Rational Surfaces, Simple Lie Algebras and Flat G Bundles over Elliptic Curves

ZHANG, Jiajin

A Thesis Submitted in Partial Fulfilment
of the Requirements for the Degree of
Doctor of Philosophy
in
Mathematics

July 2007

Thesis/Assessment Committee

Chair :	Prof. AU Kwok Keung Thomas;
Thesis Advisor :	Prof. LEUNG Nai Chung Conan;
Committee Member :	Prof. Wang Xiaowei;
External Examiner :	Prof. Li Wei-Ping, Hong Kong University of Science and Technology.

Abstract

It is well-known that del Pezzo surfaces of degree $9-n$ are in one-to-one correspondence to flat E_n bundles over elliptic curves which are anti-canonical curves of such surfaces. In my thesis, we study a broader class of rational surfaces which are called ADE surfaces. We construct Lie algebra bundles of any type on these surfaces, and extend the above correspondence to flat G bundles over elliptic curves, where G is a simple, compact and simply-connected Lie group of *any type*. Concretely, we establish a natural identification between the following two very different moduli spaces for a Lie group G of *any* type: the moduli space of rational surfaces with G-configurations and the moduli space of flat G-bundles over a fixed elliptic curve.

摘 要

衆所周知，次數 9-n 的 del Pezzo 曲面和一個固定的橢圓曲綫上面的平坦主 E_n-叢是一一對應的。在我的論文裏面，我們研究一類更廣泛的有理曲面類，稱之爲 ADE 曲面。我們構造這些曲面上的任意類型的李代數叢，並且擴充上述的對應到橢圓曲綫上面的平坦 G-叢，這裡 G 是任意類型的單、緊緻、單連通李群。更具體地，我們建立了下面兩個看起來很不同的模空間的一個同構：一個是具有 G-configurations 的有理曲面的模空間，另一個是橢圓曲綫上的平坦 G-叢的模空間。

ACKNOWLEDGMENTS

I wish to express my heartfelt gratitude to my supervisors Prof. Nai-chung Conan Leung. During my Ph.D. program, Prof. Leung gives me great help and very patient instruction. I learned much including mathematics from him, and without his direction, I can't complete this thesis. Thank Prof. Zhi-Jie Chen, Prof. Sheng-Li Tan and Prof. Kang Zuo (my M. Phil. supervisor), from who I learnt elementary algebraic geometry; especially, I owe much to Prof. Sheng-Li Tan and Prof. Kang Zuo. Thank Prof. Brendan Hassett, Prof. Wei-Ping Li, and Prof. Xiaowei Wang, Prof. Zhouping Xin, from whom I received many helps in mathematics. Thank Mr. Kwok-Wai Chan, Mr. Yong-Dong Huang , Mr. Chang-Zheng Li and Mr. Mao Sheng for our many useful discussions. Thank Ms. Lily Chan and Ms. Caris Chung for many helps. Thank the Institute of Mathematical Science and the Department of Mathematics of the Chinese University of Hong Kong for providing me such a good opportunity to complete my Ph.D. program. At last, I want to dedicate this thesis to my family, especially my parents and my wife Ms. Mang Xu, whose consistent encouragement is necessary for me to complete my study. I also learned much about Kac-Moody theory from my wife.

Contents

Chapter 1

Introduction

Let S be a smooth rational surfaces. If the anti-canonical line bundle $-K_S$ is ample, then S is called a *del Pezzo surface*. It is well-known that a del Pezzo surface can be classified as a blow-up of $\mathbb{CP}^2$ at $n(n \leq 8)$ points in general position or $\mathbb{CP}^1 \times \mathbb{CP}^1$. When these blown-up points are in *almost general position*, such a surface is called a *generalized del Pezzo surface*, according to Demazure [7]. It is also well-known that the sub-lattice $K_S^\perp$ of $Pic(S)$ is a root lattice of type E_n. For more results on (generalized) del Pezzo surfaces one can see [7] and [24]. Thus there is a natural Lie algebra bundle of type E_n over S. By restriction to a fixed smooth anti-canonical curve Σ, one obtains a flat E_n bundle over Σ. Moreover, Donagi [8] [9] and Friedman-Morgan-Witten [13] [14] prove that the moduli space of del Pezzo surfaces with fixed anti-canonical curve Σ can be identified with the moduli space of flat E_n bundles over the elliptic curve Σ.

In this thesis, we will extend this correspondence to all compact, simple, and simply connected Lie groups and to a broader class of rational surfaces, which are called ADE surfaces. Next we sketch the contents briefly.

In Chapter 2, we first analyze the structure of the Picard lattice of a rational surface which is a blow-up of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or the Hirzebruch surface $\mathbb{F}_1$ at some points. We shall see that there is a canonical sub-lattice of the Picard lattice which is a root lattice of ADE-type. Next we generalize the definition of del Pezzo surfaces to that of ADE surfaces, where an E_n surface is just a del Pezzo surface of degree $9-n$. Roughly speaking, an ADE surface S is a rational surface with a smooth rational curve C on S such that the sub-lattice $\langle K_S, C\rangle^{\perp}$ of $Pic(S)$ is an irreducible root lattice (see Definition 2.5). The condition in Definition 2.5 implies that $C^2 = -1, 0$ or 1, and that the sub-lattice $\langle K_S, C\rangle^{\perp}$ is a root lattice of type E_n, D_n, or A_n respectively (Proposition 2.6). Therefore such a surface is called a rational surface of E_n-type, D_n-type, or A_n-type accordingly.

Note that the definition of an E_n surface implies that after blowing down the (-1) curve C, the anti-canonical line bundle $-K$ will be ample. So the resulting surface is just a del Pezzo surface. Thus the definition of ADE surfaces naturally generalizes that of del Pezzo surfaces.

After this, we prove that an ADE surface is nothing but a blow-up of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at some points in general position. This gives us an explicit construction for any ADE surface.

In Section 2, we construct Lie algebra bundles of ADE-type, and their natural representation bundles over those surfaces discussed in Section 1. By a Lie algebra bundle over a surface S, we mean a vector bundle which has a fiberwise Lie algebra structure, and this structure is compatible with any trivialization. Similarly, by a representation bundle, we mean a vector bundle which is a fiberwise representation of a Lie algebra bundle, and this fiberwise representation is compatible with any trivialization.

More precisely, let S be an ADE surface. Since the sub-lattice $\langle K_S, C\rangle^{\perp}$ of $Pic(S)$ is a root lattice, we can explicitly construct a natural Lie algebra bundle of corresponding type over S, using the root system of the root lattice $\langle K_S, C\rangle^{\perp}$. Using the lines and rulings on S, we can also construct natural fundamental representation bundles over S.

In Chapter 3, we relate the above Lie algebra bundles of ADE-type over ADE rational surfaces to flat G bundles over an elliptic curve Σ, where G is a compact Lie group of corresponding type. If an ADE rational surface S contains a fixed smooth elliptic curve Σ as an anti-canonical curve, then by restriction, one obtains flat ADE-bundles over Σ.

Given Σ, the embedding of Σ as an anti-canonical curve is the following. We first embed Σ into $\mathbb{P}^2$ as an anti-canonical curve, using the projective embedding ϕ determined by the linear system $|3(0)|$ where (0) is the divisor of the identity element of Σ, and assume that all these blown up points $x_i \in \Sigma$ for $i = 1, \cdots, n$, and that $0, x_1, \cdots, x_n$ are in general position. Moreover, we blow up $\mathbb{P}^2$ at 0 to obtain the embedding of Σ into $\mathbb{F}_1$ as an anti-canonical curve, and take the exceptional curve l_0 as the section s for the ruled surface $\mathbb{F}_1$.

We can prove this restriction identifies the moduli space of flat ADE bundles over Σ and the moduli space of the pairs $(S, \Sigma \in |-K_S|)$ with extra structure ζ_G which is called a *G-configuration* (Definition 3.4). One of the main results in this paper is the following theorem.

Theorem 1.1 *Let Σ be a fixed elliptic curve, and let G be a simple, compact, simply laced and simply connected Lie group. Denote $\mathcal{S}(\Sigma, G)$ the moduli space of the pairs (S, Σ), where S is an ADE rational surface with $\Sigma \in |-K_S|$. Denote $\mathcal{M}_\Sigma^G$ the moduli space of flat G-bundles over Σ. Then by restriction, we have*

(i) $\mathcal{S}(\Sigma, G)$ can be embedded into $\mathcal{M}_\Sigma^G$ as an open dense subset.

(ii) There exists a natural and explicit compactification for $\mathcal{S}(\Sigma, G)$, denoted by $\overline{\mathcal{S}(\Sigma, G)}$, such that this embedding can be extended to an isomorphism from $\overline{\mathcal{S}(\Sigma, G)}$ onto $\mathcal{M}_\Sigma^G$.

(iii) Any surface corresponding to a boundary point in $\overline{\mathcal{S}(\Sigma, G)} \backslash \mathcal{S}(\Sigma, G)$ is equip-ped with a G-configuration, and on such a surface, any smooth rational

curve has a self-intersection number at least -2. Furthermore, in E_n case, all (-2) curves form chains of ADE-type, and the anti-canonical model of such a surface admits at worst ADE-singularities.

In Chapter 4, based on the result in simply laced cases, we construct $Lie(G)$ bundles for non-simply laced Lie group G over G-surfaces, and extend the above identification to non-simply laced cases. Therefore we establish a one-to-one correspondence between flat G bundles over a fixed elliptic curve Σ and rational surfaces with Σ as an anti-canonical curve for simple Lie groups of *all* types.

A non-simply laced Lie group G is uniquely determined by a simply laced Lie group G' and its outer automorphism group. Hence it is natural to apply the previous results for the simply laced cases to the non-simply laced situation. Similar to simply-laced cases, we can define *G-surfaces* and *rational surfaces with G-configurations* (see Definition 4.11, 4.17, 4.24, 4.32). Our main result in this case is the following theorem.

Theorem 1.2 *Let Σ be a fixed elliptic curve with identity $0 \in \Sigma$, G be any simple, compact, simply connected and non-simply laced Lie group. Denote $\mathcal{S}(\Sigma, G)$ the moduli space of the pairs (S, Σ), where S is a G-surface such that $\Sigma \in |-K_S|$. Denote $\mathcal{M}_\Sigma^G$ the moduli space of flat G-bundles over Σ. Then we have*

(i) $\mathcal{S}(\Sigma, G)$ can be embedded into $\mathcal{M}_\Sigma^G$ as an open dense subset.

(ii) This embedding can be extended to an isomorphism from $\overline{\mathcal{S}(\Sigma, G)}$ onto $\mathcal{M}_\Sigma^G$ by including all rational surfaces with G-configurations, and this gives us a natural and explicit compactification $\overline{\mathcal{S}(\Sigma, G)}$ of $\mathcal{S}(\Sigma, G)$.

In the following, we illustrate briefly via pictures what G-configurations and G-surfaces are in each case and compare it with the corresponding case that G' is simply-laced.

1.1 B_n-configurations and D_{n+1}-configurations

In these cases we consider rational surfaces with fibration structure and a fixed smooth anti-canonical curve Σ. A B_n-configuration comes from a D_{n+1}-configuration. Roughly speaking, by saying that a rational surface S has a D_{n+1}-configuration $(l_1, \cdots, l_{n+1})$, we mean that S can be considered as a blow-up of $\mathbb{F}_1$ (a *Hirzebruch surface*) at $n+1$ points on $\Sigma \in |-K_{\mathbb{F}_1}|$, such that $l_1, \cdots, l_{n+1}$ are the corresponding exceptional classes (Chapter 3, Definition 3.4). When these blown up points are *in general position*, S is called a $G = D_{n+1}$-surface. See the following picture for a surface with a D_{n+1}-configuration.

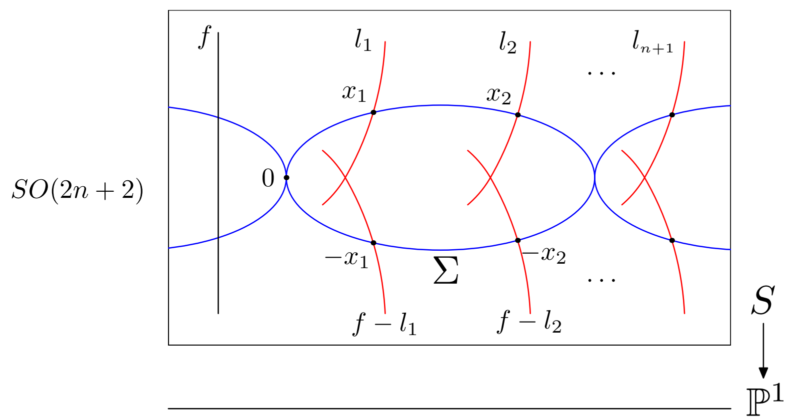

Figure 1. A surface with a D_{n+1}-configuration $(l_1, \cdots, l_{n+1})$.

Given a surface S with a D_{n+1}-configuration $\zeta = (l_1, \cdots, l_{n+1})$, if it satisfies the condition $x_1 = l_1 \cap \Sigma$ is the identity element 0 of the elliptic curve Σ, then ζ is a B_n-configuration on S (Definition 4.11). If all blown up points but x_1 are *in general position*, S is called a B_n-surface. See Figure 2 for a surface with a B_n-configuration.

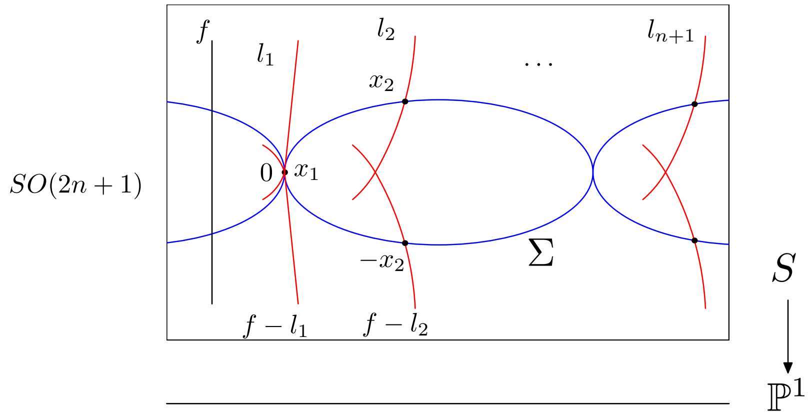

Figure 2. A surface with a B_n-configuration $(l_1, l_2, \cdots, l_{n+1})$, where $x_1 = l_1 \cap \Sigma = 0$.

1.2 C_n-configurations and A_{2n-1}-configurations

In these cases, we consider rational surfaces with fibration and section structure and a fixed smooth anti-canonical curve Σ.

A C_n-configuration comes from an A_{2n-1}-configuration. By saying a rational surface S has an A_{2n-1}-configuration $(l_1, \cdots, l_{2n})$, we mean that S can be considered as a blow-up of $\mathbb{F}_1$ at $2n$ points on $\Sigma \in |-K_{\mathbb{F}_1}|$ which sum to zero, such that $l_1, \cdots, l_{2n}$ are the corresponding exceptional classes (see Chapter 3). When these blown up points are *in general position*, S is called an A_{2n-1}-surface. See the following picture for a surface with an A_{2n-1}-configuration.

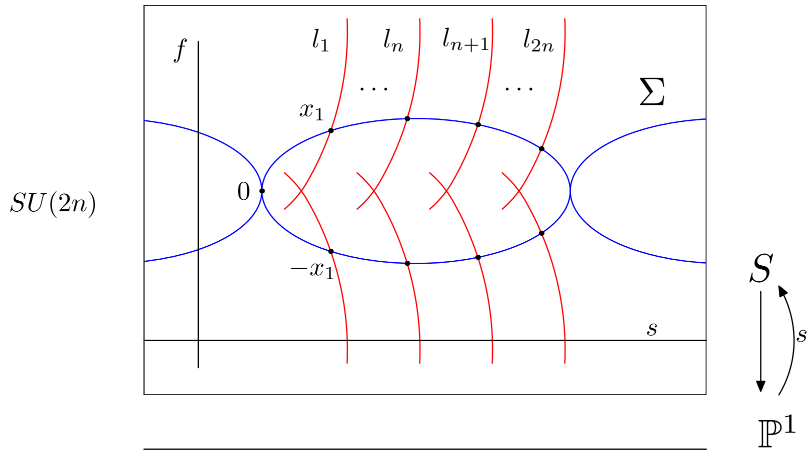

Figure 3. A surface with an A_{2n-1}-configuration $(l_1, \cdots, l_{2n})$.

Given a surface S with an A_{2n-1}-configuration $\zeta = (l_1, \cdots, l_{2n})$, if it satisfies the condition $x_i = -x_{2n+1-i}$ with $x_i = l_i \cap \Sigma$, for $i = 1, \cdots, n$, then ζ is called a C_n-configuration on S (Definition 4.17). If all blown up points are *in general position*, S is called a C_n-surface. See Figure 4 for a surface with a C_n-configuration.

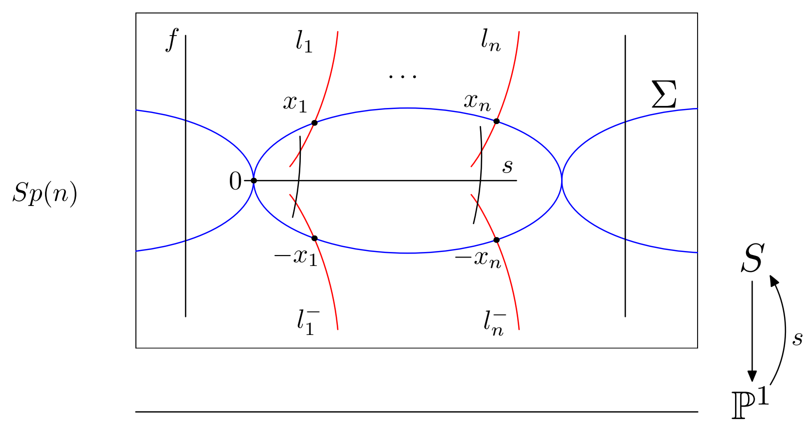

Figure 4. A surface with a C_n-configuration $(l_1, \cdots, l_n, l_n^-, \cdots, l_1^-)$.

1.3 G_2-configurations and D_4-configurations

In these cases we still consider rational surfaces with fibration structure and a fixed smooth anti-canonical curve Σ.

A G_2-configuration comes from a D_4-configuration. We have seen what a D_4-configuration is from Subsection 0.1. Roughly speaking, by saying a rational surface S has a D_4-configuration $(l_1, \cdots, l_4)$, we mean that S can be considered as a blow-up of $\mathbb{F}_1$ at 4 points on $\Sigma \in |-K_{\mathbb{F}_1}|$, such that $l_1, \cdots, l_4$ are the corresponding exceptional classes (Chapter 3). When these blown up points are *in general position*, S is called a $G = D_4$-surface. See Figure 5 for a surface with a D_4-configuration.

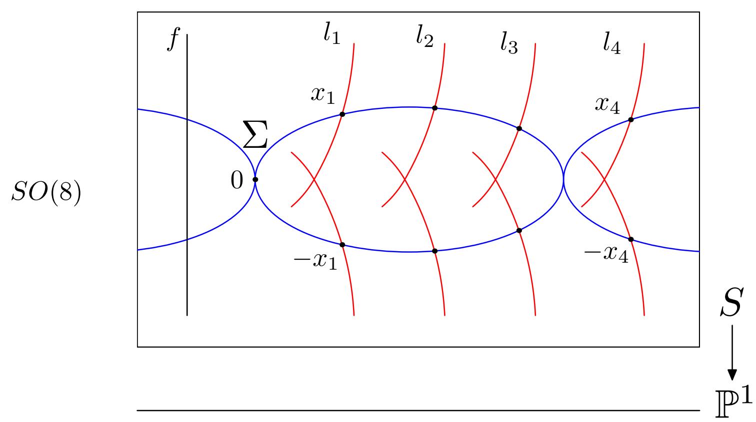

Figure 5. A surface with a D_4-configuration $(l_1, \cdots, l_4)$.

Given a surface S with a D_4-configuration $\zeta = (l_1, \cdots, l_4)$, if it satisfies these two conditions $x_1 = 0$ and $x_4 = x_2 + x_3$, where $x_i = l_i \cap \Sigma$, then ζ is called a G_2-configuration on S (Definition 4.24). If all blown up points but x_1 are *in general position*, S is called a G_2-surface. See Figure 6 for a surface with a G_2-configuration.

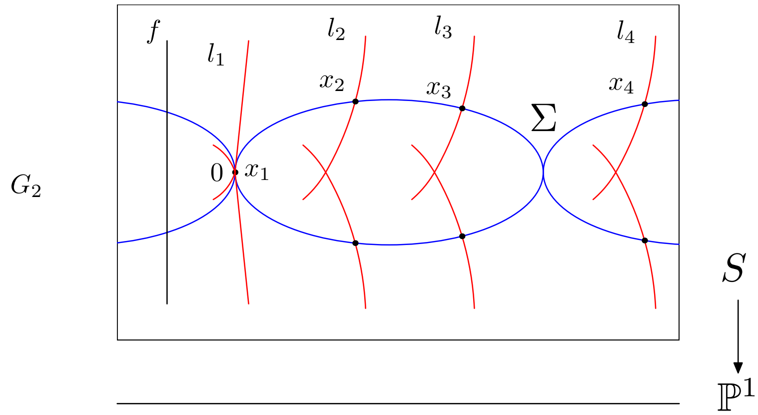

Figure 6. A surface with a G_2-configuration (l_1, l_2, l_3, l_4), where $x_1 = 0$ and $x_4 = x_2 + x_3$ with $x_i = l_i \cap \Sigma$.

1.4 F_4-configurations and E_6-configurations

In these cases we consider rational surfaces which are blow-ups of the projective plane $\mathbb{P}^2$ at 6 points in almost general position, and which contain a fixed smooth anti-canonical curve Σ (Chapter 3).

An F_4-configuration comes from an E_6-configuration. Recall that by saying a rational surface S has an E_6-configuration $(l_1, \cdots, l_6)$, we mean that S can be considered as a blow-up of $\mathbb{P}^2$ at 6 points on $\Sigma \in |-K_{\mathbb{P}^2}|$, such that $l_1, \cdots, l_6$ are the corresponding exceptional classes. When these blown up points are *in general position*, S is called an E_6-surface, which is in fact a cubic surface. See Figure 7 for a surface with an E_6-configuration.

E_6

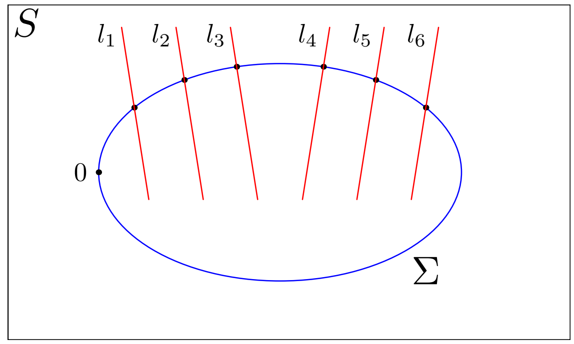

Figure 7. A surface with an E_6-configuration $(l_1, \cdots, l_6)$,

Given a surface S with an E_6-configuration $\zeta = (l_1, \cdots, l_6)$, if it satisfies the condition $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$, where $x_i = l_i \cap \Sigma$, then ζ is called an F_4-configuration on S (Definition 4.32). If all blown up points are *in general position*, S is called an F_4-surface. See Figure 8 for a surface with an F_4-configuration.

F_4

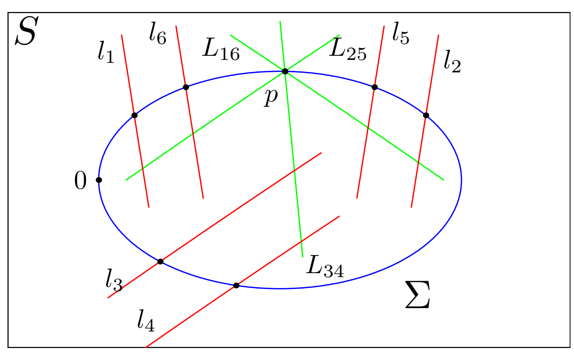

Figure 8. A surface with an F_4-configuration $(l_1, \cdots, l_6)$,
where three lines L_{16}, L_{25}, L_{34} meet at $p \in \Sigma$, or equivalently,
$x_1 + x_6 = x_2 + x_5 = x_3 + x_4$ with $x_i = l_i \cap \Sigma$.

Moreover, we can construct $\mathcal{G}$= Lie(G) bundles over S with a G-configuration. By restriction, we obtain Lie(G) bundles over Σ. And we can also constructed some natural fundamental representation bundles over Σ which have interesting geometric meanings, such that the Lie algebra bundles are the automorphism bundles of these representation bundles preserving certain algebraic structures.

Physically, when $G = E_n$ is a simple subgroup of $E_8 \times E_8$, these G bundles are related to the duality between F-theory and string theory. Among other things, this duality predicts the moduli of flat E_n bundles over a fixed elliptic curve Σ can be identified with the moduli of del Pezzo surfaces with fixed anti-canonical curve Σ. For details, one can consult [8] [9] [13] and [14].

Notation 1.3 In this thesis, we will fix some standard notations from Lie theory. Let G be a compact, simple and simply-connected Lie group. We denote

$r(G)$: the rank of G;

$R(G)$: the root system;

$R_c(G)$: the coroot system;

$W(G)$: the Weyl group;

$\Lambda(G)$: the root lattice;

$\Lambda_c(G)$: the coroot lattice;

$\Lambda_w(G)$: the weight lattice;

$T(G)$: a maximal torus;

$ad(G)$: the adjoint group of G, i.e. $G/C(G)$ where $C(G)$ is the center of G;

$\Delta(G)$: a simple root system of G.

$Out(G)$: the outer automorphism group of G, which is defined as the quotient of the automorphism group of G by its inner automorphism group. It is well-known that $Out(G)$ is isomorphic to the diagram automorphism group of the Dynkin diagram of G.

When there is no confusion, we just ignore the letter G.

Chapter 2

Rational Surfaces and Lie Algebra Bundles

2.1 Rational Surfaces of ADE-type

Before defining what ADE surfaces are, we first give their explicit constructions.

2.1.1 E_n Sublattices

First consider the E_n case, that is, the case of del Pezzo surfaces. We start with a complex projective plane $\mathbb{P}^2$ and n points $x_1, \cdots, x_n$ on $\mathbb{P}^2$ with $n \leq 8$. Note that $x_2, \cdots, x_n$ may be *infinitely near* points. For example, we say that x_2 is *infinitely near* x_1 if x_2 lies on the exceptional curve obtained by blowing up x_1. Blowing up $\mathbb{P}^2$ at these points in turn, we obtain a rational surface, denoted $X_n(x_1, \cdots, x_n)$ or X_n for brevity.

These points are said to be *in general position* if they satisfy the following conditions:

(i) They are distinct points;

(ii) No three of them are collinear;

(iii) No six of them lie on a common conic curve;

(iv) No cubics pass through 8 points with one of them a double point.

The following result is well-known (see [7] and [24]).

Lemma 2.1 *Let $x_i \in \mathbb{P}^2, i = 1, \cdots, n, n \leq 8$. Then the following conditions are equivalent:*

(i) These points are in general position.

(ii) The self-intersection number of any rational curve on X_n is bigger than or equal to -1.

(iii) The anti-canonical class $-K_{X_n}$ is ample. □

A surface X_n is called a *del Pezzo surface* if it satisfies one of the above equivalent conditions.

We say that $x_i \in \mathbb{P}^2, i = 1, \cdots, n$ with $n \leq 8$ are in *almost general position* if any smooth rational curve on X_n has a self-intersection number at least -2, and such a surface is called a *generalized del Pezzo surface* (see [7]).

Let h be the class of lines in $\mathbb{P}^2$ and l_i be the exceptional divisor corresponding to the blow-up at $x_i \in \mathbb{P}^2, i = 1, \cdots, n$. Denote $Pic(X_n)$ the Picard group of X_n, which is isomorphic to $H^2(X_n, \mathbb{Z})$. Then $Pic(X_n)$ is a lattice with basis $h, l_1, \cdots, l_n$, of signature $(1, n)$. Let $K = -3h + l_1 + \cdots + l_n$ be the canonical class. We extend the definition of the (real) Lie algebras $E_n, n = 6, 7, 8$ to all n with $0 \leq n \leq 8$ by setting $E_0 = 0, E_1 = \mathbb{R}, E_2 = A_1 \times \mathbb{R}, E_3 = A_1 \times A_2, E_4 = A_4$ and $E_5 = D_5$.

Denote

$$\begin{aligned} P_n &= \{x \in H^2(X_n, \mathbb{Z}) \mid x \cdot K = 0\}, \\ R_n &= \{x \in H^2(X_n, \mathbb{Z}) \mid x \cdot K = 0,\ x^2 = -2\} \subset P_n, \\ I_n &= \{x \in H^2(X_n, \mathbb{Z}) \mid x^2 = -1 = x \cdot K\},\ \text{and} \\ C_n &= \{\zeta_n = (e_1, \cdots, e_n) \mid e_i \in I_n,\ e_i \cdot e_j = 0,\ i \neq j\}. \end{aligned}$$

An element of I_n is called an *exceptional divisor*, and an element $\zeta_n \in C_n$ is called an *exceptional system* (of divisors) (see [7] and [24]).

Lemma 2.2 *(i) R_n is a root system of type E_n with a system of simple roots $\alpha_1 = l_1 - l_2,\ \alpha_2 = l_2 - l_3,\ \alpha_3 = h - l_1 - l_2 - l_3,\ \alpha_4 = l_3 - l_4,\ \cdots,\ \alpha_n = l_{n-1} - l_n$. Its root lattice is just P_n, and its weight lattice is $Q_n = H^2(X_n, \mathbb{Z})/\mathbb{Z}K$. Let $l \in I_n$, then $R_n \cap l^\perp$ is a root system of type E_{n-1}, and $P_n \cap l^\perp$ is its root lattice.*

(ii) The Weyl group $W(E_n)$ acts on C_n simply transitively.

Proof. (i) For the proof that R_n is a root system of type E_n with given simple roots, see Manin's book [24]. $H^2(X_n, \mathbb{Z})$ is a lattice with $\mathbb{Z}$-basis $h, l_1, \cdots, l_n$. Obviously, $\{e_0 = l_1, e_1 = \alpha_1, \cdots, e_n = \alpha_n\}$ forms another $\mathbb{Z}$-basis. Take any $x \in P_n \subset H^2(X_n, \mathbb{Z})$. Let $x = \sum a_i \cdot e_i$. Then $x \cdot K = 0$ implies $a_0 = 0$. So P_n is the root lattice of R_n.

The natural pairing $P_n \otimes H^2(X_n, \mathbb{Z}) \to \mathbb{Z}$ induces a perfect pairing

$$P_n \otimes (H^2(X_n, \mathbb{Z})/\mathbb{Z}K) \to \mathbb{Z}.$$

So the weight lattice is just $H^2(X_n, \mathbb{Z})/\mathbb{Z}K$.

For the last assertion, we can assume $l = l_8$, then it is true obviously.

(ii) See [24]. $\square$

The Dynkin diagram is the following

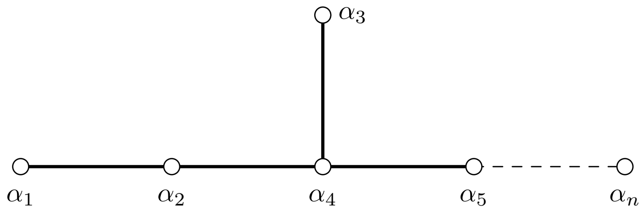

Figure 1. The root system E_n.

2.1.2 D_n Sublattices

Next we consider the D_n case. Let $Y = \mathbb{F}_1$ be a *Hirzebruch surface*, and fix the ruling f and the section s, where $s^2 = -1$. In fact $\mathbb{F}_1$ is the blow-up of $\mathbb{P}^2$ at one point x_0. Thus $f = h - l_0,\ s = l_0$ where h is the class of lines on $\mathbb{P}^2$ and l_0 is the exceptional curve. Blowing up Y at n points $x_1, \cdots, x_n$ we obtain Y_n. The Picard group of Y_n is $H^2(Y_n, \mathbb{Z})$, which is a lattice with basis $s, f, l_1, \cdots, l_n$. The canonical class $K = -(2s + 3f - \sum_{i=1}^{n} l_i)$.

Denote

$$\begin{aligned}
P_n &= \{x \in H^2(Y_n, \mathbb{Z}) \mid x \cdot K = 0 = x \cdot f\}, \\
R_n &= \{x \in H^2(Y_n, \mathbb{Z}) \mid x \cdot K = 0 = x \cdot f,\ x^2 = -2\}, \\
I_n &= \{x \in H^2(Y_n, \mathbb{Z}) \mid x^2 = -1 = x \cdot K,\ x \cdot f = 0\}, \\
C_n &= \{\zeta_n = (e_1, \cdots, e_n) \mid e_i \in I_n, e_i \cdot e_j = 0, i \neq j, \\
&\quad \sum e_i \cdot s \equiv 0 \ mod\ 2\}.
\end{aligned}$$

Similarly as before, an element $\zeta_n \in C_n$ is called an *exceptional system* (of divisors).

Lemma 2.3 *(i) R_n is a root system of type D_n with a system of simple roots $\alpha_1 = f - l_1 - l_2, \alpha_2 = l_1 - l_2, \cdots, \alpha_n = l_{n-1} - l_n$. Its root lattice is just P_n and its weight lattice is $Q_n = H^2(Y_n, \mathbb{Z})/\mathbb{Z}\langle f, K\rangle$.*

(ii) The Weyl group $W(D_n)$ acts on C_n simply transitively.

Proof. (i) $H^2(Y_n, \mathbb{Z})$ is a lattice with $\mathbb{Z}$-basis $s, f, l_i, i = 1, \cdots, n$. Let $x = as + bf + \sum c_i l_i \in R_n$ where $a, b, c_i \in \mathbb{Z}$. Then we have a system of linear equations

$$\begin{cases} x^2 = -2, \\ x \cdot K = 0 = x \cdot f. \end{cases}$$

Solving this, we obtain

$$\begin{cases} a = 0, \\ \sum c_i^2 = 2, \\ 2b = -\sum c_i. \end{cases}$$

So, $x = \pm(l_i - l_j), i \neq j$ or $x = \pm(f - l_i - l_j), i \neq j$. That is $R_n = \{\pm(l_i - l_j), \pm(f - l_i - l_j) |\ i \neq j\}$. This implies that R_n is a root system of D_n-type with indicated simple roots.

Obviously, $\{e_1 = s, e_2 = l_1, e_{i+2} = \alpha_i, i = 1, \cdots, n\}$ forms another $\mathbb{Z}$-basis. Take any $x \in P_n \subset H^2(Y_n, \mathbb{Z})$. Let $x = \sum a_i \cdot e_i$. Then $x \cdot K = 0 = x \cdot f$ implies $a_1 = a_2 = 0$. So P_n is the root lattice of R_n.

The natural pairing $P_n \otimes H^2(Y_n, \mathbb{Z}) \to \mathbb{Z}$ has kernel $\mathbb{Z}\langle f, -2s + \sum l_i \rangle = \mathbb{Z}\langle f, K \rangle$. So the pairing induces a perfect pairing $P_n \otimes (H^2(Y_n, \mathbb{Z})/\mathbb{Z}\langle f, K\rangle) \to \mathbb{Z}$. Hence the weight lattice is just $H^2(Y_n, \mathbb{Z})/\mathbb{Z}\langle f, K\rangle$.

(ii) A simple computation shows that

$$I_n = \{l_i, f - l_i | i = 1, \cdots, n\}.$$

Thus all the elements of C_n are of the form $\zeta_n = (u_1, \cdots, u_n)$ where the number of u_i's, such that $u_i = f - l_k$ for some k, is even. Then by the structure of $W(D_n)$, the result is clear. $\square$

The Dynkin diagram is the following

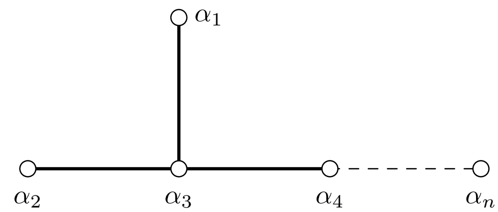

Figure 2. The root system D_n.

2.1.3 A_n Sublattices

In the following we consider the A_{n-1} case. For this, let Z_n be just the same as Y_n.

Denote

$$
\begin{aligned}
P_{n-1} &= \{x \in H^2(Z_n, \mathbb{Z}) \mid x \cdot K = x \cdot f = x \cdot s = 0\},\\
R_{n-1} &= \{x \in H^2(Z_n, \mathbb{Z}) \mid x \cdot K = x \cdot f = x \cdot s = 0,\ x^2 = -2\},\\
I_{n-1} &= \{x \in H^2(Z_n, \mathbb{Z}) \mid x^2 = -1 = x \cdot K,\ x \cdot f = 0 = x \cdot s\},\\
C_{n-1} &= \{\zeta_n = (e_1, \cdots, e_n) \mid e_i \in I_{n-1}, e_i \cdot e_j = 0, i \neq j\}.
\end{aligned}
$$

As before, an element of $\zeta_n \in C_{n-1}$ is called an *exceptional system* (of divisors).

Lemma 2.4 *(i) R_{n-1} is a root system of type A_{n-1} with a system of simple roots $\alpha_1 = l_1 - l_2, \cdots, \alpha_{n-1} = l_{n-1} - l_n$. Its root lattice is just P_{n-1} and its weight lattice is $H^2(Z_n, \mathbb{Z})/\mathbb{Z}\langle f, s, K\rangle$.*

(ii) The Weyl group $W(A_{n-1})$ acts on C_{n-1} simply transitively. In fact, $W(A_{n-1})$ acts as the permutation group of $l_1, \cdots, l_n$.

(iii) Let e be a (-1) curve which does not meet s. Then there exist i, j with $i \neq j$ such that $e = s + f - l_i - l_j$, and when $n \geq 4$, $\langle K, s, f, e\rangle^\perp$ is a reducible root lattice of type $A_1 \times A_{n-3}$; when $n = 3$, $\langle K, s, f, e\rangle^\perp$ is not a root lattice; when $n = 2$, $\langle K, s, f, e\rangle^\perp$ is the same as P_1, which is of type A_1.

(iv) Let $e_i, 1 \leq i \leq k,\ k \geq 2$ be (-1) curves such that $s, e_i, 1 \leq i \leq k$ are disjoint pairwise. Then when $k \neq 3$, $\langle K, s, f, e_i, 1 \leq i \leq k\rangle^\perp$ is not a root lattice.

When $k=3$, (a) if $e_1 = s+f-l_{i_2}-l_{i_3},\ e_2 = s+f-l_{i_1}-l_{i_3},\ e_3 = s+f-l_{i_1}-l_{i_2}$ then $\langle K, s, f, e_1, e_2, e_3\rangle^{\perp}$ is a root lattice of A-type; (b) otherwise, $\langle K, s, f, e_1, e_2, e_3\rangle^{\perp}$ is not a root lattice.

Proof. (i) $H^2(Z_n, \mathbb{Z})$ is a lattice with $\mathbb{Z}$-basis $s, f, l_i, i = 1, \cdots, n$. A simple computation shows that

$$R_{n-1} = \{l_i - l_j \mid i \neq j\}.$$

Then it is obviously a root system of type A_{n-1} with given simple roots.

Obviously, $\{e_1 = s, e_2 = f, e_3 = l_1, e_{i+3} = \alpha_i, i = 1, \cdots, n\}$ forms another $\mathbb{Z}$-basis. Take any $x \in P_{n-1} \subset H^2(Z_n, \mathbb{Z})$. Let $x = \sum a_i \cdot e_i$. Then $x \cdot K = x \cdot f = x \cdot s = 0$ implies $a_1 = a_2 = a_3 = 0$. So P_{n-1} is the root lattice of R_{n-1}.

The natural pairing

$$P_{n-1} \otimes H^2(Z_n, \mathbb{Z}) \to \mathbb{Z}$$

has a kernel

$$\mathbb{Z}\langle f, s, \sum l_i\rangle = \mathbb{Z}\langle f, s, K\rangle.$$

So the pairing induces a perfect pairing

$$P_{n-1} \otimes (H^2(Z_n, \mathbb{Z})/\mathbb{Z}\langle f, s, K\rangle) \to \mathbb{Z}.$$

Hence the weight lattice is just $H^2(Z_n, \mathbb{Z})/\mathbb{Z}\langle f, s, K\rangle$.

(ii) In fact $I_{n-1} = \{l_1, \cdots, l_n\}$. So an element of C_{n-1} is just a permutation of $l_1, \cdots, l_n$.

(iii) Let $e = as + bf + \sum c_i l_i$, then e is a (-1) curve and $e \cdot s = 0$ imply that e must be of the form $s+f-l_i-l_j, i \neq j$. Without loss of generality, we can assume that $e = s + f - l_1 - l_2$. Then the result follows from a simple computation.

(iv) First let $k = 2$. From the proof of (iii), we know both e_1 and e_2 are the form $s + f - l_i - l_j, i \neq j$. Since $e_1 \cdot e_2 = 0$, we can assume $e_1 = s + f - l_1 - l_2$ and $e_2 = s + f - l_1 - l_3$. Then the result follows easily. For

$k = 3$, if $e_1 = s + f - l_{i_2} - l_{i_3}$, $e_2 = s + f - l_{i_1} - l_{i_3}$, $e_3 = s + f - l_{i_1} - l_{i_2}$ then $\langle K, s, f, e_1, e_2, e_3\rangle^\perp = \langle K, s, f, l_{i_1}, l_{i_2}, l_{i_3}\rangle^\perp$. We can assume that $l_{i_1} = l_1$, $l_{i_2} = l_2$, and $l_{i_3} = l_3$. Then $\langle K, s, f, l_1, l_2, l_3\rangle^\perp$ is a root lattice of A-type. Other cases are similar. □

The Dynkin diagram is the following

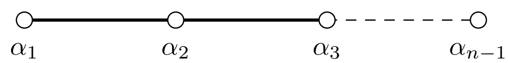

Figure 3. The root system A_{n-1}.

Note that Lemma 2.3 and Lemma 2.4 (i) (ii) are still true if we replace $\mathbb{F}_1$ by any *Hirzebruch surface* $\mathbb{F}_k (k \geq 0)$.

2.1.4 ADE Surfaces

Now we show that in a suitable sense, the converse of the above lemmas is also true. As promised in the introduction, we will see that the following definition generalizes that of *del Pezzo surfaces*.

Definition 2.5 *Let (S, C) be a pair consisting of a smooth rational surface S and a smooth rational curve $C \subset S$ with $C^2 \neq 4$. The pair (S, C) is called of ADE-type (or an ADE surface) if it satisfies the following two conditions:*

(i) Any (smooth) rational curve on S has a self-intersection number at least -1;

(ii) The sub-lattice $\langle K_S, C\rangle^\perp$ of $Pic(S)$ is an irreducible root lattice of rank equal to $rank(Pic(S)) - 2$.

The following proposition shows that such surfaces can be classified into three types.

Proposition 2.6 *Let (S, C) be a rational surface of ADE-type. And let $n = rank(Pic(S)) - 2$. Then $C^2 \in \{-1, 0, 1\}$ and*

(i) when $C^2 = -1$, $\langle K_S, C\rangle^{\perp}$ is of E_n-type, where $4 \leq n \leq 8$;

(ii) when $C^2 = 0$, $\langle K_S, C\rangle^{\perp}$ is of D_n-type, where $n \geq 3$;

(iii) when $C^2 = 1$, $\langle K_S, C\rangle^{\perp}$ is of A_n-type.

Proof. By the first condition in Definition 2.5, $C^2 \geq -1$. Therefore there are the following four cases.

Firstly, suppose $C^2 = -1$. Then we can contract C to obtain a smooth surface $\widetilde{S}$. Let $\pi : S \to \widetilde{S}$ be the blow-down. Then the projection

$$Pic(S) = Pic(\widetilde{S}) \oplus \mathbb{Z}\langle C\rangle \to Pic(\widetilde{S})$$

induces an isomorphism $\langle K_S, C\rangle^{\perp} \cong \langle K_{\widetilde{S}}\rangle^{\perp}$. But the latter is an irreducible root system if and only if $\widetilde{S}$ is a blow-up of $\mathbb{CP}^2$ at $n(4 \leq n \leq 8)$ points. At this time $\langle K_{\widetilde{S}}\rangle^{\perp}$ is a root system of E_n-type. Thus S is a blow-up of $\mathbb{CP}^2$ at $n+1(4 \leq n \leq 8)$ points.

Secondly, suppose $C^2 = 0$. Then by Riemann-Roch theorem, the linear system $|C|$ defines a ruling over $\mathbb{P}^1$ with fiber C. Contract all (-1) curves in fiber, we obtain a relatively minimal model (not unique), which is $\mathbb{P}^1 \times \mathbb{P}^1$ or the Hirzebruch surface $\mathbb{F}_1$. So, S is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at n points. And the lattice $\langle K_S, C\rangle^{\perp}$ must be of D_n-type by Lemma 2.3.

Thirdly, suppose $C^2 = 1$. Then blow up one point $p_0 \in C$, we obtain $\widetilde{S}$ which is a ruling over $\mathbb{P}^1$ with fiber $\widetilde{C} = C - E$ and section E where E is the exceptional curve associated to this blow-up. Contracting all (-1) curves in fiber which do not intersect with E, we will obtain $\mathbb{F}_1$. Thus $\widetilde{S}$ is a blow-up of $\mathbb{F}_1$ at n points. And we have $\langle K_S, C\rangle^{\perp} \cong \langle K_{\widetilde{S}}, \widetilde{C}, E\rangle^{\perp}$. Therefore the lattice is a root lattice of A_n-type by Lemma 2.4.

Finally, suppose $C^2 \geq 2$. Note that since we assume $C^2 \neq 4$, the situation of Lemma 2.4 (iv) (a) can not happen. So we only need to discuss the case where $C^2 = 2$, because the discussion on general cases is similar. Blowing up S at two points $p, q \in C, p \neq q$, we obtain $\widetilde{S}$ with exceptional curves E_p, E_q. Let $\widetilde{C} = C - E_p - E_q$ be the strict transform of C, then $|\widetilde{C}|$ defines a ruling with fiber $\widetilde{C}$ and section $s = E_p$ (fixed). Similarly as before, contracting all (-1) curves E in fiber which satisfy $E \cdot \widetilde{C} = 0 = E \cdot s$, we will obtain $\mathbb{F}_1$. Then $\widetilde{S}$ can be considered as a blow-up of $\mathbb{F}_1$ at n points. Note that $\langle K_S, C\rangle^\perp \cong \langle K_{\widetilde{S}}, \widetilde{C}, s, E_q\rangle^\perp$. We know that $\langle K_{\widetilde{S}}, \widetilde{C}, s\rangle^\perp$ is a root lattice of A_n-type from Lemma 2.4. Then the result follows also from Lemma 2.4. $\square$

Remark 2.7 We extend the definition of E_n surfaces to all n with $0 \leq n \leq 8$, by defining $E_n (n \leq 3)$ surfaces to be del Pezzo surfaces of degree $9 - n$.

Corollary 2.8 *On an ADE surface, any exceptional divisor perpendicular to C is represented by an irreducible curve. Therefore, any exceptional system consists of exceptional curves.*

Proof. In E_n case, the result follows from Proposition 2.6 and Lemma 2.1. In D_n and A_n cases, according to Proposition 2.6, the result is obvious. $\square$

In the following we generalize the definition for $n \leq 8$ points being *in general position* to any $n \geq 0$. Denote $S = \mathbb{P}^2$ (or $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$). Denote $S_n(x_1, \cdots, x_n)$ (or S_n for brevity) the blow-up of S at n points $x_1, \cdots, x_n$. We say that $x_1, \cdots, x_n$ are *in general position* if any smooth rational curve on S_n has a self-intersection number at least -1. And we say that $x_1, \cdots, x_n$ are *in almost general position* if any smooth rational curve on S_n has a self-intersection number at least -2.

Corollary 2.9 *Let (S, C) be an ADE surface.*

(i) In E_n case, blowing down the (-1) curve C, we obtain a del Pezzo surface of degree $9 - n$.

(ii) In D_n case, S is just a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at n points in general position with C as the natural ruling.

(iii) In A_n case, let $\widetilde{S}$ be the blow-up of S at a point on C, with the exceptional curve E, then $\widetilde{S}$ is a blow-up of $\mathbb{F}_1$ at $n+1$ points, and the strict transform $\widetilde{C}$ of C defines a ruling with E as the section of $\mathbb{F}_1$. □

2.2 Lie Algebra Bundles over Rational Surfaces of ADE-type and Their Representation Bundles

When G is of ADE-type, to each ADE surface S, we can construct a natural $\mathcal{G} = Lie(G)$ bundle and natural fundamental representation bundles over S, which are determined by the lines (or exceptional divisors in general) and rulings on S.

Definition 2.10 *By a Lie algebra $\mathcal{G} = Lie(G)$ bundle, we mean a vector bundle which fiberwise carries a Lie algebra structure of $\mathcal{G}$-type, and this Lie algebra structure is compatible with trivialization of this bundle. By a representation bundle of a $\mathcal{G}$ bundle, we mean a vector bundle $\mathcal{V}$ which fiberwise is a representation of $\mathcal{G}$, and the action of $\mathcal{G}$ on $\mathcal{V}$ is compatible with trivialization of them.*

We describe these bundles in the following, and give the detailed arguments just in E_n case.

2.2.1 E_n Bundles over E_n Surfaces

Let (S, C) be an E_n surface. Recall that $S = X_{n+1}(x_1, \cdots, x_{n+1})$ where C be the exceptional divisor associated to the blow-up at x_{n+1}. Denote $\widetilde{S} = X_n(x_1, \cdots, x_n)$. Since $\langle K_S, C\rangle^{\perp} \cong K_{\widetilde{S}}^{\perp}$, we can just consider the surface $\widetilde{S} = X_n(x_1, \cdots, x_n)$.

Since we have a root system of E_n-type attached to X_n, we can construct a

Lie algebra bundle over X_n as follows:

$$\mathscr{E}_n = \mathcal{O}^{\oplus n} \bigoplus_{D \in R_n} \mathcal{O}(D).$$

The fiberwise Lie algebra structure of $\mathscr{E}_n$ is defined as the following. Fix the system of simple roots of R_n as

$$\Delta(E_n) = \{\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, \alpha_3 = h - l_1 - l_2 - l_3, \cdots, \alpha_n = l_{n-1} - l_n\},$$

and take a trivialization of $\mathscr{E}_n$. Then over a trivializing open subset U, $\mathscr{E}_n|_U \cong U \times (\mathbb{C}^{\oplus n} \bigoplus_{\alpha \in R_n} \mathbb{C}_\alpha)$. Take a Chevalley basis $\{x_\alpha^U, \alpha \in R_n; h_i, 1 \leq i \leq n\}$ for $\mathscr{E}_n|_U$ and define the Lie algebra structure by the following four relations, namely, Serre's relations on Chevalley basis (see [16], p147):

(a) $[h_i h_j] = 0, 1 \leq i, j \leq n$.

(b) $[h_i x_\alpha^U] = \langle \alpha, \alpha_i \rangle x_\alpha^U, 1 \leq i \leq n, \alpha \in R_n$.

(c) $[x_\alpha^U x_{-\alpha}^U] = h_\alpha$ is a $\mathbb{Z}$-linear combination of $h_1, \cdots, h_n$.

(d) If α, β are independent roots, and $\beta - r\alpha, \cdots, \beta + q\alpha$ are the α-string through β, then $[x_\alpha^U x_\beta^U] = 0$ if $q = 0$, while $[x_\alpha^U x_\beta^U] = \pm(r+1) x_{\alpha+\beta}^U$ if $\alpha + \beta \in R_n$.

Note that $h_i, 1 \leq i \leq n$ are independent of any trivialization, so the relation (a) is always invariant under different trivializations. If $\mathscr{E}_n|_V \cong V \times (\mathbb{C}^{\oplus n} \bigoplus_{\alpha \in R_n})$ is another trivialization, and f_α^{UV} is the transition function for the line bundle $\mathcal{O}(\alpha)(\alpha \in R_n)$, that is, $x_\alpha^U = f_\alpha^{UV} x_\alpha^V$, then the relation (b) is

$$[h_i (f_\alpha^{UV} x_\alpha^V)] = \langle \alpha, \alpha_i \rangle f_\alpha^{UV} x_\alpha^V,$$

that is,

$$[h_i x_\alpha^V] = \langle \alpha, \alpha_i \rangle x_\alpha^V.$$

So (b) is also invariant. (c) is also invariant since $(f_\alpha^{UV})^{-1}$ is the transition function for $\mathcal{O}(-\alpha)(\alpha \in R_n)$. Finally, (d) is invariant since $f_\alpha^{UV} f_\beta^{UV}$ is the transition function for $\mathcal{O}(\alpha + \beta)(\alpha, \beta \in R_n)$.

Therefore, the Lie algebra structure is compatible with the trivialization. Hence it is well-defined. In other words, we can construct globally a Lie algebra bundle over a surface once we are given a root system consisting of divisors on this surface.

The following relations are intricate. One is the relation between I_n (the set of all exceptional divisors) and the fundamental representation associated to the highest weight λ_n which is dual to the simple root α_n (see Figure 1). Another one is the relation between the set of rulings and the fundamental representation associated to the highest weight λ_1 which is dual to the simple root α_1 (Figure 1). We explain the relations in the following.

Let $\mathbb{L}_n$ be the fundamental representation with the highest weight λ_n. Then we have:

n	1	2	3	4	5	6	7	8
$dim\ \mathbb{L}_n$	1	3	6	10	16	27	56	248
$\lvert I_n \rvert$	1	3	6	10	16	27	56	240

Denotes Ru_n the set of all rulings on X_n. Let $\mathbb{R}_n$ be the fundamental representation with the highest weight λ_1. Then we have:

n	1	2	3	4	5	6	7	8
$dim\ \mathbb{R}_n$	1	2	3	5	10	27	133	3875
$\lvert Ru_n \rvert$	1	2	3	5	10	27	126	2120

Inspired by these, we can construct a fundamental representation bundle $\mathscr{L}_n$ (respectively $\mathscr{R}_n$) using the exceptional divisors (respectively the rulings) on X_n as follows.

$$\mathscr{L}_n = \bigoplus_{l \in I_n} \mathcal{O}(l) \text{ when } n \leq 7,$$
$$\mathscr{L}_8 = \bigoplus_{l \in I_8} \mathcal{O}(l) \oplus \mathcal{O}(-K)^{\oplus 8}.$$

Respectively,

$$\mathscr{R}_n = \bigoplus_{R \in Ru_n} \mathcal{O}(R) \text{ when } n \leq 6,$$
$$\mathscr{R}_7 = \bigoplus_{R \in Ru_7} \mathcal{O}(R) \oplus \mathcal{O}(-K)^{\oplus 7}.$$

The fiberwise action is defined naturally, which is in fact compatible with any trivialization.

For example we consider the bundle $\mathscr{L}_n$ and suppose $n \leq 7$. Take U, V as before, and suppose they also trivialize $\mathscr{L}_n$, that is $\mathscr{L}_n|_U \cong U \times (\bigoplus_{l \in I_n} \mathbb{C}_l)$ and $\mathscr{L}_n|_V \cong V \times (\bigoplus_{l \in I_n} \mathbb{C}_l)$. Take e_l^U (resp. $e_l^V = g^{VU} e_l^U$) to be the basis of $\mathbb{C}_l$ over U (resp. V). Then define $x_\alpha^U . e_l^U$ to be equal to $e_{l'}^U$ if $l' = \alpha + l \in I_n$ and be equal to 0 otherwise. And define $h_\alpha . e_l^U = (\alpha \cdot l) e_l^U$.

Note that the situation here is slightly different from some standard usage, for example [3] [16], since the self-intersection number of an element of R_n or I_n is negative. But this does not matter if we take the simple root system to be $\{-\alpha_1, \cdots, -\alpha_n\}$, and take the pairing to be $(x, y) := -(x \cdot y)$. Firstly since $\lambda_n(-\alpha_i) = (-\alpha_i, l_n) = \alpha_i \cdot l_n = \delta_{in}$, we have $\lambda_n \cong (\cdot, l_n)$. Secondly the action is irreducible since the Weyl group acts on I_n transitively. Lastly $e_{l_n}^U$ is the maximal vector of weight λ_n. Therefore this fiberwise action does define the highest weight module with the highest weight λ_n (see [16]).

Obviously, this fiberwise Lie algebra action is compatible with the trivialization.

For $\mathscr{L}_8$, note that the bijection $I_8 \to R_8$ given by $l \mapsto l + K$ induces an isomorphism

$$\mathscr{E}_8 \cong \mathscr{L}_8 \otimes \mathcal{O}(K).$$

This implies $\mathscr{L}_8$ is just the adjoint representation bundle.

Similarly, $\mathscr{R}_n$ is the fundamental representation bundle with the highest weight $\lambda_1 \cong (\cdot, h - l_1)$ and the maximal vector $e^U_{h-l_1}$, where the simple root system and the pairing are defined as above. We also have that $\mathscr{R}_7 \otimes \mathcal{O}(K) \cong \mathscr{E}_7$ is the adjoint representation bundle.

Example 2.11 Let us look at the $sl(2)$ sub-bundle

$$\mathcal{O} \oplus \mathcal{O}(\alpha) \oplus (-\alpha),$$

where $\alpha = l_1 - l_2$. Then the bundle $\mathcal{O}(l_1) \oplus \mathcal{O}(l_2)$ is the standard representation bundle. And the line bundle $\mathcal{O}(h - l_1 - l_2)$ is a trivial representation.

In fact, the Lie algebra bundle $\mathscr{E}_n$ is uniquely determined by its representation bundles $\mathscr{L}_n$ and $\mathscr{R}_n$, according to [1]. Concretely (see [20] for more details),

(i) $\mathscr{E}_4$ is the automorphism bundle of $\mathscr{R}_4$ preserving $\wedge^5 \mathscr{R}_4 \cong \mathcal{O}(-2K)$.

(ii) $\mathscr{E}_5$ is the automorphism bundle of $\mathscr{R}_5$ preserving $q_5 : \ \mathscr{R}_5 \otimes \mathscr{R}_5 \to \mathcal{O}(-K)$, where q_5 is defined by $\mathcal{O}(R') \otimes \mathcal{O}(R'') \to \mathcal{O}(-K)$ if $R' + R'' = -K$, and 0 otherwise.

(iii) $\mathscr{E}_6$ is the automorphism bundle of $\mathscr{R}_6$ and $\mathscr{L}_6$ preserving

$$\begin{cases} c_6 : \ \mathscr{L}_6 \otimes \mathscr{L}_6 \to \mathscr{R}_6, \ \textit{and} \\ c_6^* : \ \mathscr{R}_6 \otimes \mathscr{R}_6 \to \mathscr{L}_6 \otimes \mathcal{O}(-K), \end{cases}$$

where c_6 is defined by the map $(l_i, \ l_j) \mapsto 2h - \sum\limits_{k \neq i,j} l_k$ and c_6^* is defined by the map $(h - l_i, h - l_j) \mapsto h - l_i - l_j$.

(iv) $\mathscr{E}_7$ is the automorphism bundle of $\mathscr{L}_7$ preserving

$$f_7 : \ \mathscr{L}_7 \otimes \mathscr{L}_7 \otimes \mathscr{L}_7 \otimes \mathscr{L}_7 \to \mathcal{O}(-2K),$$

where f_7 is defined by the map $(C_1, C_2, C_3, C_4) \mapsto -2K$ if $C_1 + C_2 + C_3 + C_4 = -2K$, 0 otherwise.

(v) $\mathscr{E}_8$ is the automorphism bundle of $\mathscr{L}_8$ preserving

$$\mathscr{L}_8 \wedge \mathscr{L}_8 \to \mathscr{L}_8 \otimes \mathcal{O}(-K).$$

For X_6, the bijection $Ru_6 \to I_6$ defined by $R \mapsto -(R+K)$ induces an isomorphism $\mathscr{R}_6 \cong \mathscr{L}_6^* \otimes \mathcal{O}(-K)$, which is consistent with the duality between $\mathbb{L}_6$ and $\mathbb{R}_6$ for the Lie group E_6.

2.2.2 D_n Bundles over Rational Ruled Surfaces

Let (S, C) be a D_n surface. By Proposition 2.6, S dominates $\mathbb{F}_1$ or $\mathbb{F}_0(= \mathbb{P}^1 \times \mathbb{P}^1)$ with ruling C. We can suppose that S dominates $\mathbb{F}_1$ since for another case the arguments is the same. Thus $S = Y_n(x_1, \cdots, x_n)$ is the blow-up of $\mathbb{F}_1$ at n points $x_i, i = 1, \cdots, n$, where for any i, x_i does not lie on the section s.

Since R_n is a root system of type D_n, the Lie algebra bundle can be constructed as follows.

$$\mathscr{D}_n = \mathcal{O}^{\oplus n} \bigoplus_{D \in R_n} \mathcal{O}(D).$$

Recall that in D_n case,

$$\begin{aligned} I_n &= \{C \mid C^2 = C \cdot K = -1, C \cdot f = 0\} \\ &= \{l_i, f - l_i \mid i = 1, \cdots, n\}. \end{aligned}$$

The fundamental representation with the highest weight λ_n, where λ_n is the fundamental weight corresponding to $\alpha_n = l_{n-1} - l_n$, is

$$\mathscr{W}_n = \bigoplus_{C \in I_n} \mathcal{O}(C).$$

In fact, $\mathscr{W}_n$ is the standard representation bundle of $\mathscr{D}_n$.

Note that there are n singular fibers, and each singular fiber is of the form $l_i + l_i'$ where $l_i' = f - l_i, i = 1, \cdots, n$. The relation

$$\mathcal{O}(l_i) \otimes \mathcal{O}(l_i') = \mathcal{O}(f)$$

implies we can define a non-degenerated fiberwise quadratic form

$$q_n : \mathscr{W}_n \otimes \mathscr{W}_n \to \mathcal{O}(f).$$

The two spinor bundles are defined as

$$\mathcal{S}_n^+ = \bigoplus_{S^2 = S\cdot K = -1, S\cdot f = 1} \mathcal{O}(S) \text{ and } \mathcal{S}_n^- = \bigoplus_{T^2 = -2, T\cdot K = 0, T\cdot f = 1} \mathcal{O}(T).$$

Moreover, there are all kinds of structures on these representation bundles, for example, the Clifford multiplication:

$$\mathcal{S}_n^+ \otimes \mathscr{W}_n^* \to \mathcal{S}_n^- \text{ and } \mathcal{S}_n^- \otimes \mathscr{W}_n \to \mathcal{S}_n^+.$$

When $n = 2m - 1$ is odd, we have isomorphism

$$(\mathcal{S}_n^+)^* \otimes \mathcal{O}_{Y_n}((m-4)f - K) \cong \mathcal{S}_n^-.$$

When $n = 2m$ is even, we have isomorphisms

$$\begin{aligned} (\mathcal{S}_n^+)^* \otimes \mathcal{O}_{Y_n}((m-3)f - K) &\cong \mathcal{S}_n^+, \\ (\mathcal{S}_n^-)^* \otimes \mathcal{O}_{Y_n}((m-4)f - K) &\cong \mathcal{S}_n^-. \end{aligned}$$

For more details, see [20].

2.2.3 A_{n-1} Bundles and Their Representation Bundles

Let S be an A_{n-1} surface. By Proposition 2.6, we can assume that $S = Z_n(x_1, \cdots, x_n)$ be the blow-up of $\mathbb{F}_1$ at n points $x_i, i = 1, \cdots, n$, where for any i, x_i does not lie on the section s. Recall that

$$R_{n-1} = \{l_i - l_j |\ i \neq j\} \text{ and}$$
$$I_{n-1} = \{l_1, \cdots, l_n\}.$$

Since R_{n-1} is a root system of A_{n-1}-type, the Lie algebra bundle can be constructed as

$$\mathscr{A}_{n-1} = \mathcal{O}^{\oplus n-1} \bigoplus_{D \in R_{n-1}} \mathcal{O}(D).$$

And the standard representation bundle is

$$\mathcal{V}_{n-1} = \bigoplus_{C \in I_{n-1}} \mathcal{O}(C) = \bigoplus_{i=1}^{n} \mathcal{O}(l_i).$$

The k^{th} fundamental representation bundle is just

$$\wedge^k(\mathcal{V}_{n-1}) \cong \bigoplus_{i_1 < \cdots < i_k} \mathcal{O}(l_{i_1} + \cdots + l_{i_k}).$$

We also have $\mathscr{A}_{n-1} = \mathcal{E}nd_0(\mathcal{V}_{n-1})$.

We summarize the content of this section as the following form.

Conclusion 2.12 *For every ADE surface S, there is a natural Lie algebra bundle of corresponding ADE-type over S. Furthermore, we can construct two natural fundamental representation bundles over S, using lines and rulings on S. Moreover, the Lie algebra bundle can be considered as the automorphism (Lie algebra) bundle of these fundamental representation bundles preserving natural structures.* $\square$

Chapter 3

Flat G-bundles over Elliptic Curves and Rational Surfaces: Simply Laced Cases

In the following, we study the relation between ADE surfaces and flat G bundles over elliptic curves, where G is compact, simple and simply connected Lie group of ADE-type (the type is the same as that of the corresponding surface).

3.1 Flat G-bundles over Elliptic Curves

In this section we review some well-known results about flat G bundles over elliptic curves.

Let Σ be an elliptic curve with identity element 0. The fundamental group $\pi_1(\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$. Let G be a compact, simple and simply connected Lie group of rank r with root system R, coroot system R_c, Weyl group $W(G)$, root lattice Λ, coroot lattice Λ_c and maximal torus T. The dual lattice $\Lambda_c^\vee$ of Λ_c is the weight lattice. We denote the moduli space of flat G-bundles over Σ by $\mathcal{M}_\Sigma^G$. It is well-known that we have the following isomorphisms.

$$\begin{aligned}\mathcal{M}_\Sigma^G &\cong Hom(\pi_1(\Sigma), G)/ad(G)\\ &\cong Hom(\pi_1(\Sigma), T)/W\\ &\cong T \times T/W\\ &\cong \Sigma \otimes_{\mathbb{Z}} \Lambda_c/W.\end{aligned}$$

The second isomorphism is because of Borel's theorem [5] which says that a commuting pair of elements in G can be diagonalized simultaneously. The last isomorphism comes from

$$Hom(\pi_1(\Sigma), T) = Hom(\pi_1(\Sigma), U(1) \otimes_{\mathbb{Z}} \Lambda_c) \cong Hom(\pi_1(\Sigma), U(1)) \otimes_{\mathbb{Z}} \Lambda_c$$

and

$$Hom(\pi_1(\Sigma), U(1)) \cong Pic^0(\Sigma) \cong \Sigma.$$

A famous theorem [21][22] of Looijenga's says that

$$\Sigma \otimes_{\mathbb{Z}} \Lambda_c/W \cong \mathbb{WP}^r_{s_0=1,s_1,\cdots,s_r},$$

where the latter is the weighted projective space with weights s_i's, and $s_1, \cdots, s_r$ are the coefficients of the highest coroot of R_c.

One element of $Hom(\Lambda, \Sigma)/W$ can only determine a flat $ad(G) = G/C(G)$ bundle in general. For the adjoint group $ad(G)$, the moduli space of flat $ad(G)$ bundles $\mathcal{M}_\Sigma^{ad(G)}$ contains $Hom(\Lambda, \Sigma)/W$ as a connected component (see [13]). On the other hand, we have the following short exact sequences:

$$0 \to \Lambda \to \Lambda_c^\vee \to \Gamma \to 0$$

and

$$0 \to Hom(\Gamma, \Sigma) \to Hom(\Lambda_c^\vee, \Sigma) \to Hom(\Lambda, \Sigma) \to 0.$$

Here Γ is a finite abelian group. The second sequence is exact since Σ is a divisible abelian group. It follows that $Hom(\Lambda, \Sigma)$ and $\Sigma \otimes_{\mathbb{Z}} \Lambda_c$ are isogenous as abelian varieties. Let d be the exponent of the finite group Γ. If we fix a d^{th} root of $\Sigma \cong Jac(\Sigma)$ then we can extend uniquely a homomorphism $f_0 \in Hom(\Lambda, \Sigma)$ to a homomorphism $f \in Hom(\Lambda_c^\vee, \Sigma)$. Hence when we fix a d^{th} root of Σ, we obtain the following isomorphism

$$\mathcal{M}_\Sigma^G \cong Hom(\Lambda, \Sigma)/W.$$

When G is of ADE-type, the root lattice and the coroot lattice coincide, hence the weight lattice is just the dual lattice of the root lattice.

Remark 3.1 We have constructed ADE (Lie algebra) bundles over ADE rational surfaces. For such a bundle, taking the compact form of its automorphism bundle, we obtain the adjoint Lie group bundle P. When the surface S has a smooth anti-canonical curve Σ, restricting P to Σ (fixing the identity element $0 \in E$), we shall obtain a flat $ad(G)$ bundle of ADE-type over Σ. We can also first restrict the Lie algebra bundles to Σ, and then take the compact form. We still obtain the same flat $ad(G)$ bundle over Σ. To obtain a simple Lie group E_n (resp. D_n), we need to assume that $4 \leq n \leq 8$ (resp. $n \geq 3$).

3.2 The Identification of Moduli Spaces in ADE Cases

From this section on, we fix our ADE surface S to be the rational surface $X_n(x_1, \cdots, x_n)$, $Y_n(x_1, \cdots, x_n)$, or $Z_n(x_1, \cdots, x_n)$. For X_n, we assume $n \leq 8$.

In last section, we saw that once we are given a root system of type E_n (respectively $D_n,\ A_{n-1}$) in the Picard lattice of X_n (respectively $Y_n,\ Z_n$), we can construct a Lie algebra bundle of that type and its natural fundamental representation bundles over this surface. Furthermore, we can construct an adjoint compact Lie group bundle over this surface. To obtain the corresponding Lie group bundle over the fixed elliptic curve Σ by restriction, we need to assume that Σ is an anti-canonical curve of our rational surfaces. That is, we first embed Σ into $\mathbb{P}^2$ as an anti-canonical curve, using the projective embedding ϕ determined by the linear system $|3(0)|$ where (0) is the divisor of the identity element of Σ, and assume that all these blown up points $x_i \in \Sigma$ for $i = 1, \cdots, n$, and that $0, x_1, \cdots, x_n$ are in general position. Moreover, we blow up $\mathbb{P}^2$ at 0 to obtain the embedding of Σ into $\mathbb{F}_1$ as an anti-canonical curve, and take the exceptional curve l_0 as the section s for the ruled surface $\mathbb{F}_1$.

Convention 3.2 *In Z_n case, it is well-known that in order to obtain a flat $SU(n)$-bundle over Σ we need one more assumption:*

$$\sum x_i = 0 \ in \ \Sigma.$$

We explain how the moduli space $\mathcal{M}_\Sigma^G$ is related to the moduli space of rational surfaces of the above types. Denote $\mathcal{S}(\Sigma, G)$ the moduli space of the pairs (S, Σ), where S is an ADE rational surface of type the same as that of G and $\Sigma \in |-K_S|$.

Proposition 3.3 *There exists a well-defined map*

$$\phi : \ \mathcal{S}(\Sigma, G) \to Hom(\Lambda, \Sigma)/W,$$

where Λ is the lattice P_n or P_{n-1} defined in Section 1.

Proof. First we consider the case where $S = X_n$ is a Del Pezzo surface, that is, all blown up points are in general position. Suppose we are given the pair $(X_n, \Sigma \in | -K_{X_n}| \ )$. For each element $y \in P_n$, y stands for a holomorphic line bundle over S. Restricting y to Σ, we obtain a holomorphic line bundle over Σ, denoted by $\mathcal{L}_y$. The degree of $\mathcal{L}_y$ is

$$deg(\mathcal{L}_y) = y \cdot (-K) = 0.$$

So $\mathcal{L}_y$ is an element of the Jacobian of Σ, which is canonically isomorphic to Σ since the identity element of Σ is given. Thus we obtain a map from P_n to

$\Sigma : y \mapsto \mathcal{L}_y$, which is obviously a homomorphism of abelian groups. But for one pair (X_n, Σ), we can have different choices of simple roots in order to identify P_n with the root lattice of E_n, and all choices are only differed by the action of the Weyl group $W(E_n)$. So finally we obtain a well-defined map from the moduli space $\mathcal{S}(\Sigma, E_n)$ of such pairs (X_n, Σ) to the projective variety $Hom(P_n, \Sigma)/W(E_n)$.

The other two cases are similar. Roughly speaking, given a pair (Y_n, Σ) (resp. (Z_n, Σ)), we obtain an element in

$$Hom(P_n, \Sigma)/W(D_n) \text{ (resp. } Hom(P_{n-1}, \Sigma)/W(A_{n-1})). \qquad \square$$

In fact we can prove a theorem of Torelli type for the above correspondings. Roughly speaking, the moduli space of the pairs (S, Σ) is isomorphic to

$$Hom(\Lambda, \Sigma)/W,$$

where Λ is our root lattice.

Definition 3.4 *Let $S = X_n, Y_n,$ or Z_n. An exceptional system $\zeta_n = (e_1, \cdots, e_n) \in C_n$ on X_n (resp. Y_n, Z_n) is called a G-configuration for $G = E_n$ (resp. D_n, A_{n-1}) if e_n is a (-1) curve, and after blowing down e_n, e_{n-1} is a (-1) curve. And this process can be proceeded successively until after blowing down e_1, we obtain $\mathbb{P}^2$ (resp. $\mathbb{F}_1$) for $G = E_n$ (resp. D_n and A_{n-1}). Denote ζ_G a G-configuration. When S is equipped with a G-configuration ζ_G, and S has Σ as an anti-canonical curve, we call S a rational surface with G-configuration and denote it by a pair (S, G).*

Equivalently, a G-configuration ζ_{E_n} (resp. ζ_{D_n} or $\zeta_{A_{n-1}}$) on $S = X_n$ (resp. Y_n, Z_n), means that S could be considered as the blow-up of $\mathbb{P}^2$ (resp. $\mathbb{F}_1$, $\mathbb{F}_1$) at n (maybe not distinct) points $y_1, \cdots, y_n \in S$ successively, such that $e_1, \cdots, e_n$ are the corresponding exceptional divisors.

Lemma 3.5 *Let S be a surface with G-configuration. Then any smooth rational curve on S has a self-intersection number at least -2. Furthermore, in E_n case, all these (-2) curves form chains of ADE-type.*

Proof. Let L be a smooth rational curve on S. Then $L \cdot \Sigma \geq 0$. By adjoint formula, we have $-2 = L^2 + L \cdot K_S$. Since Σ is linearly equivalent to $-K_S$, we have $L^2 \geq -2$. For the last assertion, see [7]. □

On an ADE surface, by Corollary 2.8, any exceptional system is an ADE-configuration. Thus, we can restate the result of Lemma 2.2 (ii), Lemma 2.3 (ii) and Lemma 2.4 (ii) as follows.

Proposition 3.6 *For an ADE surface, $W(G)$ acts on the set of all G-configurations simply transitively.* □

This proposition implies that a G-configuration determines exactly an isomorphism from P_n (or P_{n-1} for A_{n-1}) to the corresponding root lattice $\Lambda(G)$.

An A_{n-1}-configuration on Z_n is illustrated in the following figure

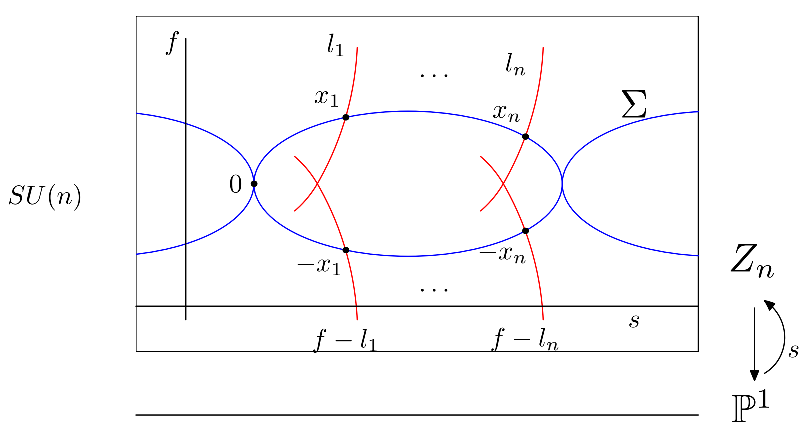

Figure 4. A surface with an A_{n-1}-configuration $(l_1, \cdots, l_n)$.

A D_n-configuration on Y_n is illustrated in the following figure

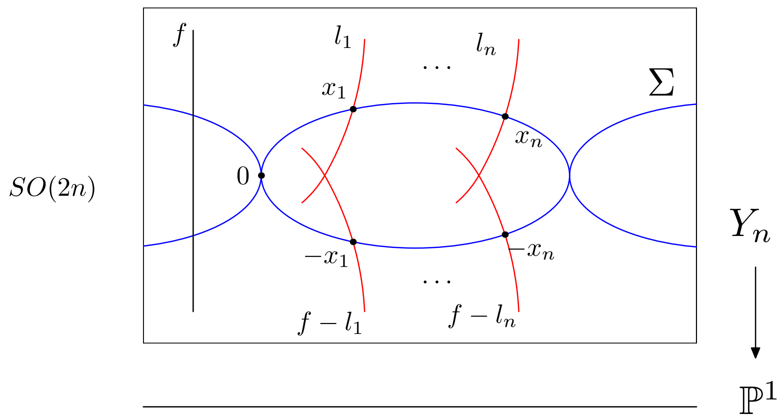

Figure 5. A surface with a D_n-configuration $(l_1, \cdots, l_n)$.

And an E_n-configuration on X_n is illustrated in the following figure

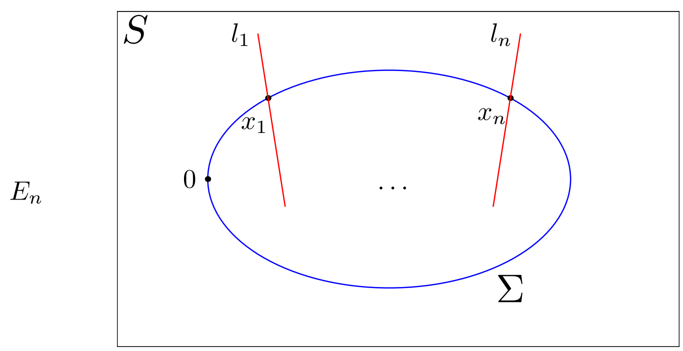

Figure 6. A surface with an E_n-configuration $(l_1, \cdots, l_n)$,

Recall the definition $\zeta_{D_n} = (e_1, \cdots, e_n)$ where $e_i \cdot K_{Y_n} = -1,\ e_i \cdot f = 0,\ e_i \cdot e_j = \delta_{ij}$ and $\sum e_i \cdot s \equiv 0 \ mod\ 2$. Next we explain geometrically why we need to assume that $\sum e_i \cdot s \equiv 0\ mod\ 2$.

Definition 3.7 *Let $C \subset \mathbb{P}^2$ be a curve of degree d. A point $P \in C$ is called a ordinary k-$fold$ point of C if P is a k-fold singular point and C has k distinct tangent directions at P.*

Lemma 3.8 *Let C be a plane curve of degree d with an ordinary $(d-1)$-fold point P. Then*

(i) P is the only singular point of C.

(ii) The normalization of C is a smooth rational curve.

(iii) Fix a point $P \in \mathbb{P}^2$. Then the variety of all plane curves of degree d with P as an ordinary $(d-1)$-fold point is of dimension $2d$.

(iv) Given P and other $2d$ generic points, there exists a unique curve $C \subset \mathbb{P}^2$ of degree d, such that C has P as an ordinary $(d-1)$-fold point and passes through these $2d$ generic points.

Proof. (i) Apply Bezout's theorem. (ii) Apply the genus formula. (iii) Let $[x, y, z]$ be the homogenous coordinates of $\mathbb{P}^2$, and $P = [1, 0, 0]$. Then C is defined by the polynomial

$$f(x, y, z) = g(y, z) + \prod_{i=1}^{d-1}(a_i y - b_i z)x,$$

where $deg(g) = d$. Therefore, the dimension is $2d$. □

Proposition 3.9 *Let Σ be embedded into $\mathbb{F}_1$ (with section s) as a smooth anti-canonical curve and $x_1, \cdots, x_n$ are distinct points of Σ. Blowing up $\mathbb{F}_1$ at x_i's we obtain Y_n with corresponding exceptional curves $l_i, i = 1, \cdots, n$.*

(i) When $n = 2k$, if $x_1, \cdots, x_n$ are in general position, then after contracting $f - l_1, \cdots, f - l_n$, we still obtain the surface $\mathbb{F}_1$. In other words, we obtain the same surface Y_{2k} by blowing up either $\{x_1, \cdots, x_n\}$, or $\{-x_1, \cdots, -x_n\}$.

(ii) When $n = 2k + 1$, if $x_1, \cdots, x_n$ are in general position, then after contracting $f - l_1, \cdots, f - l_n$, the resulting surface is $\mathbb{P}^1 \times \mathbb{P}^1$, but not $\mathbb{F}_1$.

Proof. Let C be a negative rational curve in Y_n which doesn't intersect $f - l_i, i = 1, \cdots, n$. Then C satisfies the following equations

$$\begin{cases} C \cdot C = -m, m > 0; \\ C \cdot K = m - 2; \\ C \cdot (f - l_i) = 0, i = 1, \cdots, n. \end{cases}$$

Since C is a rational curve and $\Sigma \in |-K|$, $C \cdot (-K) \geq 0$. So $m \leq 2$. Then $m = 1$ or 2. Considering $\mathbb{F}_1$ as the blow-up of $\mathbb{P}^2$ at $0 \in \Sigma$ with exceptional curve s, we can assume $C = a \cdot h - b \cdot s - \sum c_i \cdot l_i, a \geq 0, b \geq 0, c_i \geq 0$. Solving the system of equations, we obtain

$$\begin{cases} m = 1 \text{ or } 2, \\ b = a - 1, \\ c_i = 1, i = 1, \cdots, n, \\ a = (n - 1 + m)/2. \end{cases}$$

For $m = 1$, $n = 2a$ is even. The class

$$C = ah - (a-1)s - \sum_{i=1}^{n=2a} l_i = af + s - \sum_{i=1}^{2a} l_i.$$

This means that all of the points $0, x_1, \cdots, x_n$ lie on the curve $\pi(C)$, where $\pi : Y_n \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ successively at $0, x_1, \cdots, x_n$. There exists exactly one such curve C for generic $x_1, \cdots, x_n$, and it is smooth, by Lemma 3.8. Hence, after contracting $f - l_1, \cdots, f - l_{2a}$, we still obtain $\mathbb{F}_1$.

For $m = 2$, $n = 2a + 1$ is odd. The class

$$C = ah - (a-1)s - \sum_{i=1}^{n=2a+1} l_i = af + s - \sum_{i=1}^{2a+1} l_i.$$

This means that all of the points $0, x_1, \cdots, x_n$ lie on the curve $\pi(C)$, where $\pi : Y_n \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ successively at $0, x_1, \cdots, x_n$. There exists no such curves for generic $x_1, \cdots, x_n$, by Lemma 3.8. Hence, after contracting $f - l_1, \cdots, f - l_{2a+1}$, no rational curves with negative self-intersection number can survive. Therefore the resulting surface is $\mathbb{P}^1 \times \mathbb{P}^1$, but not $\mathbb{F}_1$. $\square$

Example 3.10 Blowing up $\mathbb{F}_1$ at 2 points x_1, x_2 we obtain Y_2. Contracting $f - l_1$ and $f - l_2$, or contracting l_1 and l_2, we always obtain the surface $\mathbb{F}_1$. But contracting $f - l_1$ and l_2, we just obtain the surface $\mathbb{P}^1 \times \mathbb{P}^1$, not $\mathbb{F}_1$!

Remark 3.11 (i) Lemma 3.8 has a corresponding version for $\mathbb{P}^1 \times \mathbb{P}^1$.

(ii) A G-configuration $\zeta_G = (e_1, \cdots, e_n)$ for $S = X_n$ (resp. Y_n, Z_n) just means that after blowing down $e_n, e_{n-1}, \cdots, e_1$ successively, we still obtain $\mathbb{P}^2$ (resp. $\mathbb{F}_1$, $\mathbb{F}_1$).

Let S be an ADE surface equipped with a G-configuration ζ_G. we denote the moduli space of the pairs (S, Σ) by $\mathcal{S}(\Sigma, G)$, where two pairs (S, Σ) and (S', Σ') are equivalent if and only if there is an isomorphism π from S to S' such that $\pi|_\Sigma$ is also an isomorphism from Σ to Σ'.

We show that $\mathcal{S}(\Sigma, G)$ is isomorphic to an open dense subset U of the variety $Hom(\Lambda, \Sigma)/W$. In fact, for any element $\theta \in (Hom(\Lambda, \Sigma)/W)\backslash U$, the boundary component, we can find possibly non-equivalent pairs (S, Σ) such that θ comes from the restriction. Thus, we can complete $\mathcal{S}(\Sigma, G)$ by adding these pairs and identifying them as one point. Denote the completion by $\overline{\mathcal{S}(\Sigma, G)}$. Then we can identify $\overline{\mathcal{S}(\Sigma, G)}$ with the projective variety $Hom(\Lambda, \Sigma)/W$. This provides a natural compactification for the moduli space $\mathcal{S}(\Sigma, G)$.

More precisely, let $S = X_n$ (respectively, Y_n, Z_n) be an ADE surface and Λ be the root lattice of E_n (respectively, D_n, A_{n-1}) with corresponding Weyl group W. And we fix a 3^{rd} (respectively, 2^{nd}, n^{th}) root of Σ in E_n (respectively, D_n, A_{n-1}) case. Then we have

Theorem 3.12 *(i) There is an injective map ϕ from the moduli space $\mathcal{S}(\Sigma, G)$ onto an open dense subset of $Hom(\Lambda, \Sigma)/W$.*

(ii) ϕ can be extended to a bijective map from the completion $\overline{\mathcal{S}(\Sigma, G)}$ onto $Hom(\Lambda, \Sigma)/W$.

(iii) Moreover, the completion is obtained by including all rational surfaces with G-configurations to $\mathcal{S}(\Sigma, G)$. Any smooth rational curve on a surface corresponding to a boundary point has a self-intersection number at least -2, and in E_n case these (-2) curves form chains of ADE-type.

Proof. First we suppose $S = X_n$. We have constructed the map ϕ in Proposition 3.3. We prove the injectivity. Fix a G-configuration $\zeta_G = (l_1, \cdots, l_n)$ on X_n, and a simple root system $\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, \alpha_3 = h - l_1 - l_2 - l_3, \alpha_4 = l_3 - l_4, \cdots, \alpha_n = l_{n-1} - l_n$. Blowing down $l_n, l_{n-1}, \cdots, l_1$ successively, we obtain $\mathbb{P}^2$ with Σ as an anti-canonical curve. For all $i = 1, \cdots, n$, let $x_i \in X_n$ be the unique intersection points of l_i and Σ. Then X_n can be considered as a blow-up of $\mathbb{P}^2$ at these n points $x_i \in \Sigma, i = 1, \cdots, n$ with exceptional curves $l_i, i = 1, \cdots, n$.

According to previous arguments, we have a homomorphism $g \in Hom(\Lambda, \Sigma)$. Let $g(\alpha_i) = p_i \in \Sigma$, then we have the following equations by the group law of Σ as an abelian group

$$\begin{cases} x_1 - x_2 = p_1, \\ x_2 - x_3 = p_2, \\ -x_1 - x_2 - x_3 = p_3, \\ x_{k-1} - x_k = p_k, k = 4, \cdots, n. \end{cases}$$

The determinant of the coefficient matrix of this system of linear equations is ± 3. So it has unique solution (if we fix a 3^{rd} root of Σ). That is, x_i's are uniquely determined by g up to Weyl group actions. The Weyl group actions just lead to choices of other G-configurations. By Proposition 3.6, this doesn't change the pair (X_n, Σ). Hence, ϕ is injective. These points x_i's are not ″in general position″ if and only if p_i's will satisfy some (finitely many) equations. That means the image of ϕ must be open dense in $Hom(\Lambda, \Sigma)/W$. The extendability of ϕ is also because of the existence and uniqueness of the solution of the above equations.

For the cases of Y_n and Z_n, the arguments is similar. It is easy to see that the map ϕ is well defined in both cases. For Y_n, the system of linear equations is

$$\begin{cases} -x_1 - x_2 = p_1, \\ x_{k-1} - x_k = p_k, k = 2, \cdots, n. \end{cases}$$

The determinant is ± 2. So the solution is uniquely determined (if we fix a 2^{nd} root of Σ). The remained arguments is just like the first case. At last, for the case of Z_n, the system of equations is

$$\begin{cases} \sum x_i = 0, \\ x_{k-1} - x_k = p_{k-1}, k = 2, \cdots, n. \end{cases}$$

The determinant is $\pm n$. Then the solution is uniquely determined (if we fix an n^{th} root of Σ). The remaining arguments are just the same as that in the E_n case. These prove (i) and (ii).

As for (iii), the result follows from Lemma 3.5. $\square$

In ADE case, when the finite group $\Lambda_c^\vee/\Lambda$ (the fundamental group of the Lie group $ad(G)$) is non-trivial, and of exponent d, a homomorphism ϕ_0 from Λ to Σ would determine only an $ad(G)$ bundle. For a given pair (S, Σ), suppose we are given a distinguished d^{th} root of $\Sigma \cong Jac(\Sigma)$. Then there is a homomorphism ϕ from the weight lattice to Σ, which extending and determined uniquely by ϕ_0. And ϕ will determine a G bundle on Σ. Thus we can still identify the moduli space of pairs and the moduli space of flat G bundles. Precisely, the construction is as following. In E_n case, we fix a $d^{th}(d = 9 - n)$ root of Σ and take the d^{th} root $\mathcal{N}_0$ of the line bundle $\mathcal{O}_\Sigma(-K)$ and define the homomorphism by $\mu \in Hom(\Lambda_c^\vee, \Sigma) : y \mapsto L_y \otimes \mathcal{N}_0^{y \cdot K}$. In D_n case, when n is even, we fix a 2^{nd} root of Σ, and take the 2^{nd} root $\mathcal{N}_0$ of the line bundle $\mathcal{O}_\Sigma(f)$ and a 2^{nd} root $\mathcal{N}_1$ of the line bundle $\mathcal{O}_\Sigma(K + (4 - n/2)f)$, and define the homomorphism by $\mu \in Hom(\Lambda_c^\vee, \Sigma) : y \mapsto L_y \otimes \mathcal{N}_0^{y \cdot K} \otimes \mathcal{N}_1^{y \cdot f}$; when n is odd, we fix a 4^{th} root of Σ, and take the 2^{nd} root $\mathcal{N}_0$ of the line bundle $\mathcal{O}_\Sigma(f)$ and a 4^{th} root $\mathcal{N}_1$ of the line bundle $\mathcal{O}_\Sigma(2K + (8 - n)f)$, and define the homomorphism by $\mu \in Hom(\Lambda_c^\vee, \Sigma) : y \mapsto L_y \otimes \mathcal{N}_0^{y \cdot K} \otimes \mathcal{N}_1^{y \cdot f}$. In A_{n-1} case, we fix an n^{th} root of $Jac(\Sigma) = \Sigma$ and take the n^{th} root $\mathcal{N}_0$ of the line bundle $\mathcal{O}_\Sigma(K+2s+2f)$ and define the homomorphism by $\mu \in Hom(\Lambda_c^\vee, \Sigma) : y \mapsto L_y \otimes \mathcal{N}_0^{y(K+2f+2s)} \otimes \mathcal{O}(-s)^{y \cdot f} \otimes \mathcal{O}(-f)^{y \cdot s}$. To apply Theorem 3.12, we need to fix an e^{th} root of Σ where $e = LCM(3, d)$. Then we obtain the following identification.

Theorem 3.13 *Under the above construction, we have a bijection*

$$\overline{\mathcal{S}(\Sigma, G)} \xrightarrow{\sim} Hom(\Lambda_c^\vee, \Sigma)/W \cong (\Lambda_c \otimes \Sigma)/W \cong \mathcal{M}_\Sigma^G.$$

Proof. We just prove it for D_n (n is odd) case. For other cases, the proof is similar. Since the exponent of Γ is $d = 4$, we fix a 4^{th} root of $Jac(\Sigma)$. Since the degree of $\mathcal{O}_\Sigma(f)$ is $f \cdot (-K) = 2$, there is a point $0 \in \Sigma$ determined uniquely by the 4^{th} root of $Jac(\Sigma)$, such that $\mathcal{O}_\Sigma(f) = \mathcal{O}_\Sigma(2(0)) = \mathcal{O}_\Sigma((0))^2$. Take $\mathcal{N}_0 = \mathcal{O}_\Sigma((0))$. Since the degree of the bundle $\mathcal{O}_\Sigma(2K + (8-n)f)$ is $2(n-8) + 2(8-n) = 0$, it is an element of $Jac(\Sigma) \cong \Sigma$. Of course $\mathcal{O}_\Sigma(2K + (8-n)f)$ is 4-divisible. The degree of $\mu(y)$ is 0, so μ is a homomorphism from $\Lambda_c^\vee$ to $Jac(\Sigma) \cong \Sigma$ induced by the homomorphism $\mu_0(y) = L_y$. And μ is uniquely determined by μ_0. $\square$

Remark 3.14 [29][13][14]. The moduli space of flat A_n bundles over Σ is exactly the ordinary projective space $\mathbb{CP}^n$. This can be described as follows: a flat $SU(n+1)$ bundle is determined uniquely by $n+1$ points on Σ with sum equal to 0, up to isomorphism. And $n+1$ points on Σ with sum equal to 0 are determined uniquely by a global section $H^0(\Sigma, \mathcal{O}_\Sigma(n(0)))$ up to scalar, where (0) is the divisor of the identity element 0. So the moduli space of flat $SU(n+1)$ bundles is isomorphic to $\mathbb{P}(H^0(\Sigma, \mathcal{O}_\Sigma((n+1)P))) = \mathbb{P}^n$. From this we see that the moduli space of pairs $(S,\ \Sigma)$ is just the ordinary complex projective space $\mathbb{P}^n$.

Example 3.15 Let us look at what the pre-image of a trivial G-bundle is. For example, in E_8 case, the trivial bundle means the element $0 \in Hom(\Lambda(E_8), \Sigma)/W(G)$. By the above correspondence, all $x_i = 0$ in Σ. This means that we can blow up $\mathbb{P}^2$ at the identity element 0 (an inflection point) eight times to obtain the surface represented by this pre-image, which is a boundary point in the moduli space $\overline{\mathcal{S}(\Sigma, G)}$. Blowing up once more, we obtain an elliptic fibration with a singular fiber of $\widetilde{E_8}$-type [4].

Chapter 4

Flat G-bundles over Elliptic Curves and Rational Surfaces: Non-simply Laced Cases

In previous chapters, we constructed ADE bundles over ADE-surfaces, and established a identification for the moduli space of flat G bundles over a fixed elliptic curve Σ and the moduli space of the pairs (S, Σ) with $\Sigma \in |-K_S|$, where G is any simply laced (that is, of ADE-type), simple, compact and simply connected Lie group, and S is an ADE-surface with Σ as a smooth anti-canonical curve. This identification generalized the one for the moduli space of flat E_n bundles over Σ and the moduli space of del Pezzo surfaces of degree $9 - n$ which contain Σ as an anti-canonical curve. In the remaining part, we construct $Lie(G)$ bundles for non-simply laced Lie group G over G-surfaces, and extend the above identification to non-simply laced cases. Therefore we establish a one-to-one correspondence between flat G bundles over a fixed elliptic curve Σ and rational surfaces with Σ as an anti-canonical curve for simple Lie groups of *all* types.

4.1 Reductions of Non-simply Laced Cases to Simply Laced Cases

From now on, we always assume that G is a compact, simple, simply-connected Lie group of non-simply laced type, that is, of type B_n, C_n, F_4, G_2. There are two natural approaches to reduce situations to simply laced cases. One is embedding G into a simply laced Lie group G' such that G is the subgroup fixed by the outer automorphism group of G'. Another is taking the simply laced subgroup G'' of maximal rank.

In the following we explain the first reduction. The following result is well-known.

Proposition 4.1 *Let G be a compact, non-simply laced, simple, and simply connected Lie group. There exists a simple, simply connected and simply laced Lie group G', s.t. $G \subset G'$ and $G = (G')^\rho$, where ρ is an outer automorphism of G' of order 3 for $G' = D_4$, and of order 2 otherwise.*

Proof. By the functorial property, we just need to prove it in the Lie algebra level. For the construction of $\mathcal{G} = Lie(G)$ and $\mathcal{G}' = Lie(G')$, one can see [18] for the details, where the construction of Lie algebras is determined by the construction of root systems. $\square$

Remark 4.2 For later use, we list the construction of non-simply laced root systems via simply laced root systems.

1. $G = C_n = Sp(n)$, $G' = A_{2n-1} = SU(2n)$.
   $\Delta(G') = \{\alpha_i, i = 1, \cdots, 2n-1\}$.
   $Out(G') = \{1, \rho\} \cong \mathbb{Z}_2$, *where* $\rho(\alpha_i) = \alpha_{2n-i}, i = 1, \cdots, n-1$, *and* $\rho(\alpha_n) = \alpha_n$.
   $\Delta(G) = \{\beta_i = \frac{1}{2}(\alpha_i + \alpha_{2n-i}), i = 1, \cdots, n-1, \beta_n = \alpha_n\}$.

2. $G = B_n = Spin(2n+1)$, $G' = D_{n+1} = Spin(2n+2)$.

$\Delta(G') = \{\alpha_i, i = 1, \cdots, n+1\}$.

$Out(G') = \{1, \rho\} \cong \mathbb{Z}_2$, *where* $\rho(\alpha_i) = \alpha_i, i = 3, \cdots, n+1$, $\rho(\alpha_1) = \alpha_2$, $\rho(\alpha_2) = \alpha_1$.

$\Delta(G) = \{\beta_1 = \frac{1}{2}(\alpha_1 + \alpha_2), \beta_i = \alpha_{i+1}, i = 2, \cdots, n\}$.

3. $G = F_4$, $G' = E_6$.

$\Delta(G') = \{\alpha_i, i = 1, \cdots, 6\}$.

$Out(G') = \{1, \rho\} \cong \mathbb{Z}_2$, *where* $\rho(\alpha_i) = \alpha_{6-i}, i = 1, \cdots, 5$, *and* $\rho(\alpha_6) = \alpha_6$.

$\Delta(G) = \{\beta_1 = \frac{1}{2}(\alpha_1 + \alpha_5), \beta_2 = \frac{1}{2}(\alpha_2 + \alpha_4), \beta_3 = \alpha_3, \beta_4 = \alpha_6\}$.

4. $G = G_2$, $G' = D_4 = Spin(8)$.

$\Delta(G') = \{\alpha_i, i = 1, \cdots, 4\}$.

$Out(G') = \langle \rho_1, \rho_2 \rangle \cong S_3$, *where* ρ_1 *interchanges* α_1 *and* α_2, *and* ρ_2 *interchanges* α_1 *and* α_4.

$\Delta(G) = \{\beta_1 = \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_4), \beta_2 = \alpha_3\}$.

The Dynkin diagrams of G and G' are as the following:

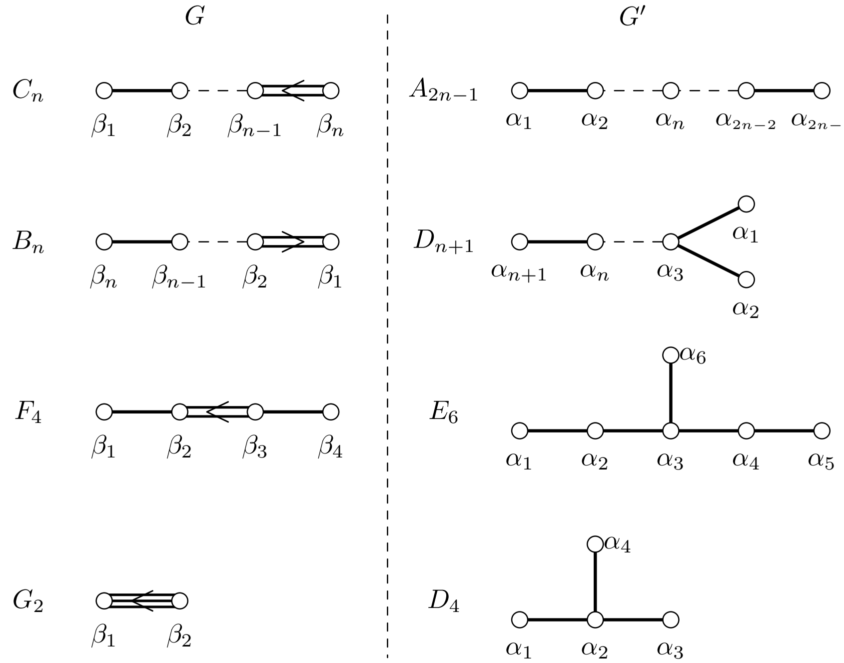

Figure 9. Non-simply laced G reduced to simply laced G'.

Remark 4.3 Note that $W(G)$ is the subgroup of $W(G')$ fixing the root system $R(G)$, and also the subgroup pointwise fixed by $Out(G')$. For a root α, let $S_\alpha \in W(G)$ be the reflection with respect to α, that is, $S_\alpha(x) = x + (x,\alpha)\alpha$. Thus as a subgroup of $W(A_{2n-1})$, $W(C_n)$ is generated by $S_{\alpha_i} \circ S_{\alpha_{2n-i}}$ for $i = 1, \cdots, n-1$ and S_{α_n}. As a subgroup of $W(D_{n+1})$, $W(B_n)$ is generated by $S_{\alpha_1} \circ S_{\alpha_2}$ and S_{α_i} for $i = 3, \cdots, n+1$. As a subgroup of $W(E_6)$, $W(F_4)$ is generated by $S_{\alpha_1} \circ S_{\alpha_5}$, $S_{\alpha_2} \circ S_{\alpha_4}$, S_{α_3} and S_{α_6}. As a subgroup of $W(D_4)$, $W(G_2)$ is generated by $S_{\alpha_1} \circ S_{\alpha_2} \circ S_{\alpha_4}$ and S_{α_3}.

In the following we let Σ be a fixed elliptic curve with identity element 0, and we fix a primitive d^{th} root of $\Sigma \cong Jac(\Sigma)$, where $d = 2$ for D_n case, $d = 9 - n$ for E_n case, and $d = n + 1$ for A_n case, respectively (see Chapter 3). Recall that for any compact, simple and simply-connected Lie group H, the moduli space of flat H bundles over Σ is

$$\mathcal{M}_\Sigma^H \cong (\Lambda_c(H) \otimes \Sigma)/W(H).$$

For G', the group $Out(G')$ acts on

$$(\Lambda_c(G') \otimes \Sigma)/W(G')$$

naturally.

Let χ be the natural map from $(\Lambda_c(G) \otimes \Sigma)/W(G)$ to the fixed part

$$((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}.$$

The image of χ is contained in a connected component of the fixed part.

Lemma 4.4 *The map*

$$\chi : (\Lambda_c(G) \otimes \Sigma)/W(G) \to ((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}$$

is injective.

Proof. It suffices to prove that for any $x, y \in \Lambda(G) \otimes \Sigma$, if $\exists\ w' \in W(G')$, such that $w'(x) = y$, then $\exists\ w \in W(G)$, such that $w(x) = y$. For A_n and D_n cases, this is obvious if we check the root lattices. For E_6 case, we can also check it directly with the help of computer. Of course we can also check this case by hand following the discussion in Section 2.4.1. $\square$

Corollary 4.5 *(i) The fixed part $((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}$ is determined by the condition $\rho(x) = x$, up to $W(G')$-action, where $x \in \Lambda_c(G') \otimes \Sigma$, and ρ is a generator of $Out(G')$, of order 3 for $G' = D_4$ and order 2 for $G' = A_n,\ E_n$.*

(ii) The moduli space $\mathcal{M}^G_\Sigma \cong (\Lambda_c(G) \otimes \Sigma)/W(G)$ is a connected component of the fixed part

$$(\mathcal{M}^{G'}_\Sigma)^{Out(G')} \cong ((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}$$

containing the trivial G' bundle.

Proof. (i) For any $x \in \Lambda_c(G') \otimes \Sigma$, denote $\bar{x}$ the class in $(\Lambda_c(G') \otimes \Sigma)/W(G')$. Then $\rho(\bar{x}) = \bar{x}$ if and only if there exists $w \in W(G')$, such that $\rho(x) = w(x)$.

Thus $w^{-1}\rho(x) = x$. But $w^{-1}\rho \in Out(G')$ since $Out(G') = Aut(G')/W(G')$. Thus we can take a new simple root system such that $w^{-1}\rho$ is the generator of the diagram automorphism (the automorphism of order 3 for D_4).

(ii) By (i), $(\Lambda_c(G) \otimes \Sigma)/W(G)$ and $(\mathcal{M}_\Sigma^{G'})^{Out(G')}$ are both orbifolds with the same dimension. Thus the result follows from Lemma 4.4. $\square$

If we express the moduli space of flat G bundles over Σ as $(T \times T)/W(G)$, where T is a maximal torus of G, then we have the following corollary.

Corollary 4.6 *If two elements of $T \times T$ are conjugate under $W(G')$, then they are also conjugate under $W(G)$.* $\square$

Another method is to reduce G to its simply-laced subgroup G'' of maximal rank, and apply the results for simply laced cases to current situation. In another occasion we will discuss our moduli space of G-bundles from this aspect in detail. Here we just mention the following well-known fact from Lie theory.

Proposition 4.7 *There exists canonically a simply laced Lie subgroup G'' of G, which is of maximal rank, that is, G'' and G share a common maximal torus. And there is a short exact sequence*

$$1 \to W(G'') \to W(G) \to Out(G'') \to 1,$$

where $Out(G'')$ is the outer automorphism group of G''. Thus, if we write the moduli space as $\mathcal{M}_\Sigma^G = (T \times T)/W$, then

$$\mathcal{M}_\Sigma^G = \mathcal{M}_\Sigma^{G''}/Out(G''). \qquad \square$$

Remark 4.8 *We give this construction of G'' in each case.*

(1) For $G = Sp(n)$, $G'' = SU(2)^n$. $Out(G'')$ is the group S_n of permutations of the n copies of $SU(2)$ in G''.

(2) For $G = G_2$, $G'' = SU(3)$. $Out(G'')$ *is the group* $\mathbb{Z}_2$ *that exchanges the* 3*-dimensional representation of* $SU(3)$ *with its dual.*

(3) For $G = Spin(2n+1)$, $G'' = Spin(2n)$. $Out(G'')$ *is the group* $\mathbb{Z}_2$ *that exchanges the two spin representations of* $Spin(2n)$*.*

(4) For $G = F_4$, $G'' = Spin(8)$. $Out(G'')$ *is the triality group* S_3 *that permutes the three* 8*-dimensional representations of* $Spin(8)$*.*

4.2 The Identification of Moduli Spaces in Non-simply Laced Cases

In this section, we study case by case the G bundles over elliptic curves and rational surfaces for a non-simply laced Lie group G.

4.2.1 The $B_n (n \geq 2)$ Bundles

According to the arguments of last section, for $G = Spin(2n+1)$ we can take $G' = Spin(2n+2)$, such that $G = (G')^{Out(G')}$.

Let $S = Y_{n+1}$ be a rational surface with a D_{n+1}-configuration which contains Σ as a smooth anti-canonical curve. Recall that Y_{n+1} is a blow-up of $\mathbb{F}_1$ at $n+1$ points $x_1, \cdots, x_{n+1}$ on Σ, with corresponding exceptional classes $l_1, \cdots, l_{n+1}$. Let f be the class of fibers in $\mathbb{F}_1$, and s be the section such that $0 = s \cap \Sigma$ is the identity element of Σ. The Picard group of Y_{n+1} is $H^2(Y_{n+1}, \mathbb{Z})$, which is a lattice with basis $s, f, l_1, \cdots, l_{n+1}$. The canonical line bundle $K = -(2s + 3f - \sum_{i=1}^{n+1} l_i)$.

We know from Chapter 2 that

$$P_{n+1} := \{x \in H^2(Y_{n+1}, \mathbb{Z}) \mid x \cdot K = x \cdot f = 0\}$$

is a root lattice of D_{n+1} type. We take a simple root system of G' as

$$\Delta(D_{n+1}) = \{\alpha_1 = l_1 - l_2, \alpha_2 = f - l_1 - l_2, \alpha_3 = l_2 - l_3, \cdots, \alpha_{n+1} = l_n - l_{n+1}\}.$$

Let ρ be the generator of $Out(G') \cong \mathbb{Z}_2$, such that $\rho(\alpha_1) = \alpha_2, \rho(\alpha_2) = \alpha_1$ and $\rho(\alpha_i) = \alpha_i$ for $i = 3, \cdots, n+1$.

From Chapter 3 we know that the pair (S, Σ) determines a homomorphism

$$u \in Hom(\Lambda(G'), \Sigma)$$

which is given by the restriction map:

$$u(\alpha) = \mathcal{O}(\alpha)|_{\Sigma}.$$

Lemma 4.9 *Let $u \in Hom(\Lambda(G'), \Sigma)$ correspond to a pair (S, Σ), where S is a surface with a D_{n+1}-configuration. Then $\rho \cdot u = u$ if and only if $2x_1 = 0$.*

Proof. Since u is the restriction map: $\alpha_i \mapsto \mathcal{O}(\alpha_i)|_{\Sigma}$, $u(\alpha_1) = \mathcal{O}(l_1 - l_2)|_{\Sigma} = x_1 - x_2$, and $u(\alpha_2) = \mathcal{O}(f - l_1 - l_2)|_{\Sigma} = -x_1 - x_2$. Hence $\rho \cdot u = u \Leftrightarrow u(\alpha_1) = u(\alpha_2) \Leftrightarrow x_1 - x_2 = -x_1 - x_2 \Leftrightarrow 2x_1 = 0 \Leftrightarrow x_1$ is one of the 4 points of order 2 on the elliptic curve Σ. $\square$

As in Chapter 3, we denote $\mathcal{S}(\Sigma, G')$ the moduli space of $G' = D_{n+1}$-surfaces with a fixed anti-canonical curve Σ, and $\overline{\mathcal{S}(\Sigma, G')}$ the natural compactification by including all rational surfaces with D_{n+1}-configurations (Figure 1). We know that $\phi : \overline{\mathcal{S}(\Sigma, G')} \xrightarrow{\sim} \mathcal{M}_{\Sigma}^{G'}$ is an isomorphism.

Corollary 4.10 *For $u \in \mathcal{M}_{\Sigma}^{G} \hookrightarrow (\mathcal{M}_{\Sigma}^{G'})^{Out(G')}$, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')}$ represents a class of surfaces $Y_{n+1}(x_1, \cdots, x_{n+1})$ with $x_1 = 0$, and such a surface corresponds to a boundary point in the moduli space, that is, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')} \backslash \mathcal{S}(\Sigma, G')$.*

Proof. By Lemma 4.9, $u \in (\mathcal{M}_{\Sigma}^{G'})^{Out(G')}$ if and only if $2x_1 = 0$. There are 4 connected components corresponding to 4 points of order 2 on Σ. Since $\mathcal{M}_{\Sigma}^{G}$ is the component containing the trivial G' bundle, we have $x_1 = 0$. Recall that $Y_{n+1}(x_1, \cdots, x_{n+1}) \in \mathcal{S}(\Sigma, G')$ if and only if $0, x_1, \cdots, x_{n+1}$ are in general position, which implies in particular $x_1 \neq 0$. Hence $\phi^{-1}(u)$ corresponds to a boundary point. $\square$

Denote $S = Y'_{n+1}(x_1 = 0, x_2, \cdots, x_{n+1})$ (or Y'_{n+1} for brevity) the blow-up of $\mathbb{F}_1$ at $n+1$ points $x_1 = 0, x_2, \cdots, x_{n+1}$ on Σ, with exceptional divisors $l_1, l_2, \cdots, l_{n+1}$, where $\Sigma \in |-K_S|$. Similar to the simply laced cases, we give the following definition.

Definition 4.11 A *B_n-exceptional system* on S is an n-tuple $(e_1, e_2, \cdots, e_{n+1})$ where e_i's are exceptional divisors such that $e_i \cdot e_j = 0 = e_i \cdot f, i \neq j$ and $y_1 = e_1 \cap \Sigma = 0$ is the identity of Σ. A *B_n-configuration* on S is a B_n-exceptional system $\zeta_{B_n} = (e_1, e_2, \cdots, e_{n+1})$ such that we can consider S as a blow-up of $\mathbb{F}_1$ at $n+1$ points $y_1 = 0, y_2, \cdots, y_{n+1}$ on Σ, that is $S = Y'_{n+1}(y_1 = 0, y_2, \cdots, y_{n+1})$, with corresponding exceptional divisors $e_1, e_2, \cdots, e_{n+1}$. When S has a B_n-configuration, we call S a *(rational) surface with a B_n-configuration* (see Figure 2).

When $x_2, \cdots, x_{n+1} \in \Sigma$ with $x_i \neq 0$ for all i are in general position, any B_n-exceptional system on S consists of exceptional curves. Such a surface is called a *B_n-surface.* So a B_n-surface must have a B_n-configuration.

Lemma 4.12 *(i) Let S be a rational surface with a B_n-configuration. Then the Weyl group $W(B_n)$ acts on all B_n-exceptional systems on S simply transitively.*

(ii) Let S be a B_n-surface. Then the Weyl group $W(B_n)$ acts on all B_n-configurations simply transitively.

Proof. It suffices to prove (i). Let $(e_1, e_2, \cdots, e_{n+1})$ be a B_n-exceptional system on S. By Definition 4.11, $e_i = l_{\sigma(i)}$ or $f - l_{\sigma(i)}$ for $i \neq 1$, where σ is a permutation of $\{2, \cdots, n+1\}$. Note that according to Remark 4.3, the Weyl group $W(B_n)$ acts as the group generated by permutations of the n pairs $\{(l_i, f - l_i) \mid i = 2, \cdots, n+1\}$ and interchanging of l_i and $f - l_i$ in each pair $(l_i, f - l_i)_{i \geq 2}$. Then the result follows. $\square$

Let $\mathcal{S}(\Sigma, B_n)$ be the moduli space of pairs (S, Σ) where S is a B_n-surface (so the blown-up points $x_1 = 0, x_2, \cdots, x_{n+1}$ are in general position), and $\Sigma \in |-K_S|$. Denote $\mathcal{M}_\Sigma^{B_n}$ the moduli space of flat B_n bundles over Σ. Then applying Corollary 4.10 we have the following identification.

Proposition 4.13 *(i) $\mathcal{S}(\Sigma, B_n)$ is embedded into $\mathcal{M}_\Sigma^{B_n}$ as an open dense subset.*

(ii) Moreover, this embedding can be extended naturally to an isomorphism

$$\overline{\mathcal{S}(\Sigma, B_n)} \cong \mathcal{M}_\Sigma^{B_n},$$

by including all rational surfaces with B_n-configurations.

Proof. The proof is similar to that in ADE cases. Firstly, we have $\mathcal{M}_\Sigma^{B_n} \cong \Lambda_c(B_n) \otimes_{\mathbb{Z}} \Sigma / W(B_n)$, and $\Lambda_c(B_n) \otimes_{\mathbb{Z}} \Sigma / W(B_n) \cong Hom(\Lambda(B_n), \Sigma)/W(B_n)$ when we fixed the square root of unity of $Jac(\Sigma) \cong \Sigma$.

Secondly, the restriction from S to Σ induces a map (again denoted by ϕ)

$$\phi : \mathcal{S}(\Sigma, B_n) \to Hom(\Lambda(B_n), \Sigma)/W(B_n).$$

This map is well-defined, since by Lemma 4.12, choosing and fixing a B_n-configuration on S is equivalent to choosing and fixing a system of simple roots $\Delta(B_n)$.

Thirdly, the map ϕ is injective. For this, we take a simple root system of B_n as

$$\beta_1 = f - 2\ l_2 \text{ and } \beta_k = 2\ \alpha_{k+1} \text{ for } 2 \le k \le n.$$

Then the restriction induces an element $u \in Hom(\Lambda(B_n), \Sigma)$, which satisfies the following system of linear equations

$$\begin{cases} -2\ x_2 = p_1, \\ 2(x_k - x_{k+1}) = p_k,\ \ k = 2, \cdots, n. \end{cases}$$

where $p_i = u(\beta_i)$. Obviously, the solution of this system of linear equations exists uniquely for given p_i with $1 \leq i \leq n$.

Finally, the statement (ii) comes from Corollary 4.10 and the existence of the solutions to the above system of linear equations. $\square$

Remark 4.14 The situation here is very similar to that in the compactification theory of the moduli space of (projective) $K3$ surfaces. A natural question there is how to extend the global Torelli theorem to the boundary components of a compactification [10][19][26][6]. If we consider the map $\phi : \mathcal{S}(\Sigma, G) \to \mathcal{M}_\Sigma^G$ [**?**] for $G = A_n, D_n$ or E_n as a type of period map, then the main result of Chapter 3 is a type of global Torelli theorem. And Proposition 4.13 implies that we can extend the theorem of Torelli type in D_{n+1} case to a boundary component of the natural compactification.

In the following, we let $S = Y_{n+1}(x_1, \cdots, x_{n+1})$ be the blow-up of $\mathbb{F}_1$ at $n+1$ points. We can construct a Lie algebra bundle on S. Here we don't need the existence of the anti-canonical curve Σ. According to Section 2, we have a root system of B_n type consisting of divisors on S:

$$R(B_n) \triangleq \{\pm(f - 2\ l_i), 2(l_i - l_j), \pm 2(f - l_i - l_j)\ \mid\ i \neq j, 2 \leq i, j \leq n+1\}.$$

Thus we can construct a Lie algebra bundle of B_n-type over S:

$$\mathscr{B}_n \triangleq \mathcal{O}^{\bigoplus n} \bigoplus_{D \in R(B_n)} \mathcal{O}(D).$$

The fiberwise Lie algebra structure of $\mathscr{B}_n$ is defined as follows (the argument here is the same as that in Chapter 3.

Fix the system of simple roots of R_n as

$$\Delta(B_n) = \{\alpha_1 = f - 2l_2, \alpha_2 = 2(l_2 - l_3), \cdots, \alpha_n = 2(l_n - l_{n+1})\},$$

and take a trivialization of $\mathscr{B}_n$. Then over a trivializing open subset U, $\mathscr{B}_n|_U \cong U \times (\mathbb{C}^{\oplus n} \bigoplus_{\alpha \in R_n} \mathbb{C}_\alpha)$. Take a Chevalley basis $\{x_\alpha^U, \alpha \in R_n; h_i, 1 \leq i \leq n\}$ for

$\mathscr{B}_n|_U$ and define the Lie algebra structure by the following four relations, namely, Serre's relations on Chevalley basis (see [16], p147):

(a) $[h_i h_j] = 0, 1 \leq i, j \leq n$.

(b) $[h_i x_\alpha^U] = \langle \alpha, \alpha_i \rangle x_\alpha^U, 1 \leq i \leq n, \alpha \in R_n$.

(c) $[x_\alpha^U x_{-\alpha}^U] = h_\alpha$ is a $\mathbb{Z}$-linearly combination of $h_1, \cdots, h_n$.

(d) If α, β are independent roots, and $\beta - r\alpha, \cdots, \beta + q\alpha$ are the α-string through β, then $[x_\alpha^U x_\beta^U] = 0$ if $q = 0$, while $[x_\alpha^U x_\beta^U] = \pm(r+1)x_{\alpha+\beta}^U$ if $\alpha + \beta \in R_n$.

Note that $h_i, 1 \leq i \leq n$ are independent of any trivialization, so the relation (a) is always invariant under different trivializations. If $\mathscr{B}_n|_V \cong V \times (\mathbb{C}^{\oplus n} \bigoplus_{\alpha \in R_n})$ is another trivialization, and f_α^{UV} is the transition function for the line bundle $\mathcal{O}(\alpha)(\alpha \in R_n)$, that is, $x_\alpha^U = f_\alpha^{UV} x_\alpha^V$, then the relation (b) is

$$[h_i(f_\alpha^{UV} x_\alpha^V)] = \langle \alpha, \alpha_i \rangle f_\alpha^{UV} x_\alpha^V,$$

that is,

$$[h_i x_\alpha^V] = \langle \alpha, \alpha_i \rangle x_\alpha^V.$$

So (b) is also invariant. (c) is also invariant since $(f_\alpha^{UV})^{-1}$ is the transition function for $\mathcal{O}(-\alpha)(\alpha \in R_n)$. Finally, (d) is invariant since $f_\alpha^{UV} f_\beta^{UV}$ is the transition function for $\mathcal{O}(\alpha + \beta)(\alpha, \beta \in R_n)$.

Therefore, the Lie algebra structure is compatible with the trivialization. Hence it is well-defined.

When the surface S contains Σ as an anti-canonical curve, restricting the above bundle to this anti-canonical curve Σ, we obtain a Lie algebra bundle of B_n-type over Σ, which determines uniquely a flat B_n bundle over Σ. On the other hand, when $x_1 = 0$, we can identify these two line bundles $\mathcal{O}_\Sigma(l_1)$ and $\mathcal{O}_\Sigma(f - l_1)$

when restricting them to Σ. Recall the spinor bundles S^+_{n+1} and S^-_{n+1} of D_{n+1} are defined as follows[20] (here we omit the subscription $n+1$ for brevity)

$$\begin{aligned}\mathcal{S}^+ &= \bigoplus_{D^2=D\cdot K=-1, D\cdot f=1} \mathcal{O}(D) \text{ and} \\ \mathcal{S}^- &= \bigoplus_{T^2=-2, T\cdot K=0, T\cdot f=1} \mathcal{O}(T).\end{aligned}$$

The identification of $\mathcal{O}_\Sigma(l_1) \cong \mathcal{O}_\Sigma(f-l_1)$ induces an identification of these two spinor bundles S^+ and S^-, which is given by (of course, when restricted to Σ)

$$S^+ \otimes \mathcal{O}(-l_1) \cong S^-.$$

From representation theory, we know this determines a flat B_n bundle over Σ.

Conversely, if $\mathcal{S}^+|_\Sigma \cong \mathcal{S}^-|_\Sigma$, then we must have $x_1 = 0$ (up to renumbering). For example, we consider the $n=2$ case. Note that

$$\begin{aligned}\mathcal{S}^+|_\Sigma &\otimes \mathcal{O}(-(0)) = \mathcal{O} \oplus \mathcal{O}((-x_1-x_2)-(0)) \oplus \mathcal{O}((-x_1-x_3)-(0)) \\ & \oplus \mathcal{O}((-x_2-x_3)-(0)), \\ \mathcal{S}^-|_\Sigma &= \mathcal{O}((0)-(x_1)) \oplus \mathcal{O}((0)-(x_2)) \oplus \mathcal{O}((0)-(x_3)) \\ & \oplus \mathcal{O}(3(0)-(x_1)-(x_2)-(x_3)).\end{aligned}$$

Where for a point $x \in \Sigma$, (x) means the divisor of degree one, and $\mathcal{O}((x))$ means the line bundle determined by this divisor. Thus, $\mathcal{S}^+_\Sigma \otimes \mathcal{O}(-(0)) = \mathcal{S}^-_\Sigma$ implies that $x_1 = 0$ (up to renumbering). The general case follows from similar arguments.

4.2.2 The C_n Bundles

We take $G = C_n \subset G' = A_{2n-1}$, where $C_n = Sp(n)$ and $A_{2n-1} = SU(2n)$. They satisfy the relation $G = (G')^{Out(G')}$.

Let $S = Z_{2n}$ be a rational surface with an A_{2n-1}-configuration (see Figure 3) which contains Σ as a smooth anti-canonical curve. Recall that Z_{2n} is a (successive) blow-up of $\mathbb{F}_1$ at $2n$ points $x_1, \cdots, x_{2n}$ on Σ, with corresponding exceptional classes $l_1, \cdots, l_{2n}$. Let f be the class of fibers in $\mathbb{F}_1$, and s be the section such that $0 = s \cap \Sigma$ is the identity element of Σ. The Picard group of Z_{2n} is $H^2(Z_{2n}, \mathbb{Z})$, which is a lattice with basis $s, f, l_1, \cdots, l_{2n}$. The canonical line bundle $K = -(2s + 3f - \sum_{i=1}^{2n} l_i)$.

Recall

$$P_{2n-1} := \{x \in H^2(Z_{2n}, \mathbb{Z}) \mid x \cdot K = x \cdot f = x \cdot s = 0\}$$

is a root lattice of A_{2n-1} type. And we can take a simple root system of A_{2n-1} as

$$\Delta(A_{2n-1}) = \{\alpha_i = l_i - l_{i+1} \mid 1 \leq i \leq 2n-1\}.$$

Note that we have used the convention that $\sum_{i=1}^{2n} x_i = 0$.

Let ρ be the generator of $Out(G') \cong \mathbb{Z}_2$, such that $\rho(\alpha_i) = \alpha_{2n-i}$ for $i = 1, \cdots, 2n-1$.

When the above simple root system is chosen, the pair (S, Σ) determines a homomorphism $u \in Hom(\Lambda(G'), \Sigma)$ which is given by the restriction map

$$u(\alpha) = \mathcal{O}(\alpha)|_{\Sigma}.$$

Lemma 4.15 *Let $u \in Hom(\Lambda(G'), \Sigma)$ be an element corresponding to a pair (S, Σ), where S is a surface with an A_{2n-1}-configuration. Then $\rho \cdot u = u$ if and only if $n(x_i + x_{2n+1-i}) = 0$ for $i = 1, \cdots, n$.*

Proof. Since u is the restriction map: $\alpha_i \mapsto \mathcal{O}(\alpha_i)|_{\Sigma}$, $u(\alpha_i) = \mathcal{O}(l_i - l_{i+1})|_{\Sigma} = x_i - x_{i+1}$ for $i = 1, \cdots, 2n-1$. Hence $\rho \cdot u = u \Leftrightarrow u(\alpha_i) = u(\alpha_{2n-i}) \Leftrightarrow x_i - x_{i+1} = x_{2n-i} - x_{2n-i+1} \Leftrightarrow n(x_i + x_{2n-i+1}) = 0$ since $\sum_{i=1}^{2n} x_i = 0$. $\square$

As in Chapter 3, we denote $\mathcal{S}(\Sigma, G')$ the moduli space of $G' = A_{2n-1}$-surfaces with a fixed anti-canonical curve Σ, and $\overline{\mathcal{S}(\Sigma, G')}$ the natural compactification by including all rational surfaces with A_{2n-1}-configurations. We know that there is an isomorphism $\phi : \overline{\mathcal{S}(\Sigma, G')} \xrightarrow{\sim} \mathcal{M}_{\Sigma}^{G'}$.

Corollary 4.16 *For $u \in \mathcal{M}_\Sigma^G \hookrightarrow (\mathcal{M}_\Sigma^{G'})^{Out(G')}$, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')}$ represents a class of surfaces $Z_{2n}(x_1, \cdots, x_{2n})$ with $x_i + x_{2n+1-i} = 0$ for $i = 1, \cdots, n$, and such a surface corresponds to a boundary point in the moduli space, that is, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')} \backslash \mathcal{S}(\Sigma, G')$.*

Proof. By Lemma 4.15, $u \in (\mathcal{M}_\Sigma^{G'})^{Out(G')}$ if and only if $n(x_i + x_{2n+1-i}) = 0$ for $i = 1, \cdots, n$. There are n^2 connected components corresponding to n^2 points of order n on Σ. Since $\mathcal{M}_\Sigma^G$ is the component containing the trivial G' bundle, we have $x_i + x_{2n+1-i} = 0$ for $i = 1, \cdots, n$. Recall that $Z_{2n}(x_1, \cdots, x_{2n}) \in \mathcal{S}(\Sigma, G')$ if and only if $0, x_1, \cdots, x_{2n}$ are in general position, which implies in particular $x_i \neq -x_{2n+1-i}$. Hence $\phi^{-1}(u)$ corresponds to a boundary point. □

Denote $S = Z'_{2n}(\pm x_1, \cdots, \pm x_n)$ the blow-up of $\mathbb{F}_1$ at n pairs of points $(x_1, -x_1)$, $\cdots$, $(x_n, -x_n)$ on Σ, with n pairs of corresponding exceptional divisors (l_1, l_1^-), $\cdots$, (l_n, l_n^-), where l_i (resp. l_i^-) is the exceptional divisor corresponding to the blowing up at x_i (resp. $-x_i$). Similar to the other cases, we give the following definitions.

Definition 4.17 A C_n-*exceptional system* on S is an n-tuple of pairs

$$((e_1, e_1^-), \cdots, (e_n, e_n^-))$$

where $(e_i, e_i^-) = (l_{\sigma(i)}, l_{\sigma(i)}^-)$ or $(l_{\sigma(i)}^-, l_{\sigma(i)})$, $i = 1, \cdots, n$, with σ is a permutation of $1, \cdots, n$. A C_n-*configuration* on S is a C_n-exceptional system $\zeta_{C_n} = ((e_1, e_1^-), \cdots, (e_n, e_n^-))$ such that we can blow down successively e_1^-,$\cdots$,e_n^-, e_n,$\cdots$,e_1 such that the resulting surface is $\mathbb{F}_1$ (see Figure 4).

We say that $x_1, x_2, \cdots, x_n \in \Sigma \subset \mathbb{F}_1$ are n points *in general position*, if they satisfy

(i) they are distinct points, and

(ii) for any i, j, $x_i + x_j \neq 0$.

Equivalently, $x_1, x_2, \cdots, x_n \in \Sigma \subset \mathbb{F}_1$ are in general position if and only if any C_n-exceptional system on $S = Z'_{2n}(\pm x_1, \cdots, \pm x_n)$ consists of smooth exceptional curves. Such a surface is called a *C_n-surface*. Thus a C_n-surface must have a C_n-configuration.

Lemma 4.18 *(i) Let S be a surface with a C_n-configuration. Then the Weyl group $W(C_n)$ acts on all C_n-exceptional systems on S simply transitively.*

(ii) Let S be a C_n-surface. Then the Weyl group $W(C_n)$ acts on all C_n-configurations on S simply transitively.

Proof. It suffices to prove (i). According to Remark 4.3, the Weyl group $W(C_n)$ acts as the group generated by permutations of the n pairs $\{(l_i, l_i^-) \mid i = 1, \cdots, n\}$ and interchanging of l_i and l_i^- for each i. From this, we see that $W(C_n)$ acts on all G-configurations simply transitively. $\square$

Denote $\mathcal{S}(\Sigma, C_n)$ the moduli space of pairs (Z'_{2n}, Σ), where Z'_{2n} is a C_n-surface, that is, the blow-up of $\mathbb{F}_1$ at $2n$ points $\pm x_1, \cdots, \pm x_n$ such that $x_1, \cdots, x_n$ are in general position. Denote $\mathcal{M}_\Sigma^{C_n}$ the moduli space of flat C_n bundles over Σ. By Corollary 4.16 we have the following identification.

Proposition 4.19 *(i) $\mathcal{S}(\Sigma, C_n)$ is embedded into $\mathcal{M}_\Sigma^{C_n}$ as an open dense subset.*

(ii) Moreover, this embedding can be extended naturally to an isomorphism

$$\overline{\mathcal{S}(\Sigma, C_n)} \cong \mathcal{M}_\Sigma^{C_n},$$

by including all rational surfaces with C_n-configurations.

Proof. The proof is basically the same as that in B_n case. We only need to replace the corresponding parts by the following two things. Firstly, according to Section 2, we can take a simple root system as

$$\Delta(C_n) = \{\beta_k = \varepsilon_k - \varepsilon_{k+1},\ 1 \leq k \leq n-1,\ \beta_n = 2\varepsilon_n\},$$

where $\varepsilon_k = l_k - l_k^-,\ 1 \le k \le n$.

Secondly, the restriction map gives us the following system of linear equations:

$$\begin{cases} 4x_n = p_n, \\ 2(x_k - x_{k+1}) = p_k,\ k = 1, \cdots, n-1. \end{cases}$$

The solution of this system exists uniquely. $\square$

Remark 4.20 As in B_n case (Remark 4.14), the above proposition is also similar to extending the Torelli theorem to a certain boundary component.

Remark 4.21 Obviously, this description in Proposition 4.19 coincides with the well-known description of flat C_n bundles over elliptic curves [13]. A flat $C_n = Sp(n)$ bundle over Σ corresponds to n pairs (unordered) of points $(x_i, -x_i), i = 1, \cdots, n$ on Σ, uniquely up to isomorphism. And one pair $(x_i, -x_i)$ will determine exactly one point on $\mathbb{CP}^1$, since the rational map determined by the linear system $|2(0)|$ induces a double covering from Σ onto $\mathbb{CP}^1$. So the moduli space of flat C_n bundles over Σ is just isomorphic to $S^n(\mathbb{CP}^1) = \mathbb{CP}^n$, the ordinary projective n space.

As in B_n case, we construct a Lie algebra bundle of C_n type over Z'_{2n}:

$$\mathscr{C}_n = \mathcal{O}^{\bigoplus n} \bigoplus_{D \in R(C_n)} \mathcal{O}(D),$$

where $R(C_n)$ is the root system of C_n according to Section 2:

$$R(C_n) = \{\pm 2(l_i - l_i^-), \pm((l_i - l_i^-) \pm (l_j - l_j^-)) \mid i \neq j, 1 \le i, j \le n\}.$$

Recall the first fundamental representation bundle of $\mathscr{A}_{2n-1}$ is

$$\mathcal{V}_{2n-1} = \bigoplus_{i=1}^{2n} \mathcal{O}(l_i).$$

The condition that $x_i + x_{2n+1-i} = 0, 1 \le i \le n$ is equivalent to an identification of the following two fundamental representation bundles $\wedge^i(\mathcal{V}_{2n-1})$ and $\wedge^{2n-i}(\mathcal{V}_{2n-1})$ with $i = 1, \cdots, n-1$, which is given by (of course, when restricted to Σ)

$$(\wedge^i(\mathcal{V}_{2n-1}))^* \otimes det(\mathcal{V}_{2n-1}) \cong \wedge^{2n-i}(\mathcal{V}_{2n-1}).$$

Note that when restricted to Σ, the line bundle $det(\mathcal{V}_{2n-1}) = \mathcal{O}(l_1 + \cdots l_{2n})$ is isomorphic to $\mathcal{O}(nf)|_\Sigma = \mathcal{O}_\Sigma(2n(0))$, by our assumption that $\sum x_i = 0$. This identification determines uniquely a flat C_n bundle over Σ.

4.2.3 The G_2 Bundles

For $G = G_2$, we take $G' = D_4 = Spin(8)$ such that $G = (G')^{Out(G')}$.

Let $S = Y_4$ be a rational surface with a D_4-configuration which contains Σ as a smooth anti-canonical curve. Recall (Figure 5) that Y_4 is a (successive) blow-up of $\mathbb{F}_1$ at 4 points $x_1, \cdots, x_4$ on Σ, with corresponding exceptional classes $l_1, \cdots, l_4$. Let f be the class of fibers in $\mathbb{F}_1$, and s be the section such that $0 = s \cap \Sigma$ is the identity element of Σ. The Picard group of Y_4 is $H^2(Y_4, \mathbb{Z})$, which is a lattice with basis $s, f, l_1, \cdots, l_4$. The canonical line bundle $K = -(2s + 3f - \sum_{i=1}^{4} l_i)$.

Recall

$$P_4 := \{x \in H^2(Y_4, \mathbb{Z}) \mid x \cdot K = x \cdot f = 0\}$$

is a root lattice of D_4-type. And we can take a simple root system of D_4 as

$$\Delta(D_4) = \{\alpha_1 = l_1 - l_2, \alpha_2 = f - l_1 - l_2, \alpha_3 = l_2 - l_3, \alpha_4 = l_3 - l_4\}.$$

Let $\rho \in Out(G') \cong S_3$ (the permutation group of 3 letters) be the triality automorphism of order 3, such that $\rho(\alpha_1) = \alpha_2$, $\rho(\alpha_2) = \alpha_4$, $\rho(\alpha_4) = \alpha_1$, and $\rho(\alpha_3) = \alpha_3$.

When the above simple root system is chosen, the pair (S, Σ) determines a homomorphism $u \in Hom(\Lambda(G'), \Sigma)$ which is given by the restriction map

$$u(\alpha) = \mathcal{O}(\alpha)|_\Sigma.$$

Lemma 4.22 *Let $u \in Hom(\Lambda(G'), \Sigma)$ correspond to the pair (S, Σ), where S is a surface with a D_4-configuration. Then $\rho \cdot u = u$ if and only if $2x_1 = 0$ and $x_1 + x_4 = x_2 + x_3$.*

Proof. Since u is the restriction map: $\alpha_i \mapsto \mathcal{O}(\alpha_i)|_\Sigma$, $u(\alpha_1) = \mathcal{O}(l_1 - l_2)|_\Sigma = x_1 - x_2$, $u(\alpha_2) = -x_1 - x_2$, $u(\alpha_4) = x_3 - x_4$, and $u(\alpha_3) = x_2 - x_3$. Hence $\rho \cdot u = u \Leftrightarrow u(\alpha_1) = u(\alpha_2) = u(\alpha_4) \Leftrightarrow x_1 - x_2 = -x_1 - x_2 = x_3 - x_4 \Leftrightarrow 2x_1 = 0$ and $x_1 + x_4 = x_2 + x_3$. $\square$

Denote $\mathcal{S}(\Sigma, G')$ the moduli space of $G' = D_4$-surfaces with a fixed anti-canonical curve Σ, and $\overline{\mathcal{S}(\Sigma, G')}$ the natural compactification by including all rational surfaces with D_4-configurations. We know that $\overline{\mathcal{S}(\Sigma, G')} \xrightarrow{\sim} \mathcal{M}_\Sigma^{G'}$. Let ϕ be the isomorphism.

Corollary 4.23 *For $u \in \mathcal{M}_\Sigma^G \hookrightarrow (\mathcal{M}_\Sigma^{G'})^{Out(G')}$, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')}$ represents a class of surfaces $Y_4(x_1, \cdots, x_4)$ with $x_1 = 0$ and $x_4 = x_2 + x_3$, and such a surface corresponds to a boundary point in the moduli space, that is, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')} \backslash \mathcal{S}(\Sigma, G')$.*

Proof. By Lemma 4.22, $u \in (\mathcal{M}_\Sigma^{G'})^{Out(G')}$ if and only if $2x_1 = 0$ and $x_1 + x_4 = x_2 + x_3$. There are 4 connected components corresponding to 4 points of order 2 on Σ. Since $\mathcal{M}_\Sigma^G$ is the component containing the trivial G' bundle, we have $x_1 = 0$ and $x_4 = x_2 + x_3$. Recall that $Y_4(x_1, \cdots, x_4) \in \mathcal{S}(\Sigma, G')$ if and only if $0, x_1, \cdots, x_4$ are in general position, which implies in particular $x_1 \neq 0$. Hence $\phi^{-1}(u)$ corresponds to a boundary point. $\square$

Denote $S = Y_4'(x_1, \cdots, x_4)$ the blow-up of $\mathbb{F}_1$ at 4 points $x_1, \cdots, x_4$ on Σ, with $x_1 = 0$ and $x_4 = x_2 + x_3$. Let $l_1, \cdots, l_4$ be the corresponding exceptional classes. We give the following definition.

Definition 4.24 A G_2-*exceptional system* on S is an ordered triple (e_1, e_2, e_3, e_4) of exceptional divisors such that $e_i \cdot e_j = 0 = e_i \cdot f, i \neq j$ and $y_1 = 0,\ y_4 = y_2 + y_3$ where $y_i = e_i \cdot \Sigma$. A G_2-*configuration* on S is a G_2-exceptional system $\zeta_{G_2} = (e_1, e_2, e_3, e_4)$ such that we can consider S as a blow-up of $\mathbb{F}_1$ at these 4 points $y_1 =$

$0, y_2, y_3, y_4$ on Σ, that is $S = Y_4'(y_1 = 0, y_2, y_3, y_4)$, with corresponding exceptional divisors e_1, e_2, e_3, e_4. When S has a G_2-configuration (of course $\Sigma \in |-K_S|$), we call S a *(rational) surface with a G_2-configuration*. For $S = Y_4'(x_1, \cdots, x_4)$ with $x_1 = 0$ and $x_4 = x_2 + x_3$, when $x_1, \pm x_2, \pm x_3, \pm x_4$ are distinct points on Σ, any G_2-exceptional system on S consists of exceptional curves. Such a surface is called a *G_2-surface*. So a G_2-surface must have a G_2-configuration. These four points $x_1, x_2, x_3, x_4 \in \Sigma$ are said to be *in general position*.

A G_2-configuration is illustrated in Figure 6.

Lemma 4.25 *(i) Let $S = Y_4'(x_1, \cdots, x_4)$ with $x_1 = 0$ and $x_4 = x_2 + x_3$ be a surface with a G_2-configuration. Then the Weyl group $W(G_2)$ acts on all G_2-exceptional systems on S simply transitively.*

(ii) Let S be a G_2-surface. Then the Weyl group $W(G_2)$ acts on all G_2-configurations on S simply transitively.

Proof. It suffices to prove (i). By an explicit computation, there are 12 G_2-configurations: (l_1, l_2, l_3, l_4), $(f - l_1, f - l_2, f - l_3, f - l_4)$, $(f - l_1, f - l_2, l_4, l_3)$, $(f - l_1, l_4, f - l_2, l_3)$, and so on. The rule is keeping the relation $x_2 + x_3 = x_4$ fixed. The Weyl group $W(G_2)$ is the automorphism group of the sub-root system A_2 with simple roots $\{3(l_2 - l_3), 3(l_3 - (f - l_4))\}$, so $W(G_2) \cong \mathbb{Z}_2 \rtimes W(A_2) = \mathbb{Z}_2 \rtimes S_3$. We can also consider $W(G_2)$ as the subgroup of $W(D_4)$ generated by two elements $S_{\alpha_1} S_{\alpha_2} S_{\alpha_4}$ and S_{α_3}, where S_α means the reflection with respect to a root α of D_4, according to Remark 4.3. Thus we can directly check that $W(G_2)$ acts on all G_2-exceptional systems simply transitively. $\square$

Proposition 4.26 *Let $\mathcal{S}(\Sigma, G_2)$ be the moduli space of pairs (Y_4', Σ) where Y_4' is a G_2-surface, and $\mathcal{M}_\Sigma^{G_2}$ be the moduli space of flat G_2 bundles over Σ. Then we have*

(i) $\mathcal{S}(\Sigma, G_2)$ is embedded into $\mathcal{M}_\Sigma^{G_2}$ as an open dense subset.

(ii) Moreover, this embedding can be extended naturally to an isomorphism

$$\overline{\mathcal{S}(\Sigma, G_2)} \cong \mathcal{M}_{\Sigma}^{G_2},$$

by including all rational surfaces with G_2-configurations.

Proof. We just note that only the following two things are different from their counterparts of the proofs in B_n, C_n cases.

(i) Take a simple root system of G_2 as (Remark 4.2)

$$\Delta(G_2) = \{\beta_1 = f - 2l_2 + l_3 - l_4,\ \beta_2 = 3(l_2 - l_3)\}.$$

(ii) Then the restriction to Σ gives us the following system of linear equations:

$$\begin{cases} 3x_2 = -p_1, \\ 3(x_2 - x_3) = p_2. \end{cases} \qquad \square$$

As before, we construct a Lie algebra bundle of G_2-type over $S = Y_4'$. For brevity, denote $\varepsilon_1 = l_2,\ \varepsilon_2 = l_3,$ and $\varepsilon_3 = f - l_4$. Then

$$\mathscr{G}_2 = \mathcal{O}^{\bigoplus 2} \bigoplus_{D \in R(G_2)} \mathcal{O}(D),$$

where $R(G_2)$ is the root system of G_2:

$$R(G_2) = \{\pm 3(\varepsilon_i - \varepsilon_j), \pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k) \mid i \neq j \neq k, 1 \leq i, j, k \leq 3\},$$

according to Remark 4.2.

Recall [20] the 3 fundamental representation bundles of rank 8 of D_4 are defined as:

$$\begin{cases} \mathscr{W}_4 = \bigoplus_{C^2 = C \cdot K = -1, C \cdot f = 0} \mathcal{O}(C), \\ \mathcal{S}_4^+ = \bigoplus_{D^2 = D \cdot K = -1, D \cdot f = 1} \mathcal{O}(D), \\ \mathcal{S}_4^- = \bigoplus_{T^2 = -2, T \cdot K = 0, T \cdot f = 1} \mathcal{O}(T). \end{cases}$$

These conditions $x_1 = 0,\ x_4 = x_2 + x_3$ enable us to identify S_4^+, S_4^- and $\mathscr{W}_4$ when restricted to Σ, by

$$S_4^+ \otimes \mathcal{O}(-l_1) \cong S_4^- \text{ and } S_4^+ \cong \mathscr{W}_4 \otimes \mathcal{O}(s).$$

And these identifications determine uniquely a flat G_2 bundle over Σ. Conversely, the identification of these three bundles restricted to Σ implies the conditions $x_1 = 0$ and $x_4 = x_2 + x_3$ (up to renumbering). Note that

$$\begin{aligned}
\mathscr{W}_4|_\Sigma &= \bigoplus \mathcal{O}_\Sigma(l_i) \bigoplus \mathcal{O}_\Sigma(f - l_i) = \bigoplus \mathcal{O}((x_i)) \bigoplus \mathcal{O}((-x_i)),\\
\mathcal{S}_4^-|_\Sigma &= \bigoplus_i \mathcal{O}((0) - (x_i)) \bigoplus_j \mathcal{O}(3(0) - \sum_{i \neq j}(x_i)), \text{ and}\\
\mathcal{S}_4^+|_\Sigma &= \mathcal{O}((0)) \bigoplus_{i \neq j} \mathcal{O}((-x_i - x_j)) \bigoplus \mathcal{O}((-\sum x_i)).
\end{aligned}$$

So $\mathscr{W}_4|_\Sigma = \mathcal{S}_4^-$ implies $x_1 = 0$, and $\mathscr{W}_4|_\Sigma = \mathcal{S}_4^+$ implies $x_4 = x_2 + x_3$.

4.2.4 The F_4 Bundles

First we recall some fundamental facts on E_6 root systems and cubic surfaces, which are of independent interest.

The Root System of E_6, Revisited

The relation between the root system of E_6-type and smooth cubic surfaces in $\mathbb{CP}^3$ has been studied for a very long time [15][7][24]. There are 27 *lines* on such a cubic surface S (a curve on S is a line if and only if it is an exceptional curve). And every E_6-*exceptional system* on S is an ordered 6-tuples of lines $(e_1, \cdots, e_6)$ which are pairwise disjoint. The Weyl group $W(E_6)$ is the symmetry group of all E_6-exceptional systems, that is, $W(E_6)$ acts simply transitively on the set of all E_6-exceptional systems. Now we consider the unordered 6-tuple $L = \{e_1, \cdots, e_6\}$.

There are 72 such 6-tuples. This corresponds to 36 *Schläfli's double-sixes* $\{L; L'\}$ [15]. In the following we consider a cubic surface S as the blow-up of $\mathbb{P}^2$ at 6 points $x_1, \cdots, x_6$ in general position, that is $S = X_6(x_1, \cdots, x_6)$, with corresponding exceptional curves $l_1, \cdots, l_6$. Fix a simple root system of E_6 as

$$\Delta(E_6) = \{\alpha_1, \cdots, \alpha_6\},$$

where $\alpha_1 = l_1 - l_2$, $\alpha_2 = l_2 - l_3$, $\alpha_3 = h - l_1 - l_2 - l_3$, and $\alpha_i = l_{i-1} - l_i$, for $i = 4, 5, 6$.

Lemma 4.27 *One double-six $\{L; L'\}$ corresponds to exactly one positive root of E_6.*

Proof. First take $L_0 = \{l_1, \cdots, l_6\}$, then $L'_0 = \{l'_1, \cdots, l'_6\} = s_{\alpha_0}(L_0)$ where $\alpha_0 = 2h - \sum l_i$ is a positive root and $l'_i = s_{\alpha_0}(l_i) = 2h - \sum_{j \neq i} l_j$. $\{L_0; L'_0\}$ forms a double-six and $\alpha_0 (\succ 0)$ is uniquely determined by $\{L_0; L'_0\}$, since $W(E_6)$ acts simply and transitively. If $L = g(L_0)$ with $g \in W(E_6)$, then $\{g(L_0); g(L'_0)\}$ is also a double-six. Let $g(L'_0) = S_\alpha(g(L_0))$, then $L'_0 = (g^{-1} S_\alpha g)(L_0)$. So $g^{-1} S_\alpha g = S_{\alpha_0}$. Then $S_\alpha = g S_{\alpha_0} g^{-1} = S_{g(\alpha_0)}$. This implies $\alpha = \pm g(\alpha_0)$. Take $\alpha \succ 0$. Now if $\alpha = \alpha_0$, then by a result in page 44 of [17], $g \in S_6$, that is, g is a permutation of the six lines l_i's. Thus $\{L; L'\}$ and $\{L_0; L'_0\}$ are the same one. □

Remark 4.28 Let ρ be an outer automorphism of E_6 of order 2, such that $\rho(\alpha_1) = \alpha_6, \rho(\alpha_2) = \alpha_5$ and ρ fixes other simple roots. Consider F_4 as the fixed part of E_6 by ρ. Then the coroot lattice $\Lambda_c(F_4)$ of F_4 is

$$\begin{aligned}
\Lambda_c(F_4) &= \Lambda_c(E_6)^\rho \\
&= \Lambda(E_6)^\rho \\
&= \{ah + \sum a_i l_i \mid a_1 + a_6 = a_2 + a_5 = a_3 + a_4 = -a\} \\
&= \mathbb{Z}\langle h - l_1 - l_2 - l_3, l_1 - l_6, l_2 - l_5, l_3 - l_4 \rangle \\
&= \Lambda(D_4).
\end{aligned}$$

And the Weyl group of F_4 is

$$\begin{aligned} W(F_4) &= \{w \in W(E_6) \mid w \text{ preserves } \Lambda_c(F_4) = \Lambda(D_4)\} \\ &= Aut(\Lambda(D_4)) \\ &= S_3 \rtimes W(D_4). \end{aligned}$$

Remark 4.29 If 3 lines e_1, e_2, e_3 pairwise intersect, we say that they form a *triangle*. Denote by $\Delta = \{e_1, e_2, e_3\}$ a (unordered) triangle, and by $\overrightarrow{\Delta} = (e_1, e_2, e_3)$ an ordered triangle. Every line belongs to 5 triangles, so there are $27 \cdot 5/3 = 45$ triangles. And if $\{e_1, e_2, e_3\}$ is a triangle, then $-K = e_1 + e_2 + e_3$. $W(E_6)$ acts on all these 45 triangles transitively, and $W(F_4)$ is the isotropy subgroup of the triangle $\Delta_0 = \{h - l_1 - l_6, h - l_2 - l_5, h - l_3 - l_4\}$. Moreover $W(D_4)$ is the isotropy subgroup of the ordered triangle $\overrightarrow{\Delta}_0 = (h - l_1 - l_6, h - l_2 - l_5, h - l_3 - l_4)$. The reason is the following:

Let $\Delta = \{e_1, e_2, e_3\}$ and $\Delta' = \{f_1, f_2, f_3\}$ be any two triangles. Since $K^2 = 3$, the position of these two triangles must be one of the following two cases. (1) They have a common edge and other edges don't intersect. (2) Each edge of Δ intersects with exactly one edge of Δ'. So we just check two special triangles in above cases. what remains to do is a direct checking.

From above we can easily write down the 45 (left or right) cosets of $W(F_4)$ in $W(E_6)$.

F_4 Bundles and Rational Surfaces

For $G = F_4$ we take $G' = E_6$, such that $F_4 = (E_6)^{Out(E_6)}$.

Let $S = X_6(x_1, \cdots, x_6)$ be a surface with an E_6-configuration (Figure 7), that is, S is a blow-up of $\mathbb{P}^2$ at 6 points $x_1, \cdots, x_6 \in \Sigma$, where $\Sigma \in |-K_S|$. Take the simple root system $\Delta(E_6)$ and $\rho \in Out(E_6)$ just as in Section 2.4.1.

Once a simple root system is fixed, the restriction from S to Σ induces a homomorphism $u \in Hom(\Lambda(E_6), \Sigma)$.

Lemma 4.30 *Let $u \in Hom(\Lambda(E_6), \Sigma)$ be an element corresponding to a pair (S, Σ), where S is a surface with an E_6-configuration. Then $\rho \cdot u = u$ if and only if $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$.*

Proof. Since u is induced by the restriction to Σ, $u(\alpha_1) = \mathcal{O}(l_1 - l_2)|_\Sigma = x_1 - x_2$, $u(\alpha_2) = x_2 - x_3$, $u(\alpha_5) = x_4 - x_5$, $u(\alpha_6) = x_5 - x_6$. Therefore $\rho \cdot u = u \Leftrightarrow u(\alpha_1) = u(\alpha_6), u(\alpha_2) = u(\alpha_5) \Leftrightarrow x_1 + x_6 = x_2 + x_5 = x_3 + x_4$. □

Denote $\mathcal{S}(\Sigma, E_6)$ the moduli space of $G' = E_6$-surfaces with a fixed anti-canonical curve Σ, and $\overline{\mathcal{S}(\Sigma, E_6)}$ the natural compactification by including all rational surfaces with E_6-configurations. From Chapter 3 we know that there is an isomorphism $\phi : \overline{\mathcal{S}(\Sigma, E_6)} \xrightarrow{\sim} \mathcal{M}_\Sigma^{E_6}$. Thus we have

Corollary 4.31 *For $u \in \mathcal{M}_\Sigma^{F_4} \subset (\mathcal{M}_\Sigma^{E_6})^{Out(E_6)}$, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, E_6)}$ represents a class of surfaces $X_6(x_1, \cdots, x_6)$ with $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$.* □

Denote $S = X_6'(x_1, \cdots, x_6)$ the blow-up of $\mathbb{P}^2$ at 6 points $x_1, \cdots, x_6$ on Σ which satisfies the condition $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$, with corresponding exceptional classes $l_1, \cdots, l_6$. The condition $x_1 + x_6 = x_2 + x_5 = x_3 + x_4 := p$ implies that the three lines L_{16}, L_{25} and L_{34} in $\mathbb{P}^2$ intersect at one points $-p \in \Sigma$, where L_{ij} means the line in $\mathbb{P}^2$ passing through these two points x_i and x_j. So after blowing up $\mathbb{P}^2$ at $x_i \in \Sigma, 1 \leq i \leq 6$, the three (-1) curves $h - l_1 - l_6, h - l_2 - l_5$ and $h - l_3 - l_4$ intersect at one points $-p \in \Sigma$. So they form a special triangle (see Section 2.4.1). As before, we give the following definition.

Definition 4.32 An F_4-*exceptional system* on $S = X_6'$ is a 6-tuple $(e_1, \cdots, e_6)$ consisting of 6 exceptional divisors which are pairwise disjoint, such that $y_1 + y_6 = y_2 + y_5 = y_3 + y_4$, where $\mathcal{O}_\Sigma(y_i) = \mathcal{O}(e_i)|_\Sigma$. And an F_4-*configuration* $\zeta_{F_4} = (e_1, \cdots, e_6)$ just means an F_4-exceptional system on S such that we can consider S as a blow-up of $\mathbb{P}^2$ at 6 points $y_1, \cdots, y_6$ with corresponding exceptional divisors

$e_1, \cdots, e_6$. For $S = X_6'(x_1, \cdots, x_6)$, when $x_1, \cdots, x_6$ are in general position, any F_4-exceptional system on S consists of exceptional curves. Such a surface is called an *F_4-surface.*

So an F_4-surface is automatically an E_6-surface (namely, a del Pezzo surface of degree 3). And any F_4-exceptional system on an F_4-surface is always an F_4-configuration. See Figure 8 for an F_4-configuration.

According to the discussions in Section 2.4.1, the Weyl group $W(F_4)$ is the automorphism group of the sub-root system of type D_4 with simple roots $\{l_1 - l_6, l_2 - l_5, l_3 - l_4, h - l_1 - l_2 - l_3\}$, and $W(F_4) \cong S_3 \rtimes W(D_4)$. Therefore we have

Lemma 4.33 *(i) Let $S = X_6'$ be a surface with an F_4-configuration. Then the Weyl group $W(F_4)$ acts on all F_4-exceptional systems on S simply transitively.*

(ii) Moreover, if S is an F_4-surface, then the Weyl group $W(F_4)$ acts on all F_4-configurations on S simply transitively. □

Proposition 4.34 *Let $\mathcal{S}(\Sigma, F_4)$ be the moduli space of pairs (X_6', Σ) where X_6' is an F_4-surface containing Σ as an anti-canonical curve, and $\mathcal{M}_\Sigma^{F_4}$ be the moduli space of flat F_4 bundles over Σ. Then we have*

(i) $\mathcal{S}(\Sigma, F_4)$ is embedded into $\mathcal{M}_\Sigma^{F_4}$ as an open dense subset.

(ii) Moreover, this embedding can be extended naturally to an isomorphism

$$\overline{\mathcal{S}(\Sigma, F_4)} \cong \mathcal{M}_\Sigma^{F_4},$$

by including all rational surfaces with F_4-configurations.

Proof. Firstly, we can take the simple root system of F_4 as

$$\Delta(F_4) = \{\beta_1, \beta_2, \beta_3, \beta_4\},$$

where $\beta_1 = l_1 - l_2 + l_5 - l_6$, $\beta_2 = l_2 - l_3 + l_4 - l_5$, $\beta_3 = 2(h - l_1 - l_2 - l_3)$, and $\beta_4 = 2(l_3 - l_4)$, according to Remark 4.2.

Secondly, the restriction to Σ induces the following system of linear equations:

$$\begin{cases} x_1 - x_2 + x_5 - x_6 = p_1, \\ x_2 - x_3 + x_4 - x_5 = p_2, \\ 2(-x_1 - x_2 - x_3) = p_3, \\ 2(x_3 - x_4) = p_4, \\ x_1 + x_6 = x_2 + x_5 = x_3 + x_4. \end{cases}$$

Since the determinant is non-zero, the result follows by the same argument as in B_n case. $\square$

The Lie algebra bundle of type F_4 over X_6' can be constructed as (for brevity, we denote $\varepsilon_1 = l_2 - l_3 + l_4 - l_5$, $\varepsilon_2 = l_2 + l_3 - l_4 - l_5$, $\varepsilon_3 = 2h - 2l_1 - l_2 - l_3 - l_4 - l_5$, and $\varepsilon_4 = 2h - 2l_6 - l_2 - l_3 - l_4 - l_5$)

$$\mathscr{F}_4 = \mathcal{O}^{\bigoplus 4} \bigoplus_{D \in R(F_4)} \mathcal{O}(D),$$

where $R(F_4)$ is the root system of F_4:

$$R(F_4) = \{\pm\varepsilon_i,\ \pm(\varepsilon_i \pm \varepsilon_j),\ \pm\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \mid i \neq j\}.$$

Remark 4.35 The 27 lines determine the 27-dimensional fundamental representation of E_6. Restricted to Σ, they give us a representation bundle of rank 27 (of $\mathscr{F}_4$) over Σ. The weights associated to the 3 special lines $h - l_1 - l_6, h - l_2 - l_5, h - l_3 - l_4$ restrict to zero and these 3 weights add to zero before restriction (since $(h - l_1 - l_6) + (h - l_2 - l_5) + (h - l_3 - l_4) = -K$). The remaining 24 weights associated to other 24 lines restrict to the 24 short roots of $\mathscr{F}_4$. The 24 lines and a rank 2 bundle V determine the 26-dimensional irreducible fundamental representation U of $\mathscr{F}_4$. Here V is determined as follows. Since $\mathcal{O}_\Sigma(h - l_1 - l_6) = \mathcal{O}_\Sigma(h - l_2 - l_5) = \mathcal{O}_\Sigma(h - l_3 - l_4) = \mathcal{O}_\Sigma((-p))$, taking the trace, we have the following exact sequence:

$$0 \to ker(tr) \to \mathcal{O}_\Sigma((-p))^{\bigoplus 3} \to \mathcal{O}_\Sigma((-p)) \to 0.$$

Then we take $V = ker(tr)$.

For more details on the 26-dimensional fundamental representation of F_4, one can consult [1].

4.3 Conclusion

Let G be any simple, compact and simply connected Lie group. Then G is classified into the following 7 types according to its Lie algebra.

(1) A_n-type, $G = SU(n+1)$;

(2) B_n-type, $G = Spin(2n+1)$;

(3) C_n-type, $G = Sp(n)$;

(4) D_n-type, $G = Spin(2n)$;

(5) E_n-type, $n = 6, 7, 8$;

(6) F_4-type;

(7) G_2-type.

Among these, A_n, D_n and E_n are called of simply laced type, while B_n, C_n, F_4 and G_2 are called of non-simply laced type. And A_n, B_n, C_n, D_n are called classic Lie groups, while E_n, F_4 and G_2 are called exceptional Lie groups.

We summarize our results in this thesis as follows. Let Σ be a fixed elliptic curve with identity $0 \in \Sigma$. Let G be any compact, simple and simply connected Lie groups, simply laced or not. Denote $\mathcal{S}(\Sigma, G)$ the moduli space of G-surfaces containing a fixed anti-canonical curve Σ. Denote $\mathcal{M}_\Sigma^G$ the moduli space of flat G bundles over Σ. Then we have

Theorem 4.36 *(i) We can construct Lie algebra $Lie(G)$-bundles over each G-surface.*

(ii) The restriction of these Lie algebra bundles to the anti-canonical curve Σ induces an embedding of $\mathcal{S}(\Sigma, G)$ into $\mathcal{M}_\Sigma^G$ as an open dense subset.

(iii) This embedding can be extended to an isomorphism from $\overline{\mathcal{S}(\Sigma, G)}$ *onto* $\mathcal{M}_{\Sigma}^{G}$*, where* $\overline{\mathcal{S}(\Sigma, G)}$ *is a natural and explicit compactification of* $\mathcal{S}(\Sigma, G)$*, by including all rational surfaces with* G*-configurations.* □

Remark 4.37 (i) The result is known for $G = E_n$ case (see [8][9][11][13]).

(ii) We have mentioned in the beginning of § 1 that there is another reduction of the non-simply laced cases to simply laced cases. In fact, using this reduction, we will obtain the same result, just following the steps as above.

According to Looijenga's theorem [21][22], the moduli space $\mathcal{S}(\Sigma, G)$ is a weighted projective space. Thus the compactification $\overline{\mathcal{S}(\Sigma, G)}$ is a weighted projective space. Conversely, we believe that the above identification between $\mathcal{S}(\Sigma, G)$ and $\overline{\mathcal{S}(\Sigma, G)}$ will give us another proof for Looijenga's theorem. This is already done in E_n case by [8][9][11][13] and so on.

Bibliography

[1] J.F. Adams, *Lectures on Exceptional Lie Groups*, The University of Chicago Press, 1996.

[2] M. F. Atiyah, *Vector Bundles over an Elliptic Curve*, Proc. London Math. Soc. (3), 7 (1957), 414–452.

[3] N. Bourbaki, *Lie groups and Lie algebras*, Chapters 4–6, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002.

[4] W. Barth, K. Hulek, C. Peters, A. Van de Ven, *Compact complex surfaces*, Second edition, Springer-Verlag, Berlin, 2004.

[5] A. Borel, *Sous-groupes Commutatifs et Torsion des Groupes de Lie Compacts Connexes*, Tohoku Journal Ser. 2 (1961), 13, 216–240.

[6] A. Clingher, J.W. Morgan, Mathematics underlying the F-theory/heterotic string duality in eight dimensions, Comm. Math. Phy. 254 (2005), no. 3, 513–563.

[7] M. Demazure, *Séminaire sur les Singularités des Surfaces*, LNM 777, Springer, Berlin, 1980.

[8] R. Donagi, *Principal Bundles on Elliptic Fibrations*, Asian J. Math. 1 (1997), no. 2, 214–223.

[9] R. Donagi, *Taniguchi lectures on principal bundles on elliptic fibrations*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 33–46, World Sci. Publ., River Edge, NJ, 1998.

[10] R. Friedman, A new proof of the global Torelli theorem for $K3$ surfaces, Ann. of Math. (2), 120 (1984), no. 2, 237–249.

[11] R. Friedman, J. Morgan, *Holomorphic principal bundles over elliptic curves*, preprint, math.AG/9811130.

[12] R. Friedman, J. Morgan, *Exceptional groups and del Pezzo surfaces*, Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000), 101–116, Contemp. Math., 312, Amer. Math. Soc., Providence, RI, 2002.

[13] R. Friedman, J. Morgan, E. Witten, *Vector Bundles and F-theory*, Comm. Math. Phys. 187 (1997), no. 3, 679–743.

[14] R. Friedman, J. Morgan, E. Witten, *Principal G-bundles over elliptic curves*, Math. Res. Lett. 5 (1998), no. 1-2, 97–118.

[15] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.

[16] J.E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York-Berlin, 1978.

[17] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, Cambridge, 1990.

[18] V.G. Kac, *Infinite-dimensional Lie algebras*, Cambridge University Press, Cambridge, 1985.

[19] V.S. Kulikov, Degenerations of $K3$ surfaces and Enriques surfaces, Izv. Akad. Nauk SSSR ser. Mat., 41(5) (1977), 1008–1042.

[20] N.C. Leung, *ADE-bundle over rational surfaces, configuration of lines and rulings*, preprint, arXiv: math.AG/0009192.

[21] E. Looijenga, *Root systems and elliptic curves*, Invent. Math. 38 (1976/77), no. 1, 17–32.

[22] E. Looijenga, *Invariant theory for generalized root systems*, Invent. Math. 61 (1980), no. 1, 1–32.

[23] E. Looijenga, C. Peters, Torelli theorem for Kähler $K3$ surfaces, Compositio Math., 42(2) (1980/81), 145–186.

[24] Y.I. Manin, *Cubic forms: algebra, geometry, arithmetic*, American Elsevier Publishing Co., New York, 1974.

[25] M.S. Narasimhan, C.S. Seshadri, *Moduli of vector bundles on a compact Riemann surface*, Ann. of Math. (2) 89 (1969) 14–51.

[26] U. Persson, H. Pinkham, Degeneration of surfaces with trivial canonical bundle, Ann. of Math. (2), 113(1) (1981), 45–66.

[27] A. Ramanathan, *Stable principal bundles on a compact Riemann surface*, Math. Ann. 213 (1975), 129–152.

[28] I.I. Pjateckiĭ-Šapiro, I.R. Šafarecič, Torelli's theorem for algebraic surfaces of type $K3$, Izv. Akad. Nauk SSSR ser. Mat., 35 (1971), 530–572.

[29] L. Tu, *Semistable bundles over an elliptic curve*, Adv. Math., 98(1993), 1–26.