_{2\nu-1} + Q_{2\nu}) + i(P_{2\nu} - Q_{2\nu-1})\right]\overline{\phi}_\nu\ .
\end{aligned}$$

Hence, one concludes that

$$\partial\Theta = \frac{1}{2}\sum_\nu\left[(P_{2\nu-1} - Q_{2\nu}) + i(P_{2\nu} + Q_{2\nu-1})\right]\phi_\nu\ .$$

Now we can prove the following.

Theorem 4.5.   *Let* L^{2n} *be a hyper-Lagrangian submanifold in a hyperkähler manifold* M^{4n} *and for any* $p \in L$, $J(p)$ *be the almost complex structure at* p *such that* $\Omega_J(p)|_{T_pL} = 0$. *Then*

$$H \lrcorner \Omega_J + 2i\,\partial J = 0\ ,$$

where H is the mean curvature vector of L in M.

Proof.   We continue to work on the local representation Θ of the map $J : L \to \mathbb{S}^2$. The mean curvature vector is

$$
\begin{aligned}
H &= -\sum_{j=1}^{2n} \left(\sum_{k=1}^{2n} h_{jkk} \right) f_j \\
&= -\sum_{\mu=1}^{n} \sum_{k=1}^{2n} \left[(h_{2\mu-1,kk} - i h_{2\mu,kk}) f'_\mu + (h_{2\mu-1,kk} + i h_{2\mu,kk}) f''_\mu \right] .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
H \lrcorner \Omega_J &= i H \lrcorner \sum_{\mu=1}^{n} \phi_\mu \wedge \psi_\mu \\
&= -i \sum_{\mu=1}^{n} \sum_{k=1}^{2n} \phi_\mu (h_{2\mu-1,kk} - i h_{2\mu,kk}) \\
&= -i \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} \phi_\mu \left[(h_{2\mu-1,2\nu-1,2\nu-1} + h_{2\mu-1,2\nu,2\nu}) - i (h_{2\mu,2\nu-1,2\nu-1} + h_{2\mu,2\nu,2\nu}) \right] .
\end{aligned}
$$

If $\mu \neq \nu$, then by Corollary 4.4 and the symmetry of the last two index, one has

$$
h_{2\mu-1,2\nu-1,2\nu-1} + h_{2\mu-1,2\nu,2\nu} = h_{2\mu,2\nu-1,2\nu-1} + h_{2\mu,2\nu,2\nu} = 0 .
$$

Hence, all the cross terms in the sum vanish and we have

$$
H \lrcorner \Omega_J = -i \sum_{\mu=1}^{n} \phi_\mu \left[(h_{2\mu-1,2\mu-1,2\mu-1} + h_{2\mu-1,2\mu,2\mu}) - i (h_{2\mu,2\mu-1,2\mu-1} + h_{2\mu,2\mu,2\mu}) \right] .
$$

Note that for any μ,

$$
\begin{aligned}
P_{2\mu-1} - Q_{2\mu} &= (h_{2\mu-1,2\mu-1,2\mu-1} + h_{2\mu,2\mu,2\mu-1}) - (-h_{2\mu-1,2\mu,2\mu} + h_{2\mu,2\mu-1,2\mu}) \\
&= h_{2\mu-1,2\mu-1,2\mu-1} + h_{2\mu-1,2\mu,2\mu} ,
\end{aligned}
$$

and

$$
\begin{aligned}
P_{2\mu} + Q_{2\mu-1} &= (h_{2\mu-1,2\mu-1,2\mu} + h_{2\mu,2\mu,2\mu}) + (h_{2\mu,2\mu-1,2\mu-1} - h_{2\mu-1,2\mu,2\mu-1}) \\
&= h_{2\mu,2\mu-1,2\mu-1} + h_{2\mu,2\mu,2\mu} .
\end{aligned}
$$

Therefore, we have

$$
H \lrcorner \Omega_J + 2i \, \partial \Theta = 0 . \qquad \square
$$

By the theorem, we immediately conclude the following.

Corollary 4.6.   *Let L^{2n} be a hyper-Lagrangian submanifold in a hyperkähler manifold M^{4n} and for any $p \in L$, $J(p)$ be the almost complex structure at p such that $\Omega_J(p)|_{T_p L} = 0$. Then L is minimal if and only if J is anti-holomorphic.*

This corollary includes the case of complex Lagrangian submanifolds. In this case, J is a constant.

In the case that L is in fact a Lagrangian with respect to a fixed ω_{J_o}. We may choose J_2 to be the fixed J_o. Then at the point $p \in L$, there is a function $\psi : L \to \mathbb{R}$ with $\psi(p) = \pi/2$ such that $\Theta(x) = \sin\psi(x)J_1 + \cos\psi(x)J_3$. It follows immediately that $d\Theta = -d\psi$ at $x = p$. These imply that $Q_k = 0$ for all k and $d\psi = -\sum_k P_k\zeta_k$. On the other hand, the holomorphic symplectic form at p is given by $\Omega_J = \omega_3 - i\omega_2$. So $H \lrcorner \Omega_J = H \lrcorner \omega_3 - i H \lrcorner \omega_2$ and by Theorem 4.5, we have

$$
\begin{aligned}
&-H \lrcorner \omega_3 + i\,(H \lrcorner \omega_2) \\
&= \; 2i\,\partial\Theta \\
&= \; i\sum_\nu (P_{2\nu-1} - i P_{2\nu})\phi_\nu \\
&= \; i\sum_\nu (P_{2\nu-1} - i P_{2\nu})(\zeta_{2\nu-1} + i\zeta_{2\nu}) \\
&= \; \sum_\nu \left[(P_{2\nu}\zeta_{2\nu-1} - P_{2\nu-1}\zeta_{2\nu}) + i(P_{2\nu-1}\zeta_{2\nu-1} + P_{2\nu}\zeta_{2\nu}) \right].
\end{aligned}
$$

Therefore

$$
H \lrcorner \omega_2 + d\psi = 0 ,
$$

which implies that ψ is in fact the phase map up to a constant. Moreover, we have an additional equation

$$
H \lrcorner \omega_3 = \sum_\nu (P_{2\nu-1}\zeta_{2\nu} - P_{2\nu}\zeta_{2\nu-1}) .
$$

5. A remark on regularity

Finally, we would like to make one remark about the regularity of the mean curvature flow of hyper-Lagrangian submanifolds under a natural condition on the associated map J. It was proved by M. T. Wang in [13] that a symplectic surface in a four-dimensional Kähler-Einstein manifold remains symplectic and does not develop any Type I singularity along the mean curvature flow. If the Kähler-Einstein is in fact hyperkähler, then a surface being symplectic is equivalent to the condition that the image of the map J is contained in a hemisphere. To see this, we observe that any surface L in a four-dimensional hyperkähler manifold is always hyper-Lagrangian. Therefore, there is a $J(x) = \sum_{\alpha=1}^{3} a_\alpha(x)J_\alpha$, with $a = (a_1, a_2, a_3) \in \mathbb{S}^2$ and an orthogonal set of parallel complex structures $\{J_\alpha\}_{\alpha=1}^{3}$ such that $J(T_x L) = T_x L$ for all $x \in L$. In a neighborhood of any point $x \in L$, we may choose an orthonormal frame for TL of the form $\{e_1, e_2 = Je_1\}$. Then

$$
\omega_{1,12} = \bar{g}(J_1 e_1, e_2) = a_1 \quad \forall x \in L .
$$

Therefore, $a_1 > 0$ if and only if L is symplectic with respect to ω_1. This shows the required equivalence. Therefore Wang's result can be rephrased as: If the image of the map J on L contained in a hemisphere, then the image J on L_t remains in the same hemisphere and does not develop any Type I singularity along the mean curvature flow. In the following, we will see that the proof in [13] actually can be applied to the higher dimensional hyper-Lagrangian submanifolds and gives the following.

Theorem 5.1. *Let $(M^{4n}, \bar{g})$ be a hyperkähler manifold. Let $L_t \subset M^{4n}$, $t \in [0, T)$ for some $T > 0$, be a family of middle dimensional submanifolds given by the mean curvature flow. Suppose that $L_0 \subset M$ is hyper-Lagrangian and the image of the corresponding map $J : L_0 \to \mathbb{S}^2$*

is contained in a hemisphere. Then the image of $J : [0, T) \times L_0 \to \mathbb{S}^2$ is contained in the same hemisphere and the mean curvature flow does not develop any Type I singularity.

Proof.   By Theorems 3.4 and 3.6, L_t is hyper-Lagrangian for each $t \in [0, T)$ and $J(t, x)$ satisfies the harmonic map heat flow which can be written as

$$(\partial_t - \triangle)a_\alpha = |\nabla a|^2 a_\alpha, \quad \alpha = 1, 2, 3,$$

where a_α are components of J as a map into $\mathbb{S}^2 \subset \mathbb{R}^3$. If $J(L_0)$ is contained in a hemisphere which may be assumed to be $\{a \in \mathbb{S}^2 \,|\, a_1 > 0\}$. Then we have $a_1(x) > 0$ for all $x \in L_0$. Applying maximum principle to the harmonic map heat flow, we conclude that $a_1(t, x) > 0$ for all $(t, x) \in [0, T) \times L_0$. This proves the first part of the theorem.

To see the second part, we observed that L_t is hyper-Lagrangian for all t. Theorem 4.5 implies that

$$H \lrcorner \Omega_J + 2i \partial \Theta = 0 .$$

Hence,

$$|H|^2 = 2|\partial \Theta|^2 \leq 2|\nabla a|^2 .$$

This estimate implies

$$(\partial_t - \triangle)a_1 \geq \frac{1}{2}|H|^2 a_1 .$$

Then standard argument using $a_1 > 0$ imply the nonexistence of Type I singularity, see, for example, [5], or the proof of Proposition 5.2 of [13]. □

The same argument is also applied to Lagrangian submanifold in a Calabi-Yau manifold (M, ω, Ω) with $*\mathrm{Re}\,\Omega > 0$. Firstly, it was proved by Smoczyk [11] that it remains Lagrangian along the mean curvature flow. Then Wang showed in [13], see also [14], that the condition $*\mathrm{Re}\,\Omega > 0$ is preserved and the flow does not develop any Type I singularity using the same argument.

Acknowledgments

References

[1] Atiyah, M. Hyper-Kähler manifolds, in *Complex Geometry and Analysis,* (Pisa, 1988), 1–13, Lecture Notes in Math., 1422, Springer, Berlin, (1990).

[2] Chen, J. Deforming surfaces in four-dimensional manifolds, *Symplectic and Contact Topology: Interactions and Perspectives,* (Toronto, ON/Montreal, QC, 2001), 79–88, Fields Inst. Commun., 35, Amer. Math. Soc., Providence, RI, (2003).

[3] Chen, J. and Li, J. Mean curvature flow of surface in 4-manifolds, *Adv. Math.* **163**(2), 287–309, (2001).

[4] Chen, J. and Tian, G. Moving symplectic curves in Kähler-Einstein surfaces, *Acta Math. Sinica* **16**(4), 541–548, (2000).

[5] Ecker, K. and Huisken, G. Mean curvature evolution of entire graphs, *Ann. of Math. (2)* **130**(3), 453–471, (1989).

[6] Hitchin, N. J., Karlhede, A., Lindström, U., and Roček, M. Hyper-Kähler metrics and supersymmetry, *Comm. Math. Phys.* **108**(4), 535–589, (1987).

[7] Hitchin, N. J. Hyper-Kähler manifolds, Séminaire Bourbaki, *Astérisque* **206**, 1991/92, Exp. No. 748, 3, 137–166, (1992).

[8] Hitchin, N. J. The moduli space of complex Lagrangian submanifolds, Sir Michael Atiyah: A great mathematician of the twentieth century, *Asian J. Math.* **3**(1), 77–91, (1999).

[9]   Leung, N. C. Lagrangian submanifolds in hyperKähler manifolds, Legendre transformation, *J. Differential Geom.* **61**(1), 107–145, (2002).

[10] Leung, N. C. Riemannian geometry over different normed division algebras, *J. Differential Geom.* **61**(2), 289–333, (2002).

[11] Smoczyk, K. A canonical way to deform a Lagrangian submanifold, dg-ga/9605005.

[12] Thomas, R. P. and Yau, S.-T. Special Lagrangians, stable bundles and mean curvature flow, *Comm. Anal. Geom.* **10**(5), 1075–1113, (2002).

[13] Wang, M. T. Mean curvature flow of surfaces in Einstein four-manifolds, *J. Differential Geom.* **57**(2), 301–338, (2001).

[14] Wang, M. T. Mean curvature flow in high codimension, in Proceedings of ICCM2001, *Stud. Adv. Math.* International Press, 275–283, (2004).

Department of Mathematics and Institute of Mathematical Sciences,
The Chinese University of Hong Kong, Shatin, N. T., Hong Kong
e-mail: leung@ims.cuhk.edu.hk

Department of Mathematics, The Chinese University of Hong Kong, Shatin, N. T., Hong Kong
e-mail: tomwan@math.cuhk.edu.hk