

Higher dimensional knot spaces for manifolds with vector cross product

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Abstract

Vector cross product structures on manifolds include symplectic, volume, G_2 - and $Spin(7)$ -structures. We show that their knot spaces have natural symplectic structures, and we relate instantons and branes in these manifolds with holomorphic disks and Lagrangian submanifolds in their knot spaces.

For the complex case, the holomorphic volume form on a Calabi-Yau manifold defines a complex vector cross product structure. We show that its isotropic knot space admits a natural holomorphic symplectic structure. We also relate the Calabi-Yau geometry of the manifold to the holomorphic symplectic geometry of its isotropic knot space.

1 Introduction

In our earlier paper [8] we studied the geometry of manifolds with vector cross product (abbrev. VCP) structure and manifolds with complex vector cross product (abbrev. \mathbb{C} -VCP) structure. For example manifolds with closed 1-fold (resp. 2-fold, 3-fold) VCP include Kähler manifolds (resp. G_2 -manifolds, $Spin(7)$ -manifolds). They play important roles in String theory (resp. M-theory, F-theory). For instance, instantons and branes for VCP geometry are supersymmetric cycles in the corresponding physical theory. For closed \mathbb{C} -VCP we proved that their geometries are either Calabi-Yau geometry or hyperkähler geometry. These geometries have special duality symmetries, called the Mirror Symmetry. From a holonomy point of view, this gives a coherent description of special holonomy structures on Riemannian manifolds, as classified by Berger.

In this article we will show that (i) the geometry of r -fold VCP on M can be realized as the symplectic geometry of the knot space $\mathcal{K}_\Sigma M$ of $(r-1)$ -dimensional knots in M and (ii) the geometry of $(n-1)$ -fold \mathbb{C} -VCP geometry on a Calabi-Yau manifold M can be realized as the complex symplectic geometry of the isotropic knot space $\hat{\mathcal{K}}_\Sigma M$.

The idea of using infinite dimensional mapping spaces to study the geometry of the underlying manifold M had proved to be very useful in many situations. For instance, Morse used the loop space to study the topology of M , Witten used the loop space to interpret the Atiyah-Singer index theorem [13] and the elliptic genus [14], Floer used the Morse theory on the loop space to study the Lagrangian intersection theory of a symplectic manifold M and defined his Floer homology group [4], Hitchin used the moment map on the mapping space of a Calabi-Yau manifold M to study the moduli space of special Lagrangian submanifolds in M [6]. The space of knots in three manifolds was used very successfully by Brylinski [2] to study the knot theory and Chern-Simons theory. Millson and Zombro found a Kähler structure on this knot space when $M = \mathbb{R}^3$ [10]

For VCP on a manifold M , we prove that its knot space $\mathcal{K}_\Sigma M$ has a natural symplectic structure. Furthermore branes (resp. instantons) in M correspond to Lagrangian submanifolds (resp. holomorphic curves) in $\mathcal{K}_\Sigma M$. For instances, quantum intersection theory of coassociative submanifolds in a seven dimensional G_2 -manifold M , as studied by Wang and the second author in [9], would correspond to Floer's Lagrangian intersection theory in $\mathcal{K}_\Sigma M$. Such an interpretation is useful in understanding the geometry of G_2 -manifolds and ultimately the M-theory. We summarize our results on knot spaces below.

Theorem 1 *Suppose M is a Riemannian manifold with a closed differential form ϕ . Then we have*

(1) *ϕ is a VCP form on M if and only if its transgression defines an almost Kähler structure on $\mathcal{K}_\Sigma M$;*

(2) *For a normal disk D in $\text{Map}(\Sigma, M)_{emb}$, \hat{D} is a holomorphic disk in $\mathcal{K}_\Sigma M$ if and only if $D \tilde{\times} \Sigma$ gives a tight instanton in M as above;*

(3) $\mathcal{K}_\Sigma C$ is a Lagrangian submanifold in $\mathcal{K}_\Sigma M$ if and only if C is a brane in M .

When we consider the complex analog of the above theorem, a holomorphic volume form on a Kahler manifold M defines a \mathbb{C} -VCP on M , thus M is a Calabi-Yau n -fold. We choose any smooth manifold Σ of dimension $n-2$, at first sight, one might expect that $Map(\Sigma, M)_{emb}/Diff(\Sigma) \otimes \mathbb{C}$, if exists, is hyperkähler. Since the complexification of $Diff(\Sigma)$ does not exist, we should interpret the above quotient as a symplectic quotient $Map(\Sigma, M)_{emb}/Diff(\Sigma)$, if it exists. The problem arises because one needs to fix a background volume form on Σ to define a symplectic structure on $Map(\Sigma, M)_{emb}$. We will explain in section five how to resolve this issue and prove the following theorem.

Theorem 2 *Suppose M is a Calabi-Yau n -fold and Σ is a closed manifold of dimension $n-2$. Then the isotropic knot space $\hat{\mathcal{K}}_\Sigma M$ has a natural holomorphic symplectic structure.*

Notice that $\hat{\mathcal{K}}_\Sigma M$ consists only of *isotropic* knots because of the moment condition, which was needed in the symplectic reduction construction.

Furthermore the Calabi-Yau geometry on M can be interpreted as the holomorphic symplectic geometry on $\hat{\mathcal{K}}_\Sigma M$. Namely we prove the following theorem in section 6.

Theorem 3 *Suppose M is a Calabi-Yau n -fold. We have*

(1) $\hat{\mathcal{K}}_\Sigma C$ is a J -complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$ if and only if C is a complex hypersurface in M ;

(2) $\hat{\mathcal{K}}_\Sigma C$ is a K -complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$ if and only if C is a special Lagrangian submanifold in M with phase $-\pi/2$.

Even though complex hypersurfaces and special Lagrangian submanifolds look very different inside a Calabi-Yau manifold, their isotropic knot spaces are both complex Lagrangian submanifolds in $\hat{\mathcal{K}}_\Sigma M$. The key reason is that any knot inside a special Lagrangian submanifold is automatically isotropic. To prove this theorem, we need to construct carefully certain appropriate deformations of isotropic knots inside C so that $\hat{\mathcal{K}}_\Sigma C$ being a J -complex Lagrangian (resp. I -complex Lagrangian) implies that the dimension of C is at least $2n-2$ (resp. at most n).

For example, when M is a Calabi-Yau threefold, $\Sigma \subset M$ is simply a loop, or a string in physical language, and the isotropic condition is automatic. In this case $\hat{\mathcal{K}}_\Sigma M$ is the space of loops in M modulo the equivalence relations generated by deforming Σ along *complex* directions in M . It would be desirable to interpret the mirror symmetry for Calabi-Yau manifolds as the twistor rotation for their isotropic knot spaces (see remark 18 for details).

The organization of this paper is as follow: In section two, we recall some basic results from [8] on the geometry of manifolds M with VCP structure. In section three, we show that the knot space of M admits a natural symplectic

structure. Moreover branes and instantons in M correspond to Lagrangian submanifolds and holomorphic curves in $\mathcal{K}_\Sigma M$. In section four, we recall some basic results from [8] on the geometry of manifolds M with \mathbb{C} -VCP. In section five, we construct the isotropic knot space $\hat{\mathcal{K}}_\Sigma M$ using a generalization of the symplectic quotient construction. Moreover we prove that it has a natural complex symplectic structure and a compatible almost hyperkähler structure. In section six, we prove that both special Lagrangian submanifolds and complex hypersurfaces in M give complex Lagrangian submanifolds in $\hat{\mathcal{K}}_\Sigma M$, but with respect to different complex structures in the twistor family.

2 Geometry of Vector Cross Product

In this section, we review the definition, basic properties and classifications of manifolds with vector cross product, as studied in [8]. VCP structure on manifolds was introduced by Gray [?] and it is a natural generalization of the vector product, or sometimes called the cross product, on \mathbb{R}^3 .

Definition 4 *Suppose that M is an n -dimensional manifold with a Riemannian metric g .*

(1) *An r -fold Vector Cross Product (VCP) is a smooth bundle map,*

$$\chi : \wedge^r TM \rightarrow TM$$

satisfying

$$\begin{cases} g(\chi(v_1, \dots, v_r), v_i) = 0, & (1 \leq i \leq r) \\ g(\chi(v_1, \dots, v_r), \chi(v_1, \dots, v_r)) = \|v_1 \wedge \dots \wedge v_r\|^2 \end{cases}$$

where $\|\cdot\|$ is the induced metric on $\wedge^r TM$. We also write

$$v_1 \times \dots \times v_r = \chi(v_1, \dots, v_r).$$

(2) *We define the VCP form $\phi \in \Omega^{r+1}(M)$ as*

$$\phi(v_1, \dots, v_{r+1}) = g(\chi(v_1, \dots, v_r), v_{r+1}).$$

(3) *The VCP χ is called closed (resp. parallel) if ϕ is a closed (resp. parallel) differential form.*

The classification of the linear VCP's on a vector space V with positive definite inner product g , by Brown and Gray [1], can be summarized in the following.

(i) $r = 1$: A 1-fold VCP is a Hermitian complex structure on V and its VCP form is the Kähler form. For $V = \mathbb{C}^m$ we have

$$\phi = dx^1 \wedge dy^1 + \dots + dx^m \wedge dy^m.$$

(ii) $r = n - 1$: The VCP form is the induced volume form Vol_V on V . For $V = \mathbb{R}^n$ we have

$$\phi = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

(iii) $r = 2$: a 2-fold VCP on a 7-dimensional vector space \mathbb{R}^7 is a cross product defined as $a \times b = \text{Im}(ab)$ for any a, b in $\mathbb{R}^7 = \text{Im}\mathbb{O}$, where \mathbb{O} is the Octonions, or the Cayley numbers. For coordinates (x_1, \dots, x_7) on $\text{Im}\mathbb{O}$, the VCP form is

$$\phi = dx^{123} - dx^{167} + dx^{145} + dx^{257} + dx^{246} - dx^{356} + dx^{347}$$

where $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$.

(iv) $r = 3$: a 3-fold VCP on an 8-dimensional vector space $\mathbb{R}^8 = \mathbb{O}$ is a cross product defined as $a \times b \times c = \frac{1}{2}(a(\bar{b}c) - c(\bar{b}a))$ for any a, b and c in \mathbb{O} . Its VCP form is

$$\begin{aligned} \phi = & -dx^{1234} - dx^{5678} - (dx^{21} + dx^{34})(dx^{65} + dx^{78}) \\ & - (dx^{31} + dx^{42})(dx^{75} + dx^{86}) - (dx^{41} + dx^{23})(dx^{85} + dx^{67}). \end{aligned}$$

Remark that the existence of a VCP on a Riemannian manifold M is equivalent to the reduction of the structure group of the frame bundle from $O(n)$ to the groups of real-linear transformation preserving both g and VCP. These groups are $U(m)$, $SO(n)$, G_2 and $Spin(7)$, for $r = 1, n - 1, 2$ and 3 respectively. For simplicity we assume that the VCP structure on M is always parallel in this paper.

Next we introduce the natural class of submanifolds in M , called instantons.

Definition 5 Let M be a Riemannian manifold with a closed r -fold VCP χ . An $r + 1$ dimensional submanifold A is called an **instanton** if it is preserved by χ .

Instantons are always calibrated submanifolds of M . In particular they are absolute minimal submanifolds in M . It is not difficult to identify instantons for r -fold VCP with various r : When $r = 1$, instantons are holomorphic curves in Kähler manifolds. When $r = n - 1$, instantons are open domains in M . When $r = 2$, instantons are associative submanifolds in G_2 -manifolds. When $r = 3$, instantons are Cayley submanifolds in $Spin(7)$ -manifolds (see [5]). In [8] we also discuss different characterizations of instantons and the deformation theory of them.

The natural boundary condition for an instanton is to require its boundary to lie on a *brane*.

Definition 6 Suppose M is an n -dimensional manifold with a closed VCP form ϕ of degree $r + 1$. A submanifold C is called a **brane** if

$$\left\{ \begin{array}{l} \phi|_C = 0 \\ \dim C = (n + r - 1)/2. \end{array} \right.$$

When $r = 1$, branes are Lagrangian submanifolds in Kähler manifolds. When $r = n - 1$, branes are hypersurfaces. When $r = 2$, branes are coassociative submanifolds in G_2 -manifolds. When $r = 3$, brane does not exist at all. The deformation theory of branes is also studied in [8].

The intersection theory branes with instanton corrections is an important topic. When $r = 1$ this is the theory of Fukaya-Floer category. When $r = 2$, the second author and X. Wang [9] found a relationship between this and the Seiberg-Witten theory.

We summarize various structures associated to VCP in the following table.

Manifolds M ($\dim M$)	VCP form ϕ (degree of ϕ)	Instanton A ($\dim A$)	Brane C ($\dim C$)
Oriented mfd. (n)	Volume form (n)	Open Subset (n)	Hypersurface ($n - 1$)
Kähler mfd. ($2m$)	Kähler form (2)	Holomorphic Curve (2)	Lagrangian Submanifold (m)
G_2 -manifold (7)	G_2 -form (3)	Associative Submanifold (3)	Coassociative Submanifold (4)
$Spin(7)$ -mfd. (8)	Cayley form (4)	Cayley submfd. (4)	N/A

3 Symplectic Geometry on Knot Spaces

Recall that a 1-fold VCP on M is a symplectic structure. In general an r -fold VCP form on M induces a symplectic structure on the space of embedded submanifolds Σ in M of dimension $r - 1$, which we simply call a (multi-dimensional) knot space $\mathcal{K}_\Sigma M = \text{Map}(\Sigma, M) / \text{Diff}(\Sigma)$. For instance when M is an oriented three manifold, $\mathcal{K}_\Sigma M$ is the space of knots in M . In this case Brylinski [2] showed that $\mathcal{K}_\Sigma M$ has a natural complex structure which makes it an infinite dimensional Kähler manifold and used it to study the problem of geometric quantization. For general M with a VCP, we will relate the symplectic geometry of $\mathcal{K}_\Sigma M$ to the geometry of branes and instantons in M .

3.1 Symplectic Structure on Knot Spaces

Let (M, g) be an n -dimensional Riemannian manifold M with a closed VCP form ϕ of degree $r + 1$. Suppose Σ is any $(r - 1)$ -dimensional oriented closed manifold Σ , we consider the mapping space of embeddings from Σ to M ,

$$\text{Map}(\Sigma, M) = \{f : \Sigma \rightarrow M \mid f \text{ is an embedding.}\}.$$

Let

$$ev : \Sigma \times \text{Map}(\Sigma, M) \rightarrow M$$

be the evaluation map $ev(x, f) = f(x)$ and p_1, p_2 be the projection map from $\Sigma \times Map(\Sigma, M)$ to its first and second factor respectively. We define a two form ω_{Map} on $Map(\Sigma, M)$ by taking the transgression of the VCP form ϕ ,

$$\omega_{Map} = (p_2)_*(ev)^*\phi = \int_{\Sigma} ev^*\phi.$$

Explicitly, suppose u and v are tangent vectors to $Map(\Sigma, M)$ at f , that is $u, v \in \Gamma(\Sigma, f^*(T_M))$, we have

$$\omega_{Map}(u, v) = \int_{\Sigma} \iota_{u \wedge v} \phi.$$

Since $(\iota_{u \wedge v} \phi)|_{\Sigma}$ can never be a top degree form if u is tangent to Σ , ω_{Map} degenerates along tangent directions to the orbits of the natural action of $Diff(\Sigma)$ on $Map(\Sigma, M)$. Thus it descends to a two form $\omega^{\mathcal{K}}$ on the quotient space

$$\mathcal{K}_{\Sigma}M = Map(\Sigma, M) / Diff(\Sigma),$$

the space of submanifolds in M . For simplicity, we call it a (multi-dimensional) *knot space*. Note that tangent vectors to $\mathcal{K}_{\Sigma}M$ are sections of the normal bundle of Σ in M .

On $\mathcal{K}_{\Sigma}M$ there is a natural L^2 -metric given as follow: Suppose $[f] \in \mathcal{K}_{\Sigma}M$ and $u, v \in \Gamma(\Sigma, N_{\Sigma/M})$ are tangent vectors to $\mathcal{K}_{\Sigma}M$ at $[f]$, then

$$\mathbf{g}_{[f]}^{\mathcal{K}}(u, v) = \int_{\Sigma} g(u, v) \nu_{\Sigma},$$

where ν_{Σ} is the volume form of Σ with respect to the induced metric on Σ . We define an endomorphism $\mathbf{J}^{\mathcal{K}}$ on the tangent space of $\mathcal{K}_{\Sigma}M$ as follow: Suppose $[f] \in \mathcal{K}_{\Sigma}M$ and $u, v \in \Gamma(\Sigma, N_{\Sigma/M})$ are tangent vectors to $\mathcal{K}_{\Sigma}M$ at $[f]$, then

$$\omega_{[f]}^{\mathcal{K}}(u, v) = \mathbf{g}_{[f]}^{\mathcal{K}}\left(\mathbf{J}_{[f]}^{\mathcal{K}}(u), v\right).$$

Proposition 7 *Suppose (M, g) is a Riemannian manifold with a VCP. Then $\mathbf{J}^{\mathcal{K}}$ is a Hermitian almost complex structure on $\mathcal{K}_{\Sigma}M$, i.e. a 1-fold VCP.*

Proof. Suppose $[f] \in \mathcal{K}_{\Sigma}M$ and $u, w \in \Gamma(\Sigma, N_{\Sigma/M})$ are tangent vectors to $\mathcal{K}_{\Sigma}M$ at $[f]$. Let e_i 's be an oriented orthonormal base of Σ at a point x . Then

$$\begin{aligned} \iota_{u \wedge w}(ev^*\phi) &= \phi(e_1, \dots, e_{r-1}, u, w) \nu_{\Sigma} \\ &= g(\chi(e_1, \dots, e_{r-1}, u), w) \nu_{\Sigma}. \end{aligned}$$

From the relationship $\omega_{[f]}^{\mathcal{K}}(u, v) = \mathbf{g}_{[f]}^{\mathcal{K}}\left(\mathbf{J}_{[f]}^{\mathcal{K}}(u), v\right)$ and the definitions of $\omega_{[f]}^{\mathcal{K}}$ and $\mathbf{g}_{[f]}^{\mathcal{K}}$, we conclude that $\mathbf{J}^{\mathcal{K}}$ on the tangent space of $\mathcal{K}_{\Sigma}M$ at $[f]$ is given by

$$\begin{aligned} \mathbf{J}^{\mathcal{K}} : \Gamma(\Sigma, N_{\Sigma/M}) &\rightarrow \Gamma(\Sigma, N_{\Sigma/M}) \\ \mathbf{J}^{\mathcal{K}}(u) &= \chi(e_1, \dots, e_{r-1}, u). \end{aligned}$$

On the other hand, from the remark ?? on the dimension of branes, we know that if χ is an r -fold VCP on $T_x M$, then $\chi(e_1, \dots, e_{r-1}, \cdot)$ defines a 1-fold VCP on the orthogonal complement to any oriented orthonormal vectors e_i 's at $T_x M$. This implies that $\mathbf{J}^{\mathcal{K}}$ is a 1-fold VCP on $\mathcal{K}_\Sigma M$. ■

Remark: The above proof actually shows that if $\mathbf{J}^{\mathcal{K}}$ on $\mathcal{K}_\Sigma M$ is induced by an arbitrary differential form ϕ on M , then $\mathbf{J}^{\mathcal{K}}$ is an Hermitian almost complex structure on $\mathcal{K}_\Sigma M$ if and only if ϕ is a r -fold VCP form on M .

When the VCP form ϕ on M is closed and Σ is a closed manifold, then ω_{Map} on $Map(\Sigma, M)$ is also closed because

$$d\omega_{Map} = d \int_{\Sigma} ev^* \phi = \int_{\Sigma} ev^* d\phi = 0.$$

Therefore ω_{Map} descends to a closed 1-fold VCP form on $\mathcal{K}_\Sigma M$.

Conversely the closedness of ω_{Map} on $Map(\Sigma, M)$, or on $\mathcal{K}_\Sigma M$, implies the closedness of ϕ on M . Since this type of localization arguments will be used several times in this paper, we include the proof of the following standard lemma.

Lemma 8 (Localization) *Let Σ be an s -dimensional manifold without boundary. A form η of degree $k > s$ on a manifold M vanishes if the corresponding $(k - s)$ -form on $Map(\Sigma, M)$ obtained by transgression vanishes.*

Proof. We need to show that

$$\eta(p)(v_1, v_2, \dots, v_k) = 0$$

for any fixed $p \in M$ and any fixed $v_i \in T_p M$. For simplicity, we may choose v_i 's to be orthonormal vectors. We can find $f \in Map(\Sigma, M)$ with $p \in f(\Sigma)$ such that v_1, \dots, v_s are along the tangential directions, and v_{s+1}, \dots, v_k are along the normal directions at p in $f(\Sigma)$. Moreover, we can choose sections $\tilde{v}_{s+1}, \tilde{v}_{s+2}, \dots, \tilde{v}_k \in \Gamma(\Sigma, f^*(T_M))$ which equal v_{s+1}, \dots, v_k at p respectively. By multiplying \tilde{v}_{s+1} with a sequence of functions on Σ approaching the delta function at p , we obtain sections $(\tilde{v}_{s+1})_\varepsilon$ which approach $\delta(p)v_{s+1}$ as $\varepsilon \rightarrow 0$ where $\delta(p)$ is Dirac delta function. Therefore,

$$\eta(p)(v_1, v_2, \dots, v_k) = \lim_{\varepsilon \rightarrow 0} \left(\int_{\Sigma} ev^* \eta \right) ((\tilde{v}_{s+1})_\varepsilon, \tilde{v}_{s+2}, \dots, \tilde{v}_k).$$

From the given condition $\int_{\Sigma} ev^* \eta = 0$, we conclude that

$$\eta(p)(v_1, v_2, \dots, v_k) = 0.$$

Hence the result. ■

As a corollary of the above lemma and discussions, we have the following result.

Theorem 9 *Suppose (M, g) is a Riemannian manifold with a differential form ϕ of degree $r + 1$. Then ϕ is a closed r -fold VCP form on M if and only if $\omega^{\mathcal{K}}$ is an almost Kähler structure on $\mathcal{K}_{\Sigma}M$, i.e. a closed 1-fold VCP form.*

Remark: In general the transgression of any closed form ϕ always gives a closed form on the knot space $\mathcal{K}_{\Sigma}M$ of degree $r + 1 - s$ where $\dim \Sigma = s$. However this can never be a VCP form unless $s = r - 1$. To see this, we can use the above localization method and the fact that there is no VCP form of degree bigger than two on any vector space with sufficiently large dimension.

When $r = 1$, that is M is a symplectic manifold, $Map(\Sigma, M)$ is the same as M and therefore it is symplectic by trivial reasons.

Remark: Given any ϕ -preserving vector field $v \in Vect(M, \phi)$, it induces a vector field V of $\mathcal{K}_{\Sigma}M$ preserving $\omega^{\mathcal{K}}$. Furthermore, if v is ϕ -Hamiltonian, that is

$$\iota_v \phi = d\eta,$$

for some $\eta \in \Omega^r(M)$, then we can define a function on the knot space,

$$F_{\eta} : \mathcal{K}_{\Sigma}M \rightarrow \mathbb{R}$$

$$F_{\eta}(f) = \int_{\Sigma} f^* \eta.$$

It is easy to see that F_{η} is a Hamiltonian function on the symplectic manifold $\mathcal{K}_{\Sigma}M$ whose Hamiltonian vector field equals V .

3.2 Holomorphic Curves and Lagrangians in Knot Spaces

In this subsection, we are going to show that holomorphic disks (resp. Lagrangian submanifolds) in the knot space $\mathcal{K}_{\Sigma}M$ of M correspond to instantons (resp. branes) in M . More generally the geometry of vector cross products on M should be closely related to the symplectic geometry of its knot space $\mathcal{K}_{\Sigma}M$. A natural problem is to understand the analog of the Floer's Lagrangian intersection theory for manifolds with vector cross products. Note that if two branes C_1 and C_2 intersect transversely along a submanifold Σ , then the dimension of Σ equals $r - 1$ and $[\Sigma]$ represents a transverse intersection point of (Lagrangians) $\mathcal{K}_{\Sigma}C_1$ and $\mathcal{K}_{\Sigma}C_2$ in $\mathcal{K}_{\Sigma}M$. The converse is also true.

Because $\mathcal{K}_{\Sigma}M$ is of infinite dimensional, a Lagrangian submanifold is defined as a subspace in $\mathcal{K}_{\Sigma}M$ where the restriction of $\omega^{\mathcal{K}}$ vanishes and with the property that any vector field $\omega^{\mathcal{K}}$ -orthogonal to L is a tangent vector field along L , see [11] and [2]. In fact, it is easy to see that the latter condition for a submanifold L to be Lagrangian is equivalent to the statement that $\omega^{\mathcal{K}}$ will not vanish on any bigger space containing L . So, we refer the this condition as the **maximally self $\omega^{\mathcal{K}}$ -perpendicular** condition.

Theorem 10 *Suppose that M is an n -dimensional manifold with a closed VCP form ϕ of degree $r + 1$ and C is a submanifold in M . Then $\mathcal{K}_{\Sigma}C$ is a Lagrangian submanifold in $\mathcal{K}_{\Sigma}M$ if and only if C is a brane in M .*

Proof. For the *only if* part, we suppose that $\mathcal{K}_\Sigma C$ is a Lagrangian in $\mathcal{K}_\Sigma M$ and we want to show that ϕ vanishes along C and $\dim C = (n + r - 1)/2$. For any fixed $[f] \in \mathcal{K}_\Sigma C$, we have

$$0 = \omega_{[f]}^{\mathcal{K}}(u, v) = \int_{\Sigma} \iota_{u \wedge v} (ev^* \phi)$$

where $u, v \in \Gamma(\Sigma, N_{\Sigma/C})$. By applying the localization method as in lemma 8 along C , we obtain the following: For any x in Σ ,

$$\phi(u(x), v(x), e_1, \dots, e_{r-1}) = 0,$$

where e_1, \dots, e_{r-1} are any oriented orthonormal vectors of $T_x \Sigma$. By varying u, v, f to cover T_C . One can show $\phi|_C = 0$.

Moreover, $\mathcal{K}_\Sigma C$ being $\omega^{\mathcal{K}}$ -perpendicular in $\mathcal{K}_\Sigma M$ implies that C has the biggest possible dimension with $\phi|_C = 0$. As explained in the remark following the definition of branes that this gives $\dim C = (n + r - 1)/2$.

For the *if* part, we assume that C is any brane in M . The condition $\phi|_C = 0$ implies that $\omega^{\mathcal{K}}|_{\mathcal{K}_\Sigma C} = 0$. To show that $\mathcal{K}_\Sigma C$ is a Lagrangian in $\mathcal{K}_\Sigma M$, we need to verify the maximally self $\omega^{\mathcal{K}}$ -perpendicular condition in $\mathcal{K}_\Sigma M$. Recall that the tangent space at any point $[f] \in \mathcal{K}_\Sigma C$ is $\Gamma(\Sigma, N_{\Sigma/C})$. Suppose there is a section v in $\Gamma(\Sigma, N_{\Sigma/M})$ but not in $\Gamma(\Sigma, N_{\Sigma/C})$ such that it is $\omega^{\mathcal{K}}$ -perpendicular to $\Gamma(\Sigma, N_{\Sigma/C})$. By the localization arguments as in lemma 8, given any point $x \in f(\Sigma) \subset C$, ϕ vanishes on the linear space spanned by $v(x)$ and $T_x C$. This contradicts to the fact that C is a brane. Hence the result. ■

Remark: ϕ -Hamiltonian deformation of a brane C in M corresponds to Hamiltonian deformation of the corresponding Lagrangian submanifold $\mathcal{K}_\Sigma C$ in the symplectic manifold $\mathcal{K}_\Sigma M$. More precisely, if v is a normal vector field to C satisfying

$$\iota_v \phi = d\eta,$$

for some $\eta \in \Omega^{r-1}(C)$, then the transgression of η defines a function on $\mathcal{K}_\Sigma C$ which generates a Hamiltonian deformation of $\mathcal{K}_\Sigma C$.

Next we discuss holomorphic disks, i.e. instantons, in $\mathcal{K}_\Sigma M$. We consider a two dimensional disk D in $Map(\Sigma, M)$ such that for each tangent vector $v \in T_{[f]} D$, the corresponding vector field in $\Gamma(\Sigma, f^* T_M)$ is normal to Σ . We call such a disk D as a **normal disk**. For simplicity we assume that the $r + 1$ dimensional submanifold

$$A = \bigcup_{f \in D} f(\Sigma) \subset M,$$

is an embedding. This is always the case if D is small enough. Notice that A is diffeomorphic to $D \times \Sigma$. We will denote the corresponding disk in $\mathcal{K}_\Sigma M$ as $\hat{D} =: \pi(D)$. We remark that the principal fibration

$$Diff(\Sigma) \rightarrow Map(\Sigma, M) \xrightarrow{\pi} \mathcal{K}_\Sigma M$$

has a canonical connection (see [2]) and D being a normal disk is equivalent to it being an integral submanifold for the horizontal distribution of this connection.

In the following theorem, we describe the relation between a disk \hat{D} in $\mathcal{K}_\Sigma M$ given above and the corresponding $r + 1$ dimensional subspace A in M .

Theorem 11 *Suppose that M is a manifold with a closed r -fold VCP form ϕ and $\mathcal{K}_\Sigma M$ is its knot space as before. For a normal disk D in $\text{Map}(\Sigma, M)$, $\hat{D} =: \pi(D)$ is a $J^\mathcal{K}$ -holomorphic disk in $\mathcal{K}_\Sigma M$, i.e. calibrated by $\omega^\mathcal{K}$, if and only if A is an instanton in M and $A \rightarrow D$ is a Riemannian submersion.*

We call such an A a *tight instanton*.

Proof. For a fixed $[f] \in \hat{D}$, we consider $\nu, \mu \in T_{[f]}(\hat{D}) \subset \Gamma(\Sigma, N_{\Sigma/A})$. Since ϕ is a calibrating form, we have,

$$\phi(\nu, \mu, e_1, \dots, e_{r-1}) \leq \text{Vol}_A(\nu, \mu, e_1, \dots, e_{r-1}) = |\nu \wedge \mu|$$

where e_1, \dots, e_{r-1} is any orthonormal frame on $f(\Sigma)$. In particular we have

$$\int_{f(\Sigma)} \iota_{\nu \wedge \mu} (ev^* \phi) \leq \int_{f(\Sigma)} |\nu \wedge \mu| \text{vol}_\Sigma,$$

and the equality sign holds for every $[f] \in \hat{D}$ if and only if A is an instanton in M . We will simply denote $\int_{f(\Sigma)}$ by \int_Σ . Notice that the symplectic form on $\mathcal{K}_\Sigma M$ is given by,

$$\omega_{[f]}^\mathcal{K}(\nu, \mu) = \int_\Sigma \iota_{\nu \wedge \mu} (ev^* \phi).$$

Since $\omega^\mathcal{K}$ is a 1-fold VCP form on $\mathcal{K}_\Sigma M$, we have

$$\omega_{[f]}^\mathcal{K}(\nu, \mu) \leq \left(|\nu|_\mathcal{K}^2 |\mu|_\mathcal{K}^2 - \langle \nu, \mu \rangle_\mathcal{K}^2 \right)^{1/2}$$

where $\langle \nu, \mu \rangle_\mathcal{K} =: g_\mathcal{K}(\nu, \mu)$ and $|a|_\mathcal{K}^2 =: g_\mathcal{K}(a, a)$. Furthermore the equality sign holds when \hat{D} is a $J^\mathcal{K}$ -holomorphic disk in $\mathcal{K}_\Sigma M$.

To prove the *only if* part, we suppose that \hat{D} is a $J^\mathcal{K}$ -holomorphic disk in $\mathcal{K}_\Sigma M$. From above discussions, we have

$$\begin{aligned} \int_\Sigma |\nu \wedge \mu| &\geq \int_\Sigma \iota_{\nu \wedge \mu} (ev^* \phi) = \left(|\nu|_\mathcal{K}^2 |\mu|_\mathcal{K}^2 - \langle \nu, \mu \rangle_\mathcal{K}^2 \right)^{1/2} \\ &= \left(\int_\Sigma |\nu|^2 \int_\Sigma |\mu|^2 - \left(\int_\Sigma \langle \nu, \mu \rangle \right)^2 \right)^{1/2}. \end{aligned}$$

Combining with the Hölder inequality,

$$\left(\int_\Sigma |\nu \wedge \mu| \right)^2 + \left(\int_\Sigma \langle \nu, \mu \rangle \right)^2 \leq \int_\Sigma |\nu|^2 \int_\Sigma |\mu|^2,$$

we obtain

$$\begin{aligned} \text{(i)} \quad & \int_\Sigma \iota_{\nu \wedge \mu} (ev^* \phi) = \int_\Sigma |\nu \wedge \mu| \text{vol}_\Sigma \quad \text{and} \\ \text{(ii)} \quad & \left(\int_\Sigma |\nu \wedge \mu| \right)^2 + \left(\int_\Sigma \langle \nu, \mu \rangle \right)^2 = \int_\Sigma |\nu|^2 \int_\Sigma |\mu|^2. \end{aligned}$$

Condition (i) says that A is an instanton in M . Condition (ii) implies that given any $[f]$, there exists constant C_1 and C_2 such that for any $x \in \Sigma$,

$$|\nu(x)| = C_1 |\mu(x)|, \quad \angle(\nu(x), \mu(x)) = C_2.$$

This implies that $A \rightarrow D$ is a *Riemannian submersion*.

For the *if* part, we notice that A being an instanton in M implies that

$$\begin{aligned} \int_{\Sigma} |\nu \wedge \mu| &= \int_{\Sigma} \iota_{\nu \wedge \mu} (ev^* \phi) = \omega_{[f]}^{\mathcal{K}}(\nu, \mu) \leq \left(|\nu|_{\mathcal{K}}^2 |\mu|_{\mathcal{K}}^2 - \langle \nu, \mu \rangle_{\mathcal{K}}^2 \right)^{1/2} \\ &= \left(\int_{\Sigma} |\nu|^2 \int_{\Sigma} |\mu|^2 - \left(\int_{\Sigma} \langle \nu, \mu \rangle \right)^2 \right)^{1/2}. \end{aligned}$$

Recall that the Riemannian submersion condition implies an equality ,

$$\left(\int_{\Sigma} |\nu \wedge \mu| \right)^2 + \left(\int_{\Sigma} \langle \nu, \mu \rangle \right)^2 = \int_{\Sigma} |\nu|^2 \int_{\Sigma} |\mu|^2$$

so the above inequality turns into an equality so that it gives

$$\omega_{[f]}^{\mathcal{K}}(\nu, \mu) = \int_{\Sigma} \iota_{\nu \wedge \mu} (ev^* \phi) = \left(|\nu|_{\mathcal{K}}^2 |\mu|_{\mathcal{K}}^2 - \langle \nu, \mu \rangle_{\mathcal{K}}^2 \right)^{1/2}$$

i.e. \hat{D} is $J^{\mathcal{K}}$ holomorphic in $\mathcal{K}_{\Sigma}M$. ■

4 Geometry of Complex Vector Cross Product

In this section we review the *complex* version of the VCP geometry from [8]. The definition of instantons depends on a phase θ and there are two types of branes corresponding to the Dirichlet and Neumann boundary conditions for instantons. In the case of M being a Calabi-Yau manifold, D-branes (resp. N-branes) are special Lagrangian submanifolds (resp. complex hypersurfaces) in M . We will see in the next two sections that all these geometric structures will have a beautiful interpretation in terms of the hyperkähler geometry of the isotropic knot space of M .

Definition 12 *On a Kähler manifold (M, g, J) of complex dimension n , an r -fold **complex vector cross product** (abbrev. \mathbb{C} -VCP) is a holomorphic form ϕ of degree $r + 1$ satisfying*

$$|\iota_{e_1 \wedge e_2 \wedge \dots \wedge e_r}(\phi)| = 2^{(r+1)/2},$$

for any orthonormal tangent vectors $e_1, \dots, e_r \in T_x^{1,0}M$, for any x in M .

A \mathbb{C} -VCP is called *closed* (resp. *parallel*) if ϕ is closed (resp. parallel with respect to the Levi-Civita connection) form.

There are only two classes of \mathbb{C} -VCP corresponding to the holomorphic volume form ($r = n - 1$) and the holomorphic symplectic form ($r = 1$). The corresponding manifolds M are Calabi-Yau manifolds and hyperkähler manifolds. In particular, the analog of G_2 - and $Spin(7)$ -manifolds do not exist.

For VCP, instantons are submanifolds calibrated by the VCP form. In the complex case, the \mathbb{C} -VCP form is a complex differential form and we have a circle family of calibrated forms $\text{Re}(e^{i\theta}\phi)$ parametrized by the phase θ .

Definition 13 *On a Kähler manifold M with an r -fold \mathbb{C} -VCP ϕ , an $r + 1$ dimensional submanifold A is called an **instanton** with phase $\theta \in \mathbb{R}$ if it is calibrated by $\text{Re}(e^{i\theta}\phi)$, i.e.*

$$\text{Re}(e^{i\theta}\phi)|_A = \text{vol}_A.$$

As explained in [LL1] we can impose either the Dirichlet or the Neumann boundary conditions for instantons. Thus we have the following definitions.

Definition 14 *On a Kähler manifold (M, ω) of complex dimension n with an r -fold \mathbb{C} -VCP $\phi \in \Omega^{r+1,0}(M)$,*

(i) *a submanifold C is called a **Neumann brane** (abbrev. N -brane) if $\dim(C) = n + r - 1$ and*

$$\phi|_C = 0,$$

(ii) *an n -dimensional submanifold C is called a **Dirichlet brane** (abbrev. D -brane) with phase $\theta \in \mathbb{R}$ if*

$$\omega|_C = 0, \text{Re}(e^{i\theta}\phi)|_C = 0$$

Instantons and branes in CY manifold : (n-1)-fold \mathbb{C} -VCP

A Kähler manifold of complex dimension n with a $(n - 1)$ -fold \mathbb{C} -VCP ϕ is a Calabi-Yau n -fold M . An instanton in M is a special Lagrangian submanifold with phase θ since it is calibrated by $\text{Re}(e^{i\theta}\phi)$ and a D -brane is a special Lagrangian submanifold with phase $\theta - \pi/2$, because the definition of D -brane implies it is calibrated by $\text{Re}(e^{i(\theta-\pi/2)}\phi) = \text{Im}(e^{i\theta}\phi)$. An N -brane in M is a complex hypersurface.

Instantons and branes in hyperkähler manifold : 1-fold C -VCP

A Kähler manifold of complex dimension $2n$ with a Kähler form ω_J and a 1-fold C -VCP $\phi =: \omega_I - \sqrt{-1}\omega_K$ is a hyperkähler manifold M . Here, J , I and K are the complex structure corresponding to Kähler structures ω_J , ω_I and ω_K , respectively. Note that by putting $e^{i\theta}\phi =: \omega_{J_\theta} - \sqrt{-1}\omega_{J_{\theta+\pi/2}}$ in place of ϕ , one can obtain another Kähler structure $\text{Re} e^{i\theta}\phi$ with a complex structure $J_\theta =: \cos\theta I + \sin\theta K$.

An instanton in M is a J_θ -holomorphic curve since it is calibrated by $\text{Re} e^{i\theta}\phi$, namely preserved by J_θ . A D -brane is a real $2n$ -dimensional submanifold where ω and $\text{Re}(e^{i\theta}\phi)$ vanish which implies it is preserved by $J_{\theta+\pi/2}$ the almost complex structure corresponding to $-\text{Im}(e^{i\theta}\phi)$, i.e. $\omega_{J_{\theta+\pi/2}}$. Therefore, a D -brane

is a $J_{\theta+\pi/2}$ -complex Lagrangian. An N -brane in M is a real $2n$ -dimensional submanifold where $\phi = \omega_I - \sqrt{-1}\omega_K$ vanishes, i.e. calibrated by ω_J . So it is a J -complex Lagrangian which is a complex submanifold preserved by a complex structure J with complex dimension n .

The above classification of instantons, N -branes and D -branes in manifolds with \mathbb{C} -VCP is summarized in the following table.

Manifolds w/ \mathbb{C} -VCP	Calabi-Yau manifolds	hyperkähler manifolds
Instantons	Special Lagrangian $_{\theta=0}$	I -holomorphic curves
N -Branes	Complex Hypersurfaces	J -complex Lagrangians
D -Branes	Special Lagrangian $_{\theta=-\pi/2}$	K -complex Lagrangians

5 Isotropic Knot Spaces of CY manifolds

Recall that any volume form on a manifold M of dimension n determines a closed 1-fold VCP on the knot space $\mathcal{K}_\Sigma M = \text{Map}(\Sigma, M) / \text{Diff}(\Sigma)$ where Σ is any closed manifold of dimension $n - 2$. It is natural to expect that the holomorphic volume form on any Calabi-Yau n -fold M would determine a closed 1-fold \mathbb{C} -VCP on the symplectic quotient $\text{Map}(\Sigma, M) // \text{Diff}(\Sigma)$. In particular we obtain a hyperkähler knot space.

There are some complications to this naive approach. First we need to choose a background volume form σ on Σ to construct a symplectic structure in $\text{Map}(\Sigma, M)$ which is only invariant under $\text{Diff}(\Sigma, \sigma)$, the group of volume preserving diffeomorphisms of Σ ([3][6]). Therefore we can only construct the symplectic quotient $\text{Map}(\Sigma, M) // \text{Diff}(\Sigma, \sigma) = \mu^{-1}(0) / \text{Diff}(\Sigma, \sigma)$. Second, this larger space is not hyperkähler because the holomorphic two form $\int_\Sigma \Omega_M$ degenerates. We will show that a further symplectic reduction will produce a Hermitian integrable complex manifold $\hat{\mathcal{K}}_\Sigma M$ with a 1-fold \mathbb{C} -VCP, in particular a holomorphic symplectic structure. This may not be hyperkähler because even the Hermitian complex structure is integrable on $\hat{\mathcal{K}}_\Sigma M$, its Kähler form may not be closed. $\hat{\mathcal{K}}_\Sigma M$ should be regarded as a modified construction for the non-existing space $\text{Map}(\Sigma, M) // \text{Diff}(\Sigma)$. We call this an *isotropic knot space* of M for reasons which will become clear later.

Furthermore we will relate instantons, N -branes and D -branes in a Calabi-Yau manifold M to holomorphic curves and complex Lagrangian submanifolds in the holomorphic symplectic manifold $\hat{\mathcal{K}}_\Sigma M$. These constructions are of particular interest when M is a Calabi-Yau threefold (see remark 18 for details).

5.1 Holomorphic Symplectic Structures on Isotropic Knot Spaces

Let M be a Calabi-Yau n -fold with a holomorphic volume form Ω_M , i.e. a closed $(n-1)$ -fold \mathbb{C} -VCP. To obtain a 1-fold \mathbb{C} -VCP on a certain knot space by transgression, we first fix an $n-2$ dimensional manifold Σ without boundary and we let $Map(\Sigma, M)$ be the space of embeddings from Σ to M as before. For simplicity we assume that the first Betti number of Σ is zero, $b_1(\Sigma) = 0$.

If we fix a background volume form σ on Σ , then the Kähler form ω on M induces a natural symplectic form on $Map(\Sigma, M)$ as follow: for any tangent vectors X and Y at a point f in $Map(\Sigma, M)$, we define

$$\omega_{Map}(X, Y) = \int_{\Sigma} \omega(X, Y) \sigma$$

where $X, Y \in \Gamma(\Sigma, f^*(TM))$. Note that this symplectic structure on $Map(\Sigma, M)$ is not invariant under general diffeomorphisms of Σ . Instead it is preserved by the natural action of $Diff(\Sigma, \sigma)$, the group of volume preserving diffeomorphisms on (Σ, σ) .

As studied by Donaldson in [3] and Hitchin in [6], this action is Hamiltonian on the components of $Map(\Sigma, M)$ consisting of those f 's satisfying

$$f^*([\omega]) = 0 \in H^2(\Sigma, \mathbb{R}),$$

and the moment map is given by

$$\begin{aligned} \mu : \quad Map_0(\Sigma, M) &\rightarrow \Omega^1(\Sigma) / d\Omega^0(\Sigma) \\ f &\mapsto \mu(f) = \alpha \end{aligned}$$

for any one form $\alpha \in \Omega^1(\Sigma)$ satisfying $d\alpha = f^*\omega$. Note that the dual of the Lie algebra of $Diff(\Sigma, \sigma)$ can be naturally identified with $\Omega^1(\Sigma) / d\Omega^0(\Sigma)$. In particular $\mu^{-1}(0)$ consists of isotropic embeddings of Σ in M . Therefore the symplectic quotient

$$Map(\Sigma, M) // Diff(\Sigma, \sigma) = \mu^{-1}(0) / Diff(\Sigma, \sigma)$$

is almost the same as the moduli space of isotropic submanifolds in M , which is $\mu^{-1}(0) / Diff(\Sigma)$. Observe that the definition of the map μ is independent of the choice of σ and therefore $\mu^{-1}(0)$ is preserved by the action of $Diff(\Sigma)$.

Remark: For the moment map μ to be well-defined, the condition $b_1(\Sigma) = 0$ is necessary. However even in the case of $b_1(\Sigma) \neq 0$, which always happens when M is a Calabi-Yau threefold, there is modification of the symplectic quotient construction and we can obtain a symplectic manifold which is a torus bundle over the moduli space of isotropic submanifolds in M with fiber dimension $b_1(\Sigma)$ ([3][6]).

In order to construct a holomorphic symplectic manifold, we first consider a complex closed 2-form Ω_{Map} on $Map(\Sigma, M)$ induced from the holomorphic

volume form Ω_M on M by transgression.

$$\begin{aligned}\Omega_{Map} &= \int_{\Sigma} ev^* \Omega_M \\ &= \int_{\Sigma} ev^* \operatorname{Re} \Omega_M + \sqrt{-1} \int_{\Sigma} ev^* \operatorname{Im} \Omega_M \\ &= \omega_I - \sqrt{-1} \omega_K,\end{aligned}$$

where $\omega_I = \int_{\Sigma} ev^* (\operatorname{Re} \Omega_M)$ and $\omega_K = - \int_{\Sigma} ev^* (\operatorname{Im} \Omega_M)$. We also define endomorphisms I and K on $T_f(\mu^{-1}(0))$ as follows:

$$\omega_I(A, B) = g(IA, B), \quad \omega_K(A, B) = g(KA, B)$$

where $A, B \in T_f(\mu^{-1}(0))$ and g is the natural L^2 -metric on $\mu^{-1}(0)$. Both ω_I and ω_K are degenerated two forms. We will define the isotropic knot space as the symplectic reduction of $\mu^{-1}(0)$ with respect to either ω_I or ω_K and show that it has a natural holomorphic symplectic structure.

To do that, we define a distribution \mathcal{D} on $\mu^{-1}(0)$ by

$$\mathcal{D}_f = \{X \in T_f(\mu^{-1}(0)) \subset \Gamma(\Sigma, f^*(T_M)) \mid \iota_X \omega_{I,f} = 0\} \subset T_f(\mu^{-1}(0)),$$

for any $f \in \mu^{-1}(0)$.

Lemma 15 *For any $f \in \mu^{-1}(0) \subset \operatorname{Map}(\Sigma, M)$, we have,*

$$\mathcal{D}_f = \Gamma(\Sigma, T_{\Sigma} + JT_{\Sigma}).$$

Proof. First, it is easy to see that $\mathcal{D}_f \supset \Gamma(\Sigma, T_{\Sigma})$ since for any X in $\Gamma(\Sigma, T_{\Sigma})$ and any Y in $\Gamma(\Sigma, T_M)$, $\iota_{X \wedge Y} \operatorname{Re} \Omega_M$ can not be a top degree form on Σ . By similar reasons, \mathcal{D}_f is preserved by the Hermitian complex structure J on M . Because f is isotropic, we have $\mathcal{D}_f \supset \Gamma(\Sigma, T_{\Sigma} + JT_{\Sigma})$. Now, as in lemma 8, we consider localization of

$$0 = \iota_X \omega_{I,f} = \int_{\Sigma} \iota_X ev^* \operatorname{Re} \Omega_M,$$

at x in Σ by varying Σ and we obtain

$$0 = \iota_{X(x) \wedge E_1 \wedge \dots \wedge E_{n-2}} \operatorname{Re} \Omega_M$$

where E_1, \dots, E_{n-2} is an orthonormal oriented basis of $(T_{\Sigma})_x$. This implies that

$$\mathcal{D}_f = \Gamma(\Sigma, T_{\Sigma} + JT_{\Sigma}),$$

because for any X in $T_M \setminus (T_{\Sigma} + JT_{\Sigma})$, there is x in Σ such that

$$\iota_{X(x) \wedge E_1 \wedge \dots \wedge E_{n-2}} \operatorname{Re} \Omega_M \neq 0.$$

Note that the same construction applied to ω_K , instead of ω_I , will give another distribution which is identical to \mathcal{D}_f because f is isotropic. ■

Observe that, for each $f \in \mu^{-1}(0)$, the rank of the subbundle $T_\Sigma + JT_\Sigma$ in f^*T_M is $2(n-2)$, i.e. constant rank on $\mu^{-1}(0)$. That is ω_I is a closed two form of constant rank on $\mu^{-1}(0)$. From the standard theory of symplectic reduction, the distribution \mathcal{D} is integrable and the space of leaves has a natural symplectic form descended from ω_I . We call this space as the **isotropic knot space** of M and we denote it as

$$\hat{\mathcal{K}}_\Sigma M = \mu^{-1}(0) / \langle \mathcal{D} \rangle$$

where $\langle \mathcal{D} \rangle$ are equivalence relations generated by the distribution \mathcal{D} .

Remark: The isotropic knot space $\hat{\mathcal{K}}_\Sigma M$ is a quotient space of $\mu^{-1}(0) / Diff(\Sigma)$, the space of isotropic submanifolds in M . If $f : \Sigma \rightarrow M$ parametrizes an isotropic submanifold in M , then deforming Σ along T_Σ directions simply changes the parametrization of the submanifold $f(\Sigma)$, namely the equivalence class in $\mu^{-1}(0) / Diff(\Sigma)$ remains unchanged. However if we deform Σ along $T_\Sigma \otimes \mathbb{C}$ directions, then their equivalence classes in $\hat{\mathcal{K}}_\Sigma M$ remains constant. Notice that the isotropic condition on Σ implies that $T_\Sigma \otimes \mathbb{C} \cong T_\Sigma \oplus J(T_\Sigma) \subset f^*T_M$. Therefore, roughly speaking, $\hat{\mathcal{K}}_\Sigma M$ is the space of isotropic submanifolds in M divided by $Diff(\Sigma) \otimes \mathbb{C}$. In particular, the tangent space of $\hat{\mathcal{K}}_\Sigma M$ is given by

$$T_{[f]}(\hat{\mathcal{K}}_\Sigma M) = \Gamma(\Sigma, f^*(TM) / (T_\Sigma + JT_\Sigma))$$

for any $[f]$ in $\hat{\mathcal{K}}_\Sigma M$. Note a leaf is preserved by the induced almost Hermitian structure on $\mu^{-1}(0)$ from the Hermitian complex structure J on M , because a tangent vector on it is a section to the subbundle $T_\Sigma + JT_\Sigma$. However $\hat{\mathcal{K}}_\Sigma M$ may not be symplectic, because the symplectic form ω_{Map} on $\mu^{-1}(0)$ descends to a 2-form $\omega^\mathcal{K}$ on $\hat{\mathcal{K}}_\Sigma M$ which may not be closed.

We will show that $\hat{\mathcal{K}}_\Sigma M$ admits three almost complex structures $I^\mathcal{K}$, $K^\mathcal{K}$ and $J^\mathcal{K}$ satisfying the Hamilton relation

$$(I^\mathcal{K})^2 = (J^\mathcal{K})^2 = (K^\mathcal{K})^2 = I^\mathcal{K}J^\mathcal{K}K^\mathcal{K} = -id.$$

Furthermore the associated Kähler forms $\omega_I^\mathcal{K}$ and $\omega_K^\mathcal{K}$ are closed. In [7], Hitchin showed that existence of such structures on any finite dimensional Riemannian manifold implies that its almost complex structure $J^\mathcal{K}$ is *integrable*. Namely we obtain a 1-fold \mathbb{C} -VCP on a Hermitian complex manifold. If, in addition, $\omega_J^\mathcal{K}$ is closed then $I^\mathcal{K}$, $J^\mathcal{K}$ and $K^\mathcal{K}$ are all integrable complex structures and we have a hyperkähler manifold.

We begin with the following lemma which holds true both on $\mu^{-1}(0)$ and on $\hat{\mathcal{K}}_\Sigma M$.

Lemma 16 *On $\mu^{-1}(0) \subset Map(\Sigma, M)$, the endomorphisms I , J and K satisfy the following relations,*

$$\begin{aligned} IJ &= -JI \text{ and } KJ = -JK, \\ I &= -KJ \text{ and } K = IJ. \end{aligned}$$

Proof: The formula $IJ = -JI$ can be restated as follows: for any $f \in \mu^{-1}(0)$ and for any tangent vectors of $Map(\Sigma, M)$ at f , $A, B \in \Gamma(\Sigma, f^*(T_M))$ we have

$$g_f(I_f J_f(A), B) = g_f(-J_f I_f(A), B).$$

For simplicity, we ignore the subscript f . Since $J^2 = -id$, we have

$$\begin{aligned} & g(IJ(A), B) - g(-JI(A), B) \\ &= g(IJ(A), B) - g(I(A), JB) \\ &= \omega_I(JA, B) - \omega_I(A, JB) \\ &= \int_{\Sigma} \iota_{JA \wedge B} \operatorname{Re} \Omega_M - \int_{\Sigma} \iota_{A \wedge JB} \operatorname{Re} \Omega_M \\ &= \int_{\Sigma} \iota_{(JA \wedge B - A \wedge JB)} \frac{1}{2} (\Omega_M + \bar{\Omega}_M) = 0. \end{aligned}$$

The last equality follows from the fact that Ω_M and $\bar{\Omega}_M$ vanish when each is contracted by an element in $\wedge^{1,1} T_M$. By replacing $\operatorname{Re} \Omega_M$ with $\operatorname{Im} \Omega_M$ in the above calculations, we also have $KJ = -JK$.

To prove the others formulas, we consider

$$\begin{aligned} & g(IA, B) - g(-KJA, B) \\ &= \omega_I(A, B) + \omega_K(JA, B) \\ &= \int_{\Sigma} \iota_{A \wedge B} \operatorname{Re} \Omega_M + \int_{\Sigma} \iota_{JA \wedge B} (-\operatorname{Im} \Omega_M) \\ &= \int_{\Sigma} \iota_{A \wedge B} \frac{1}{2} (\Omega_M + \bar{\Omega}_M) - \int_{\Sigma} \iota_{JA \wedge B} \frac{-\sqrt{-1}}{2} (\Omega_M - \bar{\Omega}_M) \\ &= \frac{1}{2} \int_{\Sigma} \iota_{(A + \sqrt{-1}JA) \wedge B} \Omega_M + \frac{1}{2} \int_{\Sigma} \iota_{(A - \sqrt{-1}JA) \wedge B} \bar{\Omega}_M = 0 \end{aligned}$$

since $i_{(A + \sqrt{-1}JA)} \Omega_M = 0$ and taking complex conjugation. This implies that $g(IA, B) - g(-KJA, B) = 0$ and therefore $I = -KJ$. Finally $J^2 = -Id$ and $I = -KJ$ imply that $K = IJ$. Hence the results. ■

Theorem 17 *Suppose that M is a Calabi-Yau n -fold M . For any $n-2$ dimensional closed manifold Σ with $b_1(\Sigma) = 0$, the isotropic knot space $\hat{\mathcal{K}}_{\Sigma} M$ is an infinite dimensional integrable complex manifold with a natural 1-fold \mathbb{C} -VCP structure, in particular a natural holomorphic symplectic structure.*

Proof. From the construction of $\hat{\mathcal{K}}_{\Sigma} M$, it has a Hermitian Kähler form $\omega^{\mathcal{K}}$ induced from that of M and a closed holomorphic symplectic form $\Omega^{\mathcal{K}}$ given by the transgression of the closed $(n-1)$ -fold \mathbb{C} -VCP form Ω_M on M . As we have seen above, the induced holomorphic symplectic form is closed but the induced Hermitian Kähler form may not be closed. If their corresponding almost complex structures satisfy the Hamilton relation then this implies that

$J^\mathcal{K}$ is integrable [7]. Namely $\hat{\mathcal{K}}_\Sigma M$ is a Hermitian integrable complex manifold with a 1-fold \mathbb{C} -VCP structure. In order to verify the Hamilton relation,

$$(I^\mathcal{K})^2 = (J^\mathcal{K})^2 = (K^\mathcal{K})^2 = I^\mathcal{K} J^\mathcal{K} K^\mathcal{K} = -id,$$

we only need to show that $(I^\mathcal{K})^2 = -Id$ and $(K^\mathcal{K})^2 = -Id$ because of the previous lemma. Namely, $I^\mathcal{K}$ and $K^\mathcal{K}$ are almost complex structures on $\hat{\mathcal{K}}_\Sigma M$.

We consider a fixed $[f]$ in $\hat{\mathcal{K}}_\Sigma M$ and by localization method as in the proof of lemma 8, we can reduce the identities to the tangent space of a point x in Σ . The transgression

$$\int_\Sigma ev^* \Omega_M = \omega_{I,[f]}^\mathcal{K} - \sqrt{-1} \omega_{K,[f]}^\mathcal{K}$$

is descended to

$$\iota_{E_1 \wedge \dots \wedge E_{n-2}} \Omega_M = \omega_{I,[f],x}^\mathcal{K} - \sqrt{-1} \omega_{K,[f],x}^\mathcal{K}$$

where E_1, \dots, E_{n-2} is an orthonormal oriented basis $(T_\Sigma)_x$. Since f is isotropic, the complexified vectors $E_i - \sqrt{-1} J E_i$ of E_i can be defined over $(T_\Sigma + J T_\Sigma)_x$, therefore the above equality is equivalent to

$$\iota_{(E_1 - \sqrt{-1} J E_1)/2 \wedge \dots \wedge (E_{n-2} - \sqrt{-1} J E_{n-2})/2} \Omega_M = \omega_{I,[f],x}^\mathcal{K} - \sqrt{-1} \omega_{K,[f],x}^\mathcal{K},$$

i.e. a 1-fold \mathbb{C} -VCP on $f^*(TM) / (T_\Sigma + J T_\Sigma) |_x$ which is $T_{[f],x} \hat{\mathcal{K}}_\Sigma M$. Since this 1-fold \mathbb{C} -VCP gives a hyperkähler structure on $T_{[f],x} \hat{\mathcal{K}}_\Sigma M$, $I_{[f]}^\mathcal{K}$ and $K_{[f]}^\mathcal{K}$ satisfy the Hamilton relation at x in Σ . That is

$$\left(I_{[f],x}^\mathcal{K} \right)^2 = -Id \text{ and } \left(K_{[f],x}^\mathcal{K} \right)^2 = -Id.$$

Therefore we have $(I^\mathcal{K})^2 = -Id$ and $(K^\mathcal{K})^2 = -Id$. Hence the result. ■

Remark 18 *In String theory we need to compactify a ten dimensional space-time on a Calabi-Yau threefold. When M is a Calabi-Yau threefold, then Σ is an one dimensional circle and therefore $b_1(\Sigma)$ is nonzero. In general, as discussed in [3] and [6], the symplectic quotient construction for $\text{Map}(\Sigma, M) // \text{Diff}(\Sigma, \sigma)$ can be modified to obtain a symplectic structure on a rank $b_1(\Sigma)$ torus bundle over the space of isotropic submanifolds in M . Roughly speaking this torus bundle is the space of isotropic submanifolds coupled with flat rank one line bundles (or gerbes) in M . In the Calabi-Yau threefold case, every circle Σ in M is automatically isotropic. Therefore $\hat{\mathcal{K}}_\Sigma M$ is the space of loops (or string) coupled with flat line bundles in M , up to deformations of strings along their complexified tangent directions. We wonder whether this infinite dimensional holomorphic symplectic manifold $\hat{\mathcal{K}}_\Sigma M$ has any natural physical interpretations.*

6 Complex Lagrangians in Isotropic Knot Spaces

In this subsection we relate the geometry of \mathbb{C} -VCP of a Calabi-Yau manifold M to the holomorphic symplectic geometry of its isotropic knot space $\hat{\mathcal{K}}_\Sigma M$. For example both N-branes (i.e. complex hypersurfaces) and D-branes (i.e. special Lagrangian submanifold with phase $-\pi/2$) in M correspond to complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$, but for different almost complex structures in the twistor family. First we discuss the correspondence for instantons.

In the following proposition, we use $e^{i\theta}\Omega_M$ instead of Ω_M to get a 1-fold \mathbb{C} -VCP on $\hat{\mathcal{K}}_\Sigma M$, and also $\text{Re}(e^{i\theta}\Omega_M)$ gives another symplectic form $\omega_{I,\theta}^{\mathcal{K}}$ and corresponding Hermitian almost complex structure $J_\theta^{\mathcal{K}}$ on $\hat{\mathcal{K}}_\Sigma M$ defined as $J_\theta^{\mathcal{K}} = \cos\theta I^{\mathcal{K}} + \sin\theta K^{\mathcal{K}}$.

Proposition 19 *Suppose that M is a Calabi-Yau n -fold. Let D be a normal disk in $\mu^{-1}(0)$ with an n -dimensional submanifold A defined as $A = D \times \Sigma$, and assume $A \rightarrow D$ is a Riemannian submersion. We denote the reduction of D in $\hat{\mathcal{K}}_\Sigma M$ as \hat{D} .*

Then \hat{D} is an instanton i.e. a $J_\theta^{\mathcal{K}}$ -holomorphic curve in $\hat{\mathcal{K}}_\Sigma M$ if and only if A is an instanton i.e. a special Lagrangian with phase θ in M .

Proof. In the proposition 11, VCP form ϕ plays a role of calibrating form rather than that of a VCP form. So by replacing ϕ by $\text{Re}(e^{i\theta}\Omega_M)$ in proposition 11, readers can see this theorem can be proved in essentially the same manner. But since proposition 11 is given for the parametrized knot space and this theorem is given by a reduction, we need to check \hat{D} is a disk in $\hat{\mathcal{K}}_\Sigma M$. Let f be the center of disk D . Since D is a normal disk, a tangent vector v at f along $D \subset \mu^{-1}(0)$ is in $\Gamma(\Sigma, N_{\Sigma/M})$. And since f is isotropic, Jv is also in $\Gamma(\Sigma, N_{\Sigma/M})$ equivalently v is in $\Gamma(\Sigma, N_{J\Sigma/M})$. So $v \in \Gamma(\Sigma, f^*(TM)/(T_\Sigma + JT_\Sigma))$. This implies $T_f D$ can be identified with $T_{[f]}\hat{D}$, i.e. \hat{D} is a disk in $\hat{\mathcal{K}}_\Sigma M$. ■

Next we are going to relate N- and D-branes in Calabi-Yau manifolds M to complex Lagrangian submanifolds in $\hat{\mathcal{K}}_\Sigma M$.

Definition 20 *For any submanifold C in a Calabi-Yau manifold M , we define a subspace $\hat{\mathcal{K}}_\Sigma C$ in $\hat{\mathcal{K}}_\Sigma M = \mu^{-1}(0)/\langle \mathcal{D} \rangle$ as follow: The equivalent relation $\langle \mathcal{D} \rangle$ on $\mu^{-1}(0)$ restrict to one on $\text{Map}(\Sigma, C) \cap \mu^{-1}(0)$ and we define*

$$\hat{\mathcal{K}}_\Sigma C = \{ \text{Map}(\Sigma, C) \cap \mu^{-1}(0) \} / \langle \mathcal{D} \rangle .$$

In the following theorem, we see that C being a N-brane in M corresponds to $\hat{\mathcal{K}}_\Sigma C$ being a $J^{\mathcal{K}}$ -complex Lagrangian in $\hat{\mathcal{K}}_\Sigma M$ which means that $\hat{\mathcal{K}}_\Sigma C$ is both maximally self $\omega_I^{\mathcal{K}}$ -perpendicular and maximally self $\omega_K^{\mathcal{K}}$ -perpendicular in $\hat{\mathcal{K}}_\Sigma M$.

Theorem 21 *Let C be a connected analytic submanifold in a Calabi-Yau manifold M . Then the following two statements are equivalent:*

- (i) C is an N -brane (i.e. a complex hypersurface) in M ;
- (ii) $\hat{\mathcal{K}}_\Sigma C$ is a $J^\mathcal{K}$ -complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$.

Proof: For the *if* part, we assume that $\hat{\mathcal{K}}_\Sigma C$ is a $J^\mathcal{K}$ -complex Lagrangian in $\hat{\mathcal{K}}_\Sigma M$. First we need to show that the dimension of C is at least $2n - 2$ where n is the complex dimension of M .

Before proving this claim for the general case, let us discuss a simplified linear setting where some of the key arguments become more transparent. Suppose that M is a *linear* Calabi-Yau manifold, that is $M \cong \mathbb{C}^n$ with the standard Kähler structure and $\Omega_M = dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n$. Let Σ be a $(n - 2)$ -dimensional isotropic linear subspace in M lying inside another linear subspace C in M . For simplicity we assume that Σ is the linear span of x^1, x^2, \dots, x^{n-2} . Of course $M/\Sigma \oplus J\Sigma \cong \mathbb{C}^2$ is a linear holomorphic symplectic manifold with

$$\int_\Sigma \Omega_M = dz^{n-1} \wedge dz^n,$$

which is the standard holomorphic symplectic form on \mathbb{C}^2 . Suppose that

$$C/(\Sigma \oplus J\Sigma) \cap C \subset M/\Sigma \oplus J\Sigma$$

is a complex Lagrangian subspace. Then there is a vector in C perpendicular to both Σ and $J\Sigma$, say x^{n-1} . If we denote the linear span of $x^2, \dots, x^{n-2}, x^{n-1}$ as Σ' , then Σ' is another $(n - 2)$ -dimensional isotropic linear subspace in M lying inside C . Furthermore x^1 is a normal vector in C perpendicular to $\Sigma' \oplus J\Sigma'$. This implies that $y^1 = Jx^1$ also lie in C . This is because $C/(\Sigma' \oplus J\Sigma') \cap C \subset M/\Sigma' \oplus J\Sigma'$ being a complex Lagrangian subspace implies that it is invariant under the complex structure on $M/\Sigma' \oplus J\Sigma'$ induced by J on M . By the same reasoning, y^j also lie in C for $j = 1, 2, \dots, n - 2$. That is C contains the linear span of $\{x^j, y^j\}_{j=1}^{n-2}$. On the other hand it also contain x^{n-1} and y^{n-1} and therefore $\dim C \geq 2n - 2$.

We come back to the general situation where $\hat{\mathcal{K}}_\Sigma C$ is a $J^\mathcal{K}$ -complex Lagrangian in $\hat{\mathcal{K}}_\Sigma M$. One difficulty is to rotate the isotropic submanifold Σ to Σ' inside M .

We observe that the tangent space to $\hat{\mathcal{K}}_\Sigma C$ at any point $[f]$ is given by

$$T_{[f]}(\hat{\mathcal{K}}_\Sigma C) = \Gamma\left(\Sigma, \frac{f^*T_C}{T_\Sigma \oplus JT_\Sigma \cap f^*T_C}\right).$$

Since any $J^\mathcal{K}$ -complex Lagrangian submanifold is indeed an integrable complex submanifold, $T_{[f]}(\hat{\mathcal{K}}_\Sigma C)$ is preserved by $J^\mathcal{K}$. This implies that the complex structure on T_M induces a complex structure on the quotient bundle $f^*T_C/[T_\Sigma \oplus JT_\Sigma \cap f^*T_C]$. For any $\nu \in T_{[f]}(\hat{\mathcal{K}}_\Sigma C)$ we can regard it as a section of f^*T_C over Σ , perpendicular to T_Σ and $JT_\Sigma \cap f^*T_C$. In particular $J\nu$ is also such a section.

We need the following lemma which will be proven later.

Lemma 22 *In the above situation, we have $JT_\Sigma \subset f^*T_C$.*

This lemma implies that the restriction of T_C on Σ contains the linear span of T_Σ , JT_Σ , ν and $J\nu$, which has rank $2n - 2$. (More precisely we consider the restriction to the complement of the zero set of ν in Σ .) Therefore $\dim C \geq 2n - 2$.

On the other hand, if $\dim C > 2n - 2$, then by using localizing method as in lemma 8, $\hat{\mathcal{K}}_\Sigma C$ would be too large to be a complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$. That is $\dim C = 2n - 2$.

In particular, this implies that f^*T_C is isomorphic to the linear span of T_Σ , JT_Σ , ν and $J\nu$ outside the zero set of ν . Therefore T_C is preserved by the complex structure of M along Σ . By varying the isotropic submanifold Σ in C , we can cover an open neighborhood of Σ in C (for example using the gluing arguments as in the proof of the above lemma). This implies that an open neighborhood of Σ in C is a complex submanifold in M . By the analyticity of C , the submanifold C is a complex hypersurface in M .

For the *only if* part, we suppose C is complex hypersurface in M , then it is clear that $\omega_I^\mathcal{K}$ and $\omega_K^\mathcal{K}$ vanish along $\hat{\mathcal{K}}_\Sigma C$ since Ω_M vanishes along C . Using similar arguments as in the proof of proposition 10, it is not difficult to verify that $\hat{\mathcal{K}}_\Sigma C$ is maximally self $\omega_I^\mathcal{K}$ -perpendicular and maximally self $\omega_K^\mathcal{K}$ -perpendicular in $\hat{\mathcal{K}}_\Sigma M$. That is $\hat{\mathcal{K}}_\Sigma C$ is a $J^\mathcal{K}$ -complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$. Hence the theorem. ■

We suspect that the analyticity assumption on C is unnecessary. All we need is to deform the isotropic submanifold Σ of M inside C to cover every point in C .

Proof of lemma: For any tangent vector u of Σ at a point p , i.e. $u \in T_p\Sigma \subset T_pC$, we need to show that $Ju \in T_pC$. We can assume that $\nu(p)$ has unit length. First we choose local holomorphic coordinate z^1, z^2, \dots, z^n near p satisfying the following properties at p :

$$\begin{aligned} z^i(p) &= 0 \text{ for all } i = 1, \dots, n, \\ \Omega_M(p) &= dz^1 \wedge dz^2 \wedge \dots \wedge dz^n, \\ T_p\Sigma &\text{ is spanned by } \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^{n-2}}, \\ \nu(p) &= \frac{\partial}{\partial x^{n-1}} \text{ and } u = \frac{\partial}{\partial x^1}. \end{aligned}$$

where $z^j = x^j + iy^j$ for each j . We could use x^1, \dots, x^{n-2} to parametrize Σ near p .

Recall that

$$\nu \in T_{[f]}(\hat{\mathcal{K}}_\Sigma C) = \Gamma\left(\Sigma, \frac{f^*T_C}{T_\Sigma \oplus JT_\Sigma \cap f^*T_C}\right).$$

For any smooth function $\alpha(x^1, \dots, x^{n-2})$ on Σ , $\alpha\nu$ is again a tangent vector to $\hat{\mathcal{K}}_\Sigma C$ at $[f]$. We are going to construct a particular α supported on a small

neighborhood of p in Σ satisfying

$$\begin{aligned}\frac{\partial\alpha}{\partial x^1}(0) &= \infty, \\ \frac{\partial\alpha}{\partial x^j}(0) &= 0, \text{ for } j = 2, \dots, n-2.\end{aligned}$$

For simplicity we assume that this small neighborhood contain the unit ball in Σ . To construct α , we fix a smooth even cutoff function $\eta : \mathbb{R} \rightarrow [0, 1]$ with $\text{supp}(\eta) \in (-1, 1)$, $\eta(0) = 1$ and $\eta'(0) = 0$. We define $\alpha : \Sigma \rightarrow \mathbb{R}$ as follow:

$$\alpha(x^1, x^2, \dots, x^{n-2}) = (x^1)^{1/3} \cdot \eta(x^1) \cdot \eta\left(\sum_{j=2}^{n-2} (x^j)^2\right).$$

For any small real number ε , we write

$$\nu_\varepsilon = \varepsilon\alpha\nu \in T_{[f]}(\hat{\mathcal{K}}_\Sigma C)$$

and we denote the corresponding family of isotropic submanifolds of M in C by $f_\varepsilon : \Sigma_\varepsilon \rightarrow M$. From $\alpha(0) = 0$, there is a family of points $p_\varepsilon \in \Sigma_\varepsilon$ with the property that

$$|p_\varepsilon - p| = O(\varepsilon^2).$$

Using the property $\partial\alpha/\partial x^1(0) = \infty$, we can find a family of normal vectors at p_ε ,

$$u_\varepsilon \in N_{\Sigma_\varepsilon/C},$$

with the property

$$\left|u_\varepsilon - \frac{\partial}{\partial x^1}\right| = O(\varepsilon^2).$$

Since $T_{[f_\varepsilon]}(\hat{\mathcal{K}}_\Sigma C)$ is preserved by $J^\mathcal{K}$, $f_\varepsilon^*T_C/[T_{\Sigma_\varepsilon} \oplus JT_{\Sigma_\varepsilon} \cap f_\varepsilon^*T_C]$ is preserved by J , we have $Ju_\varepsilon \in T_{p_\varepsilon}C$. By letting ε goes to zero, we have $Ju \in T_pC$. Hence the lemma. ■

In the following theorem, we see that relationship between a D -brane L in M with respect to $e^{i\theta}\Omega_M$ and a $J_{\theta+\frac{\pi}{2}}^\mathcal{K}$ complex Lagrangian $\hat{\mathcal{K}}_\Sigma L$ in $\hat{\mathcal{K}}_\Sigma M$, i.e. it is maximally self $\omega_{T,\theta}^\mathcal{K}$ -perpendicular and maximally self $\omega^\mathcal{K}$ -perpendicular.

Theorem 23 *Let L be a connected analytic submanifold of a Calabi-Yau manifold M , then the following two statements are equivalent:*

- (i) L is a D -brane with phase θ , (i.e. a special Lagrangian with phase $\theta - \frac{\pi}{2}$);
- (ii) $\hat{\mathcal{K}}_\Sigma L$ is a $J_{\theta+\frac{\pi}{2}}^\mathcal{K}$ -complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$.

Proof: By replacing the holomorphic volume form on M from Ω_M to $e^{i\theta}\Omega_M$ if necessary, we can assume that θ is zero.

For the *only if* part, we assume that L is a special Lagrangian submanifold in M with phase $-\pi/2$. Because the Kähler form ω and $\text{Re}\Omega_M$ of M

vanish along L , it is clear that $\omega^{\mathcal{K}}$ and $\omega_{T,0}^{\mathcal{K}}$ vanishes along $\hat{\mathcal{K}}_{\Sigma}L$. Notice that L being a Lagrangian submanifold in M implies that any submanifold in L is automatically isotropic in M , and moreover the equivalent relation $\langle \mathcal{D} \rangle$ on $\mu^{-1}(0)$ restricted to $Map(\Sigma, L)$ is trivial locally. That is $\hat{\mathcal{K}}_{\Sigma}L$ is the same as $\mathcal{K}_{\Sigma}L = Map(\Sigma, L)/Diff(\Sigma)$ at least locally.

We claim that $\hat{\mathcal{K}}_{\Sigma}L$ is maximally self $\omega^{\mathcal{K}}$ -perpendicular in $\hat{\mathcal{K}}_{\Sigma}M$. Otherwise there is normal vector field $\nu \in \Gamma(\Sigma, N_{\Sigma/M})$ not lying in $\Gamma(\Sigma, N_{\Sigma/L})$ such that $\omega(\nu, u) = 0$ for any $u \in \Gamma(\Sigma, f^*T_L)$. Suppose that $\nu(p) \notin N_{\Sigma/L,p}$, then this implies that ω vanishes on the linear span of T_pL and $\nu(p)$ inside T_pM . This is impossible because the dimension of the linear span is bigger than n .

Similarly $\hat{\mathcal{K}}_{\Sigma}L$ is maximally self $\omega_{T,0}^{\mathcal{K}}$ -perpendicular in $\hat{\mathcal{K}}_{\Sigma}M$. Otherwise $Re\Omega_M$ vanishes on the linear span of T_pL and $\nu(p)$ inside T_pM . This is again impossible because $\Omega_{M,p}$ is a complex volume form on $T_pM \cong \mathbb{C}^n$ and therefore can not vanish on any co-isotropic subspaces other than Lagrangians. Hence $\hat{\mathcal{K}}_{\Sigma}L$ is a $J_{\pi/2}^{\mathcal{K}}$ -complex Lagrangian in $\hat{\mathcal{K}}_{\Sigma}M$.

For the *if* part, we assume that $\hat{\mathcal{K}}_{\Sigma}L$ is a $K^{\mathcal{K}}$ -complex Lagrangian submanifold in $\hat{\mathcal{K}}_{\Sigma}M$. For any $[f] \in \hat{\mathcal{K}}_{\Sigma}L$, the tangent space of $\hat{\mathcal{K}}_{\Sigma}L$ (resp. $\hat{\mathcal{K}}_{\Sigma}M$) at $[f]$ is the section of the bundle $f^*T_L/T_{\Sigma} \oplus JT_{\Sigma} \cap f^*T_L$ (resp. $f^*(T_M)/(T_{\Sigma} + JT_{\Sigma})$) over Σ . Let $v \in T_{[f]}(\hat{\mathcal{K}}_{\Sigma}L)$, we can regard v as a section of f^*T_L over Σ , orthogonal to $T_{\Sigma} \oplus JT_{\Sigma} \cap f^*T_L$.

Since $f^*(T_M)/(T_{\Sigma} + JT_{\Sigma})$ is a rank four bundle and $\hat{\mathcal{K}}_{\Sigma}L$ is a Lagrangian in $\hat{\mathcal{K}}_{\Sigma}M$, this implies that $f^*T_L/T_{\Sigma} \oplus JT_{\Sigma} \cap f^*T_L$ must be a rank two bundle over Σ . Therefore

$$\dim L \geq n,$$

with the equality sign holds if and only if $JT_{\Sigma} \cap f^*T_L$ is trivial. Suppose that ν is a tangent vector of $\hat{\mathcal{K}}_{\Sigma}L$ at f

Assume that $JT_{\Sigma} \cap f^*T_L$ is not trivial, we can find a tangent vector u to $f(\Sigma)$ at a point p such that $Ju \in T_pL$. For simplicity we assume that ν has unit length at the point p .

As in the proof of the previous theorem, we can choose local holomorphic coordinates z^j 's of M around p such that

$$\begin{aligned} z^i(p) &= 0 \text{ for all } i = 1, \dots, n, \\ \Omega_M(p) &= dz^1 \wedge dz^2 \wedge \dots \wedge dz^n, \\ T_p\Sigma &\text{ is spanned by } \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^{n-2}}, \\ v(p) &= \frac{\partial}{\partial x^{n-1}}, u = \frac{\partial}{\partial x^1} \text{ and } Ju = \frac{\partial}{\partial y^1}. \end{aligned}$$

We choose a function $\alpha(x)$ as in the proof of the previous theorem, write

$$\nu_{\varepsilon} = \varepsilon\alpha\nu \in T_{[f]}(\hat{\mathcal{K}}_{\Sigma}L)$$

for any small real number ε , and we denote the corresponding family of isotropic submanifolds of M in L by $f_\varepsilon : \Sigma_\varepsilon \rightarrow M$ as before. From $\alpha(0) = 0$, there is a family of points $p_\varepsilon \in \Sigma_\varepsilon$ with the property that

$$|p_\varepsilon - p| = O(\varepsilon^2).$$

Using the property $\partial\alpha/\partial x^1(0) = \infty$, we can find two family of normal vectors at p_ε ,

$$u_\varepsilon, w_\varepsilon \in N_{\Sigma_\varepsilon/L},$$

with the property

$$\left| u_\varepsilon - \frac{\partial}{\partial x^1} \right| = O(\varepsilon^2) \quad \text{and} \quad \left| w_\varepsilon - \frac{\partial}{\partial y^1} \right| = O(\varepsilon^2).$$

However, using localization arguments as before, $\hat{K}_\Sigma L$ being an $\omega^\mathcal{K}$ -Lagrangian submanifold in $\hat{K}_\Sigma M$ implies that $\omega(u_\varepsilon, w_\varepsilon) = 0$. By letting ε goes to zero, we have $0 = \omega(\partial/\partial x^1, \partial/\partial y^1) = -1$, a contradiction. Hence $JT_\Sigma \cap f^*T_L$ is trivial and $\dim L = n$.

Since $\Gamma(\Sigma, f^*T_L/T_\Sigma \oplus JT_\Sigma \cap f^*T_L)$ is preserved by $K^\mathcal{K}$, we have $K^\mathcal{K}\nu(p) \in f^*T_L$. Note that $K^\mathcal{K}\nu(p)$ is the tangent vector of M which is the metric dual of the one form,

$$\begin{aligned} \nu \lrcorner \omega_K^\mathcal{K}(p) &= \left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-2}} \wedge \nu \right) \lrcorner \text{Im } \Omega_M(p) \\ &= \pm dy^n, \end{aligned}$$

since $\nu(p) = \partial/\partial x^{n-1}$. This implies that $T_p L$ is spanned by $\partial/\partial x^j$'s for $j = 1, 2, \dots, n-1$ and $\partial/\partial y^n$, that is a special Lagrangian subspace of phase $-\pi/2$.

As in the proof of the previous theorem, by deforming the isotropic submanifold Σ in L and using the fact that L is an analytic submanifold of M , we conclude that L is a special Lagrangian submanifold in M with zero phase.

Hence the theorem. ■

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