MIYAOKA-YAU TYPE INEQUALITIES FOR KÄHLER-EINSTEIN MANIFOLDS

KWOKWAI CHAN AND NAICHUNG CONAN LEUNG

ABSTRACT. We investigate Chern number inequalities on Kähler-Einstein manifolds and their relation to uniformization. For Kähler-Einstein manifolds with $c_1 > 0$, we prove certain Chern number inequalities in the toric case. For Kähler-Einstein manifolds with $c_1 < 0$, we propose a series of Chern number inequalities, interpolating Yau's and Miyaoka's inequalities.

1. INTRODUCTION

Yau's celebrated solution to Calabi's conjecture [17] says that a compact Kähler n-fold X with $c_1(X) < 0$ or $c_1(X) = 0$ always admit Kähler-Einstein metrics. As an application, he proved in [16] that for the case $c_1(X) < 0$, we have the following Chern number inequality[1]:

$$(1.1) \qquad c_1^2(X)|c_1(X)|^{n-2} \leq \frac{2(n+1)}{n} c_2(X)|c_1(X)|^{n-2}$$

with equality holds if and only if X is uniformized by the unit ball in $\mathbb{C}^n$.

It turns out that (see e.g. [12]) Yau's inequality (1.1) actually follows from the (Higgs-)stability of the bundle $T_X^* \oplus \mathcal{O}_X$ on X. On the other hand, as a generalization of the Miyaoka-Yau inequality for surfaces [6], [16], Miyaoka proved in [7] the following inequality:

$$(1.2) \qquad c_1^2(X)H^{n-2} \leq 3c_2(X)H^{n-2}$$

for any nef class H on a compact Kähler n-fold X with $c_1(X) < 0$. The relation between this inequality and stability also deserves investigation.

In 1990, Yau [18] conjectured that the existence of Kähler-Einstein metrics should be equivalent to some notion of stability. This vast program of Yau would reveal deep relations between the existence of Kähler-Einstein metrics, stability, Chern number inequalities and uniformization. However, despite the much work done by Yau, Tian, Donaldson and others [14], [18],

[1]Throughout this paper, by $|c_1(X)|$ we mean $-c_1(X)$ (resp. $c_1(X)$) if $c_1(X) < 0$ (resp. $c_1(X) > 0$).

[13], [3], these relations are still not completely clear, especially for $c_1 > 0$ case.

In view of Yau's program and the inequalities (1.1) and (1.2), we propose the following series of Miyaoka-Yau type Chern number inequalities for the $c_1 < 0$ case:

$$c_1^2(X)|c_1(X)|^{n-2} \leq \frac{2(n+1)}{n} c_2(X)|c_1(X)|^{n-2}$$

$$\vdots$$

$(*)$ $$c_1^2(X)|c_1(X)|^{k-2}H^{n-k} \leq \frac{2(k+1)}{k} c_2(X)|c_1(X)|^{k-2}H^{n-k}$$

$$\vdots$$

$$c_1(X)^2 H^{n-2} \leq 3c_2(X)H^{n-2}.$$

This, in particular, will provide a link between the inequalities (1.1) and (1.2). We also infer that an equality will lead to uniformization. More precisely, we propose the following:

Conjecture 1.1. *Let X be a compact Kähler n-fold with $c_1(X) < 0$. Then the above series of inequalities $(*)$ hold for any nef class H on X. Moreover, equality holds for the k-th level inequality with H^{n-k} effective if and only if X is a fibration whose general fiber is uniformized by the k-dimensional unit ball and H comes from the base.*

In the next section, we will investigate the intimate relation between these inequalities and stability, and explain why such a conjecture is conceivable.

For the $c_1 > 0$ case, both the existence of Kähler-Einstein metrics and stability become much more subtle, as mentioned above. Nevertheless, the situation becomes considerably better in the toric case. Recall that for a toric Fano manifold X, there is an associated reflexive polytope P. Recently Wang and Zhu [15] proved that a toric Fano manifold admits Kähler-Einstein metrics if and only if it is *balanced*, i.e. the center of mass of P, denoted by center(P), is at the origin, . In view of this, we consider Kähler-Einstein manifolds with $c_1 > 0$ which are toric, and prove, in §3 and §4, the following Chern number inequalities:

Theorem 1.1. *Let X be a Kähler-Einstein toric Fano n-fold. Then for any nef class H, we have:*

$$c_1^2(X)H^{n-2} \leq 3c_2(X)H^{n-2}$$

provided that either

(i) $n = 2, 3, 4$, or

(ii) each facet of P contains a lattice point in its interior.

Theorem 1.1, together with the discussion at the end of §2, yield the following

Corollary 1.1. *Let X be a product of Kähler-Einstein toric Fano manifolds of dimensions less than or equal to 3. Then the whole set of inequalities (∗) hold for any nef class H on X.*

We will also discuss how the proposed inequalities (∗) can give new combinatorial information on balanced reflexive polytopes. Let P be an n-dimensional balanced reflexive polytope such that the corresponding toric variety X is smooth. Denote by $\#(kP)$ the number of lattice points in kP. The *normalized Ehrhart polynomial* of P:

$$\widetilde{E}(P) := \frac{\#(kP)}{\mathrm{Vol}(P)} = b_n k^n + b_{n-1} k^{n-1} + b_{n-2} k^{n-2} + \dots$$

measures the growth of the number of lattice points. Now Yau's inequality gives the following *asymptotic lower bound*:

Proposition 1.1. *For k large, we have*

$$\widetilde{E}(P) \geq \widetilde{E}(P_n)$$

with equality holds if and only if $P = P_n$, where P_n denotes $(n+1)$ times of the standard simplex, which is the polytope of $\mathbb{P}^n$.

Other inequalities of (∗) can give asymptotic *upper* bounds on Ehrhart polynomials. These will be discussed in the final section.

2. MANIFOLDS WITH $c_1 < 0$

In this section, we first review how Yau's inequality (1.1) can be deduced from stability. Then we explain, along similar lines, how Conjecture 1.1 comes out.

Let X be a compact Kähler n-fold with $c_1(X) < 0$. The direct sum $E := T_X^* \oplus \mathcal{O}_X$ of the holomorphic cotangent bundle and the trivial line bundle is of rank $n+1$. Let $\theta \in \Omega^{1,0}(\mathrm{End}(E))$ be the *Higgs field* given by the identity map $T_X^* \to T_X^* = \mathcal{O}_X \otimes T_X^*$. Then it can be shown that (E, θ) is *Hermitian-Yang-Mills* and hence *Higgs-stable* with respect to the polarization $|c_1(X)|$ (see [12]). Bogomolov's inequality then give Yau's inequality:

$$c_1^2(X)|c_1(X)|^{n-2} \leq \frac{2(n+1)}{n} c_2(X)|c_1(X)|^{n-2}$$

and equality implies that X is uniformized by the unit ball. In view of this, we suggest that Miyaoka's inequality:

$$c_1^2(X)H^{n-2} \leq 3c_2(X)H^{n-2}$$

also comes from the (Higgs-)stability of some rank 3 bundle, which, analogous to the above situation, should be the direct sum of a rank 2 bundle and a trivial bundle. A natural choice of this is described as follows.

We begin with the rank n bundle T_X^* over X. Let $Gr(2, T_X^*)$ be the Grassmannian bundle of 2-planes in T_X^*, $\pi : Gr(2, T_X^*) \to X$ the projection map. Let S_2 be the universal rank 2 sub-bundle of $\pi^*T_X^*$. It is natural to expect that Miyaoka's inequality is related to the stability of $S_2 \oplus \mathcal{O}$. More generally, for $k = 3, \ldots, n$, let $Gr(k, T_X^*)$ be the Grassmannian bundle of k-planes and S_k the universal rank k bundle. We observe the following relation between the Chern classes of S_k and X:

Lemma 2.1.

$$c_1^2(X)c_1^{k-2}(X)H^{n-k} = c_1^2(S_k)c_1^{k-2}(X)H^{n-k} \cdot c_k^{n-k}(S_k)$$
$$c_2(X)c_1^{k-2}(X)H^{n-k} = c_2(S_k)c_1^{k-2}(X)H^{n-k} \cdot c_k^{n-k}(S_k)$$

where on the right hand sides, we have suppressed the pull-back.

Proof. Let $Q_k := \pi^*T_X^*/S_k$ be the universal quotient bundle. Then $c_{n-r}(X) = c_{n-r-k}(Q_k)c_k(S_k) + \ldots + c_{n-r-m}(Q_k)c_m(S_k) + \ldots$, for $r = 0, 1, \ldots, n - k$. Intersecting with $\alpha := c_1^{k-2}(X)H^{n-k}$, all except the first two Chern classes of X become zero. For $r = 0$, $c_{n-k}(Q_k)c_k(S_k)\alpha = 0$. For $r = 1$, $c_{n-k-1}(Q_k)c_k^2(S_k)\alpha = -c_{n-k}(Q_k)c_k(S_k)c_{k-1}(S_k)\alpha = 0$. Recursively, we have

$$c_1(Q_k)c_k^{n-k}(S_k)\alpha = 0, \ c_2(Q_k)c_k^{n-k-1}(S_k)\alpha = 0.$$

Since $c_1(Q_k) = c_1(X) - c_1(S_k)$, the first equation gives $c_1^2(X)c_k^{n-k}(S_k)\alpha = c_1^2(S_k)c_k^{n-k}(S_k)\alpha$. Then by $c_2(Q_k) = c_2(X) - c_2(S_k) - c_1(S_k)c_1(X) + c_1^2(S_k)$, $c_2(X)c_k^{n-k}(S_k)\alpha = c_2(S_k)c_k^{n-k}(S_k)\alpha$. Finally, if $\widetilde{S}_k$ is the universal bundle over the ordinary Grassmannian $Gr(k, n)$, then the intersection number $c_k^{n-k}(\widetilde{S}_k) = 1$. This completes the proof of the lemma. $\square$

Since $c_k^{n-k}(\widetilde{S}_k) = 1$, $\widetilde{X} := c_k^{n-k}(S_k)$ *looks like* a section of $Gr(k, T_X^*)$ over X. Suppose that this is really the case and $\widetilde{X}$ is a genuine manifold. Restrict the bundle S_k to $\widetilde{X}$, which we still denote by S_k. Also denote by $\pi : \widetilde{X} \to X$ the restriction of the projection map. Consider the direct sum $E_k = S_k \oplus \mathcal{O}_{\widetilde{X}}$. Let $\theta_k \in \Omega^{1,0}(\text{End}(E_k))$ be the Higgs-field given by the inclusion $S_k \hookrightarrow T_{\widetilde{X}}^* = \mathcal{O}_{\widetilde{X}} \otimes T_{\widetilde{X}}^*$. Suppose that (E_k, θ_k) is Hermitian-Yang-Mills and hence Higgs-stable with respect to the polarization $\pi^*|c_1(X)|$. Let H be an ample class on X. We can assume that H is very ample by replacing

it with a sufficiently high multiple if necessary. Then choosing $n - k$ sections of $\pi^* H$ will generically cut out a k-dimensional submanifold of $\widetilde{X}$. When restricted to this k-dimensional submanifold, (E_k, θ_k) remains Higgs-stable. Therefore we have

$$c_1^2(S_k)|c_1(X)|^{k-2}H^{n-k} \leq \frac{2(k+1)}{k}c_2(S_k)|c_1(X)|^{k-2}H^{n-k}$$

on $\widetilde{X} = c_k^{n-k}(S_k)$. By the above lemma, the following inequality holds

$$c_1^2(X)|c_1(X)|^{k-2}H^{n-k} \leq \frac{2(k+1)}{k}c_2(X)|c_1(X)|^{k-2}H^{n-k}.$$

Moreover, equality means that any submanifold cut out by a generic choice of $n - k$ sections of $\pi^* H$ is uniformized by the k-dimensional unit ball. Hence $\widetilde{X}$ and X should be fibred by these ball-quotients.

Remark 2.1. *Note that the above is not a proof; we aim to show that Conjecture 1.1 is conceivable and intimately related to stability.*

As a test of $(*)$, let's look at $X = M \times N$, a product of manifolds with $c_1 < 0$ of dimensions m and n respectively. Let H_1 and H_2 be ample classes on M and N respectively. Then by a direct computation, we have that, for $k = 2, \ldots, m + n$, the intersection number

$$\left(\frac{2(k+1)}{k}c_2(X) - c_1^2(X) \right)|c_1(X)|^{k-2}(H_1 + H_2)^{m+n-k}$$

is greater than or equal to the following sum

$$\sum_{p=2}^{min(k,m)} A_p \left(\frac{2(p+1)}{p}c_2(M) - c_1^2(M) \right)|c_1(M)|^{p-2}H_1^{m-p}|c_1(N)|^{k-p}H_2^{n-(k-p)}$$

$$+ \sum_{p=2}^{min(k,n)} B_p \left(\frac{2(p+1)}{p}c_2(N) - c_1^2(N) \right)|c_1(N)|^{p-2}H_2^{n-p}|c_1(M)|^{k-p}H_1^{m-(k-p)}$$

where A_p, B_p are nonnegative numbers. It follows that if both M and N satisfy the series of inequalities $(*)$, then the same is true for X (for classes of the form $H_1 + H_2$). Moreover, if, say, M is a ball-quotient, then an equality would result for classes H_2 on N, which agrees with our conjecture.

3. Kähler-Einstein toric Fano manifolds

Next we turn to Kähler-Einstein manifolds with $c_1 > 0$ which are toric. In this section, we give a proof of:

Theorem 3.1 (=2nd half of Theorem 1.1). *Let X be a Kähler-Einstein toric Fano n-fold, or more generally, toric Fano manifold with reductive automorphism group. Then for any nef class H, we have the inequality:*

$$c_1^2(X)H^{n-2} \leq 3c_2(X)H^{n-2}$$

provided that each facet of P contains a lattice point in its interior.

We fix our notations first. For details of toric Fano varieties and polytopes, please refer to [1], [4] and [10]. Let $N \cong \mathbb{Z}^n$ be a lattice and $M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^n$ the dual lattice with $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ the natural pairing. Denote by $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ (resp. $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$) the real (resp. rational) scalar extensions. For a subset S in a real vector space, we denote by $\operatorname{conv}(S)$ (resp. $\operatorname{aff}(S)$) the convex (resp. affine) hull of S.

Let $P \subset M_{\mathbb{R}}$ be a reflexive polytope and $X = X_P$ the corresponding toric Fano variety. We assume that X is smooth. Denote by $\mathcal{F}(P) = \{F_1, F_2, \ldots, F_d\}$ the facets of P, and by $D_1, D_2, \ldots, D_d$ the corresponding torus invariant prime divisors on X. Then the first and second Chern classes are given by:

$$c_1(X) = \sum_{i=1}^{d} D_i, \ c_2(X) = \sum_{i<j} D_i \cdot D_j.$$

We then compute the codimension 2 cycle

$$3c_2(X) - c_1^2(X) = 3\sum_{i<j} D_i \cdot D_j - \Big(\sum_{i=1}^{d} D_i\Big)^2 = \sum_{i<j} D_i \cdot D_j - \sum_{i=1}^{d} D_i^2.$$

By general theory of toric varieties, each $u \in M_{\mathbb{Q}}$ determines a $\mathbb{Q}$-divisor by:

$$\operatorname{div}(\chi^u) = \sum_{i=1}^{d} \langle u, v_i \rangle D_i.$$

We can move each D_i by choosing an element $u_{F_i} \in \operatorname{aff}(F_i) \cap M_{\mathbb{Q}}$, so that $\langle u_{F_i}, v_i \rangle = -1$, and we have

$$D_i^2 \sim_{\mathbb{Q}} D_i \cdot \Big(D_i + \sum_{j=1}^{d} \langle u_{F_i}, v_j \rangle D_j\Big) = \sum_{j \neq i} \langle u_{F_i}, v_j \rangle D_i \cdot D_j$$

where $\sim_{\mathbb{Q}}$ denotes $\mathbb{Q}$-rationally equivalence. Summing up all gives

$$\sum_{i=1}^{d} D_i^2 \sim_{\mathbb{Q}} \sum_{i=1}^{d} \sum_{j \neq i} \langle u_{F_i}, v_j \rangle D_i \cdot D_j = \sum_{i<j} (\langle u_{F_i}, v_j \rangle + \langle u_{F_j}, v_i \rangle) D_i \cdot D_j.$$

Hence we have

$$3c_2(X) - c_1^2(X) \sim_{\mathbb{Q}} \sum_{i<j} (1 - \langle u_{F_i}, v_j \rangle - \langle u_{F_j}, v_i \rangle) D_i \cdot D_j.$$

We formulate the following

Conjecture 3.1. *If $center(P) = 0$, then there exists $u_{F_i} \in aff(F_i) \cap M_{\mathbb{Q}}$ for $i = 1, 2, \ldots, d$ such that*

$$\langle u_{F_i}, v_j \rangle \leq \frac{1}{2}$$

for any adjacent facets F_i and F_j.

Since this would imply that the cycle class $[3c_2(X) - c_1^2(X)]$ is effective, the inequality (1.2) for Kähler-Einstein toric Fano manifolds:

$$(3c_2(X) - c_1^2(X))H^{n-2} \geq 0$$

for any nef class H, would follow. The question is now in purely combinatorial terms.

To prove Theorem 3.1, we recall that a *root* of a reflexive polytope P is a lattice point contained in the relative interior of some facet. We denote the set of all roots of P by $\mathcal{R}(P)$. P is said to be *semisimple* if $\mathcal{R}(P) = -\mathcal{R}(P)$. By results of Demazure (see [10]), the automorphism group of $X = X_P$, i.e. $\mathrm{Aut}(X)$, is *reductive* if and only if P is semisimple. While by Matsushima's obstruction to the existence of Kähler-Einstein metrics, if X has Kähler-Einstein metrics, then $\mathrm{Aut}(X)$ is reductive. Hence, if $center(P) = 0$, then P is semisimple. Recently, Nill gave a purely convex-geometrical proof of this in [9].

We will need the following powerful results on the geometry of semisimple reflexive polytopes proved by Nill (see [9]):

Lemma 3.1 (see [9], Lemma 2.4.)**.** *Let $P \subset M_{\mathbb{R}}$ be a reflexive polytope. For any $F \in \mathcal{F}(P)$ and $m \in F \cap M$, there is a $\mathbb{Z}$-basis $\{e_1^*, e_2^*, \ldots, e_n^*\}$ of M such that $m = e_n^*$ and $F \subset \{u \in M_{\mathbb{R}} : u^n = \langle u, e_n \rangle = 1\}$.*

Lemma 3.2 (see [9], Lemma 5.9.)**.** *Let $P \subset M_{\mathbb{R}}$ be a n-dimensional reflexive polytope. Let $m \in \mathcal{R}(P)$ be contained in F_m which is associated to the vertex $v_m \in \mathcal{V}(P^*)$. Then*

1. *$P \subset F_m - \mathbb{R}_{\geq 0}m$, $P \cap M \subset (F \cap M) - \mathbb{N}m$, $\{v \in P^* \cap N : \langle m, v \rangle < 0\} = \{v_m\}$.*
2. *$P = conv(F_m, F_{m'})$ if and only if there is only one facet $F_{m'}$ which is associated to $v_{m'} \in \mathcal{V}(P^*)$ with $\langle m, v_{m'} \rangle > 0$.*
3. *$-m \in \mathcal{R}(P)$ if and only if the previous condition is satisfied and $\langle m, v_{m'} \rangle = 1$.*

In this case $F_{m'} = F_{-m}$. Furthermore F_m and F_{-m} are naturally isomorphic as lattice polytopes and $\{v \in P^* \cap N : \langle m, v \rangle \neq 0\} = \{v_m, v_{-m}\}$.

Now Theorem 3.1 is an immediate consequence of the following lemma:

Lemma 3.3. *Suppose that a facet, say $F_1 \in \mathcal{F}(P)$, contains a root $m \in \mathcal{R}(P)$ such that $-m \in \mathcal{R}(P)$. Then there exists $u_{F_1} \in \mathrm{aff}(F_1) \cap M_{\mathbb{Q}}$ such that*

$$\langle u_{F_1}, v_j \rangle \leq \frac{1}{2}$$

for any facet F_j adjacent to F_1.

Proof. Since $-m \in \mathcal{R}(P)$, by Lemma 3.2, there is a unique facet, say $F_2 \in \mathcal{F}(P)$ such that $\langle m, v_2 \rangle = 1$ and

$$\{v \in P^* \cap N : \langle m, v \rangle \neq 0\} = \{v_1, v_2\}.$$

Suppose that F_2 is not adjacent to F_1. Then we can simply take $u_{F_1} = m$ and we have

$$\langle u_{F_1}, v_j \rangle = 0 < \frac{1}{2}$$

for any F_j adjacent to F_1.

If F_2 is adjacent to F_1, then we choose a $\mathbb{Z}$-basis $\{e_1^*, e_2^*, \ldots, e_n^*\}$ of M such that $m = e_n^*$ and $F \subset \{u \in M_{\mathbb{R}} : u^n = \langle u, e_n \rangle = 1\}$ by Lemma 3.1. In particular, we have $v_1 = -e_n$. Suppose $F_2, F_3, \ldots, F_l$ are all the facets adjacent to F_1. By Lemma 3.2, $P = \mathrm{conv}(F_1, F_2)$. We observe that $v_2 = z_2 + e_n$ and $v_j = z_j$ for $j = 3, \ldots, l$, where $z_2, z_3, \ldots, z_l$ are elements in the span of $\{e_1, \ldots, e_{n-1}\}$. Regard F_1 as a polytope in the affine hyperplane $H_1 = \{u \in M_{\mathbb{R}} : u^n = \langle u, e_n \rangle = 1\}$ with respect to the lattice $\mathrm{aff}(F_1) \cap M$ and with origin m. Using the coordinates with respect to the basis $\{e_1^*, \ldots, e_{n-1}^*\}$, F_1 is given by

$$F_1 = \{y \in H_1 : \langle y, z_2 \rangle \geq -2, \ \langle y, z_j \rangle \geq -1 \text{ for } j = 3, \ldots, l\}.$$

Also we have

$$F_1 \cap F_2 \subset \{y \in H_1 : \langle y, z_2 \rangle = -2\} \text{ and}$$
$$F_1 \cap F_j \subset \{y \in H_1 : \langle y, z_j \rangle = -1\}$$

for $j = 3, \ldots, l$. Consider the polytope $Q \subset F_1$ defined by

$$Q = \{y \in F_1 : \langle y, z_2 \rangle \leq 1, \ \langle y, z_j \rangle \leq \frac{1}{2} \text{ for } j = 3, \ldots, l\}.$$

In fact, $Q = (-1/2)F_1$. Q is not empty since the origin (i.e. m) is contained in it. Moreover, there is a positive constant $\epsilon > 0$ such that the ball

$$B_\epsilon = \{y \in H_1 : |y| \leq \epsilon\}$$

is contained in Q. We claim that there exists $y \in Q$ such that

$$\langle y, z_2 \rangle \leq -\frac{1}{2}.$$

Suppose not, then any vertex of Q, say w, will satisfy

$$\langle w, z_2 \rangle > -\frac{1}{2}.$$

Since $Q = (-1/2)F_1$ and F_1 is integral, this implies that

$$\langle w, z_2 \rangle \geq 0.$$

But Q is the convex hull of its vertices, so we have

$$\langle y, z_2 \rangle \geq 0$$

for all $y \in Q$. This is a contradiction since B_ϵ contains points with $\langle y, z_2 \rangle < 0$. This proves our claim. It follows that there exists $y \in Q$, with rational coordinates such that

$$\langle y, z_2 \rangle \leq -\frac{1}{2} \text{ and } \langle y, z_j \rangle \leq \frac{1}{2} \text{ for } j = 3, \ldots, l.$$

If we set $u_{F_1} = y + e_n$, then this is equivalent to

$$\langle u_{F_1}, v_j \rangle \leq \frac{1}{2}$$

for $j = 2, 3, \ldots, l$. $\qquad\square$

Remark 3.1. *The difficulty in removing the last condition in Theorem 3.1 can be described as follows. For a facet not containing a root, there are examples, such as the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, -1))$, such that the possible values of $u_{F_i} \in \operatorname{aff}(F_i) \cap M_{\mathbb{Q}}$, though form a sub-polytope of F_i, is of dimension less than that of F_i, in other words, the sub-polytope has empty relative interior. It is then more difficult to locate the required u_{F_i}. Indeed, we don't see why such a sub-polytope is always non-empty.*

4. Further evidence

In this section, we will prove:

Theorem 4.1 (=1st half of Theorem 1.1)**.** *Let X be a Kähler-Einstein toric Fano surface, 3-fold or 4-fold. Then we have*

$$c_1^2(X)H^{n-2} \leq 3c_2(X)H^{n-2}$$

for any nef class H.

Since Kähler-Einstein toric n-folds for $n \leq 4$ are completely classified, we can and will prove Theorem 4.1 by directly checking the inequality for all cases along the lines of Conjecture 3.1.

For any complex surface, it is well known that the Miyaoka-Yau inequality $c_1^2 \leq 3c_2$ is true, so we are done. In fact there are exactly three Kähler-Einstein toric Fano surfaces, namely,

$$\mathbb{P}^2, \ \mathbb{P}^1 \times \mathbb{P}^1, \ S_3$$

where S_3 is blow-up of $\mathbb{P}^2$ at three points, which is a del Pezzo surface of degree 6. It is easy to check the inequality directly.

For 3-folds and 4-folds, we need a lemma:

Lemma 4.1. *Let $X = Y \times Z$ be a product of varieties. Suppose $[3c_2(Y) - c_1^2(Y)]$ and $[3c_2(Z) - c_1^2(Z)]$ are effective (resp. positive) cycle classes on Y and Z respectively. Then $[3c_2(X) - c_1^2(X)]$ is also effective (resp. positive).*

Proof. Note that effectivity and positivity of a cycle class is preserved under pull-back via a projection. Now the lemma follows from $3c_2(X) - c_1^2(X) = (3c_2(Z) - c_1^2(Z)) + c_1(Y)c_1(Z)$ if $\dim(Y) = 1$ and $\dim(Z) \geq 2$, and from $3c_2(X) - c_1^2(X) = (3c_2(Y) - c_1^2(Y)) + (3c_2(Z) - c_1^2(Z)) + c_1(Y)c_1(Z)$ if both $\dim(Y), \dim(Z) \geq 2$. Here we have suppressed the pull-backs. $\square$

Kähler-Einstein toric Fano 3-folds are classified by Mabuchi (see [5]). There are five examples. The first four are $\mathbb{P}^3$ and products of lower dimensional smooth toric Fano varieties:

$$\mathbb{P}^2 \times \mathbb{P}^1, \ \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \ \mathbb{P}^1 \times S_3.$$

By the above lemma, Conjecture 3.1 is true for these four examples. The remaining one is the $\mathbb{P}^1$-bundle

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, -1)).$$

The verification for this will be carried out in the appendix.

Kähler-Einstein toric Fano 4-folds are classified by Nakagawa and Batyrev-Selivanova (see [8], [2]). There are twelve of them. The first nine are $\mathbb{P}^4$ and the following eight products of lower dimensional varieties:

$$\mathbb{P}^1 \times \mathbb{P}^3, \qquad \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2,$$
$$\mathbb{P}^2 \times \mathbb{P}^2, \ \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$
$$\mathbb{P}^2 \times S_3, \ \mathbb{P}^1 \times \mathbb{P}^1 \times S_3, \ S_3 \times S_3,$$
$$\mathbb{P}^1 \times \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, -1)).$$

Applying Lemma 4.1 again, Conjecture 3.1 holds for these examples. The remaining three are, in the notations of [2], V_2, W_2 and $X_{1,1}$. We again check the inequality for them in the appendix. This completes the proof of Theorem 4.1.

Remark 4.1. *There is a notion of symmetric toric Fano manifolds intro-
duced by Batyrev-Selivanova in* [2]. *Namely, the toric Fano manifold X
associated to a polytope P, is called symmetric if the group* $Aut_M(P)$ *of lat-
tice automorphisms leaving P invariant has only the origin as a fix point. As
noticed by Batyrev-Selivanova, all known examples of Kähler-Einstein toric
Fano manifolds (including all examples of dimension 2, 3 and 4) are sym-
metric. It is not known whether the inequality (1.2) is true for symmetric
toric Fano manifolds.*

5. Applications to counting lattice points

In this section, we apply our results to give new asymptotic bounds on
counting lattice points of balanced reflexive polytopes.

Let $X = X_P$ be a toric Fano variety associated to a reflexive polytope
$P \subset \mathbb{R}^n$. Let H be an ample divisor on X and $P_H \subset \mathbb{R}^n$ be the corresponding
polytope. Suppose that X is smooth, then kH is very ample for every $k \geq 1$.
The higher cohomology groups of the line bundle $\mathcal{O}(kH)$ vanish and the
sections precisely correspond to the lattice points in kP_H. Hence, by the
Riemann-Roch formula, the number of lattice points of kP_H is given by the
Ehrhart polynomial (see [4], section 5.3):

$$
\begin{aligned}
E(P_H) &= \#(kP_H) = \sum_{\nu=0}^{n} a_\nu k^\nu \\
&= a_n k^n + a_{n-1} k^{n-1} + a_{n-2} k^{n-2} + O(k^{n-3})
\end{aligned}
$$

where $a_\nu = \frac{1}{\nu!}(H^\nu \cdot \mathrm{Td}_\nu(X))$ and $\mathrm{Td}_\nu(X)$ denotes the ν-th Todd class of X.
The first few Todd classes are given by

$$
\mathrm{Td}_n(X) = 1, \ \mathrm{Td}_{n-1}(X) = \frac{1}{2}c_1(X), \ \mathrm{Td}_{n-2}(X) = \frac{1}{12}(c_1^2(X) + c_2(X)).
$$

Hence we have

$$
a_n = \frac{H^n}{n!} = \mathrm{Vol}(P_H), \ a_{n-1} = \frac{1}{2}\frac{H^{n-1}c_1(X)}{(n-1)!} = \frac{1}{2}\mathrm{Vol}(\partial P_H).
$$

Since $A_*(X)_{\mathbb{Q}}$ is spanned by the classes $[V(\tau)]$, $\tau \in \Delta_P$, the ν-th Todd
class can be written as

$$
\mathrm{Td}_\nu(X) = \sum_{\tau \in \Delta_P(\nu)} r_\tau [V(\tau)]
$$

for some rational numbers r_τ. A famous question raised by Danilov is that
given a lattice M (or N), whether we can choose the r_τ's uniformly, inde-
pendent of the toric variety X. Of course, by the above, we have uniformly,
$r_\tau = 1/2$, for all $\tau \in \Delta_P(1)$. There has been much work on this question
(see [11]).

Applying the inequality (1.2), we can get a uniform bound on r_τ for $\tau \in \Delta_P(2)$. Namely, if the inequality (1.2) is true for X, then we have

$$\mathrm{Td}_{n-2}(X)H^{n-2} = \frac{1}{12}(c_1^2(X) + c_2(X))H^{n-2} \le \frac{1}{3}c_2(X)H^{n-2}$$

from which follows that

$$r_\tau \le \frac{1}{3}$$

for any $\tau \in \Delta_P(2)$. Also, for the coefficient a_{n-2} in the above formula for the number of lattice points, we have

$$a_{n-2} = \frac{1}{(n-2)!}(H^{n-2} \cdot \mathrm{Td}_{n-2}(X)) \le \frac{1}{3}\frac{H^{n-2}c_2(X)}{(n-2)!}.$$

The quantity $H^{n-2}c_2(X)/(n-2)!$ is the sum of volumes of codimension 2 faces of P_H. If we denote by $P_H(2)$ the union of codimension 2 faces of P_H, then we get the following *asymptotic upper bound* for the growth of number of lattice points of kP_H:

$$E(kP_H) \le \mathrm{Vol}(P_H)k^n + \frac{1}{2}\mathrm{Vol}(\partial P_H)k^{n-1} + \frac{1}{3}\mathrm{Vol}(P_H(2))k^{n-2} + O(k^{n-3})$$

provided that the inequality (1.2) holds for X. Hence by the results of previous sections, we have

Proposition 5.1. *If P is balanced, then the following asymptotic upper bound holds:*

$$E(kP_H) \le \mathit{Vol}(P_H)k^n + \frac{1}{2}\mathit{Vol}(\partial P_H)k^{n-1} + \frac{1}{3}\mathit{Vol}(P_H(2))k^{n-2} + O(k^{n-3})$$

provided that either

(i) $n = 2, 3, 4$, or
(ii) each facet of P contains a root.

On the other hand, taking H to be the anticanonical divisor $K^{-1} = c_1(X)$, then the normalized Ehrhart polynomial of P is given by

$$\widetilde{E}(P) := \frac{\#(kP)}{\mathrm{Vol}(P)} = b_n k^n + b_{n-1}k^{n-1} + b_{n-2}k^{n-2} + \ldots$$

where in general we have $b_n = 1$, $b_{n-1} = 1/2n$, and

$$b_{n-2} = \frac{a_{n-2}}{\mathrm{Vol}(P)} = \frac{1}{12(n-2)!\mathrm{Vol}(P)}(c_1^2(X) + c_2(X))c_1^{n-2}(X).$$

By Yau's inequality (1.1), we have

Proposition 5.2 (=Proposition 1.1). *If P is balanced, then for k large, we have*

$$\widetilde{E}(P) \ge \widetilde{E}(P_n)$$

with equality holds if and only if $P = P_n$ where P_n is the polytope of $\mathbb{P}^n$.

Appendix A. The last four examples

In this appendix, we check that Conjecture 3.1 is true for the remaining low dimension examples. In the following, $\{e_1, e_2, \ldots, e_n\}$ will always denote the standard basis of the lattice $N \cong \mathbb{Z}^n$ and P the corresponding polytope. Recall that we have to prove that there exists $u_{F_i} \in \mathrm{aff}(F_i) \cap M_{\mathbb{Q}}$ for $i = 1, 2, \ldots, d$ such that

$$\langle u_{F_i}, v_j \rangle \le \frac{1}{2}$$

for any adjacent facets F_i and F_j.

For $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, -1))$, The 6 vertices of P^* are given by

$$\mathcal{V}(P^*) = \{e_1, e_2, e_3, -e_3, -e_1 - e_3, -e_2 + e_3\}$$

which correspond to the 6 facets of P:

$$
\begin{aligned}
F_1 &= \mathrm{conv}(u_1, u_2, u_3, u_4), & F_2 &= \mathrm{conv}(u_1, u_2, u_5, u_6), \\
F_3 &= \mathrm{conv}(u_1, u_3, u_5, u_7), & F_4 &= \mathrm{conv}(u_2, u_4, u_6, u_8), \\
F_5 &= \mathrm{conv}(u_5, u_6, u_7, u_8), & F_6 &= \mathrm{conv}(u_3, u_4, u_7, u_8),
\end{aligned}
$$

where $u_1, u_2, \ldots, u_8$ are vertices of P given, with respect to the dual basis $\{e_1^*, e_2^*, e_3^*\}$ of M, by the column vectors of the following matrix

$$
\begin{pmatrix}
-1 & -1 & -1 & -1 & 2 & 0 & 2 & 0 \\
-1 & -1 & 0 & 2 & -1 & -1 & 0 & 2 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{pmatrix}
$$

respectively. We can choose $u_{F_1}, u_{F_2}, u_{F_3}, u_{F_4}, u_{F_5}, u_{F_6}$ to be the column vectors of

$$
\begin{pmatrix}
-1 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\
0 & -1 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 \\
0 & 0 & -1 & 1 & 0 & 0
\end{pmatrix}
$$

respectively.

V_2 is the so-called del Pezzo variety of dimension 4, which is centrally symmetric. The 10 vertices of P^* are given by

$$\mathcal{V}(P^*) = \{e_1, e_2, e_3, e_4, -(e_1 + e_2 + e_3 + e_4), -e_1, -e_2, -e_3, -e_4, e_1 + e_2 + e_3 + e_4\}$$

which correspond to the 10 facets of P:

$$
\begin{aligned}
F_1 &= \operatorname{conv}(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}), \\
F_2 &= \operatorname{conv}(u_1, u_2, u_3, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}, u_{18}, u_{19}, u_{20}, u_{21}), \\
F_3 &= \operatorname{conv}(u_4, u_5, u_6, u_{13}, u_{14}, u_{15}, u_{22}, u_{23}, u_{24}, u_{25}, u_{26}, u_{27}), \\
F_4 &= \operatorname{conv}(u_7, u_8, u_9, u_{16}, u_{17}, u_{18}, u_{22}, u_{23}, u_{24}, u_{28}, u_{29}, u_{30}), \\
F_5 &= \operatorname{conv}(u_{10}, u_{11}, u_{12}, u_{19}, u_{20}, u_{21}, u_{25}, u_{26}, u_{27}, u_{28}, u_{29}, u_{30}), \\
F_6 &= \operatorname{conv}(u_{13}, u_{14}, u_{16}, u_{17}, u_{19}, u_{20}, u_{22}, u_{23}, u_{25}, u_{26}, u_{28}, u_{29}), \\
F_7 &= \operatorname{conv}(u_4, u_5, u_7, u_8, u_{10}, u_{11}, u_{22}, u_{24}, u_{25}, u_{27}, u_{28}, u_{30}), \\
F_8 &= \operatorname{conv}(u_1, u_2, u_7, u_9, u_{10}, u_{12}, u_{16}, u_{18}, u_{19}, u_{21}, u_{29}, u_{30}), \\
F_9 &= \operatorname{conv}(u_1, u_3, u_4, u_6, u_{11}, u_{12}, u_{13}, u_{15}, u_{20}, u_{21}, u_{26}, u_{27}), \\
F_{10} &= \operatorname{conv}(u_2, u_3, u_5, u_6, u_8, u_9, u_{14}, u_{15}, u_{17}, u_{18}, u_{23}, u_{24}),
\end{aligned}
$$

where $u_1, u_2, \ldots, u_{30}$ are the 30 vertices of P. The first one-third are given, with respect to the dual basis $\{e_1^*, e_2^*, e_3^*, e_4^*\}$ of M, by the column vectors of the following matrix

$$
\begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & -1 & -1 & -1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & -1 & -1 & -1 & 0
\end{pmatrix}
$$

respectively, the second one-third are given by the columns of

$$
\begin{pmatrix}
-1 & -1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 1 & -1 & -1 & -1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & -1 & -1 & -1 & 0 & 1
\end{pmatrix}
$$

respectively, and the last one-third are given by the columns of

$$
\begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
-1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & 1 \\
1 & -1 & -1 & -1 & 0 & 1 & 1 & -1 & -1 & -1
\end{pmatrix}
$$

respectively. We can choose $u_{F_i} \in \operatorname{aff}(F_i) \cap M_{\mathbb{Q}}$, $i = 1, 2, \ldots, 10$, to be given by the column vectors of the matrix

$$
\begin{pmatrix}
-1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

respectively.

Geometrically, W_2 is the blown-up of $\mathbb{P}^2 \times \mathbb{P}^2$ along 3 codimension 2 subvarieties $Z_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ defined by the equations $z_i = 0$, $z_i' = 0$, $i = 0, 1, 2$. Combinatorially, the vertices of P^* are given by the following 9 vectors

$$\mathcal{V}(P^*) = \{e_1, e_2, e_3, e_4, -e_1 - e_2, -e_3 - e_4, -e_1 - e_2 - e_3 - e_4, e_1 + e_3, e_2 + e_4\}$$

which correspond to the 9 facets of P:

$$
\begin{aligned}
F_1 &= \operatorname{conv}(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}), \\
F_2 &= \operatorname{conv}(u_1, u_2, u_3, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}), \\
F_3 &= \operatorname{conv}(u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{18}, u_{19}, u_{20}, u_{21}, u_{22}), \\
F_4 &= \operatorname{conv}(u_4, u_5, u_6, u_7, u_8, u_{18}, u_{19}, u_{20}, u_{23}, u_{24}), \\
F_5 &= \operatorname{conv}(u_4, u_5, u_9, u_{11}, u_{12}, u_{16}, u_{18}, u_{19}, u_{21}, u_{23}), \\
F_6 &= \operatorname{conv}(u_1, u_2, u_6, u_7, u_{10}, u_{13}, u_{14}, u_{17}, u_{22}, u_{24}), \\
F_8 &= \operatorname{conv}(u_1, u_3, u_5, u_8, u_9, u_{10}, u_{14}, u_{15}, u_{18}, u_{20}, u_{21}, u_{22}), \\
F_9 &= \operatorname{conv}(u_2, u_3, u_7, u_8, u_{12}, u_{15}, u_{16}, u_{17}, u_{19}, u_{20}, u_{23}, u_{24}),
\end{aligned}
$$

where $u_1, u_2, \ldots, u_{24}$ are the 24 vertices of P. The first half are given, with respect to the dual basis $\{e_1^*, e_2^*, e_3^*, e_4^*\}$ of M, by the column vectors of the following matrix

$$
\begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 \\
-1 & -1 & -1 & 2 & 2 & 1 & 0 & 0 & 2 & 1 & -1 & -1 \\
0 & 1 & 0 & 1 & 0 & 2 & 2 & 0 & 0 & 0 & -1 & -1 \\
1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & 1 & 0
\end{pmatrix}
$$

respectively, while the second half are given by the columns of

$$
\begin{pmatrix}
1 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
-1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 2 \\
2 & 2 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 2 & -1 & -1
\end{pmatrix}
$$

respectively. We can choose $u_{F_i} \in \operatorname{aff}(F_i) \cap M_{\mathbb{Q}}$, $i = 1, 2, \ldots, 9$, to be given by the column vectors of the matrix

$$
\begin{pmatrix}
-1 & \frac{1}{2} & \frac{3}{5} & -\frac{3}{10} & \frac{1}{2} & -\frac{3}{10} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\
\frac{1}{2} & -1 & -\frac{3}{10} & \frac{3}{5} & \frac{1}{2} & -\frac{3}{10} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \\
\frac{3}{5} & -\frac{3}{10} & -1 & \frac{1}{2} & -\frac{3}{10} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\
-\frac{3}{10} & \frac{3}{5} & \frac{1}{2} & -1 & -\frac{3}{10} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2}
\end{pmatrix}
$$

respectively.

For $X_{1,1}$, the 10 vertices of P^* are given by

$$\mathcal{V}(P^*) = \{e_1, e_2, e_3, -e_3, e_4, -e_4, e_3 - e_4, -e_3 + e_4, -e_1 + e_3, -e_2 - e_3\}$$

which correspond to the 10 facets of P:

$$
\begin{aligned}
F_1 &= \operatorname{conv}(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}) \\
F_2 &= \operatorname{conv}(u_1, u_2, u_3, u_4, u_5, u_6, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}, u_{18}) \\
F_3 &= \operatorname{conv}(u_1, u_2, u_7, u_8, u_{13}, u_{14}, u_{19}, u_{20}) \\
F_4 &= \operatorname{conv}(u_3, u_4, u_9, u_{10}, u_{15}, u_{16}, u_{21}, u_{22}) \\
F_5 &= \operatorname{conv}(u_1, u_5, u_7, u_{11}, u_{13}, u_{17}, u_{19}, u_{23}) \\
F_6 &= \operatorname{conv}(u_3, u_6, u_9, u_{12}, u_{15}, u_{18}, u_{21}, u_{24}) \\
F_7 &= \operatorname{conv}(u_2, u_6, u_8, u_{12}, u_{14}, u_{18}, u_{20}, u_{24}) \\
F_8 &= \operatorname{conv}(u_4, u_5, u_{10}, u_{11}, u_{16}, u_{17}, u_{22}, u_{23}) \\
F_9 &= \operatorname{conv}(u_{13}, u_{14}, u_{15}, u_{16}, u_{17}, u_{18}, u_{19}, u_{20}, u_{21}, u_{22}, u_{23}, u_{24}) \\
F_{10} &= \operatorname{conv}(u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{19}, u_{20}, u_{21}, u_{22}, u_{23}, u_{24})
\end{aligned}
$$

where $u_1, u_2, \ldots, u_{24}$ are the 24 vertices of P. The first half are given, with respect to the dual basis $\{e_1^*, e_2^*, e_3^*, e_4^*\}$ of M, by the column vectors of the following matrix

$$
\begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 0 & 0 & 1 & 1 \\
-1 & -1 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & 1
\end{pmatrix}
$$

respectively, while the second half are given by the columns of

$$
\begin{pmatrix}
0 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 2 & 2 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 0 & 0 & 1 & 1 \\
-1 & -1 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & 1
\end{pmatrix}
$$

respectively. We can choose $u_{F_i} \in \operatorname{aff}(F_i) \cap M_{\mathbb{Q}}$, $i = 1, 2, \ldots, 10$, to be given by the column vectors of the matrix

$$
\begin{pmatrix}
-1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 1 & 0 \\
0 & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 & 1 \\
0 & 0 & -1 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} & \frac{1}{2} & -1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0
\end{pmatrix}
$$

respectively.

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The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: kwchan@math.cuhk.edu.hk

The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: leung@ims.cuhk.edu.hk