

# A quadratic inequality for sum of co-adjoint orbits

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## Abstract

We obtain an effective lower bound on the distance of sum of co-adjoint orbits from the origin. Even when the distance is zero, thus the symplectic quotient is well-defined, our result give a nontrivial constraint on these co-adjoint orbits.

In the particular case of unitary groups, we recover the quadratic inequality for eigenvalues of Hermitian matrices satisfying

$$A + B = C.$$

This quadratic inequality was obtained earlier by the authors using completely different means, namely Klyachko's theory of toric stable reflexive sheaves and the Chern number inequality for Hermitian Yang-Mills connection.

## 1 Introduction

Given any rank  $r$  Hermitian matrix  $A$ , we may order its eigenvalues in such a way that

$$\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_r(A),$$

and denote  $\lambda(A) := (\lambda_1(A), \dots, \lambda_r(A)) \in \mathbb{R}^r$  as its spectrum. In [K2] (Also see [Fu] for an excellent account of the subject), Klyachko discovered following series of linear inequalities for Hermitian matrices  $A, B, C$  satisfying  $A+B=C$  :

$$\sum_{k \in K} \lambda_k(C) \leq \sum_{i \in I} \lambda_i(A) + \sum_{j \in J} \lambda_j(B)$$

for some triple of subsets  $I, J, K \subset \{1, 2, \dots, r\}$  of the same cardinality and such that the associated Schubert cycle  $s_K$  is a component of  $s_I \cdot s_J$ . This result can be interpreted as describing the linear inequalities which determine the intersection of the sum of co-adjoint orbits  $\mathcal{O}_{\lambda(A)} + \mathcal{O}_{\lambda(B)} + \mathcal{O}_{\lambda(-C)}$  with the positive Weyl chamber for the unitary group, which is a convex polytope by Kirwan's convexity theorem. Klyachko's result was generalized to any compact Lie group by Berenstein and Sjamaar in the beautiful paper [BeS], they are

able to relate above eigenvalue inequalities to the convexity of the image of the moment map. All above inequalities are linear on eigenvalues of matrices.

In [LW], we find a natural quadratic inequality on these eigenvalues by relating the Hermitian matrices to stable reflexive sheaves over the projective spaces. More precisely, we have shown

**Theorem 1** *Suppose  $A_1, \dots, A_N$  are rank  $r$  Hermitian matrices satisfying*

$$\sum_{\alpha=1}^N A_\alpha = 0.$$

*Then the following inequality holds true*

$$\sum_{\alpha=1}^N \lambda(A_\alpha)^2 \leq \sum_{1 \leq \alpha \neq \beta \leq N} \lambda(A_\alpha) \lambda(-A_\beta). \quad (1)$$

*Moreover, the equality holds if and only if all but two of  $A_\alpha$ 's are scalar matrices.*

In this paper, we give a direct proof of the above theorem in terms of sum of co-adjoint orbits for unitary groups. In fact our result in this paper is much stronger in the sense that it applies to ANY compact Lie group.

**Main Theorem:** *Let  $\mathcal{O}_{\lambda_\alpha}$ 's be co-adjoint orbits of a compact Lie group  $G$  with  $\alpha = 1, 2, \dots, N$ . If we choose  $\lambda_\alpha$  be the unique point of the intersection of  $\mathcal{O}_{\lambda_\alpha}$  with the positive Weyl chamber. Then for any  $\lambda$  in the sum of co-adjoint orbits*

$$\lambda \in \sum_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha},$$

*we have*

$$|\lambda|^2 \geq \sum_{\alpha=1}^N |\lambda_\alpha|^2 + \sum_{\alpha \neq \beta} \langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle, \quad (2)$$

*where  $w_0$  is the longest element in the Weyl group  $W$ .*

*Furthermore the equality holds if and only if all but at most two of  $\lambda_\alpha$ 's,  $1 \leq \alpha \leq N$  are in the center of  $\mathfrak{g}$ .*

Notice that  $\sum_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha} \subset \mathfrak{g}^*$  is the image of the moment map for the diagonal  $G$ -action on  $\prod_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha}$ . If we want to construct the symplectic quotient  $(\prod_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha}) // G$ , we need the origin to be inside of the image of moment map. Thus our main theorem gives a necessary constraint for this to happen, namely  $\sum_{\alpha=1}^N |\lambda_\alpha|^2 + \sum_{\alpha \neq \beta} \langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle$  must be non-positive.

Suppose  $\sum_{\alpha=1}^N |\lambda_\alpha|^2 + \sum_{\alpha \neq \beta} \langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle$  is positive, then our main theorem gives an effective lower bound on the distance between  $\sum_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha}$  and the origin

in  $\mathfrak{g}^*$ . If we define

$$r^2 = \sum_{\alpha=1}^N |\lambda_\alpha|^2 + \sum_{\alpha \neq \beta} \langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle$$

$$R^2 = \sum_{\alpha=1}^N |\lambda_\alpha|^2 + \sum_{\alpha \neq \beta} \langle \lambda_\alpha, \lambda_\beta \rangle = \left| \sum_{\alpha=1}^N \lambda_\alpha \right|^2.$$

Then we have

$$\sum_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha} \subset B(R) - B(r) \subset \mathfrak{g}^*$$

where  $B(R)$  and  $B(r)$  are balls centered at origin of radii  $R$  and  $r$  respectively. As it was found in [LW], that the equation (1) is closely related to the Chern number inequality for stable vector bundles, the main motivation of the generalization made in this note is to obtain the analogy of Chern number inequality for the zeros of moment map.

Finally, let us close this section by introducing our notations for the remaining sections.

$G; T; Z$	compact Lie group; its maximal torus; its center
$\mathfrak{g}; \mathfrak{t}; \mathfrak{z}$	Lie algebra of $G; T; Z$
$\langle \cdot, \cdot \rangle$	bi-invariant inner product on $\mathfrak{g}$ , which identifies $\mathfrak{g}$ with $\mathfrak{g}^*$
$\mathfrak{t}_+$	positive Weyl chamber
$\mathfrak{W}; w_0$	Weyl group of $G$ ; A longest element in the Weyl group
$(\mathcal{O}_\lambda, \Omega_{\mathcal{O}_\lambda}) \subset \mathfrak{g}$ ;	adjoint orbit through $\lambda \in \mathfrak{t}_+$ with $\Omega_{\mathcal{O}_\lambda}$ being its symplectic form
$\mathfrak{c}_\lambda \subset \mathfrak{g}$	centralizer of $\lambda \in \mathfrak{t}$
$\mathfrak{c}_\lambda^{ss}, \mathfrak{z}_\lambda \subset \mathfrak{c}_\lambda$	semi-simple part of $\mathfrak{c}_\lambda$ , center of $\mathfrak{c}_\lambda$
$C_\lambda, C_\lambda^{ss} \subset G$	closed subgroups of $G$ with Lie algebra $\mathfrak{c}_\lambda$ and $\mathfrak{c}_\lambda^{ss}$
$(\cdot)^\top : \mathfrak{g} \rightarrow \mathfrak{z}_\lambda$	orthogonal projection to $\mathfrak{z}_\lambda$ with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$
$(\cdot)^\perp : \mathfrak{g} \rightarrow \mathfrak{z}_\lambda^\perp$	projection to the orthogonal complement of $\mathfrak{z}_\lambda$ with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$

## 2 Convexity of moment map

Let us consider the diagonal  $G$ -action on the symplectic manifold  $\prod_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha}$  with Kostant-Kirillov-Souriau symplectic form, then the map

$$\begin{aligned} i^* : \prod_{\alpha=1}^N \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (\xi_1, \dots, \xi_N) &\longrightarrow \sum_{\alpha=1}^N \xi_\alpha \end{aligned}$$

being dual to the diagonal embedding  $i : G \hookrightarrow G^N$  is the moment map of the diagonal  $G$ -action. We define

$$\Delta_N := i^* \left( \prod_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha} \right) \cap \mathfrak{t}_+$$

i.e.  $\Delta_N$  is the moment polytope inside the positive Weyl chamber. If we restrict the  $G$ -action to its maximal torus then the moment map for the  $T$ -action is given by  $\Pi := j^* \circ i^*|_{\prod_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha}}$ , where  $j^* : \mathfrak{g} \rightarrow \mathfrak{t}$  is the projection induced from  $j : T \rightarrow G$ . That is,

$$\begin{aligned} \Pi : \prod_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha} &\longrightarrow \mathfrak{t} \\ (\xi_1, \dots, \xi_N) &\longrightarrow j^* \left( \sum_{\alpha=1}^N \xi_\alpha \right). \end{aligned}$$

In this section, we prove

**Theorem 2** *For any  $\lambda \in \Delta_N = i^* \left( \prod_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha} \right) \cap \mathfrak{t}_+$ , we have*

$$\left\langle \lambda, \sum_{\alpha=1}^N \lambda_\alpha \right\rangle \geq |\lambda|^2. \quad (3)$$

Moreover, the equality holds if and only if

$$\lambda = \sum_{\alpha=1}^N \lambda_\alpha \text{ or } \lambda = 0$$

The proof of the above theorem based on the convexity property of moment maps. Let us first state the following elementary lemma and the proof will be left to the readers.

**Lemma 3** *For any  $\lambda, \eta \in \mathfrak{t}_+$ , we have*

$$\langle \lambda, w_0 \cdot \eta \rangle \leq \langle \lambda, w \cdot \eta \rangle \leq \langle \lambda, \eta \rangle$$

for any  $w \in \mathfrak{W}$ . Moreover, we have

1.  $\langle \lambda, \eta \rangle = \langle \lambda, w_0 \cdot \eta \rangle$  if and only if  $\eta$  or  $\lambda \in \mathfrak{z}$ .
2. If  $\lambda$  lies in the interior of  $\mathfrak{t}_+$  then  $\langle \lambda, w_0 \cdot \eta \rangle = \langle \lambda, w \cdot \eta \rangle$  if and only if  $w_0 \cdot \eta = w \cdot \eta$ .

The next lemma describes the convex hull  $Hull(\mathfrak{W} \cdot \lambda)$  of a Weyl group orbit.

**Lemma 4** *Suppose  $\lambda, \eta \in \mathfrak{t}_+$  satisfying  $\langle \lambda, \zeta \rangle = \langle \eta, \zeta \rangle$  for any  $\zeta \in \mathfrak{z}$  and  $\langle \xi, \lambda \rangle \geq \langle \xi, \eta \rangle$  for any  $\xi \in \mathfrak{t}_+$ , then*

$$\eta \in Hull(\mathfrak{W} \cdot \lambda).$$

**Proof.** First, we have a Lie algebra decomposition  $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}^\perp$ . By our assumption  $\langle \lambda, \zeta \rangle = \langle \eta, \zeta \rangle$  for any  $\zeta \in \mathfrak{z}$  and the fact that  $\mathfrak{z}$  is invariant under  $\mathfrak{W}$ , we may reduce to the case that  $\mathfrak{z} = 0$ . By assumption  $\langle \xi, \lambda - \eta \rangle \geq 0$  for any  $\xi \in \mathfrak{t}_+$ , precisely means that  $\eta$  always lies in one side of the supporting hyperplane of the convex set  $Hull(\mathfrak{W} \cdot \lambda) \subset \mathfrak{t}$ , hence

$$\eta \in Hull(\mathfrak{W} \cdot \lambda).$$

■  
 Let us recall the following convexity theorem due to Atiyah [A1], Guillemin and Sternberg [GS].

**Theorem 5** *Let  $(X, \omega)$  be a symplectic manifold with a Hamiltonian  $T$ -action such that fixed points are all isolated and  $\mu : X \rightarrow \mathfrak{t}$  be its moment map. Then*

$$\text{Im}(\mu) = \text{Hull}(\{c_1, \dots, c_p\})$$

with  $\{c_i\}$ 's being the fixed points of the  $T$ -action.

In order to apply the above theorem, we restrict our  $G$ -action to its maximal torus, then the moment map is given by  $\Pi := j^* \circ i^*|_{\prod_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha}}$ , where  $j^* : \mathfrak{g} \rightarrow \mathfrak{t}$  is the map induced from  $j : T \rightarrow G$ . That is,

$$\begin{aligned} \Pi : \prod_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha} &\longrightarrow \mathfrak{t} \\ (\xi_1, \dots, \xi_N) &\longrightarrow j^* \left( \sum_{\alpha=1}^N \xi_\alpha \right). \end{aligned}$$

Then we obtain

**Theorem 6**

$$\text{Im}\Pi = \text{Hull} \left( \mathfrak{W} \cdot \sum_{\alpha=1}^N \lambda_\alpha \right)$$

where  $\mathfrak{W} \cdot \left( \sum_{\alpha=1}^N \lambda_\alpha \right)$  is the orbit of  $\sum_{\alpha=1}^N \lambda_\alpha$  under the action of  $\mathfrak{W}$ . Furthermore, we have

$$\Pi^{-1} \left( w \cdot \sum_{\alpha=1}^N \lambda_\alpha \right) \in \prod_{\alpha=1}^N \mathfrak{t}$$

for any  $w \in \mathfrak{W}$ .

**Proof.** First, we notice that the fixed points of the adjoint action of  $T$  on  $\mathfrak{g}$  is  $\mathfrak{t}$ , this implies that the image of the fixed point set under  $\Pi$  is

$$\left\{ \sum_{\alpha=1}^N w_\alpha \cdot \lambda_\alpha \in \mathfrak{t} \mid (w_1, \dots, w_N) \in \mathfrak{W}^N \right\}.$$

By Lemma 3, we have

$$\left\langle \sum_{\alpha=1}^N \lambda_\alpha, \xi \right\rangle = \sum_{\alpha=1}^N \max_{w \in \mathfrak{W}} \langle w \cdot \lambda_\alpha, \xi \rangle = \max_{(w_1, \dots, w_N) \in \mathfrak{W}^N} \left\langle \sum_{\alpha=1}^N w_\alpha \cdot \lambda_\alpha, \xi \right\rangle$$

for  $\xi \in \mathfrak{t}_+$ . This implies that, for any  $\xi \in \mathfrak{t}_+$

$$\left\langle \sum_{\alpha=1}^N \lambda_\alpha, \xi \right\rangle \geq \left\langle \sum_{\alpha=1}^N w_\alpha \cdot \lambda_\alpha, \xi \right\rangle.$$

So we deduce

$$\text{Hull} \left( \mathfrak{W} \cdot \sum_{\alpha=1}^N \lambda_{\alpha} \right) \ni \sum_{\alpha=1}^N w_{\alpha} \cdot \lambda_{\alpha}$$

for any  $(w_1, \dots, w_N) \in \mathfrak{W}^N$  by Lemma 4. Combining this with Theorem 5, we have

$$\text{Im}\Pi = \text{Hull} \left( \sum_{\alpha=1}^N w_{\alpha} \cdot \lambda_{\alpha} \mid (w_1, \dots, w_N) \in \mathfrak{W}^N \right) = \text{Hull} \left( \mathfrak{W} \cdot \sum_{\alpha=1}^N \lambda_{\alpha} \right).$$

■

**Corollary 7** *Suppose  $\lambda \in \Delta_N$ . Then for any  $0 \neq \eta \in \mathfrak{t}_+$ , we have*

$$\left\langle \eta, \sum_{\alpha=1}^N \lambda_{\alpha} \right\rangle \geq \langle \eta, \lambda \rangle.$$

Moreover, the equality holds if and only if  $\lambda = \sum_{\alpha=1}^N \lambda_{\alpha}$ .

**Proof of Theorem 2** Let  $\eta = \lambda$  then the inequality follows from the above corollary.

### 3 Proof of the main result

By using the co-adjoint action of  $G$ , it is enough for us to prove the inequality for  $\lambda \in \Delta_N$ . Since we are going to prove the main theorem by induction, for any positive integer  $N$ , we introduce the following *statement*  $(*_N)$  to indicate its dependence on  $N$ .

**Statement  $(*_N)$ :** Let  $G$  be a compact Lie group and  $\{\lambda_1, \dots, \lambda_N\} \subset \mathfrak{t}_+$ . For any  $\lambda \in \Delta_N$ , we have

$$|\lambda|^2 \geq \sum_{\alpha=1}^N |\lambda_{\alpha}|^2 + \sum_{\alpha \neq \beta} \langle \lambda_{\alpha}, w_0 \cdot \lambda_{\beta} \rangle$$

with the equality holds if and only if there are  $w_1, \dots, w_N \in \mathfrak{W}$ , the Weyl group of  $G$ , such that

$$\lambda := \sum_{\alpha=1}^N w_{\alpha} \cdot \lambda_{\alpha}$$

and

$$\langle \lambda_{\alpha}, w_0 \cdot \lambda_{\beta} \rangle = \langle w_{\alpha} \cdot \lambda_{\alpha}, w_{\beta} \cdot \lambda_{\beta} \rangle \text{ for } 1 \leq \alpha, \beta \leq N.$$

Before we proceed let us verify that the existence of  $w_1, \dots, w_N \in \mathfrak{W}$  with

$$\langle \lambda_{\alpha}, w_0 \cdot \lambda_{\beta} \rangle = \langle w_{\alpha} \cdot \lambda_{\alpha}, w_{\beta} \cdot \lambda_{\beta} \rangle \text{ for } 1 \leq \alpha, \beta \leq N$$

is equivalent to the condition that all but at most two of  $\lambda_\alpha$ 's are in  $\mathfrak{z}$ .

For simplicity, let us assume that  $w_N = id$  and  $\lambda_N$  is semisimple, that is, it is not fixed by any element in  $\mathfrak{W}$ . The identities

$$\langle \lambda_N, w_0 \cdot \lambda_\alpha \rangle = \langle \lambda_N, w_\alpha \cdot \lambda_\alpha \rangle \text{ for } 1 \leq \alpha \leq N-1$$

and the semisimplicity of  $\lambda_N$  imply  $w_0 \cdot \lambda_\alpha = w_\alpha \cdot \lambda_\alpha$  for  $1 \leq \alpha \leq N-1$  due to Lemma 3. Hence for  $1 \leq \alpha, \beta \leq N-1$ , we have

$$\langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle = \langle w_\alpha \cdot \lambda_\alpha, w_\beta \cdot \lambda_\beta \rangle = \langle w_0 \cdot \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle = \langle \lambda_\alpha, \lambda_\beta \rangle.$$

By Lemma 3 again, this implies that all but at most one of  $\lambda_\alpha$ 's,  $1 \leq \alpha \leq N-1$  lie in  $\mathfrak{z}$ . For general  $\lambda_N$ , the idea is very similar, and we leave it to the readers.

Now our main theorem stated in the introduction is equivalent to the following

**Theorem 8** *Statement  $(*_N)$  holds true for any positive integer  $N$ .*

To make the proof more transparent, let us briefly sketch the main idea modulo the technical details. As we mentioned earlier, we will argue by mathematical induction on  $N$ . In Proposition 9 we show that that if  $\lambda = 0$  then the inequality

$$0 \geq \sum_{\alpha=1}^N |\lambda_\alpha|^2 + \sum_{\alpha \neq \beta} \langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle$$

follows from statement  $(*_{N-1})$ . So the argument boils down to reducing the general case to the case of  $\lambda = 0$ . To do that, we first show in Lemma 10 that when  $|\lambda|$  attain its minimum, its necessary that  $\lambda \in \mathfrak{g}_\xi$  where  $\xi \in i^{*-1}(\lambda)$ . Then by introducing a smaller group  $C_\lambda^{ss} \subset G$  which is essentially perpendicular to  $\lambda$ , we will be able to reduce the problem to the case with  $\lambda = 0$  but a smaller group  $C_\lambda^{ss} \subset G$  to which the Proposition 9 is applicable, hence we have finished the proof.

**Proposition 9** *Suppose statement  $(*_{N-1})$  holds true, then we have*

$$0 \geq \sum_{\alpha=1}^N |\lambda_\alpha|^2 + \sum_{\alpha \neq \beta} \langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle$$

*provided  $0 \in \Delta_N$ . Moreover, the equality holds if and only if  $\lambda_\alpha = 0$  for  $1 \leq \alpha \leq N$  or there is a  $\lambda_\nu \neq 0$  such that*

$$(1 - w_0) \cdot \lambda_\nu = \sum_{\alpha=1}^N \lambda_\alpha$$

*and*

$$\langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle = \langle \lambda_\alpha, \lambda_\beta \rangle \text{ for } \alpha, \beta \neq \nu,$$

*That is, all but two of  $\lambda_\alpha$ 's are in  $\mathfrak{z}$ .*

**Proof.** Without loss of generality, we may assume  $\lambda_N \neq 0$ . Our assumption  $\lambda = 0 \in \Delta_N$  implies  $-w_0 \cdot \lambda_N \in \Delta_{N-1}$ . Then statement  $(*_{N-1})$  says

$$|-w_0 \cdot \lambda_N|^2 \geq \sum_{\alpha=1}^{N-1} |\lambda_\alpha|^2 + \sum_{\substack{\alpha \neq \beta \\ 1 \leq \alpha, \beta \leq N-1}} \langle \lambda_\alpha, w_0(\lambda_\beta) \rangle$$

On the other hand, Theorem 2 implies

$$\left\langle -w_0 \cdot \lambda_N, \sum_{\alpha=1}^{N-1} \lambda_\alpha \right\rangle \geq |-w_0 \cdot \lambda_N|^2.$$

By adding up above inequalities, we obtain

$$0 \geq \sum_{\alpha=1}^N |\lambda_\alpha|^2 + \sum_{\substack{\alpha \neq \beta \\ 1 \leq \alpha, \beta \leq N}} \langle \lambda_\alpha, w_0(\lambda_\beta) \rangle,$$

which is precisely what we want. Moreover, by Theorem 2 the equality holds if and only if

$$-w_0 \cdot \lambda_N = \sum_{\alpha=1}^{N-1} \lambda_\alpha$$

since  $\lambda_N \neq 0$ , and statement  $(*_{N-1})$  implies that equality holds if and only if there are  $\{w_\alpha\}_{\alpha=1}^N \subset \mathfrak{W}$  satisfying

$$-w_0 \cdot \lambda_N := \sum_{\alpha=1}^{N-1} w_\alpha \cdot \lambda_\alpha$$

and

$$\langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle = \langle w_\alpha \cdot \lambda_\alpha, w_\beta \cdot \lambda_\beta \rangle \text{ for } 1 \leq \alpha, \beta \leq N-1.$$

In particular, we have  $\sum_{\alpha=1}^{N-1} w_\alpha \cdot \lambda_\alpha = \sum_{\alpha=1}^{N-1} \lambda_\alpha$ , which is only possible if  $w_\alpha \cdot \lambda_\alpha = \lambda_\alpha$  for  $1 \leq \alpha \leq N-1$  by Lemma 3. So the equality holds if and only if

$$-w_0 \cdot \lambda_N = \sum_{\alpha=1}^{N-1} \lambda_\alpha$$

and

$$\langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle = \langle \lambda_\alpha, \lambda_\beta \rangle \text{ for } 1 \leq \alpha, \beta \leq N-1,$$

which is possible only if all but at most one of  $\lambda_\alpha, 1 \leq \alpha \leq N-1$  are in  $\mathfrak{z}$ . ■

Thus all we need to do is to reduce the general case to the  $\lambda = 0$  case, in order to do so need some preparation from Lie theory and moment map theory.

**Lemma 10** *Suppose  $\min_{\eta \in \Delta_N} |\eta|^2$  is attained by  $\lambda \in \Delta_N$ . Then for any  $\xi \in i^{*-1}(\lambda) \subset \prod_{\alpha=1}^N \mathcal{O}_{\lambda_\alpha}$  we have  $\lambda \in \mathfrak{g}_\xi$ , the stabilizer of  $\xi$  under the  $G$ -action, .*



**Proof.** Notice that the minimum of  $|\eta|^2$  over  $\Delta_N = i^*(\prod_{\alpha} \mathcal{O}_{\lambda_{\alpha}}) \cap \mathfrak{t}_+$  and  $i^*(\prod_{\alpha} \mathcal{O}_{\lambda_{\alpha}})$  are the same because of the Ad-invariance of the metric  $|\cdot|$  on  $\mathfrak{g}$ . Hence  $\xi$  is a critical point for the norm squared of the moment map  $i^*$  for the  $G$ -action on  $\prod_{\alpha} \mathcal{O}_{\lambda_{\alpha}}$ . By general theory of moment map [GS],  $\lambda = i^*(\xi)$  must lie inside stabilizer of  $\xi$ . ■

From now on, let us assume  $|\lambda|^2 = \min_{\eta \in \Delta_N} |\eta|^2$  and  $\xi := (\xi_1, \dots, \xi_N) \in \prod_{\alpha=1}^N \mathcal{O}_{\lambda_{\alpha}}$  satisfying  $\lambda = i^*(\xi) \in \Delta_N$ . By Lemma 10 we have  $\lambda \in \mathfrak{g}_{\xi}$ , which is equivalent to

$$[\lambda, \xi_{\alpha}] = 0, \quad \text{for } 1 \leq \alpha \leq N.$$

That is,  $\xi_{\alpha} \in \mathfrak{c}_{\lambda}$ , the Lie algebra of the centralizer  $C_{\lambda} \subset G$  of  $\lambda$ .

In order to reduce our considerations to the  $\lambda = 0$  case, we introduce the Lie algebra decomposition

$$\mathfrak{c}_{\lambda} = \mathfrak{c}_{\lambda}^{ss} \oplus \mathfrak{z}_{\lambda}$$

where  $\mathfrak{c}_{\lambda}^{ss} \subset \mathfrak{c}_{\lambda}$  is the semi-simple part and  $\mathfrak{z}_{\lambda} \subset \mathfrak{c}_{\lambda}$  is the center. Correspondingly, for  $\forall \xi \in \mathfrak{c}_{\lambda}$ , we may write  $\xi = \xi^{\perp} + \xi^{\top}$  with  $\xi^{\perp} \in \mathfrak{c}_{\lambda}^{ss}$  and  $\xi^{\top} \in \mathfrak{z}_{\lambda}$ , note that this is an orthogonal decomposition of  $\xi$  with respect to the bi-invariant inner product. Let  $C_{\lambda}^{ss}$  be the closed subgroup of  $G$  with Lie algebra  $\mathfrak{c}_{\lambda}^{ss}$  and  $\mathcal{O}_{\lambda_{\alpha}^{\perp}}$ 's be the  $C_{\lambda}^{ss}$ -orbits of  $G$ -action restricted to  $C_{\lambda}^{ss}$ . Then

$$\left( \xi_1^{\perp}, \dots, \xi_N^{\perp} \right) \in \prod_{\alpha=1}^N \mathcal{O}_{\lambda_{\alpha}^{\perp}} = \prod_{\alpha=1}^N \mathcal{O}_{\lambda_{\alpha}} \cap (\mathfrak{c}_{\lambda}^{ss})^N \subset \mathfrak{g}^N,$$

and

$$(i^*|_{\mathfrak{c}_{\lambda}^{ss*}})^{\perp} : \prod_{\alpha=1}^N \mathcal{O}_{\lambda_{\alpha}^{\perp}} \longrightarrow \mathfrak{c}_{\lambda}^{ss*}$$

is exactly the moment map of the  $C_{\lambda}^{ss}$ -action on  $\prod_{\alpha=1}^N \mathcal{O}_{\lambda_{\alpha}^{\perp}}$ . This implies

$$(i^*|_{\mathfrak{c}_{\lambda}^{ss*}})^{\perp} \left( \xi_1^{\perp}, \dots, \xi_N^{\perp} \right) = \sum_{\alpha=1}^N \xi_{\alpha}^{\perp} = \lambda^{\perp} = 0,$$

and

$$\lambda = \left( \sum_{\alpha=1}^N \xi_{\alpha} \right)^{\top} = \sum_{\alpha=1}^N \xi_{\alpha}^{\top} \quad (4)$$

because  $\lambda \in \mathfrak{z}_{\lambda}$ . Since the maximal torus of  $C_{\lambda}$  is a subtorus of  $T$ , there are  $g_{\alpha} \in C_{\lambda}^{ss}$  for  $1 \leq \alpha \leq N$  such that  $\eta_{\alpha} := \text{Ad}_{g_{\alpha}} \xi_{\alpha} \in \mathfrak{t}_{\lambda,+}$ , the positive Weyl chamber for  $C_{\lambda}$ . However  $\mathfrak{t}_{\lambda,+}$  in general is not a subset of  $\mathfrak{t}_+$ , in any case there are  $w_{\alpha} \in \mathfrak{W}$  such that  $\eta_{\alpha} = w_{\alpha} \cdot \lambda_{\alpha}$ . For any  $\eta \in \mathfrak{z}_{\lambda}$  of unit length, we have

$$\begin{aligned} \langle \xi_{\alpha}, \eta \rangle \eta &= \left\langle \text{Ad}_{g_{\alpha}^{-1}} \eta_{\alpha}, \eta \right\rangle \eta \\ &= \langle \eta_{\alpha}, \text{Ad}_{g_{\alpha}} \eta \rangle \eta \\ &= \langle \eta_{\alpha}, \eta \rangle \eta \end{aligned}$$

which means

$$\sum_{\alpha=1}^N \eta_{\alpha}^{\top} = \sum_{\alpha=1}^N \xi_{\alpha}^{\top} = \lambda. \quad (5)$$

The following lemma helps us to reduce the general case to  $\lambda = 0$  case.

**Lemma 11** *Let  $\lambda$  be as above and  $w \in \mathfrak{W}_{C_{\lambda}^{ss}}$ , the Weyl group of  $C_{\lambda}^{ss}$ . Then*

$$\sum_{\alpha=1}^N |\eta_{\alpha}|^2 + \sum_{\alpha \neq \beta} \langle \eta_{\alpha}, w \cdot \eta_{\beta} \rangle - |\lambda|^2 = \sum_{\alpha \neq \beta} \langle \eta_{\alpha}^{\perp}, w \cdot \eta_{\beta}^{\perp} \rangle + \sum_{\alpha=1}^N |\eta_{\alpha}^{\perp}|^2 \quad (6)$$

**Proof.** The assumption  $w \in \mathfrak{W}_{C_{\lambda}^{ss}}$  implies that  $w \cdot \zeta = \zeta$  for any  $\zeta \in \mathfrak{z}_{\lambda}$ . Hence we have  $\langle w \cdot \eta, \zeta \rangle = \langle \eta, \zeta \rangle$  for any  $\eta \in \mathfrak{t}, \zeta \in \mathfrak{z}_{\lambda}$ , thus  $(w \cdot \eta)^{\top} = \eta^{\top}$ , or equivalently  $((1-w) \cdot \eta)^{\top} = 0$ . Moreover,  $w \cdot \eta^{\top} = (w \cdot \eta)^{\top}$  since  $w$  fixes  $\mathfrak{z}_{\lambda}$ . Therefore,  $w \cdot \eta^{\perp} = (w \cdot \eta)^{\perp}$ .

Let  $\Lambda := \sum_{\alpha=1}^N \eta_{\alpha}$ , we calculate, using above formulae and (5)

$$\begin{aligned} & \sum_{\alpha=1}^N |\eta_{\alpha}|^2 + \sum_{\alpha \neq \beta} \langle \eta_{\alpha}, w \cdot \eta_{\beta} \rangle \\ &= \sum_{\alpha=1}^N \langle \eta_{\alpha}, (1-w) \cdot \eta_{\alpha} \rangle + \langle \Lambda, w \cdot \Lambda \rangle \\ &= \sum_{\alpha=1}^N \langle \eta_{\alpha}^{\perp}, (1-w) \cdot \eta_{\alpha}^{\perp} \rangle + \langle \Lambda^{\perp} + \Lambda^{\top}, w \cdot (\Lambda^{\perp} + \Lambda^{\top}) \rangle \\ &= \sum_{\alpha=1}^N |\eta_{\alpha}^{\perp}|^2 - \sum_{\alpha=1}^N \langle \eta_{\alpha}^{\perp}, w \cdot \eta_{\alpha}^{\perp} \rangle + |\Lambda^{\top}|^2 + \langle \Lambda^{\perp}, w \cdot \Lambda^{\perp} \rangle \\ &= \sum_{\alpha=1}^N |\eta_{\alpha}^{\perp}|^2 - \sum_{\alpha=1}^N \langle \eta_{\alpha}^{\perp}, w \cdot \eta_{\alpha}^{\perp} \rangle + |\lambda|^2 + \langle \Lambda^{\perp}, w \cdot \Lambda^{\perp} \rangle \end{aligned}$$

■

**Proof of Theorem 1:** We prove the theorem by applying mathematical induction on  $N$ .

For  $N = 1$ , the statement is trivially true. We assume now statement  $(*_{N-1})$  holds true, and take  $G$  to be  $C_{\lambda}^{ss}$  then Proposition 9 says precisely

$$\sum_{\alpha \neq \beta} \langle \eta_{\alpha}^{\perp}, -w_0^{C_{\lambda}^{ss}} \cdot \eta_{\beta}^{\perp} \rangle \geq \sum_{\alpha=1}^N |\eta_{\alpha}^{\perp}|^2, \quad (7)$$

where  $w_0^{C_{\lambda}^{ss}}$  is the longest element of  $\mathfrak{W}_{C_{\lambda}^{ss}}$ . Moreover, the equality holds if and only if  $\eta_{\alpha}^{\perp} = 0$  for all  $1 \leq \alpha \leq N$  or  $\eta_N^{\perp} \neq 0$

$$0 \neq -w_0^{C_{\lambda}^{ss}} \eta_N^{\perp} = \sum_{\alpha=1}^{N-1} \eta_{\alpha}^{\perp} \text{ and } \langle \eta_{\alpha}^{\perp}, \eta_{\beta}^{\perp} \rangle = \langle \eta_{\alpha}^{\perp}, w_0^{C_{\lambda}^{ss}} \cdot \eta_{\beta}^{\perp} \rangle \text{ for } 1 \leq \alpha, \beta \leq N-1.$$

By applying Lemma 11 we see that the inequality (7) is equivalent to

$$\begin{aligned} |\lambda|^2 &\geq \sum_{\alpha=1}^N |\eta_\alpha|^2 + \sum_{\alpha \neq \beta} \langle \eta_\alpha, w_0^{C_\lambda^{ss}} \cdot \eta_\beta \rangle \\ &= \sum_{\alpha=1}^N |w_\alpha \cdot \lambda_\alpha|^2 + \sum_{\alpha \neq \beta} \langle w_\alpha \cdot \lambda_\alpha, w_0^{C_\lambda^{ss}} \cdot w_\beta \cdot \lambda_\beta \rangle. \end{aligned}$$

Since  $\mathfrak{W}$  acts on  $\mathfrak{t}$  isometrically, we have  $|w_\alpha \cdot \lambda_\alpha|^2 = |\lambda_\alpha|^2$  and

$$\langle w_\alpha \cdot \lambda_\alpha, w_0^{C_\lambda^{ss}} \cdot w_\beta \cdot \lambda_\beta \rangle = \langle \lambda_\alpha, w_\alpha^{-1} \cdot w_0^{C_\lambda^{ss}} \cdot w_\beta \cdot \lambda_\beta \rangle \geq \langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle \quad (8)$$

by Lemma 3. These imply that

$$|\lambda|^2 \geq \sum_{\alpha=1}^N |\lambda_\alpha|^2 + \sum_{\alpha \neq \beta} \langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle.$$

Moreover, it follows from (8) that for the equality to hold in the above inequality, one needs

$$\langle \eta_\alpha, w_0^{C_\lambda^{ss}} \cdot \eta_\beta \rangle = \langle w_\alpha \cdot \lambda_\alpha, w_0^{C_\lambda^{ss}} \cdot w_\beta \cdot \lambda_\beta \rangle = \langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle \text{ for } 1 \leq \alpha, \beta \leq N \quad (9)$$

and

$$\sum_{\alpha \neq \beta} \langle \eta_\alpha^\perp, -w_0^{C_\lambda^{ss}} \cdot \eta_\beta^\perp \rangle = \sum_{\alpha=1}^N |\eta_\alpha^\perp|^2.$$

If  $\eta_\alpha^\perp = 0$  for  $1 \leq \alpha \leq N$  then  $w_0^{C_\lambda^{ss}} \eta_\alpha = \eta_\alpha$  for all  $\alpha$ , since  $w_0^{C_\lambda^{ss}} \lambda = \lambda$ . This together with the equality (9) imply that

$$\lambda = \sum_{\alpha=1}^N \eta_\alpha^\top = \sum_{\alpha=1}^N \eta_\alpha = \sum_{\alpha=1}^N w_\alpha \cdot \lambda_\alpha$$

and

$$\langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle = \langle \eta_\alpha, w_0^{C_\lambda^{ss}} \cdot \eta_\beta \rangle = \langle \eta_\alpha, \eta_\beta \rangle = \langle w_\alpha \cdot \lambda_\alpha, w_\beta \cdot \lambda_\beta \rangle \text{ for } 1 \leq \alpha, \beta \leq N,$$

which is exactly what we need.

If  $\eta_N^\perp \neq 0$  then Proposition 9 implies that

$$-w_0^{C_\lambda^{ss}} \eta_N^\perp = \sum_{\alpha=1}^{N-1} \eta_\alpha^\perp$$

and

$$\langle \eta_\alpha^\perp, \eta_\beta^\perp \rangle = \langle \eta_\alpha^\perp, w_0^{C_\lambda^{ss}} \cdot \eta_\beta^\perp \rangle \text{ for } 1 \leq \alpha, \beta \leq N-1,$$

from which we deduce

$$\langle \eta_\beta^\perp, \eta_N^\perp \rangle = \left\langle \eta_\beta^\perp, -w_0^{C_\lambda^{ss}} \cdot \sum_{\alpha=1}^{N-1} \eta_\alpha^\perp \right\rangle = - \left\langle \eta_\beta^\perp, \sum_{\alpha=1}^{N-1} \eta_\alpha^\perp \right\rangle = \left\langle \eta_\beta^\perp, w_0^{C_\lambda^{ss}} \eta_N^\perp \right\rangle.$$

So for  $1 \leq \alpha, \beta \leq N$ , we have

$$\langle \eta_\alpha, \eta_\beta \rangle = \left\langle \eta_\alpha, w_0^{C_\lambda^{ss}} \cdot \eta_\beta \right\rangle$$

and

$$\langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle = \left\langle \eta_\alpha, w_0^{C_\lambda^{ss}} \cdot \eta_\beta \right\rangle = \langle \eta_\alpha, \eta_\beta \rangle = \langle w_\alpha \cdot \lambda_\alpha, w_\beta \cdot \lambda_\beta \rangle.$$

On the other hand,  $\eta_N^\top \in \mathfrak{z}_\lambda$  implies  $w_0^{C_\lambda^{ss}} \eta_N^\top = \eta_N^\top$ , which means

$$\begin{aligned} \lambda &= \sum_{\alpha=1}^N \eta_\alpha^\top + 0 \\ &= \sum_{\alpha=1}^{N-1} \eta_\alpha^\top + w_0^{C_\lambda^{ss}} \eta_N^\top + \sum_{\alpha=1}^{N-1} \eta_\alpha^\perp + w_0^{C_\lambda^{ss}} \eta_N^\perp \\ &= \sum_{\alpha=1}^{N-1} \eta_\alpha + w_0^{C_\lambda^{ss}} \eta_N \\ &= \sum_{\alpha=1}^{N-1} w_\alpha \cdot \lambda_\alpha + w_0^{C_\lambda^{ss}} w_N \cdot \lambda_N \end{aligned}$$

and

$$\langle \lambda_\alpha, w_0 \cdot \lambda_N \rangle = \left\langle \eta_\alpha, w_0^{C_\lambda^{ss}} \cdot \eta_N \right\rangle = \left\langle w_\alpha \cdot \lambda_\alpha, w_0^{C_\lambda^{ss}} \cdot w_N \cdot \lambda_N \right\rangle.$$

So the proof of Theorem 1 will be completed if we replace  $w_N$  by  $w_0^{C_\lambda^{ss}} \cdot w_N$ .

**Corollary 12** *Suppose*

$$\lambda \in \Delta_N$$

*Then*

$$\left\langle \lambda, \sum_{\alpha=1}^N \lambda_\alpha \right\rangle \geq \sum_{\alpha=1}^N |\lambda_\alpha|^2 + \sum_{\alpha \neq \beta} \langle \lambda_\alpha, w_0 \cdot \lambda_\beta \rangle. \quad (10)$$

*Moreover, there are two cases for the equality to hold:*

1. *Suppose  $\lambda \neq 0$ , then all but at most one of  $\lambda_\alpha$  lie in  $\mathfrak{z}$ .*
2. *Suppose  $\lambda = 0$ , then all but at most two of  $\lambda_\alpha$ 's lie in  $\mathfrak{z}$*

**Proof.** By adding up inequalities (3) and (2), we obtain

$$\left\langle \lambda, \sum_{\alpha=1}^N \lambda_{\alpha} \right\rangle \geq \sum_{\alpha=1}^N |\lambda_{\alpha}|^2 + \sum_{\alpha \neq \beta} \langle \lambda_{\alpha}, w_0 \cdot \lambda_{\beta} \rangle.$$

Now suppose the equality holds, there are two cases:

If  $\lambda \neq 0$ , then the identities  $\lambda = \sum \lambda_{\alpha}$  and  $\langle \lambda_{\alpha}, w_0 \cdot \lambda_{\beta} \rangle = \langle \lambda_{\alpha}, \lambda_{\beta} \rangle$  for all  $\alpha, \beta$  implies that all but one of  $\lambda_{\alpha}$ 's lie in  $\mathfrak{z}$ .

If  $\lambda = 0$ , without loss of generality we may assume  $\lambda_N \neq 0$ , then the inequality

$$0 \geq \sum_{\alpha=1}^N |\lambda_{\alpha}|^2 + \sum_{\alpha \neq \beta} \langle \lambda_{\alpha}, w_0 \cdot \lambda_{\beta} \rangle$$

is equivalent to

$$2 \left\langle \lambda_N, \sum_{\alpha=1}^{N-1} \lambda_{\alpha} \right\rangle \geq |\lambda_N|^2 + \sum_{\alpha=1}^{N-1} |\lambda_{\alpha}|^2 + \sum_{\substack{\alpha \neq \beta \\ 1 \leq \alpha, \beta \leq N-1}} \langle \lambda_{\alpha}, w_0 \cdot \lambda_{\beta} \rangle,$$

which can be obtained by adding up inequality (3) and (10) with  $N$  replaced by  $N - 1$ . Now the equality would imply both (3) and (10) become equality, the assumption  $\lambda_N \neq 0$  would then reduce this case to the previous one. ■

In particular, if we let the group  $G = U(r)$ , hence  $\mathfrak{W} = \mathfrak{S}_r$ , the permutation group of  $r$  letters and  $w_0 = \begin{pmatrix} 12 \cdots (r-1)r \\ r(r-1) \cdots 21 \end{pmatrix} \in \mathfrak{S}_r$ . Then Corollary 12 implies

**Corollary 13** *For any rank  $r$  Hermitian matrix  $A$ , let  $\lambda(A) := (\lambda_1(A), \dots, \lambda_r(A)) \in \mathbb{R}^r$  be the spectrum of  $A$  with*

$$\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_r(A).$$

*Suppose  $A_1, A_2, \dots, A_N$  are Hermitian matrices satisfying*

$$\sum_{\alpha=1}^N A_{\alpha} = 0.$$

*Then*

$$\sum_{\alpha=1}^N \lambda(A_{\alpha})^2 \leq \sum_{1 \leq \alpha \neq \beta \leq N} \lambda(A_{\alpha}) \lambda(-A_{\beta}).$$

*Moreover, the equality holds if and only if all but possibly two of  $A_{\alpha}$ 's are scalar matrices.*

Finally, let us finish this section by an example that the equality holds.

**Example 14** *Let*

$$A = \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} b_2 & \\ & b_1 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & \\ & c_2 \end{bmatrix}$$

such that  $A+B+C=0$  and  $a_1 \geq a_2$ ,  $b_1 \geq b_2$  and  $c_1 \geq c_2$ . Let  $\lambda(A) := (a_1, a_2)$ ,  $\lambda(B) := (b_1, b_2)$ ,  $\lambda(C) := (c_1, c_2)$  and

$$\begin{aligned} w_0 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (y, x) \end{aligned}$$

be the permutation of two elements then

$$\begin{aligned} &|\lambda(A)|^2 + |\lambda(B)|^2 + |\lambda(C)|^2 + 2\langle \lambda(A), w_0 \cdot \lambda(B) \rangle \\ &+ 2\langle \lambda(C), w_0 \cdot \lambda(B) \rangle + 2\langle \lambda(A), w_0 \cdot \lambda(C) \rangle \\ &= 2(a_1 - a_2)(c_2 - c_1) \leq 0 \end{aligned}$$

since  $c_1 \geq c_2$ . Moreover the equality holds if and only if  $a_1 = a_2$  or  $c_1 = c_2$ , without loss of generality, let us assume  $a_1 = a_2$ . If we write  $A+B=-C$  then  $\lambda(-C) = \lambda(A) + \lambda(B)$  and  $\langle \lambda(B), w_0 \cdot \lambda(A) \rangle = \langle \lambda(B), \lambda(A) \rangle$ , this will correspond to the first case in the Corollary 13. On the other hand if we write  $A+B+C=0$ , then we have  $\lambda(A) + w_0 \cdot \lambda(B) + \lambda(C) = 0$  and

$$\begin{aligned} \langle \lambda(A), w_0 \cdot \lambda(B) \rangle &= \langle \lambda(A), w_0 \cdot \lambda(B) \rangle; \\ \langle \lambda(A), w_0 \cdot \lambda(C) \rangle &= \langle \lambda(A), \lambda(C) \rangle; \\ \langle \lambda(C), w_0 \cdot \lambda(B) \rangle &= \langle \lambda(C), w_0 \cdot \lambda(B) \rangle. \end{aligned}$$

This will correspond to the second case in the Corollary 13.

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