Uniqueness Theorem of the Mean Curvature Flow

by

YIN Le

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Abstract

Mean curvature flow evolves isometrically immersed base Riemannian manifolds M in the direction of their mean curvature in an ambient manifold $\bar{M}$. We consider the classical solutions to the mean curvature flow. If the base manifold M is compact, the short time existence and uniqueness of the mean curvature flow are well-known. For complete noncompact isometrically immersed hypersurfaces M (uniformly local lipschitz) in Euclidean space, the short time existence was established by Ecker and Huisken in [10]. The short time existence and the uniqueness of the solutions to the mean curvature flow of complete isometrically immersed manifolds of arbitrary codimensions in the Euclidean space are still open questions. In this thesis, we solve the uniqueness problem affirmatively for the mean curvature flow of general codimensions and general ambient manifolds. More precisely, let $(\bar{M}, \bar{g})$ be a complete Riemannian manifold of dimension $\bar{n}$ such that the curvature and its covariant derivatives up to order 2 are bounded and the injectivity radius is bounded from below by a positive constant, we prove that the solution of the mean curvature flow with bounded second fundamental form on an isometrically immersed manifold M (may be of high codimension) is unique. In the second part of the thesis, inspired by the Ricci flow, we prove the pseudolocality theorem of mean curvature flow. As a consequence, we obtain the strong uniqueness theorem, which removes the boundedness assumption of the second fundamental form of the solution in the uniqueness theorem (only assume the second fundamental form of the initial submanifold is bounded).

摘要

本論文研究平均曲率流的經典解的唯一性問題。平均曲率流使得黎曼流形$\bar{M}$中的一個等距浸入子流形M沿著它的平均曲率向量流動。當子流形M是緊的時候，平均曲率流的短時間存在性和唯一性經已熟知。對於歐氏空間中的完備非緊的等距浸入超曲面M （並且是一致局部李普希茲連續的），短時間存在性已經由Ecker和Huisken建立。但是對於歐氏空間中的任意餘維數的完備非緊的等距浸入子流形，平均曲率流經典解的短時間存在性和唯一性問題至今仍未解決。本論文就一般黎曼流形中的任意餘維數的完備非緊子流形對此唯一性問題作了肯定回答。具體地說，設$(\bar{M}, \bar{g})$是一個n維的完備黎曼流形，它的曲率和直到二階的曲率的協變導數均有界，並且它的單射半徑有一個正的下界，本文證明，對於任意餘維數的子流形M，平均曲率流的第二基本形式有界的經典解是唯一的。在本文的第二部分，受Ricci流理論的啟發，我們將證明對應於Ricci流理論的平均曲率流的Pseudolocality定理。作為Pseudolocality定理的推論，我們得到了平均曲率流的強唯一性定理，此定理去掉了上述唯一性定理的第二基本形式有界的假設（僅須假設初始子流形的第二基本形式有界）。

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Contents

Chapter 1

Introduction

Let $(\bar{M}^{\bar{n}}, \bar{g})$ be a complete Riemannian (compact or noncompact) manifold, and $X_0 : (M^n, g) \to \bar{M}^{\bar{n}}$ be an isometrically immersed Riemannian manifold. For any fixed point $x_0 \in M^n$, $X, Y \in T_{x_0}M^n$, the second fundamental form II at x_0 is defined by $II(X,Y) = \bar{\nabla}_{\tilde{X}}\tilde{Y} - \nabla_{\tilde{X}}\tilde{Y} = (\bar{\nabla}_{\tilde{X}}\tilde{Y})^{\perp}$, where M^n is regarded as a submanifold of $\bar{M}$ locally by the immersion X_0, $\bar{\nabla}$ and ∇ are the covariant derivatives of $\bar{g}$ and g respectively, $\tilde{X}, \tilde{Y}$ are any smooth extensions of X and Y on $\bar{M}^{\bar{n}}$. In a local coordinate system $\{x^1, x^2, \cdots, x^n\}$ on M^n, denote the second fundamental form by $h_{ij} = II(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and the mean curvature by $H = g^{ij}h_{ij}$. The mean curvature flow is a deformation $X_t : M^n \to \bar{M}^{\bar{n}}$ of X_0 in the direction of the mean curvature H:

$$\frac{\partial}{\partial t}X(x,t) = H(x,t), \qquad \text{for } x \in M^n \text{ and } \ t \geq 0, \tag{1.1}$$

with $X(x,0) = X_0(x)$, where M^n is equipped with the induced metric from $X(\cdot,t) : M^n \to \bar{M}^{\bar{n}}$ and $H(x,t)$ is the corresponding mean curvature. We can write (1.1) in another form

$$\frac{\partial}{\partial t}X(x,t) = \Delta X(x,t), \qquad \text{for } x \in M^n \text{ and } \ t \geq 0, \tag{1.2}$$

where $\Delta X^{\alpha}(x,t) = g^{ij}(x,t)(\frac{\partial^2 X^{\alpha}}{\partial x^i \partial x^j} - \Gamma^k_{ij}(t)\frac{\partial X^{\alpha}}{\partial x^k} + \bar{\Gamma}^{\alpha}_{\beta\gamma}\frac{\partial X^{\beta}}{\partial x^i}\frac{\partial X^{\gamma}}{\partial x^j})$ is the harmonic map Laplacian from the manifold $(M^n, g_{ij}(\cdot,t))$ to $(\bar{M}^{\bar{n}}, \bar{g})$, and $g_{ij}(\cdot,t)$ is the induced metric from the inclusion map $X(\cdot, t)$.

Various weak solutions to the mean curvature flow have been studied in the past 30 years by many mathematicians with different approaches, e.g. Brakke solutions[1], the level set solutions (see Ref. [18]), etc. The existence, uniqueness and non-uniqueness of weak solutions for Euclidean (non)smooth hypersurface have been extensively studied ([6],[13],[14],[15],[16]).

For classical solutions to the mean curvature flow, some remarkable geometric results are obtained. The famous Huisken's theorem[21] states that if the initial hypersurface $M^n (n \geq 2)$ is compact and uniformly convex in Euclidean space, then under the mean curvature flow it shrinks to a point in finite time and the normalized flow (area is fixed) converges to a round sphere (also see Huisken[22] for some generalizations) . If the initial hypersurface is a Lipschitz entire graph, Ecker and Huisken prove that the mean curvature flow has a long time graph solution. The corresponding one dimensional version (i.e., the curve shortening flow) of the above Huisken's Theorem was proved by Gage and Hamilton[17]. Moreover, Grayson[19] proved that the curve shortening flow starting at any closed embedded curve becomes convex before it develops singularities.

For higher codimension case, little is known since it is not so easy to control the multi-component mean curvature vector. In [32] Wang Mu-Tao considered the mean curvature flow of the graph of a map between two compact constant curvature Riemannian manifolds. Under suitable conditions on the curvature of these two manifolds and the differential of the initial map, he proved the flow exists smoothly for all time and converges to a constant map as time approaches infinity. As an application, Tsui Mao-Pei and Wang Mu-Tao [30] proved that any area-decreasing map from $\mathbb{S}^n (n \geq 2)$ to $\mathbb{S}^m$ is homotopically trivial.

In this thesis, motivated by geometric applications, we consider the classical solutions of arbitrary codimension in general ambient Riemannian manifolds.

When M^n is compact, the mean curvature flow (1.1) has a unique short time solution, since (1.2) is a quasi-linear parabolic equation. When M^n is noncompact, in codimesion 1 case, Ecker and Huisken in [10] established the short time existence for complete noncompact isometrically immersed hypersurfaces (uniformly local lipschitz) in Euclidean space. For submanifolds of arbitrary codimensions in a general ambient Riemannian manifold, the short time existence and the uniqueness of (1.1) have not been established in the literature. In this thesis, we deal with the problem of uniqueness and derive the pseudolocality estimate of the mean curvature flow (1.1).

The first theorem of this thesis is the following

Theorem 1.1 *Let $(\bar{M}^{\bar{n}}, \bar{g})$ be a complete Riemannian manifold of dimension $\bar{n}$ such that the curvature and its covariant derivatives up to order 2 are bounded and the injectivity radius is bounded from below by a positive constant, i.e. there are constants $\bar{C}$ and $\bar{\delta}$ such that*

$$|\bar{R}m| + |\bar{\nabla}\bar{R}m| + |\bar{\nabla}^2\bar{R}m|(x) \leq \bar{C}, \qquad inj(\bar{M}^{\bar{n}}, x) > \bar{\delta} > 0,$$

for all $x \in \bar{M}^{\bar{n}}$. Let $X_0 : M^n \to \bar{M}^{\bar{n}}$ be an isometrically immersed Riemannian manifold with bounded second fundamental form in $\bar{M}^{\bar{n}}$. Suppose $X_1(x,t)$ and $X_2(x,t)$ are two solutions to the mean curvature flow (1.1) on $M^n \times [0,T]$ with the same X_0 as initial data and with bounded second fundamental forms on $[0,T]$. Then $X_1(x,t) = X_2(x,t)$ for all $(x,t) \in M^n \times [0,T]$.

We remark that the uniqueness of the Ricci flow has been established by Chen and Zhu in [5]. More precisely, it was proved that the solutions of the Ricci flow in the class of bounded curvature with the same initial data are unique:

Uniqueness of Ricci Flow ([5]) *Let $(M^n, g_{ij}(x))$ be a complete noncompact Riemannian manifold of dimension n with bounded curvature. Let $g_{ij}(x,t)$ and $\bar{g}_{ij}(x,t)$ be two solutions to the Ricci flow on $M^n \times [0,T]$ with the same $g_{ij}(x)$ as initial data and with bounded curvatures. Then $g_{ij}(x,t) = \bar{g}_{ij}(x,t)$ for all $(x,t) \in M^n \times [0,T]$.*

One can find that in order to prevent the surgery times from accumulation in the theory of the Ricci flow with surgery in dimension three [27][2] and four[4], it is crucial to employ the uniqueness theorem [5]. The uniqueness theorem of mean curvature flow will also play important role in the theory of the mean curvature flow with surgery. Let us briefly describe the necessity of the theory of the mean curvature flow with surgery comparing with the full developed theory of the weak solutions. This is motivated by the theory of the Ricci flow with surgery. Recall in [1], Brakke defined a canonical weak solution to (1.1) by geometric measure theory, the so-called Brakke solution. The Brakke solution may loss area instantaneously at countably times in a finite time interval. It seems that it is hard to study the geometry and the topology of the initial manifold from the Brakke solution. This suggests that we should construct a kind of "weak" solution in a controlled way, that is to say, the mean curvature flow with surgery.

The solution of the mean curvature flow with surgery is constructed as follows. Consider the mean curvature flow on a compact manifold, the mean curvature flow may develop singularities in finite time, after carefully detecting the structure of the singularities, one will eliminate the singularities by performing a surgery by cutting off the high curvature part and gluing back other standard pieces (e.g. the standard cap), then continue to run the mean curvature flow and do the same procedure again. One of the crucial questions in this theory is to control the geometry of the glued piece after surgery. This is an important step in preventing the surgery times from accumulation. The uniqueness theorem of the mean curvature flow insures that the solutions on the glued pieces are sufficiently close to a (complete noncompact) standard solution, which is the evolution of the complete noncompact standard piece (e.g. capped round cylinder). So we can appeal to the estimates of standard solutions. In the whole procedure, the employment of the uniqueness theorem is essential. So even if we consider the mean curvature flow on compact manifolds, we still have to encounter the problem of the uniqueness on noncompact manifolds.

Since the mean curvature flow is degenerate in tangent directions, it is not a strictly parabolic system. In order to apply the standard theory of strict parabolic equations, we use the De Turck trick [8]. The idea is to pull back the mean curvature flow through a family of diffeomorphisms of the base manifold M^n generated by solving a harmonic map flow coupled with the mean curvature flow; this gives us the so-called mean curvature flow in harmonic map gauge, which is a strict parabolic system. Then we apply the uniqueness of the strict parabolic system. The issue is not quite straight forward as it seems. Because before applying the uniqueness theorem of a strict parabolic system on a noncompact manifold, we encounter two analytic difficulties. The first one is that we need to establish a short time existence for the harmonic map flow between complete manifolds. The second one is to get a priori estimates for the harmonic map flow so that after pulling back, the solutions to the strictly parabolic system still satisfy suitable growth conditions.

In the classical theory of the harmonic map flow, people usually would like to impose certain convexity conditions to ensure the existence (e.g. the negative curvature condition [12] or convex condition [9]). We observed that the condition of injectivity radius bounded from below by a positive constant ensures certain uniform (local) convexity and this is sufficient to give the short time existence and a priori estimates for the harmonic map flow. Note that the mean curvature flow is a kind of harmonic map flow with varying base metrics. In order to deal with the a priori estimates for mean curvature flow and harmonic map flow coupled with mean curvature flow, we have to consider the general harmonic map flow. These estimates have been dealt with systematically in this thesis(Sections 2.1, 2.2 and 2.3).

The difference of Theorem 1.1 with [5] is between the extrinsic and intrinsic geometries. In the present case, instead of the metric as in the Ricci flow, we consider the equation of the position function.

As a direct consequence of Theorem 1.1, we have

Corollary 1.2 *Let $(\bar{M}^{\bar{n}}, \bar{g})$ be a complete Riemannian manifold of dimension $\bar{n}$ such that the curvature and its covariant derivatives up to order 2 are bounded and the injectivity radius is bounded from below by a positive constant, i.e. there are constants $\bar{C}$ and $\bar{\delta}$ such that*

$$|\bar{R}m| + |\bar{\nabla}\bar{R}m| + |\bar{\nabla}^2\bar{R}m|(x) \le \bar{C}, \qquad inj(\bar{M}^{\bar{n}}, x) > \bar{\delta} > 0,$$

for all $x \in \bar{M}^{\bar{n}}$. Let $X_0 : (M^n, g) \to (\bar{M}^{\bar{n}}, \bar{g})$ be an isometrically immersed complete Riemannian manifold with bounded second fundamental form in $\bar{M}^{\bar{n}}$. Suppose $X_t : M^n \to \bar{M}^{\bar{n}}$ is a solution to the mean curvature flow (1.1) on $M^n \times [0, T]$ with X_0 as initial data and with bounded second fundamental forms on $[0, T]$. Let $\bar{\sigma}$ be an isometry of $(\bar{M}^{\bar{n}}, \bar{g})$ such that there is an isometry σ of (M^n, g) to itself satisfying

$$(\bar{\sigma} \circ X_0)(x) = (X_0 \circ \sigma)(x) \tag{1.3}$$

for all $x \in M^n$. Then we have

$$(\bar{\sigma} \circ X_t)(x) = (X_t \circ \sigma)(x) \tag{1.4}$$

for all $(x, t) \in M^n \times [0, T]$. In particular, the isometry subgroup of (M^n, g) induced by an isometry subgroup of $(\bar{M}^{\bar{n}}, \bar{g})$ at initial time by (1.3) remains to be an isometry subgroup of (M^n, g_t) for any $t \in [0, T]$.

From the PDE point of view, it is a natural condition in Theorem 1.1 of assuming that the second fundamental form of the solution is bounded. In Chapter 3 of the thesis, we try to remove this condition. We remark that in [7], Chou and Zhu have obtained the strong uniqueness of the curve shortening flow for the locally Lipschitz continuous properly embedded curve whose two ends are presentable as graphs over semi-infinite line. Our strong uniqueness theorem is the following

Theorem 1.3 *Let $\bar{M}$ be an $\bar{n}$-dimensional complete Riemannian manifold satisfying $\sum_{i=0}^{3} |\bar{\nabla}^i \bar{R}m| \leq c_0^2$ and $inj(\bar{M}) \geq i_0 > 0$. Let $X_0 : M \to \bar{M}$ be an n-dimensional isometrically properly embedded submanifold with bounded second fundamental form in $\bar{M}$. We assume $X_0(M)$ is uniform graphic with some radius $r > 0$. Suppose $X_1(x,t)$ and $X_2(x,t)$ are two smooth solutions to the mean curvature flow (1.1) on $M \times [0, T_0]$ properly embedded in $\bar{M}$ with the same X_0 as initial data. Then there is $0 < T_1 \leq T_0$ such that $X_1(x,t) = X_2(x,t)$ for all $(x,t) \in M \times [0, T_1]$.*

Here roughly speaking, uniform graphic with radius $r > 0$ means that for any $x_0 \in X_0(M)$, $X_0(M) \cap B_{\bar{M}}(x_0, r)$ is a graph. We say a submanifold $M \subset \bar{M}$ is properly embedded in a ball $B_{\bar{M}}(x_0, r_0)$ if either M is closed or ∂M has distance $\geq r_0$ from x_0. A submanifold $M \subset \bar{M}$ is said to be properly embedded in (complete manifold)$\bar{M}$ if either M is closed or there is an $x_0 \in \bar{M}$ such that M is properly embedded in $B_{\bar{M}}(x_0, r_0)$ for any $r_0 > 0$.

The strong uniqueness theorem was proved as a consequence of Theorem 1.1 and pseudolocality theorem.

The pseudolocality theorem says that the behavior of the solution at a point can be controlled by the initial data at nearby points, whatever how the solution or initial data outside the neighborhood behaves like. Precisely the following theorem is proved in the thesis:

Theorem 1.4 *Let $\bar{M}$ be an $\bar{n}$-dimensional manifold satisfying $\sum_{i=0}^{3} |\bar{\nabla}^i \bar{R}m| \leq c_0^2$ and $inj(\bar{M}) \geq i_0 > 0$. Then for every $\alpha > 0$ there exist $\varepsilon > 0$, $\delta > 0$ depending only on the constants $\bar{n}$, c_0 and i_0 with the following property. Suppose we have a smooth solution to the mean curvature flow $M_t \subset \bar{M}$ properly embedded in $B_{\bar{M}}(x_0, r_0)$ for $t \in [0, T]$, where $0 < T \leq \varepsilon^2 r_0^2$, and assume that at time zero, M_0*

is a local δ-Lipschitz graph of radius r_0 at $x_0 \in M$ with $r_0 \leq \frac{i_0}{2}$. Then we have an estimate of the second fundamental form

$$|A|(x,t)^2 \leq \frac{\alpha}{t} + (\varepsilon r_0)^{-2}$$

on $B_{\bar{M}}(x_0, \varepsilon r_0) \cap M_t$, for any $t \in [0,T]$.

We refer the reader to see the precise definition of δ-Lipschitz graph in Chapter 3. The third covariant derivative of the curvature is a technical assumption which could be improved, we assume it only for simplicity. For most of interesting cases, we have all covariant derivative bounds.

We remark that for codimension one uniformly local Lipschitz hypersurface in Euclidean space, the estimate was firstly derived by Ecker and Huisken [10]. For higher codimension case, under an additional condition which assumes that the submanifold is compact, the estimate was proved by Wang Mu-Tao[31]. In codimension one case [10], the constant δ in Theorem 1.4 does not need to be small; however, in higher codimension case, as noted by [31], the smallness assumption is necessary in view of the example of Lawson and Osserman [24]. The strategy of the proofs of [10] [31] is to find a suitable gradient function. The philosophy is that this gradient function will serve as the lower order quantity as in the Bernstein trick, and the second fundamental form is the higher order quantity, then apply the maximum principle.

Our approach is completely different. This approach can be regarded as an integral version of Bernstein trick. It is a mean curvature flow analogue of the corresponding estimate in Ricci flow given by Perelman [26].

As a nontrivial corollary of Theorem 1.4, we have

Corollary 1.5 *Let $\bar{M}$ be an $\bar{n}$-dimensional complete manifold satisfying $\sum_{i=0}^{3} |\bar{\nabla}^i \bar{R}m| \leq c_0^2$ and $inj(\bar{M}) \geq i_0 > 0$. Let $X_0 : M \to \bar{M}$ be an n-dimensional isometrically properly embedded submanifold with bounded second fundamental form $|A| \leq c_0$ in $\bar{M}$. We assume $M_0 = X_0(M)$ is uniform graphic with some radius $r > 0$. Sup-*

pose $X(x,t)$ is a smooth solution to the mean curvature flow (1.1) on $M \times [0, T_0]$ properly embedded in $\bar{M}$ with X_0 as initial data. Then there is $0 < T_1 \leq T_0$ depending upon c_0, i_0, r and the dimension $\bar{n}$ such that

$$|A|(x,t) \leq 2c_0$$

for all $x \in M$, $0 \leq t \leq T_1$.

This thesis is organized as follows. Theorem 1.1 and Corollary 1.2 are proved in Chapter 2. Explicitly, in section 2.1 we derive the injectivity radius estimate of an immersed manifold and some preliminary estimates for a general harmonic map flow. In section 2.2, the higher derivative estimates for the mean curvature flow are derived. In Section 2.3, we study the harmonic map flow coupled with the mean curvature flow. In Section 2.4, we deal with the uniqueness theorem of the mean curvature flow in harmonic map gauge. In section 2.5, we prove the uniqueness Theorem 1.1 and Corollary 1.2. In Chapter 3, we establish the pseudolocality theorems 1.4, 1.5 of the mean curvature flow and prove the strong uniqueness theorem 1.3.

Chapter 2

Uniqueness Theorem

In this chapter, we prove Theorem 1.1 and Corollary 1.2. We divide the whole proof into five sections.

2.1 Preliminary estimates

In the first part of this section, we will derive the injectivity radius estimate for any complete isometrically immersed manifold M^n with bounded second fundamental form in a complete manifold whose curvature is bounded and the injectivity radius is bounded from blow by a positive constant. The following is a basic lemma we will use.

Klingenberg's Lemma (see for example, Corollary 5.7 in Cheeger & Ebin [3]) *Let M be a complete Riemannian manifold and let $p \in M$. Let $l_M(p)$ denote the minimal length of a nontrivial geodesic loop starting and ending at p (maybe not smooth at p). Then the injectivity radius of M at p satisfies*

$$inj(M,p) \geq min\{\frac{\pi}{\sqrt{K_{max}}}, \frac{1}{2}l_M(p)\}$$

where K_{max} is the supermum of the sectional curvature on M and we understand $\frac{\pi}{\sqrt{K_{max}}}$ to be positive infinity if $K_{max} \leq 0$. □

Theorem 2.1.1 *Let $(\bar{M}^{\bar{n}},\bar{g})$ be a complete Riemannian manifold of dimension $\bar{n}$ with bounded curvature and the injectivity radius is bounded from below by a positive constant, i.e. there are constants $\bar{C}$ and $\bar{\delta}$ such that*

$$|\bar{R}m|(x)\le\bar{C}\quad\text{ and }\quad inj(\bar{M}^{\bar{n}},x)\ge\bar{\delta}>0,\quad\text{for all } x\in\bar{M}^{\bar{n}}.\tag{2.1.1}$$

Let $X:M^n\to\bar{M}^{\bar{n}}$ be a complete isometrically immersed manifold with bounded second fundamental form $|h^{\alpha}_{ij}|\le C$ in $\bar{M}^{\bar{n}}$, then there is a positive constant $\delta=\delta(\bar{C},\bar{\delta},C,\bar{n})$ such that the injectivity radius of M^n satisfies

$$inj(M^n,x)\ge\delta>0,\quad\text{for all } x\in M^n.\tag{2.1.2}$$

Proof. Fix $x_0\in M^n$, let $\{y^1,y^2,\cdots,y^{\bar{n}}\}$ and $\{x^1,x^2,\cdots,x^n\}$ be any two local coordinates of $\bar{M}^{\bar{n}}$ and M^n at $y_0(=X(x_0))$ and x_0 respectively, recall that the second fundamental form can be written in these local coordinates in the following form

$$\begin{aligned}h^{\alpha}_{ij}&=\frac{\partial^2y^{\alpha}}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial y^{\alpha}}{\partial x^k}+\bar{\Gamma}^{\alpha}_{\beta\gamma}\frac{\partial y^{\beta}}{\partial x^i}\frac{\partial y^{\gamma}}{\partial x^j}\\&=\nabla_i\nabla_j(y^{\alpha})+\bar{\Gamma}^{\alpha}_{\beta\gamma}\frac{\partial y^{\beta}}{\partial x^i}\frac{\partial y^{\gamma}}{\partial x^j},\qquad\text{for}\quad\alpha=1,2,\cdots,\bar{n},\end{aligned}\tag{2.1.3}$$

where $\nabla_i\nabla_j(y^{\alpha})$ is the Hessian of y^{α}, which is viewed as a function of M^n near x_0. In the following argument, we denote by $\bar{C}_1$ various constants depending only on $\bar{C}$, C and $\bar{\delta}$.

Define $f(x)=\bar{d}^2(y_0,X(x))$ on $M^n\cap X^{-1}(\bar{B}(y_0,\bar{C}_1))$ for some $\bar{C}_1\le\bar{\delta}$, then $\nabla_jf=\frac{\partial f}{\partial y^{\alpha}}\frac{\partial y^{\alpha}}{\partial x^j}$ and the Hessian of f with respect to the metric g on $M^n\cap X^{-1}(\bar{B}(y_0,\bar{C}_1))$ can be computed as follows

$$\begin{aligned}\nabla_i\nabla_jf&=\frac{\partial}{\partial x^i}\nabla_jf-\Gamma^k_{ij}\nabla_kf\\&=(\frac{\partial^2f}{\partial y^{\alpha}\partial y^{\beta}}-\bar{\Gamma}^{\gamma}_{\alpha\beta}\frac{\partial f}{\partial y^{\gamma}})\frac{\partial y^{\alpha}}{\partial x^j}\frac{\partial y^{\beta}}{\partial x^i}+\frac{\partial f}{\partial y^{\alpha}}(\frac{\partial^2y^{\alpha}}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial y^{\alpha}}{\partial x^k}+\bar{\Gamma}^{\alpha}_{\beta\gamma}\frac{\partial y^{\beta}}{\partial x^i}\frac{\partial y^{\gamma}}{\partial x^j})\\&=\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}\bar{d}^2\frac{\partial y^{\alpha}}{\partial x^j}\frac{\partial y^{\beta}}{\partial x^i}+2\bar{d}\bar{\nabla}_{\alpha}\bar{d}\cdot h^{\alpha}_{ij}.\end{aligned}\tag{2.1.4}$$

Using Hessian comparison theorem on $\bar{M}^{\bar{n}}$ and choosing $\bar{C}_1$ suitable small so that $\bar{d}$ is suitable small, we get

$$\nabla_i \nabla_j f \geq \frac{1}{2} g_{ij} \tag{2.1.5}$$

on $M^n \cap X^{-1}(\bar{B}(y_0, \bar{C}_1))$. Now we claim that any closed geodesic starting and ending at x_0 on (M^n, g) must have length $\geq 2\bar{C}_1$.

We argue by contradiction. Indeed, suppose we have a closed geodesic $\gamma : [0, L] \to M^n$ of length $L < 2\bar{C}_1$, $X \circ \gamma$ must be contained in $\bar{B}(y_0, \bar{C}_1)$, then by (2.1.5), we have

$$\frac{d^2}{ds^2} f \circ \gamma(s) = \nabla^2 f(\dot{\gamma}, \dot{\gamma}) \geq \frac{1}{2}, \qquad s \in [0, L]. \tag{2.1.6}$$

By the maximum principle, we have

$$\sup_{s \in [0,L]} f \circ \gamma(s) \leq f \circ \gamma(0),$$

this implies that γ is just a point $\gamma(0)$. The contradiction proves the claim.

On the other hand, by the Gauss equation,

$$R_{ijkl} = \bar{R}_{ijkl} + (h^\alpha_{ik} h^\beta_{jl} - h^\alpha_{il} h^\beta_{kj}) \bar{g}_{\alpha\beta}(\cdot, 0),$$

we see that

$$|Rm| \leq \bar{C} + 2C^2. \tag{2.1.7}$$

Finally, by Klingenberg Lemma, the injectivity radius of (M^n, g) at x_0 is given by

$$\begin{aligned} inj(M^n, g, x_0) &= \min\{\text{the conjugate radius at } x_0, \\ &\qquad \frac{1}{2} \text{ the length of the shortest closed geodesic at } x_0\} \\ &\geq \min\{\frac{\pi}{\sqrt{\bar{C} + 2C^2}}, \bar{C}_1\}. \end{aligned}$$

The proof of the theorem is completed. □

Let N be a Riemanian manifold, the distance function $d(y_1, y_2)$ can be regarded as a function on $N \times N$. In the next theorem, we will estimate the Hession of the distance function viewed as the function of two variables. The crucial computation of the Hessian was carried out in [29].

Theorem 2.1.2 *Let N^n be a complete Riemannian manifold of dimension n with bounded curvature, and the injectivity radius is bounded from below by a positive constant,*

$$|Rm| \leq K_0, \qquad inj(N^n) \geq i_0 > 0. \tag{2.1.8}$$

Let $d(y_1, y_2)$ be the distance function regarded as a function on $N \times N$, then there is a positive constant $C(K_0, i_0)$ depending only on K_0 and i_0 such that when $d(y_1, y_2) \leq \min\{\frac{i_0}{2}, \frac{1}{4\sqrt{K_0}}\}$, we have

$(i)|\nabla^2 d^2|(y_1, y_2) \leq C(K_0, i_0)$,

$$(ii)(\nabla^2 d^2)(X, X) \geq 2|X_1 - P_\gamma^{-1} X_2|^2 - C(K_0, i_0)|X|^2 d^2 \quad \text{for all } X \in T_{(y_1,y_2)} N^n \times N^n, \tag{2.1.9}$$

where $X = X_1 + X_2$, $X_1 \in T_{y_1} N^n$, $X_2 \in T_{y_2} N^n$, ∇ is the covariant derivative of $N \times N$, γ is the unique geodesic connecting y_1 and y_2 in N^n, and P_γ is the parallel translation of N^n along γ.

Proof. Set $\psi(y_1, y_2) = d^2_{N^n}(y_1, y_2)$. Then ψ is a smooth function of (y_1, y_2) when $d(y_1, y_2) \leq \min\{\frac{i_0}{2}, \frac{1}{4\sqrt{K_0}}\}$. Recall the computation of $Hess(\psi)$ in [29]. For any $(u, v) \in D = \{(u, v) : (u, v) \in N^n \times N^n, d_{N^n}(u, v) \leq \min\{\frac{i_0}{2}, \frac{1}{4\sqrt{K_0}}\}\} \setminus \{(u, u) : u \in N^n\}$, let γ_{uv} be the minimal geodesic from u to v and $e_1 \in T_u N^n$ be the tangent vector to γ_{uv} at u. Then $e_1(u, v)$ defines a smooth vector field on D. Let $\{e_i\}$ be an orthonormal basis for $T_u N^n$ which depends on u smoothly. By parallel translation of $\{e_i\}$ along γ, we define $\{\bar{e}_i\}$ an orthonormal basis for $T_v N^n$. Thus $\{e_1, \cdots e_n, \bar{e}_1, \cdots \bar{e}_n\}$ is a local frame on D. Then for any $X = X_1 + X_2 \in T_{(u,v)} D$ with

$$X_1 = \sum_{i=1}^n \xi_i e_i \quad \text{and} \quad X_2 = \sum_{i=1}^n \eta_i \bar{e}_i,$$

by the formula (16) in [29],

$$\begin{aligned}\frac{1}{2}Hess(\psi)(X,X) =& \sum_{i=1}^{n}(\xi_i-\eta_i)^2+\int_0^r t\langle\nabla_{e_1}V,\nabla_{e_1}V\rangle+\int_0^r t\langle\nabla_{\bar{e}_1}V,\nabla_{\bar{e}_1}V\rangle\\ &-\int_0^r t\langle R(e_1,V)V,e_1\rangle-\int_0^r t\langle R(\bar{e}_1,V)V,\bar{e}_1\rangle,\end{aligned} \tag{2.1.10}$$

where V is a Jacobi field on geodesic σ (connecting (v,v) to (u,v)) and $\bar{\sigma}$ (connecting (u,u) to (u,v) of length $r=\sqrt{\psi}$) with X as the boundary values, where X is extended to be a local vector field by letting its coefficients with respect to $\{e_1,\cdots e_n,\bar{e}_1,\cdots\bar{e}_n\}$ be constant(see [29]). By the Jacobi equation, we have the estimates

$$|V|\le C(K_0,i_0)|X|,\quad r|\nabla_{e_1}V|\le C(K_0,i_0)|X|,\qquad r|\nabla_{\bar{e}_1}V|\le C(K_0,i_0)|X|$$

under the assumption $d(y_1,y_2)\le\min\{\frac{i_0}{2},\frac{1}{4\sqrt{K_0}}\}$. Thus by (2.1.10) we have

$$|Hess(\psi)|\le C(K_0,i_0),$$

this proves (i). Similarly, when $d(y_1,y_2)\le\min\{\frac{i_0}{2},\frac{1}{4\sqrt{K_0}}\}$, by (2.1.10), we have

$$\begin{aligned}\frac{1}{2}Hess(\psi)(X,X) &\ge \sum_{i=1}^{n}(\xi_i-\eta_i)^2-\int_0^r t\langle R(e_1,V)V,e_1\rangle-\int_0^r t\langle R(\bar{e}_1,V)V,\bar{e}_1\rangle\\ &\ge \sum_{i=1}^{n}(\xi_i-\eta_i)^2-C(K_0,i_0)|X|^2r^2.\end{aligned}$$

This proves (ii). The Theorem is proved. □

For future applications, in the next part of this section, we will calculate the equations of derivatives of general harmonic map flow. Since the mean curvature flow is a kind of harmonic map flow with varying base metrics evolved by mean curvature flow, the formulas computed here is important in deriving the higher derivatives estimates in section 2.2 and 2.3. The formulas are of interest in their own rights. First we fix some notations.

Let F be a map from a Riemannian manifold (M, g_{ij}) to another Riemannian manifold $(N, \bar{g}_{\alpha\beta})$, let $F^{-1}TN$ be the pull back of the tangent bundle of N, we equip the bundle $(T^*M)^{\otimes p} \otimes F^{-1}TN$ the connection and metric induced from the connections and metrics of M and N. Let u be a section of $(T^*M)^{\otimes(p-1)} \otimes F^{-1}TN$. In local coordinates $\{x^i\}$ and $\{y^\alpha\}$ of M and N with $y = F(x)$, we have $|u|^2 = u^\alpha_{i_1 i_2 \cdots i_{p-1}} u^\beta_{j_1 j_2 \cdots j_{p-1}} g^{i_1 j_1} \cdots g^{i_{p-1} j_{p-1}} \bar{g}_{\alpha\beta}$. The coefficients of the covariant derivative ∇u can be computed by the formula

$$(\nabla u)^\alpha_{i_1 i_2 \cdots i_{p-1} i_p} = \frac{\partial u^\alpha_{i_1 i_2 \cdots i_{p-1}}}{\partial x^{i_p}} - \Gamma^l_{i_p i_j} u^\alpha_{i_1 i_2 \cdots i_{j-1} l i_{j+1} \cdots i_{p-1}} + \bar{\Gamma}^\alpha_{\beta\gamma} \frac{\partial F^\beta}{\partial x^{i_p}} u^\gamma_{i_1 i_2 \cdots i_{p-1}},$$

where Γ and $\bar{\Gamma}$ are connection coefficients of M and N respectively. We can define the Laplacian of u by $\Delta u = tr_g \nabla^2 u = g^{ij} (\nabla^2 u)_{\cdots ij}$. Recall the Ricci identity

$$(\nabla^2 u)^\alpha_{\cdots ij} - (\nabla^2 u)^\alpha_{\cdots ji} = -R_{iji_m l} u^\alpha_{\cdots i_{m-1} k i_{m+1} \cdots} g^{kl} + \bar{R}_{\beta\gamma\delta\zeta} \frac{\partial F^\beta}{\partial x^j} \frac{\partial F^\gamma}{\partial x^i} \bar{g}^{\alpha\delta} u^\zeta_{\cdots}. \quad (2.1.11)$$

Note that the derivative ∇F ($\nabla_i F^\alpha = \frac{\partial F^\alpha}{\partial x^i}$) is a section of the bundle $T^*M \otimes F^{-1}TN$, the higher derivative $\nabla^p F$ is a section of $(T^*M)^{\otimes p} \otimes F^{-1}TN$.

If we have a family of metrics $g_{ij}(\cdot, t)$ on M and a family of maps $F(\cdot, t)$ from M to N, then for each time t, we can still define the bundle $(T^*M)^{\otimes p} \otimes F^{-1}TN$ and define the covariant derivative ∇. It is a useful observation that the natural time derivative $\frac{\partial}{\partial t}$ is not covariant. We define a covariant time derivative D_t as follows. For any section $u^\alpha_{i_1 \cdots i_p}$ of $(T^*M)^{\otimes p} \otimes F^{-1}TN$, we define

$$D_t u^\alpha_{i_1 \cdots i_p} = \frac{\partial}{\partial t} u^\alpha_{i_1 \cdots i_p} + \bar{\Gamma}^\alpha_{\beta\gamma} \frac{\partial F^\beta}{\partial t} u^\gamma_{i_1 \cdots i_p}.$$

It is a routine computation which shows that the operator D_t is covariant.

Proposition 2.1.3 *Let M be a manifold with a family of metrics $g_{ij}(x, t)$, $(N, \bar{g})$ a Riemannian manifold. Let $F(\cdot, t)$ be a solution to the harmonic map flow with respect to the evolving metrics g_t and $\bar{g}$*

$$\frac{\partial}{\partial t} F(x, t) = \Delta F(x, t), \qquad \text{for } x \in M^n \text{ and } t \geq 0, \quad (2.1.12)$$

where $\Delta F(x,t)$ is the harmonic map Laplacian of F defined by metrics $g_{ij}(x,t)$ and $\bar{g}$. Then we have

$$\begin{aligned}(D_t-\Delta)\nabla^k F = & \sum_{l=0}^{k-1}\nabla^l[(R_M * g^{-2}+\bar{R}_N*(\nabla F)^2*g^{-1}*\bar{g}^{-1})]*\nabla^{k-l}F \\ & +\sum_{l=1}^{k-1} g^{-1}*\nabla^l\frac{\partial g}{\partial t}*\nabla^{k-l}F,\end{aligned} \tag{2.1.13}$$

where $\nabla^l(A*B)$ represents the linear combinations of $\nabla^l A*B$,$\nabla^{l-1}A*\nabla B$, $\cdots$, $A*\nabla^l B$ with universal coefficients.

Proof. For $k=1$, by direct computation and Ricci identity, we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i F^\alpha+\bar{\Gamma}^\alpha_{\beta\gamma}F^\beta_i(\Delta F)^\gamma &= \nabla_i\Delta F^\alpha \\ &= \Delta\nabla_i F^\alpha - R^l_i\nabla_l F^\alpha+\bar{R}^\alpha_{\beta\delta\gamma}\nabla_i F^\beta\nabla_k F^\delta\nabla_l F^\gamma g^{kl}.\end{aligned}$$

For $k\geq 2$, we prove by induction. Since

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla^k F)^\alpha_{i_1\cdots i_k} = & \frac{\partial}{\partial x^{i_k}}\frac{\partial}{\partial t}(\nabla^{k-1}F)^\alpha_{i_1\cdots i_{k-1}}-\Gamma^p_{i_k i_l}\frac{\partial}{\partial t}(\nabla^{k-1}F)^\alpha_{i_1\cdots p\cdots i_{k-1}} \\ & +\bar{\Gamma}^\alpha_{\beta\gamma}F^\beta_{i_k}\frac{\partial}{\partial t}(\nabla^{k-1}F)^\gamma_{i_1\cdots i_{k-1}}+(g^{-1}*\nabla\frac{\partial g}{\partial t}*\nabla^{k-1}F)^\alpha_{i_1\cdots i_k} \\ & +\frac{\partial}{\partial y^\delta}\bar{\Gamma}^\alpha_{\beta\gamma}(\Delta F)^\delta F^\beta_{i_k}(\nabla^{k-1}F)^\gamma_{i_1\cdots i_{k-1}}+\bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial}{\partial t}F^\beta_{i_k}(\nabla^{k-1}F)^\gamma_{i_1\cdots i_{k-1}},\end{aligned}$$

we have

$$\begin{aligned}D_t(\nabla^k F)^\alpha_{i_1\cdots i_k} = & \frac{\partial}{\partial x^{i_k}}D_t(\nabla^{k-1}F)^\alpha_{i_1\cdots i_{k-1}}-\Gamma^p_{i_k i_l}D_t(\nabla^{k-1}F)^\alpha_{i_1\cdots p\cdots i_{k-1}} \\ & +\bar{\Gamma}^\alpha_{\beta\gamma}F^\beta_{i_k}D_t(\nabla^{k-1}F)^\gamma_{i_1\cdots i_{k-1}}+(g^{-1}*\nabla\frac{\partial g}{\partial t}*\nabla^{k-1}F)^\alpha_{i_1\cdots i_k} \\ & +\frac{\partial}{\partial y^\delta}\bar{\Gamma}^\alpha_{\beta\gamma}(\Delta F)^\delta F^\beta_{i_k}(\nabla^{k-1}F)^\gamma_{i_1\cdots i_{k-1}}+\bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial}{\partial t}F^\beta_{i_k}(\nabla^{k-1}F)^\gamma_{i_1\cdots i_{k-1}} \\ & -\frac{\partial}{\partial x^{i_k}}[\bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial F^\beta}{\partial t}(\nabla^{k-1}F)^\gamma_{i_1\cdots i_{k-1}}]+\Gamma^p_{i_k i_l}\bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial F^\beta}{\partial t}(\nabla^{k-1}F)^\gamma_{i_1\cdots p\cdots i_{k-1}} \\ & -\bar{\Gamma}^\alpha_{\beta\gamma}\bar{\Gamma}^\gamma_{\delta\xi}F^\beta_{i_k}\frac{\partial F^\delta}{\partial t}(\nabla^{k-1}F)^\xi_{i_1\cdots i_{k-1}}+\bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial F^\beta}{\partial t}(\nabla^k F)^\gamma_{i_1\cdots i_k}.\end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial}{\partial x^{i_k}}[\bar{\Gamma}^{\alpha}_{\beta\gamma}\frac{\partial F^{\beta}}{\partial t}(\nabla^{k-1}F)^{\gamma}_{i_1\cdots i_{k-1}}] &= \frac{\partial}{\partial y^{\beta}}\bar{\Gamma}^{\alpha}_{\delta\gamma}F^{\beta}_{i_k}\frac{\partial F^{\delta}}{\partial t}(\nabla^{k-1}F)^{\gamma}_{i_1\cdots i_{k-1}} \\
&\quad +\bar{\Gamma}^{\alpha}_{\beta\gamma}\frac{\partial}{\partial x^{i_k}}\frac{\partial F^{\beta}}{\partial t}(\nabla^{k-1}F)^{\gamma}_{i_1\cdots i_{k-1}} \\
&\quad +\bar{\Gamma}^{\alpha}_{\beta\gamma}\frac{\partial F^{\beta}}{\partial t}(\nabla^{k}F)^{\gamma}_{i_1\cdots i_k} + \Gamma^{p}_{i_k i_l}\bar{\Gamma}^{\alpha}_{\beta\gamma}\frac{\partial F^{\beta}}{\partial t}(\nabla^{k-1}F)^{\gamma}_{i_1\cdots p\cdots i_{k-1}} \\
&\quad -\bar{\Gamma}^{\alpha}_{\delta\gamma}\bar{\Gamma}^{\gamma}_{\beta\xi}F^{\beta}_{i_k}\frac{\partial F^{\delta}}{\partial t}(\nabla^{k-1}F)^{\xi}_{i_1\cdots i_{k-1}},
\end{aligned}$$

we have

$$\begin{aligned}
D_t(\nabla^k F)^{\alpha}_{i_1\cdots i_k} &= [\nabla D_t(\nabla^{k-1}F)]^{\alpha}_{i_1\cdots i_k} + (g^{-1} * \nabla\frac{\partial g}{\partial t} * \nabla^{k-1}F)^{\alpha}_{i_1\cdots i_k} \\
&\quad +\bar{R}^{\alpha}_{\delta\beta\gamma}(\Delta F)^{\delta}F^{\beta}_{i_k}(\nabla^{k-1}F)^{\gamma}_{i_1\cdots i_{k-1}}.
\end{aligned}$$

Combining with Ricci identity

$$\nabla\Delta\nabla^{k-1}F = \Delta\nabla^k F + \nabla[(R_M * g^{-2} + \bar{R}_N * (\nabla F)^2 * g^{-1} * \bar{g}^{-1}) * \nabla^{k-1}F]$$

and induction on k, we have

$$\begin{aligned}
(D_t - \Delta)(\nabla^k F) &= g^{-1} * \nabla\frac{\partial g}{\partial t} * \nabla^{k-1}F + \bar{R}_N * \nabla F * \nabla^2 F * \nabla^{k-1}F * g^{-1} * \bar{g}^{-1} \\
&\quad +\nabla[(D_t - \Delta)\nabla^{k-1}F] \\
&\quad +\nabla[(R_M * g^{-2} + \bar{R}_N * (\nabla F)^2 * g^{-1} * \bar{g}^{-1}) * \nabla^{k-1}F] \\
&= \nabla[(D_t - \Delta)\nabla^{k-1}F] \\
&\quad +\nabla[(R_M * g^{-2} + \bar{R}_N * (\nabla F)^2 * g^{-1} * \bar{g}^{-1}) * \nabla^{k-1}F] \\
&\quad +g^{-1} * \nabla\frac{\partial g}{\partial t} * \nabla^{k-1}F \\
&= \sum_{l=0}^{k-1}\nabla^l[(R_M * g^{-2} + \bar{R}_N * (\nabla F)^2 * g^{-1} * \bar{g}^{-1})] * \nabla^{k-l}F \\
&\quad +\sum_{l=1}^{k-1} g^{-1} * \nabla^l\frac{\partial g}{\partial t} * \nabla^{k-l}F.
\end{aligned}$$

We finish the proof of the proposition. □

Corollary 2.1.4 *Let $F(\cdot,t)$ be assumed as in proposition 2.1.3. Then we have*

$$(\frac{\partial}{\partial t}-\Delta)|\nabla^k F|^2 \leq -2|\nabla^{k+1}F|^2+\langle\sum_{l=0}^{k-1}\{\nabla^l[(R_M*g^{-2}+\bar{R}_N*(\nabla F)^2*g^{-1}*\bar{g}^{-1})]$$
$$+g^{-1}*\nabla^l\frac{\partial g}{\partial t}\}*\nabla^{k-l}F,\nabla^k F\rangle+g^{-(k+1)}\frac{\partial g}{\partial t}*(\nabla^k F)^2*\bar{g}. \tag{2.1.14}$$

Proof. Since $|\nabla^k F|^2=(\nabla^k F)^\alpha_{i_1\cdots i_k}(\nabla^k F)^\beta_{j_1\cdots j_k}g^{i_1j_1}\cdots g^{i_kj_k}\bar{g}_{\alpha\beta}$, and

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k F|^2 &= 2\frac{\partial}{\partial t}(\nabla^k F)^\alpha_{i_1\cdots i_k}(\nabla^k F)^\beta_{j_1\cdots j_k}g^{i_1j_1}\cdots g^{i_kj_k}\bar{g}_{\alpha\beta}\\ &\quad+\frac{\partial\bar{g}_{\alpha\beta}}{\partial y^\delta}\frac{\partial F^\delta}{\partial t}(\nabla^k F)^\alpha_{i_1\cdots i_k}(\nabla^k F)^\beta_{j_1\cdots j_k}g^{i_1j_1}\cdots g^{i_kj_k}+g^{-(k+1)}*\frac{\partial g}{\partial t}*(\nabla^k F)^2*\bar{g}\\ &= 2D_t(\nabla^k F)^\alpha_{i_1\cdots i_k}(\nabla^k F)^\beta_{j_1\cdots j_k}g^{i_1j_1}\cdots g^{i_kj_k}\bar{g}_{\alpha\beta}+g^{-(k+1)}*\frac{\partial g}{\partial t}*(\nabla^k F)^2*\bar{g},\end{aligned}$$

then (2.1.14) follows from Proposition 2.1.3. □

2.2 Higher derivative estimates for the mean curvature flow

Now we come back to the mean curvature flow, suppose $X(\cdot,t)$ is a solution to the mean curvature flow equation (1.2), $g(\cdot,t)$ is the family of the induced metrics on M^n from $(\bar{M}^{\bar{n}},\bar{g})$ by $X(\cdot,t)$, then

$$\frac{\partial}{\partial t}g_{ij}=-2H^\alpha h^\beta_{ij}\bar{g}_{\alpha\beta}. \tag{2.2.1}$$

Note that $\frac{\partial g}{\partial t}=(\nabla^2X)^2*\bar{g}*g^{-1}$ and $R_M=\bar{R}_{\bar{M}}*(\nabla X)^4+(\nabla^2X)^2*\bar{g}$. Combining with corollary 2.1.4, we have

Proposition 2.2.1 *Let $(\bar{M}^{\bar{n}},\bar{g})$ be a Riemannian manifold of dimension $\bar{n}$. Let $X_0:M^n\to\bar{M}^{\bar{n}}$ be an isometrically immersed manifold in $\bar{M}^{\bar{n}}$. Suppose $X(x,t)$*

is a solution to the mean curvature flow (1.1) on $M^n \times [0,T]$ *with* X_0 *as initial data. Then*

$$(\frac{\partial}{\partial t} - \Delta)|\nabla^k X|^2 \le -2|\nabla^{k+1}X|^2 + \langle \sum_{l=0}^{k-1} \nabla^l[(\nabla^2 X)^2 * \bar{g} * g^{-2} + \bar{R}_{\bar{M}} * (\nabla X)^4 * g^{-2} * \bar{g} * \bar{g}^{-1}] * \nabla^{k-l}X, \nabla^k X\rangle + g^{-(k+2)} * \bar{g}^2 * (\nabla^2 X)^2 * (\nabla^k X)^2. \tag{2.2.2}$$

Now we are ready to derive the higher derivatives estimates of the second fundamental form of the mean curvature flow provided that we have bounded the second fundamental form. Before the deriving of the higher derivatives estimates, we need to construct a family of cut-off functions ξ_k, which are used also in the next section. For each integer $k > 0$, let ξ_k be a smooth non-increasing function from $(-\infty, +\infty)$ to $[0,1]$ so that $\xi_k(s) = 1$ for $s \in (-\infty, \frac{1}{2} + \frac{1}{2^{k+1}}]$, and $\xi_k(s) = 0$ for $s \in [\frac{1}{2} + \frac{1}{2^k}, +\infty)$; moreover for any $\epsilon > 0$ there exists a universal $C_{k,\epsilon} > 0$ such that

$$|\xi_k'(s)| + |\xi_k''(s)| \le C_{k,\epsilon}\xi_k(s)^{1-\epsilon}. \tag{2.2.3}$$

Theorem 2.2.2 *(local estimates) Let* $(\bar{M}^{\bar{n}}, \bar{g})$ *be a complete Riemannian manifold of dimension* $\bar{n}$. *Let* $X_0 : M^n \to \bar{M}^{\bar{n}}$ *be an isometrically immersed complete manifold in* $\bar{M}^{\bar{n}}$. *Suppose* $X(x,t)$ *is a solution to the mean curvature flow (1.1) on* $M^n \times [0,T]$ *with* X_0 *as initial data and with bounded second fundamental forms* $|h_{ij}^\alpha| \le \bar{C}$ *on* $[0,T]$. *Then for any fixed* $x_0 \in M^n$ *and any geodesic ball* $B_0(x_0, a)$ *of radius* $a > 0$ *of initial metric* g_{ij}, *for any* $k \ge 3$, *we have*

$$|\nabla^k X|(x,t) \le \frac{C_k}{t^{\frac{k-2}{2}}}, \quad \text{for all } (x,t) \in B_0(x_0, \frac{a}{2}) \times [0,T], \tag{2.2.4}$$

where the constant C_k *depends on* $\bar{C}$, T, $\bar{n}$, a *and the bounds of the curvature and its covariant derivatives up to order* $k-1$ *of the ambient manifold* $\bar{M}$ *on its geodesic ball* $B_{\bar{M}}(X_0(x_0), a + 1 + \sqrt{n}\bar{C}T)$.

Proof. Since $|\frac{\partial}{\partial t}X| = |H| \le \sqrt{n}\bar{C}$, it is not hard to see that under the evolution of the mean curvature flow, at any time $t \in [0,T]$, $X_t(B_0(x_0,a))$ is contained in

$B_{\bar{M}}(X_0(x_0), a+1+\sqrt{n}\bar{C}T)$. For any fixed $a>0$, $k>0$, we denote by C_k various constants depending only on a, $\bar{C}$, T, $\bar{n}$ and the bounds of the curvature and its covariant derivatives up to order $k-1$ of the ambient manifold $\bar{M}$ on its ball $B_{\bar{M}}(X_0(x_0), a+1+\sqrt{n}\bar{C}T)$.

By Proposition 2.2.1, we have

$$\begin{aligned}(\frac{\partial}{\partial t}-\Delta)|\nabla^2 X|^2 &\leq -2|\nabla^3 X|^2 + C_2 + C_2|\nabla^3 X| \\ &\leq -|\nabla^3 X|^2 + C_2\end{aligned} \tag{2.2.5}$$

and

$$\begin{aligned}(\frac{\partial}{\partial t}-\Delta)|\nabla^3 X|^2 &\leq -2|\nabla^4 X|^2 + C_3(|\nabla^3 X|^3 + |\nabla^3 X|^2 + |\nabla^3 X| + |\nabla^4 X||\nabla^3 X|) \\ &\leq -|\nabla^4 X|^2 + C_3|\nabla^3 X|^3 + C_3.\end{aligned} \tag{2.2.6}$$

Combining (2.2.5) and (2.2.6), for any constant $A>0$ we have

$$\begin{aligned}(\frac{\partial}{\partial t}-\Delta)((A+|\nabla^2 X|^2)|\nabla^3 X|^2) \leq& (-|\nabla^3 X|^2 + C_3)|\nabla^3 X|^2 + 8|\nabla^3 X|^2|\nabla^4 X||\nabla^2 X| \\ &+ [-|\nabla^4 X|^2 + C_3|\nabla^3 X|^3 + C_3](A+|\nabla^2 X|^2).\end{aligned} \tag{2.2.7}$$

Since $|\nabla^2 X|^2$ is bounded by assumption, by choosing A suitable large, let $u = (A+|\nabla^2 X|^2)|\nabla^3 X|^2$ and $v = tu$, we have

$$(\frac{\partial}{\partial t}-\Delta)u \leq -\frac{1}{C_3}u^2 + C_3$$

and

$$(\frac{\partial}{\partial t}-\Delta)v \leq \frac{1}{t}(-\frac{1}{C_3}v^2 + C_3). \tag{2.2.8}$$

Now we need a cut-off function technique as in [5]. Let $\xi(x) = \xi_3(\frac{d_0(x,x_0)}{a})$, where ξ_3 is the cut-off function satisfying (2.2.3) for $k=3$. Then the function $\xi(x)$

satisfies

$$\begin{cases} \xi(x) = 1, & \text{for } x \in B_0(x_0, (\frac{1}{2} + \frac{1}{2^4})a), \\ \xi(x) = 0, & \text{for } x \in M \backslash B_0(x_0, a), \\ |\nabla_0 \xi|^2 \leq C_3 \xi, \\ (\nabla_0^2 \xi)_{ij} \geq -C_3 \xi^{\frac{1}{2}} g_{ij}(\cdot, 0), \end{cases} \tag{2.2.9}$$

where we used the Hessian comparison theorem. Since by Gauss equation, the curvature of the initial metric is bounded from below by a constant, which depends on $\bar{C}$ and the curvature bound on the ball $B_{\bar{M}}(X_0(x_0), a + 1 + \sqrt{n}\bar{C}T)$ of the ambient manifold. The last formula holds in the sense of support functions. Define $\phi(x,t) = \xi(x)v(x,t)$. Then we have

$$(\frac{\partial}{\partial t} - \Delta)\phi \leq \frac{1}{t}(-\frac{1}{C_3}\xi v^2 - tv\Delta\xi - 2t\nabla\xi \cdot \nabla v + C_3\xi). \tag{2.2.10}$$

Suppose $\phi(x,t)$ achieves its maximum value over $M^n \times [0,T]$ at some point $(x_1, t_1) \in B(x_0, a) \times (0, T]$, i.e.

$$\phi(x_1, t_1) = \max_{M \times [0,T]} \phi(x,t).$$

Suppose the point x_1 does not lie in the cut-locus of x_0, then

$$\frac{\partial \phi}{\partial t}(x_1, t_1) \geq 0, \quad \nabla v(x_1, t_1) = -\frac{\nabla \xi}{\xi} v, \qquad \Delta\phi(x_1, t_1) \leq 0. \tag{2.2.11}$$

By (2.2.10) and (2.2.11), at (x_1, t_1) we have

$$0 \leq -\frac{1}{C_3}\xi v^2 - t_1 v\Delta\xi + 2t_1\frac{|\nabla\xi|^2}{\xi}v + C_3\xi. \tag{2.2.12}$$

Note that the second fundamental form is bounded in $M^n \times [0,T]$, the metrics $g_{ij}(\cdot, t)$ are equivalent. Since

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = (g^{-1} * \nabla\frac{\partial g}{\partial t})_{ij}^k = g^{-2} * \bar{g} * \nabla^2 X * \nabla^3 X,$$

we have

$$
\begin{aligned}
|(\Gamma_{ij}^k(x_1,t_1)-\Gamma_0{}_{ij}^k(x_1)| &\le C(\bar{n})\bar{C}\int_0^{t_1}|\nabla^3X|dt \\
&\le C(\bar{n})\bar{C}\int_0^{t_1}(\frac{\phi}{\xi t})^{\frac{1}{2}}(x_1,t)dt \\
&\le C_3\frac{\phi(x_1,t_1)^{\frac{1}{2}}}{\xi(x_1)^{\frac{1}{2}}},
\end{aligned}
$$

where we used the fact that ϕ achieves its maximum at (x_1,t_1). Thus at (x_1,t_1), we have

$$
\begin{aligned}
-\Delta\xi &= -\, g^{ij}\nabla_i\nabla_j\xi \\
&= -\, g^{ij}(\nabla_{0i}\nabla_{0j}\xi+(\Gamma_0{}_{ij}^k-\Gamma_{ij}^k)\nabla_{0k}\xi) \\
&\le C_3+C_3\frac{\phi(x_1,t_1)^{\frac{1}{2}}}{\xi(x_1)^{\frac{1}{2}}}|\nabla\xi|,
\end{aligned}
$$

Substituting into (2.2.12), multiplying by $\xi(x_1)$ and combining with (2.2.9), we have at (x_1,t_1)

$$
\begin{aligned}
0 &\le -\frac{1}{C_3}\xi^2v^2+(C_3+C_3\phi(x_1,t_1)^{\frac{1}{2}}\frac{|\nabla\xi|}{\xi^{\frac{1}{2}}})\xi v+2\frac{|\nabla\xi|^2}{\xi}\xi v+C_3\xi^2 \\
&\le -\frac{1}{C_3}\phi^2+C_3\phi^{\frac{3}{2}}+C_3\phi+C_3.
\end{aligned}
$$

This implies

$$
\phi(x_1,t_1)\le C_3,
$$

hence we have

$$
|\nabla^3X|\le\frac{C_3}{t^{\frac{1}{2}}}
$$

on $B_0(x_0,(\frac{1}{2}+\frac{1}{2^4})a)\times[0,T]$. If x_1 lies on the cut locus of x_0, then by applying a standard support function technique as in [28], the same estimate is still valid.

For higher derivatives, we prove by induction. Fix $x_0\in M^n$, $a>0$, suppose

$$
|\nabla^kX|\le\frac{C_k}{t^{\frac{k-2}{2}}},\qquad k=3,...,m-1, \tag{2.2.13}
$$

on $B_0(x_0, (\frac{1}{2} + \frac{1}{2^{k+1}})a) \times [0, T]$. Now we prove the estimate for $k = m$.

By induction hypothesis and Proposition 2.2.1, we have

$$\begin{aligned}
(\frac{\partial}{\partial t} - \Delta)|\nabla^m X|^2 \leq & -2|\nabla^{m+1}X|^2 + \langle\sum_{l=0}^{m-1} \nabla^l[(\nabla^2 X)^2 * \bar{g} * g^{-2} + \bar{R}_{\bar{M}} * (\nabla X)^4 * g^{-2} \\
& * \bar{g} * \bar{g}^{-1}] * \nabla^{m-l}X, \nabla^m X\rangle + g^{-(m+2)} * \bar{g}^2 * (\nabla^2 X)^2 * (\nabla^m X)^2 \\
\leq & -2|\nabla^{m+1}X|^2 + C_m \sum_{l=0}^{m-1}\{\sum_{l_1+l_2=l} |\nabla^{2+l_1}X||\nabla^{2+l_2}X| \\
& + \sum_{l_1+\cdots+l_4=l} |\nabla^{l_1+1}X||\nabla^{l_2+1}X||\nabla^{l_3+1}X||\nabla^{l_4+1}X|\}|\nabla^{m-l}X||\nabla^m X| \\
\leq & -2|\nabla^{m+1}X|^2 + C_m[|\nabla^{m+1}X||\nabla^m X| + (|\nabla^3 X| + 1)|\nabla^m X|^2 \\
& + t^{-\frac{m-2}{2}}|\nabla^m X|] \\
\leq & -|\nabla^{m+1}X|^2 + \frac{C_m}{t^{\frac{1}{2}}}|\nabla^m X|^2 + C_m t^{-\frac{m-2}{2}}|\nabla^m X|
\end{aligned} \tag{2.2.14}$$

and

$$\begin{aligned}
(\frac{\partial}{\partial t} - \Delta)|\nabla^{m-1}X|^2 & \leq -|\nabla^m X|^2 + \frac{C_{m-1}}{t^{\frac{1}{2}}}|\nabla^{m-1}X|^2 + C_{m-1}t^{-\frac{m-3}{2}}|\nabla^{m-1}X| \\
& \leq -|\nabla^m X|^2 + \frac{C_{m-1}}{t^{m-3+\frac{1}{2}}}
\end{aligned} \tag{2.2.15}$$

on $B_0(x_0, (\frac{1}{2} + \frac{1}{2^m})a) \times [0, T]$.

Define

$$\psi(x,t) = (A + t^{m-3}|\nabla^{m-1}X|^2)|\nabla^m X|^2 t^{m-2}$$

for A to be determined later. Combining (2.2.14) and (2.2.15), we have for suitable large A as before

$$\begin{aligned}(\frac{\partial}{\partial t}-\Delta)\psi \le & \frac{2m-5}{t}\psi + t^{m-3}|\nabla^m X|^2 t^{m-2}(-|\nabla^m X|^2 + \frac{C_{m-1}}{t^{m-3+\frac{1}{2}}}) \\ & + t^{m-2}(A+t^{m-3}|\nabla^{m-1}X|^2)(-|\nabla^{m+1}X|^2 + \frac{C_m}{t^{\frac{1}{2}}}|\nabla^m X|^2 + C_m t^{-\frac{m-2}{2}}|\nabla^m X|) \\ & + 8t^{2m-5}|\nabla^{m-1}X||\nabla^m X|^2|\nabla^{m+1}X| \\ \le & \frac{2m-5}{t}\psi - \frac{1}{2t}[t^{m-2}|\nabla^m X|^2]^2 + \frac{C_m}{t^{\frac{1}{2}}}[t^{m-2}|\nabla^m X|^2] + C_m[t^{m-2}|\nabla^m X|^2]^{\frac{1}{2}} \\ \le & \frac{1}{t}[-\frac{1}{C_m}\psi^2 + C_m\psi + C_m\psi^{\frac{1}{2}}] \\ \le & \frac{1}{t}[-\frac{1}{C_m}\psi^2 + C_m]\end{aligned} \tag{2.2.16}$$

on $B_0(x_0,(\frac{1}{2}+\frac{1}{2^m})a)\times[0,T]$. To apply the cut-off function technique to (2.2.16) as before, we note that by the estimate for $k=3$, we know that

$$|\Gamma-\Gamma_0| \le C(\bar{n})\bar{C}\int_0^T |\nabla^3 X|dt \le C_3\int_0^T \frac{1}{\sqrt{t}}dt \le C_3.$$

By calculating the equation of $\xi_m(\frac{d_0(x_0,\cdot)}{a})\psi$ using (2.2.16), and repeating the same procedure of applying maximum principle as before, we can prove that

$$\xi_m(\frac{d_0(x_0,\cdot)}{a})\psi \le C_m \qquad \text{on} \qquad B_0(x_0,a)\times[0,T],$$

which implies

$$|\nabla^m X|(x,t) \le \frac{C_m}{t^{\frac{m-2}{2}}}, \qquad \text{for all } \ (x,t)\in B_0(x_0,(\frac{1}{2}+\frac{1}{2^{m+1}})a)\times[0,T].$$

We complete the induction step and the theorem is proved. □

Corollary 2.2.3 *Let $(\bar{M}^{\bar{n}},\bar{g})$ be a complete Riemannian manifold of dimension $\bar{n}$ with bounded curvature and its derivatives up to order 2, i.e. there is a constant $\bar{C}$ such that*

$$|\bar{\nabla}^k \bar{Rm}|(\cdot) \le \bar{C}, \qquad \textit{for} \quad k\le 2.$$

Let $X_0 : M^n \to \bar{M}^{\bar{n}}$ be an isometrically immersed complete manifold in $\bar{M}^{\bar{n}}$. Suppose $X(\cdot, t)$ is a solution to the mean curvature flow (1.1) on $M^n \times [0,T]$ with X_0 as initial data and with bounded second fundamental forms $|h^\alpha_{ij}| \leq \bar{C}$ on $[0,T]$. Then there is a constant C_1 depending only on $\bar{C}$, $\bar{n}$ and T such that

$$|\nabla Rm|(x,t) \leq \frac{C_1}{t^{\frac{1}{2}}}, \quad \textit{for all} \ \ (x,t) \in M^n \times [0,T]. \tag{2.2.17}$$

Moreover, for any fixed $x_0 \in M^n$ and any ball $B_0(x_0, a)$ of radius $a > 0$ of initial metric g_{ij}, for any $k \geq 2$, there is a constant C_k depending only on a, $\bar{C}$, $\bar{n}$, T and the bounds of the curvature and its derivatives up to order $k+1$ of the ambient manifold on its geodesic ball $B_{\bar{M}}(X_0(x_0), a + 1 + \sqrt{n}\bar{C}T)$, such that

$$|\nabla^k Rm|(x,t) \leq \frac{C_k}{t^{\frac{k}{2}}}, \quad \textit{for all} \ \ (x,t) \in B_0(x_0, \frac{a}{2}) \times [0,T]. \tag{2.2.18}$$

Proof. This follows from Gauss equation and Theorem 2.2.2. □

2.3 Harmonic map flow coupled with mean curvature flow

Let X_t be the solution to the mean curvature flow as in Theorem 1.1. Let $g_{ij}(x,t)$ be the induced metrics with $g_{ij}(x)$ as initial data, let $f : M^n \to N^m$ be a map from M^n to a fixed Riemanian manifold $(N^m, \hat{g}_{\alpha\beta})$. Then the harmonic map flow coupled with mean curvature flow is the following evolution equation of the maps

$$\begin{cases} \dfrac{\partial}{\partial t} f(x,t) = \Delta f(x,t), & \text{for } x \in M^n, t > 0, \\ \\ f(x,0) = f(x), & \text{for } x \in M^n, \end{cases}$$

where the Harmonic map Laplacian Δ is defined by using the metric $g_{ij}(x,t)$ and $\hat{g}_{\alpha\beta}(y)$, i.e.

$$\Delta f^\alpha(x,t) = g^{ij}(x,t)\nabla_i\nabla_j f^\alpha(x,t),$$

and

$$\nabla_i \nabla_j f^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f^\alpha}{\partial x^k} + \hat{\Gamma}^\alpha_{\beta\gamma} \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j}.$$

Here we use $\{x^i\}$ and $\{y^\alpha\}$ to denote the local coordinates of M^n and N^m respectively, Γ^k_{ij} and $\hat{\Gamma}^\alpha_{\beta\gamma}$ the corresponding Christoffel symbols of g_{ij} and $\hat{g}_{\alpha\beta}$.

Now we fix a metric $\hat{g} = g(\cdot, T)$ on M^n, let $(N^m, \hat{g}) = (M^n, \hat{g})$. Since the ambient manifold $(\bar{M}, \bar{g})$ in Theorem 1.1 satisfies the assumption of Corollary 2.2.3, by Corollary 2.2.3 and Theorem 2.1.1, we know that there are positive constants $\hat{C}_1$, $\hat{\delta}$ depending only on $\bar{C}$, T $\bar{n}$ and $\bar{\delta}$ such that

$$\begin{aligned} &|\hat{R}_N| + |\hat{\nabla}\hat{R}_N| \le \hat{C}_1, \\ &inj(N, \hat{g}) \ge \hat{\delta} > 0. \end{aligned} \tag{2.3.1}$$

Moreover, by (2.2.18) of Corollary 2.2.3, for any fixed $y_0 \in N$, for any $k \ge 2$, there is a constant $\hat{C}_k$ depending only on $\bar{C}$, $\bar{n}$, T and the bounds of the curvature and its derivatives up to order $k+1$ of the ambient manifold on its ball $B_{\bar{M}}(X_0(y_0), 2e^{\sqrt{n}\bar{C}^2 T} + 1 + \sqrt{n}\bar{C}T)$, such that

$$|\hat{\nabla}^k \hat{R}_N|(y) \le \hat{C}_k, \quad \text{for all } \ y \in \hat{B}(y_0, 1). \tag{2.3.2}$$

In this section, We will establish the existence theorem for the above harmonic map flow coupled with mean curvature flow. More precisely, we will prove

Theorem 2.3.1 *There exists $0 < T_0 < T$, depending only on $\bar{C}, T, \bar{n}, \bar{\delta}$, such that the harmonic map flow coupled with mean curvature flow*

$$\begin{cases} \dfrac{\partial}{\partial t} F(x,t) = \Delta F(x,t), & x \in M^n, t > 0, \\ \\ F(\cdot, 0) = Identity, & x \in M^n \end{cases} \tag{2.3.3}$$

has a solution on $M^n \times [0, T_0]$ such that the follwing estimates hold. There is a constant C_2 depending only on $\bar{C}$, $\bar{\delta}$, $\bar{n}$ and T such that

$$|\nabla F| + |\nabla^2 F| \le C_2. \tag{2.3.4}$$

For any $k \geq 3$, $B_0(x_1, 1) \subset M^n$, *there is a constant* C_k *depending only on* $\bar{C}$,$\bar{\delta}$, T, $\bar{n}$ *and* x_1 *such that*

$$|\nabla^k F| \leq C_k t^{-\frac{k-2}{2}}, \quad on \quad B_0(x_1, 1) \times [0, T_0]. \tag{2.3.5}$$

We will adapt the strategy of [5] by solving the corresponding initial-boundary value problem on a sequence of exhausted bounded domains $D_1 \subseteq D_2 \subseteq \cdots$ with smooth boundaries and $D_j \supseteq B_0(x_0, j+1)$,

$$\begin{cases} \dfrac{\partial}{\partial t} F^j(x,t) = \Delta F^j(x,t) \\ F^j(x,0) = x \qquad \text{for all } x \in D_j, \\ F^j(x,t) = x \qquad \text{for all } x \in \partial D_j, \end{cases} \tag{2.3.6}$$

and taking a convergent subsequence of F^j as $j \to \infty$, where x_0 is a fixed point in M^n.

First we need the zero order estimate for the Dirichlet problem (2.3.6).

Lemma 2.3.2 *There exist positive constants* $T_1 > 0$ *and* $C > 0$ *such that for any* j, *if* F^j *solves problem(2.3.6) on* $\bar{D}_j \times [0, T_1']$ *with* $T_1' \leq T_1$, *then we have*

$$\hat{d}(x, F^j(x,t)) \leq C\sqrt{t}$$

for any $(x,t) \in D_j \times [0, T_1']$, *where* $\hat{d}$ *is the distance with respect to the metric* $\hat{g}$.

Proof. For simplicity, we drop the superscript j. In the following argument, we denote by C various positive constants depending only on the constants $\bar{C}$, $\bar{\delta}$, T, and $\bar{n}$ in Theorem 1.1. Note that $\hat{d}(y_1, y_2)$ is the distance function on the target $(M^n, \hat{g})$, which can be regarded as a function on $M^n \times M^n$ with the product metric. Let $\varphi(y_1, y_2) = \frac{1}{2}\hat{d}^2(y_1, y_2)$ and $\rho(x,t) = \varphi(x, F(x,t))$. We compute

$$
\begin{aligned}
(\frac{\partial}{\partial t} - \Delta)\rho =& \hat{d}(x, F(x,t))(-\frac{\partial \hat{d}}{\partial y_1^\alpha}\Delta Id^\alpha) - g^{ij}\{\frac{\partial^2 \varphi}{\partial {y_1}^\alpha \partial y_1^\beta} - (\hat{\Gamma}^\gamma_{\alpha\beta}\circ Id)\frac{\partial \varphi}{\partial y_1^\gamma}\}\frac{\partial Id^\alpha}{\partial x^i}\frac{\partial Id^\beta}{\partial x^j} \\
& - 2g^{ij}\frac{\partial^2 \varphi}{\partial y_1^\alpha \partial y_2^\beta}\frac{\partial Id^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j} - g^{ij}\{\frac{\partial^2 \varphi}{\partial y_2^\alpha \partial y_2^\beta} - (\hat{\Gamma}^\gamma_{\alpha\beta}\circ F)\frac{\partial \varphi}{\partial y_2^\gamma}\}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j} \\
=& - \hat{d}\frac{\partial \hat{d}}{\partial y_1^\alpha}\Delta Id^\alpha - g^{ij}Hess(\varphi)(V_i, V_j),
\end{aligned}
$$

where

$$V_i = \frac{\partial Id^\alpha}{\partial x^i}\frac{\partial}{\partial y_1^\alpha} + \frac{\partial F^\alpha}{\partial x^i}\frac{\partial}{\partial y_2^\alpha}.$$

By Theorem 2.2.2, there is a constant C depending only on $\bar{C}$, T and $\bar{n}$ such that

$$|\frac{\partial \Gamma}{\partial t}| \le C|\nabla^3 X| \le \frac{C}{\sqrt{t}}. \tag{2.3.7}$$

Since

$$\Delta Id = g^{-1} * (\hat{\Gamma}\circ Id - \Gamma) = g^{-1} * (\Gamma(\cdot, T) - \Gamma(\cdot, t))$$

then by (2.3.7) we have $|\Delta Id| \le C$, this implies

$$(\frac{\partial}{\partial t} - \Delta)\rho \le C\hat{d} - g^{ij}Hess(\varphi)(V_i, V_j).$$

By (2.3.1), the curvature of $\hat{g}$ is bounded by some constant $\hat{K}$, the injectivity radius of $\hat{g}$ have a uniform positive lower bound $\hat{\delta}$. We claim that if $\hat{d}(x, F(x,t)) \le \min\{\hat{\delta}/2, 1/4\sqrt{\hat{K}}\}$, then

$$g^{ij}Hess(\varphi)(V_i, V_j) \ge \frac{1}{2}|\nabla F|^2 - C.$$

Firstly, by Theorem 2.1.2 (i), we have $|Hess(\varphi)| \le C$ under the assumption of the claim. On the other hand, the Hessian comparison theorem at those points not lying on the cut locus shows that

$$\frac{\partial^2 \varphi}{\partial y_2^\alpha \partial y_2^\beta} - (\hat{\Gamma}^\gamma_{\alpha\beta}\circ F)\frac{\partial \varphi}{\partial y_2^\gamma} \ge \frac{\pi}{4}\hat{g}_{\alpha\beta},$$

$$\frac{\partial^2 \varphi}{\partial y_1^\alpha \partial y_1^\beta} - (\hat{\Gamma}^\gamma_{\alpha\beta}\circ Id)\frac{\partial \varphi}{\partial y_1^\gamma} \ge \frac{\pi}{4}\hat{g}_{\alpha\beta}.$$

Combining the above inequalities, we have

$$\begin{aligned} g^{ij}Hess(\varphi)(V_i, V_j) \geq& \frac{\pi}{4}|\nabla F|^2 - C|\nabla F| - C \\ \geq& \frac{1}{2}|\nabla F|^2 - C, \end{aligned}$$

which proves the claim. Hence when $\hat{d}(x, F(x,t)) \leq \min\{\frac{\hat{\delta}}{2}, \frac{1}{4\sqrt{\hat{K}}}\}$, we have

$$(\frac{\partial}{\partial t} - \Delta)\rho \leq C\hat{d} - \frac{1}{2}|\nabla F|^2 + C. \tag{2.3.8}$$

By maximum principle we have

$$\hat{d}(x, F(x,t)) \leq C\sqrt{t} \quad \text{whenever } \hat{d}(x, F(x,t)) \leq \min\{\frac{\hat{\delta}}{2}, \frac{1}{4\sqrt{\hat{K}}}\}.$$

Therefore there exists $T_1 \leq \frac{1}{C^2}\min^2\{\frac{\hat{\delta}}{2}, \frac{1}{4\sqrt{\hat{K}}}\}$ such that

$$\hat{d}(x, F(x,t)) \leq C\sqrt{t}, \quad \text{for } t \leq T_1'(\leq T_1),$$

we have proved the lemma. □

After proving the above lemma, we can apply the standard parabolic equation theory to get a local existence for the initial-boundary value problem (2.3.6) as follows. This is similar to [5], we include the proof here for completeness.

Lemma 2.3.3 *There exists a positive constant $T_2(\leq T_1)$ depending only on the dimension n, the constants T_1 and C obtained in the previous lemma such that for each j, the initial-boundary value problem (2.3.6) has a smooth solution F^j on $\bar{D}_j \times [0, T_2]$.*

Proof. For an arbitrarily fixed point $\bar{x}$ in M^n, we consider the normal coordinates $\{x^i\}$ and $\{y^\alpha\}$ of the metric g_{0ij} and the metric $\hat{g}_{\alpha\beta}$ respectively around $\bar{x}$. Locally the equation (2.3.6) is written as a system of equations

$$\frac{\partial y^\alpha}{\partial t}(x,t) = g^{ij}(x,t)[\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k(x,t)\frac{\partial y^\alpha}{\partial x^k} + \hat{\Gamma}_{\beta\gamma}^\alpha(y^1(x,t), \cdots, y^n(x,t))\frac{\partial y^\beta}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j}]. \tag{2.3.9}$$

Note that $\hat{\Gamma}^{\alpha}_{\beta\gamma}(\bar{x}) = 0$. Since by (2.3.1) the curvature of metric $\hat{g}$ and it's first covariant derivative are bounded on the whole target manifold, by applying a result of Hamilton (Corollary 4.11 in [20]), we know that there is some uniform constant $\hat{C}$ such that if $\hat{d}(y, \bar{x}) \leq \frac{1}{\hat{C}}$, then $|\hat{\Gamma}^{\alpha}_{\beta\gamma}(y)| \leq \hat{C}\hat{d}(y, \bar{x})$. (This fact is proved essentially in [20], although it is not explicitly stated.) By Lemma 2.3.2, $\hat{d}(x, F(x,t)) \leq C\sqrt{t}$, we conclude that the coefficients of the quadratic terms on the RHS of (2.3.9) can be as small as we like provided $T_2 > 0$ sufficiently small (independent of $\bar{x}$ and j).

Now for fixed j, we consider the corresponding parabolic system of the difference of the map F^j and the identity map. Clearly the coefficients of the quadratic terms of the gradients are also very small. Thus, whenever (2.3.9) has a solution on a time interval $[0, T_2']$ with $T_2' \leq T_2$, we can argue exactly as in the proof of Theorem 6.1 in Chapter VII of the book [25] to bound the norm of ∇F^j on the time interval $[0, T_2']$ by a positive constant depending only on g_{0ij}, and $\hat{g}_{\alpha\beta}$ over the domain D_{j+1}, the L^∞ bound of F^j obtained in the previous lemma, and the boundary ∂D_j. Hence by the same argument as in the proof of Theorem 7.1 in Chapter VII of the book [25], we deduce that the initial-boundary value problem (2.3.9) has a smooth solution F^j on $\bar{D}_j \times [0, T_2]$. □

To get a convergent sequence of F^j, we need the following uniform estimates.

Lemma 2.3.4 *There exists a positive constant T_3, $0 < T_3 \leq T_2$, independent of j, such that if F^j solves*

$$\begin{cases} \dfrac{\partial}{\partial t} F^j(x,t) = \Delta F^j(x,t) & \text{on } D_j \times [0, T_3], \\ \\ F^j(x,0) = x & \text{on } D_j. \end{cases}$$

Then for any $B_0(x_1, 1) \subset D_j$, there is a positive constant $C = C(\bar{C}, \bar{\delta}, \bar{n}, T)$ such that

$$|\nabla F^j| + |\nabla^2 F^j| \leq C$$

on $B_0(x_1, \frac{1}{2}) \times [0, T_3]$, *and for any* $k \geq 3$ *there exist constants* $C_k = C(k, \bar{C}, \bar{\delta}, T, \bar{n}, x_1)$ *satisfying*

$$|\nabla^k F^j| \leq C_k t^{-\frac{k-2}{2}}$$

on $B_0(x_1, \frac{1}{2}) \times [0, T_3]$.

Proof. We drop the superscript j. We denote by C various constants depending only on $\bar{C}$, $\bar{\delta}$, T, $\bar{n}$. We first estimate $|\nabla F|$. By Corollary 2.1.4, we have

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)|\nabla F|^2 \leq -2|\nabla^2 F|^2 + \langle([R_M * g^{-2} + \hat{R}_N * (\nabla F)^2 * g^{-1} * \hat{g}^{-1}]\\ + g^{-1} * \frac{\partial g}{\partial t}) * \nabla F, \nabla F\rangle + g^{-2}\frac{\partial g}{\partial t} * (\nabla F)^2 * \hat{g}.\end{aligned}$$

Note that $\frac{\partial g}{\partial t} = (\nabla^2 X)^2 * \bar{g} * g^{-1}$, $R_M = \bar{R}_{\bar{M}} * (\nabla X)^4 + (\nabla^2 X)^2 * \bar{g}$, the second fundamental form $\nabla^2 X$ and curvature $\bar{R}_{\bar{M}}$ are bounded by assumption, we know that $|\frac{\partial g}{\partial t}|$ and $|R_M|$ are bounded. The above formula gives

$$\frac{\partial}{\partial t}|\nabla F|^2 \leq \Delta|\nabla F|^2 - 2|\nabla^2 F|^2 + C|\nabla F|^2 + C|\nabla F|^4. \tag{2.3.10}$$

On the other hand, we know from (2.3.8) that

$$\frac{\partial}{\partial t}\rho \leq \Delta\rho - \frac{1}{2}|\nabla F|^2 + C,$$

where $\rho(x,t) = \frac{1}{2}\hat{d}^2(x, F(x,t))$. For any $a > 0$ to be determined later, we compute

$$\begin{aligned}\frac{\partial}{\partial t}[(a+\rho)|\nabla F|^2] \leq &\Delta[(a+\rho)|\nabla F|^2] - 2\nabla\rho \cdot \nabla|\nabla F|^2\\ &- 2(a+\rho)|\nabla^2 F|^2 + C(a+\rho)|\nabla F|^2 + C(a+\rho)|\nabla F|^4\\ &- \frac{1}{2}|\nabla F|^4 + C|\nabla F|^2.\end{aligned}$$

Since

$$-2\nabla\rho \cdot \nabla|\nabla F|^2 \leq C\hat{d}(|\nabla F| + |\nabla F|^2)|\nabla^2 F| \leq C(|\nabla F|^2 + |\nabla F|^4)\hat{d} + C\hat{d}|\nabla^2 F|^2$$

and $\hat{d}(\cdot, F(\cdot, t)) \leq C\sqrt{t}$, by taking $a = \frac{1}{8C}$ and T_3 suitable small, we have

$$\frac{\partial}{\partial t}[(a+\rho)|\nabla F|^2] \leq \Delta[(a+\rho)|\nabla F|^2] - \frac{1}{8C}|\nabla^2 F|^2 - \frac{1}{4}|\nabla F|^4 + C$$

for $t \le T_3$. Let $u = (a+\rho)|\nabla F|^2$, then

$$\frac{\partial u}{\partial t} \le \Delta u - \frac{1}{C}u^2 + C \tag{2.3.11}$$

for $t \le T_3$. Let $\xi(x) = \xi_1(d_0(x_1, x))$ be a cut-off function, where ξ_1 is the non-increasing smooth function in (2.2.3) supported in $[0,1)$ and equal to 1 in $[0, \frac{3}{4}]$. Note that at $t = 0$, $u = ag^{ij}(\cdot, 0)g_{ij}(\cdot, T) \le C$. Then by computing the equation of ξu and applying the maximum principle as before, we have

$$\xi u(x,t) \le C \qquad \text{on} \qquad M^n \times [0, T_3],$$

this implies

$$|\nabla F| \le C \quad \text{on} \quad B_0(x_1, \frac{3}{4}) \times [0, T_3].$$

We now estimate $|\nabla^2 F|$. By Corollary 2.1.4 again

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)|\nabla^2 F|^2 \;\; \le \;\; & -2|\nabla^3 F|^2 + \langle \sum_{l=0}^{1} \{\nabla^l[(R_M * g^{-2} + \hat{R}_N * (\nabla F)^2 * g^{-1} * \hat{g}^{-1})] \\ & + g^{-1} * \nabla^l \frac{\partial g}{\partial t}\} * \nabla^{2-l} F, \nabla^2 F\rangle + g^{-(3)} \frac{\partial g}{\partial t} * (\nabla^2 F)^2 * \hat{g},\end{aligned}$$

and by (2.2.4),(2.2.17),(2.3.1), we know $\sqrt{t}|\nabla \frac{\partial g}{\partial t}| + \sqrt{t}|\nabla R_M| + |\hat{\nabla}\hat{R}_N| \le C$, and

$$\frac{\partial}{\partial t}|\nabla^2 F|^2 \le \Delta|\nabla^2 F|^2 - 2|\nabla^3 F|^2 + C|\nabla^2 F|^2 + \frac{C}{\sqrt{t}}|\nabla^2 F| \tag{2.3.12}$$

on $B_0(x_1, \frac{3}{4}) \times [0, T_3]$. This implies

$$\frac{\partial}{\partial t}|\nabla^2 F| \le \Delta|\nabla^2 F| + C|\nabla^2 F| + \frac{C}{\sqrt{t}}. \tag{2.3.13}$$

By (2.3.10) we have

$$\frac{\partial}{\partial t}|\nabla F|^2 \le \Delta|\nabla F|^2 - 2|\nabla^2 F|^2 + C.$$

Let

$$u = |\nabla^2 F| + |\nabla F|^2 - 2C\sqrt{t} + 2C\sqrt{T},$$

then

$$\frac{\partial}{\partial t}u \leq \Delta u - u^2 + C \qquad \text{on} \quad B_0(x_1, \frac{3}{4}) \times [0, T_3]. \tag{2.3.14}$$

Define the cutoff function $\xi(x) = \xi_2(d_0(x_1, x))$. Note that at $t = 0$, $|\nabla^2 F| = |\Gamma_0 - \hat{\Gamma}| \leq C$, then $u\,|_{t=0} \leq C$. Using the similar maximum principle argument as before we get

$$\xi u \leq C \qquad \text{on } B_0(x_1, \frac{1}{2} + \frac{1}{2^2}) \times [0, T_3],$$

which implies

$$|\nabla^2 F| \leq C \qquad \text{on } B_0(x_1, \frac{1}{2} + \frac{1}{2^3}) \times [0, T_3].$$

To derive the higher derivative estimates we prove by induction on k. We denote by C_k various constants depending only on $\bar{C}$, T, $\bar{\delta}$, $\bar{n}$ and the bounds of the curvature and its covariant derivatives up to order k of the ambient manifold $\bar{M}$ on its ball $B_{\bar{M}}(X_0(x_1), C)$ for suitable C.

Now suppose we have proved

$$|\nabla^l F| \leq \frac{C_l}{t^{\frac{l-2}{2}}}, \qquad l = 2, , ..., k-1 \tag{2.3.15}$$

on $B_0(x_1, (\frac{1}{2} + \frac{1}{2^k})) \times [0, T_3]$. By Corollary 2.1.4, Theorem 2.2.2, Corollary 2.2.3 and using (2.3.15), we get

$$\frac{\partial}{\partial t}|\nabla^k F|^2 \leq \Delta|\nabla^k F|^2 - 2|\nabla^{k+1} F|^2 + C_k|\nabla^k F|^2 + \frac{C_k}{t^{\frac{k-1}{2}}}|\nabla^k F|, \tag{2.3.16}$$

which implies

$$\frac{\partial}{\partial t}|\nabla^k F| \leq \Delta|\nabla^k F| + C_k|\nabla^k F| + \frac{C_k}{t^{\frac{k-1}{2}}}. \tag{2.3.17}$$

We also have

$$\frac{\partial}{\partial t}|\nabla^{k-1} F|^2 \leq \Delta|\nabla^{k-1} F|^2 - 2|\nabla^k F|^2 + \frac{C_{k-1}}{t^{k-\frac{5}{2}}}. \tag{2.3.18}$$

Let

$$u = t^{\frac{k-2}{2}}|\nabla^k F| + t^{k-3}|\nabla^{k-1} F|^2.$$

By combining (2.3.17) and (2.3.18), we obtain

$$\frac{\partial}{\partial t}u \leq \Delta u - \frac{1}{t}(u^2 + C_k) \tag{2.3.19}$$

on $B_0(x_1, (\frac{1}{2}+\frac{1}{2^k}))\times[0,T_3]$. Using the cutoff function $\xi(x) = \xi_k(d_0(x_1,x))$, (2.3.19) and applying maximum principle as before, we conclude with

$$|\nabla^k F| \leq \frac{C_k}{t^{\frac{k-2}{2}}} \qquad \text{on } B_0(x_1, (\frac{1}{2} + \frac{1}{2^{k+1}})) \times [0, T_3].$$

Therefore we complete the proof of Lemma 2.3.4. $\square$

Proof of Theorem 2.3.1

Now we combine the above three lemmas to prove Theorem 2.3.1. We have known that there is a $T_3 > 0$ such that for each j, the equation

$$\begin{cases} \dfrac{\partial}{\partial t}F^j(x,t) = \Delta F^j(x,t) \\ F^j(x,0) = x \qquad \text{for all } x \in D_j, \\ F^j(x,t) = x \qquad \text{for all } x \in \partial D_j \end{cases}$$

has a smooth solution F^j on $\bar{D}_j \times [0,T_3]$. Since $D_j \supset B_0(x_0, j+1)$, by choosing any $x_1 \in B_0(x_0, j)$ in Lemma 2.3.4 we have

$$|\nabla F^j| + |\nabla^2 F^j| \leq C$$

on $B_0(x_0, j) \times [0, T_3]$, where C depends only on $\bar{C}$, $\bar{n}$, $\bar{\delta}$, T. Moreover for any $x_1 \in B_0(x_0, j)$, $k \geq 3$, there is a C_k depending on $\bar{C}$, $\bar{\delta}$, T, $\bar{n}$ and x_1 such that

$$|\nabla^k F^j|(x_1, t) \leq C_k t^{-\frac{k-2}{2}}.$$

Then we can take a convergent subsequence of F^j (as $j \to \infty$) to get the desired F with the desired estimates. So the proof of Theorem 2.3.1 is completed. $\square$

For later pupose, now we derive some estimate of $g_{ij}(x,t)$ with respect to $F^*\hat{g}$. Let $\hat{g}_{ij} = (F^*\hat{g})_{ij}$.

Proposition 2.3.5 *Under the assumption of Theorem 2.3.1, there exist* $0 < T_4 \leq T_3$ *and* $C > 0$ *depending only on* $\bar{C}$, $\bar{n}$, $\bar{\bar{\delta}}$ *and* T *such that for all* $(x,t) \in M^n \times [0, T_4]$, *we have*

$$\frac{1}{C}\hat{g}_{ij}(x,t) \leq g_{ij}(x,t) \leq C\hat{g}_{ij}(x,t). \tag{2.3.20}$$

Proof. Note that $|\nabla F|^2 = \hat{g}_{ij}g^{ij} \leq C$, which implies $\hat{g}_{ij}(x,t) \leq Cg_{ij}(x,t)$. For the reverse inequality, since the curvature of $g_{ij}(\cdot,t)$ is bounded, we compute the equation of $\hat{g}_{ij}(x,t)$ on the domain,

$$\begin{aligned}\frac{\partial}{\partial t}\hat{g}_{ij} =& \Delta\hat{g}_{ij} - R_{ik}F^\alpha_l F^\beta_j \hat{g}_{\alpha\beta}g^{kl} - R_{jk}F^\alpha_l F^\beta_i \hat{g}_{\alpha\beta}g^{kl} + 2\hat{R}_{\alpha\beta\gamma\delta}F^\alpha_i F^\beta_k F^\gamma_j F^\delta_l g^{kl} - 2\hat{g}_{\alpha\beta}F^\alpha_{ki}F^\beta_{lj}g^{kl}\\ \geq& \Delta\hat{g}_{ij} - R_{ik}\hat{g}_{jl}g^{kl} - R_{jk}\hat{g}_{il}g^{kl} - C|\nabla F|^2\hat{g}_{ij} - 2|\nabla^2 F|^2 g_{ij}\\ \geq& \Delta\hat{g}_{ij} - Cg_{ij}.\end{aligned} \tag{2.3.21}$$

Note that for suitable large constant C, we have

$$\frac{\partial}{\partial t}g_{ij} \leq Cg_{ij}, \qquad 0 < t < T,$$

and $\hat{g}_{ij} \geq \frac{1}{C}g_{ij}$ at time 0. Thus for $t \leq 1/C^3$, we have

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)(\hat{g}_{ij} + (C^2 t - \frac{1}{C})g_{ij}) \geq& [-C + C^2 + C(C^2 t - \frac{1}{C})]g_{ij}\\ \geq& 0.\end{aligned} \tag{2.3.22}$$

Note that

$$(\hat{g}_{ij} + (C^2 t - \frac{1}{C})g_{ij}) \mid_{t=0} \geq 0.$$

Since $|\nabla^2 X| + \sqrt{t}|\nabla^3 X| \leq C$ and the curvature is bounded, then there is a smooth proper function φ with $\varphi(x) \geq 1 + d_0(x_0, x)$, $|\nabla\varphi| + |\nabla^2\varphi| \leq C$. So Hamilton's maximum principle for tensors on complete manifolds is applicable, we get

$$\hat{g}_{ij} + (C^2 t - \frac{1}{C})g_{ij} \geq 0 \quad \text{for} \quad t \leq \min\{T_3, C^{-3}\},$$

which implies

$$g_{ij} \le 2C\hat{g}_{ij}$$

for $t \le T_4 = \min\{T_3, 1/2C^3\}$.

The proof of the proposition is completed. □

As a consequence, we know that the solution of the harmonic map flow coupled with mean curvature flow is a family of diffeomorphisms.

Corollary 2.3.6 *Let $F(x,t)$ be assumed as in the previouse proposition. Then $F(\cdot,t)$ are diffeomorphisms from M to N for all $t \in [0,T_4]$.*

Proof. Note that (2.3.20) implies that F are local diffeomorphisms. For any $x_1 \neq x_2$, we claim that $F(x_1,t) \neq F(x_2,t)$ for all $t \in [0,T_4]$. Suppose not, then there is the first time $t_0 > 0$ such that $F(x_1,t_0) = F(x_2,t_0)$. Choose small $\sigma > 0$ so that there exist a neighborhood $\hat{O}$ of $F(x_1,t_0)$ and a neighborhood O of x_1 such that $F^{-1}(\cdot,t)$ is a diffeomorphism from $\hat{O}$ to O for each $t \in [t_0-\sigma, t_0]$, and let $\hat{\gamma}$ be a shortest geodesic(parametrized by arc length) on the target (with respect to the metric $\hat{g}$) with $\hat{\gamma}(0) = F(x_1,t)$, $\hat{\gamma}(l) = F(x_2,t)$ and $\hat{\gamma} \subset \hat{O}$. We compute

$$\frac{\partial}{\partial t}\hat{d}(F(x_1,t),F(x_2,t)) = \langle V(F(x_2,t)), \hat{\gamma}'(l)\rangle_{\hat{g}} - \langle V(F(x_1,t)), \hat{\gamma}'(0)\rangle_{\hat{g}}, \quad (2.3.23)$$

where $V(F(x,t)) = \frac{\partial}{\partial t}F(x,t)$. Now we pull back everything by F^{-1} to O,

$$\begin{aligned}\frac{\partial}{\partial t}\hat{d}(F(x_1,t),F(x_2,t)) &= \langle P_{-\hat{\gamma}}V - V, \hat{\gamma}'(0)\rangle_{F^*\hat{g}} \\ &\ge -\sup_{x\in F^{-1}\hat{\gamma}} |\hat{\nabla}V|(x,t)\hat{d}(F(x_1,t),F(x_2,t)),\end{aligned}$$

where $P_{\hat{\gamma}}$ is the parallel translation along $F^{-1}\hat{\gamma}$ using the connection defined by $F^*\hat{g}$. Since

$$\hat{\nabla}_k V^l = \nabla_k V^\alpha \frac{\partial x^l}{\partial y^\alpha},$$

where $\nabla_k V^\alpha$ is the covariant derivative of the section V^α of the bundle $F^{-1}TN$. Thus by (2.3.20) in proposition 2.3.5, we have

$$|\hat{\nabla}_k V^l| = [\nabla_k V^\alpha \nabla_l V^\beta \hat{g}_{\alpha\beta} \hat{g}^{kl}]^{\frac{1}{2}} \leq C|\nabla^3 F| \leq \frac{C}{\sqrt{t}},$$

where the constant C depends on the x_1 and x_2 and is independent of t by (2.3.5) of Theorem 2.3.1. Therefore, for $t \in [t_0 - \sigma, t_0]$, we have

$$\hat{d}(F(x_1,t), F(x_2,t)) \leq e^{C(\sqrt{t_0} - \sqrt{t_0 - \sigma})} \hat{d}(F(x_1,t_0), F(x_2,t_0)) = 0,$$

which contradicts with the choice of t_0. The corollary is proved. □

2.4 Mean curvature flow in harmonic map gauge

From the previous section, we know that the harmonic map flow coupled with mean curvature flow with identity as initial data has a short time solution $F(x,t)$ which maintains being a diffeomorphism with good estimates. Let $\bar{X} = X \circ F^{-1}$ be a family of maps defined from $(M^n, \hat{g}_{\alpha\beta})$ to $\bar{M}^{\bar{n}}$, then $\bar{X}$ satisfies the following mean curvature flow in harmonic map gauge

$$\frac{\partial}{\partial t}\bar{X} = g^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{X} \quad \text{ for } y \in N, \tag{2.4.1}$$

where $g^{\alpha\beta}$ is the inverse matrix of $g_{\alpha\beta}(\cdot,t) = ((F^{-1})^* g(\cdot,t))_{\alpha\beta}$ and $\hat{\nabla}$ is the covariant derivative with respect to $\hat{g}_{\alpha\beta}$. We denote the local coordinates of $\bar{M}$ by $\{\bar{z}^{\bar{\alpha}}\}$. It is not hard to see

$$g_{\alpha\beta}(y,t) = g_{ij}(x,t)\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta} = \bar{g}_{\bar{\alpha}\bar{\beta}} \frac{\partial X^{\bar{\alpha}}}{\partial x^i}\frac{\partial X^{\bar{\beta}}}{\partial x^j}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta} = \frac{\partial \bar{X}^{\bar{\gamma}}}{\partial y^\alpha} \cdot \frac{\partial \bar{X}^{\bar{\delta}}}{\partial y^\beta}\bar{g}_{\bar{\gamma}\bar{\delta}}(\bar{X}(y,t)), \tag{2.4.2}$$

this implies that the metric $g_{\alpha\beta}(y,t)$ is just the induced metric from the ambient space by the map $\bar{X}$. Since

$$\hat{\Gamma}^\gamma_{\alpha\beta}(y) - \Gamma^\gamma_{\alpha\beta}(y,t) = (\nabla^2 F)^\gamma_{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta},$$

we have

$$\begin{aligned} &\frac{1}{C}\hat{g}_{\alpha\beta}(y) \leq g_{\alpha\beta}(y,t) \leq C\hat{g}_{\alpha\beta}(y), \\ &|\hat{\Gamma}^{\gamma}_{\alpha\beta}(y) - \Gamma^{\gamma}_{\alpha\beta}(y,t)| \leq C, \end{aligned} \tag{2.4.3}$$

by Theorem 2.3.1 and Proposition 2.3.5.

Let X_1 and X_2 be two solutions to the mean curvature flow (1.1) with bounded second fundamental form and with the same initial value X_0 assumed as in the Theorem 1.1, and $g^1_{ij}(x,t)$ and $g^2_{ij}(x,t)$ are the corresponding induced metrics. As in section 2.3, we solve the harmonic map flows coupled with mean curvature flow with the same target $(M^n, \hat{g}_{\alpha\beta})$ (where $\hat{g} = g^1(T)$) respectively

$$\begin{cases} \dfrac{\partial}{\partial t}F_1 = \Delta_{g^1,\hat{g}}F_1 \\ F_1 \mid_{t=0} = \text{Identity} \qquad \text{on } M^n, \end{cases} \tag{2.4.4}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t}F_2 = \Delta_{g^2,\hat{g}}F_2 \\ F_2 \mid_{t=0} = \text{Identity} \qquad \text{on } M^n, \end{cases} \tag{2.4.5}$$

where $\Delta_{g^k,\hat{g}}$ is the harmonic map Laplacian defined by the metric $g^k_{ij}(x,t)$ and $\hat{g}_{\alpha\beta}$ for $k = 1,2$ respectively. By section 2.3, we obtain two solutions $F_1(x,t)$ and $F_2(x,t)$ such that Theorem 2.3.1 holds with $F = F_1$ and $F = F_2$. Corollary 2.3.6 says that $F_1(x,t)$ and $F_2(x,t)$ are diffeomorphisms for any $t \in [0,T_4]$. Let $g_{1\alpha\beta}(y,t) = ((F_1^{-1})^*g^1)_{\alpha\beta}(y,t)$ and $g_{2\alpha\beta}(y,t) = ((F_2^{-1})^*g^2)_{\alpha\beta}(y,t)$. Then $\bar{X}_1 = X_1 \circ F_1^{-1}$ and $\bar{X}_2 = X_2 \circ F_2^{-1}$ are two solutions to the mean curvature flow in harmonic map gauge (5.1) with the same initial value X_0,

$$\begin{cases} \dfrac{\partial}{\partial t}\bar{X}_1 = g_1^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_\beta\bar{X}_1, & \text{on } M^n \times [0,T_4], \\ \bar{X}_1 \mid_{t=0} = X_0, & \text{on } M^n, \end{cases} \tag{2.4.6}$$

$$\begin{cases} \dfrac{\partial}{\partial t}\bar{X}_2 = g_2^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_\beta\bar{X}_2, & \text{on } M^n \times [0,T_4], \\ \bar{X}_2 \mid_{t=0} = X_0, & \text{on } M^n, \end{cases} \tag{2.4.7}$$

where by (2.4.2) $g_{1\alpha\beta}$ and $g_{2\alpha\beta}$ are the corresponding induced metrics from the target $(\bar{M}^{\bar{n}}, \bar{g}_{\bar{\alpha}\bar{\beta}})$ by the maps $\bar{X}_1$ and $\bar{X}_2$.

Proposition 2.4.1 *Under the assumptions of Theorem 1.1, there is some* $T_5 > 0$ *depending only on* $\bar{C}$, $\bar{\delta}$, T *and* $\bar{n}$ *such that*

$$\bar{X}_1(y,t) = \bar{X}_2(y,t) \qquad on\ M^n \times [0, T_5]$$

for the two solutions of mean curvature flow in harmonic map gauge constructed above.

Proof. Let $\psi(\bar{z}_1, \bar{z}_2) = d^2_{\bar{M}}(\bar{z}_1, \bar{z}_2)$ be the square of the distance function on $\bar{M}$ which is viewed as a function of $(\bar{z}_1, \bar{z}_2) \in \bar{M} \times \bar{M}$. Set $u(y,t) = d^2_{\bar{M}}(\bar{X}_1(y,t), \bar{X}_2(y,t))$. Let $\Delta_k = g_k^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta$ for $k = 1, 2$. By direct computation, we have

$$\begin{aligned}
\frac{\partial}{\partial t} u(y,t) &= 2 d_{\bar{M}}(\bar{X}_1, \bar{X}_2) \frac{\partial d}{\partial \bar{z_1}^{\bar{\xi}}} \Delta_1 \bar{X}_1^{\bar{\xi}} + 2 d_{\bar{M}}(\bar{X}_1, \bar{X}_2) \frac{\partial d}{\partial \bar{z_2}^{\bar{\zeta}}} \Delta_2 \bar{X}_2^{\bar{\zeta}}, \\
g_1^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta u(y,t) &= 2 d_{\bar{M}}(\bar{X}_1, \bar{X}_2) [\frac{\partial d}{\partial \bar{z_1}^{\bar{\xi}}} \Delta_1 \bar{X}_1^{\bar{\xi}} + \frac{\partial d}{\partial \bar{z_2}^{\bar{\zeta}}} \Delta_1 \bar{X}_2^{\bar{\zeta}}] + Hess(\psi)(Z_\alpha, Z_\beta) g_1^{\alpha\beta},
\end{aligned}$$

where $Z_\alpha = \frac{\partial \bar{X}_1^{\bar{\xi}}}{\partial y^\alpha} \frac{\partial}{\partial \bar{z}_1^{\bar{\xi}}} + \frac{\partial \bar{X}_2^{\bar{\zeta}}}{\partial y^\alpha} \frac{\partial}{\partial \bar{z}_2^{\bar{\zeta}}} \in T_{(\bar{X}_1, \bar{X}_2)} \bar{M} \times \bar{M}$, for $\alpha = 1, 2 \cdots, n$ are vector fields on $\bar{M} \times \bar{M}$. Combining these two formulas, we have

$$[\frac{\partial}{\partial t} - g_1^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta] u(y,t) = -2 d_{\bar{M}}(\bar{X}_1, \bar{X}_2) \frac{\partial d}{\partial \bar{z}_2^{\bar{\zeta}}} ((\Delta_1 - \Delta_2) \bar{X}_2)^{\bar{\zeta}} - Hess(\psi)(Z_\alpha, Z_\beta) g_1^{\alpha\beta}. \tag{2.4.8}$$

Note that

$$\begin{aligned}
(\Delta_1 - \Delta_2) \bar{X}_2 &= g_1^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{X}_2 - g_2^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{X}_2 \\
&= g_1^{\alpha\gamma} g_2^{\beta\delta} (g_{2\delta\gamma} - g_{1\delta\gamma}) \hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{X}_2, \qquad (2.4.9)
\end{aligned}$$

$$\hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{X}_2 = \nabla_{2\alpha} \nabla_{2\beta} \bar{X}_2 + (\hat{\Gamma} - \Gamma_2) * \nabla \bar{X}_2,$$

where Γ_2 and ∇_2 are the christoffel symbol and the covariant derivative of the metric $g_{2\alpha\beta}(y,t)$.

For each $y \in M^n$ and $t \in [0,T]$, if $\bar{X}_1(y,t) \neq \bar{X}_2(y,t)$, denote the minimal geodesic on $\bar{M}$ from $\bar{X}_1(y,t)$ to $\bar{X}_2(y,t)$ by σ, and denote the parallel translation of $\bar{M}$ along σ by P_σ, then we have

$$\begin{aligned} g_{1\delta\gamma}(y,t) - g_{2\delta\gamma}(y,t) &= \langle \bar{X}_{1*}(\frac{\partial}{\partial y^\delta}), \bar{X}_{1*}(\frac{\partial}{\partial y^\gamma})\rangle_{\bar{g}} - \langle \bar{X}_{2*}(\frac{\partial}{\partial y^\delta}), \bar{X}_{2*}(\frac{\partial}{\partial y^\gamma})\rangle_{\bar{g}} \\ &= \langle \bar{X}_{1*}(\frac{\partial}{\partial y^\delta}), \bar{X}_{1*}(\frac{\partial}{\partial y^\gamma})\rangle_{\bar{g}} - \langle P_\sigma^{-1}(\bar{X}_{2*}(\frac{\partial}{\partial y^\delta})), P_\sigma^{-1}(\bar{X}_{2*}(\frac{\partial}{\partial y^\gamma}))\rangle_{\bar{g}} \\ &= \langle \bar{X}_{1*}(\frac{\partial}{\partial y^\delta}) - P_\sigma^{-1}(\bar{X}_{2*}(\frac{\partial}{\partial y^\delta})), \bar{X}_{1*}(\frac{\partial}{\partial y^\gamma})\rangle_{\bar{g}} \\ &\quad + \langle P_\sigma^{-1}(\bar{X}_{2*}(\frac{\partial}{\partial y^\delta})), \bar{X}_{1*}(\frac{\partial}{\partial y^\gamma}) - P_\sigma^{-1}(\bar{X}_{2*}(\frac{\partial}{\partial y^\gamma}))\rangle_{\bar{g}}. \end{aligned} \tag{2.4.10}$$

If $\bar{X}_1(y,t) = \bar{X}_2(y,t)$, $P_\sigma = Identity$, the above formula still holds.

In the following argument, we compute norms by using the metrics g_1 and $\bar{g}$. For example

$$|\hat{\Gamma} - \Gamma_2|^2 = (\hat{\Gamma} - \Gamma_2)^\gamma_{\alpha\beta}(\hat{\Gamma} - \Gamma_2)^{\gamma'}_{\alpha'\beta'} g_{1\gamma\gamma'} {g_1}^{\alpha\alpha'} {g_1}^{\beta\beta'}$$

and

$$|\nabla_2^2 \bar{X}_2|^2 = \bar{g}_{\bar{\xi}\bar{\zeta}} g_1^{\alpha\alpha'} g_1^{\beta\beta'} \nabla_{2\alpha}\nabla_{2\beta}\bar{X}_2^{\bar{\xi}} \nabla_{2\alpha'}\nabla_{2\beta'}\bar{X}_2^{\bar{\zeta}}.$$

We denote by C various constants depending only on the constants $\bar{C}$, T $\bar{n}$ and $\bar{\delta}$ in the main theorem 1.1. Then by (2.4.3), we have

$$\begin{aligned} &|\hat{\Gamma} - \Gamma_2| \le C, \\ &|\hat{\nabla}^2 \bar{X}_2| \le C|\hat{\Gamma} - \Gamma_2| + C|\nabla_2^2 \bar{X}_2| \le C, \\ &|g_2| + |g_2^{-1}| \le C, \end{aligned} \tag{2.4.11}$$

where $|\nabla_2^2 \bar{X}_2|$ is just the norm of the second fundamental form of $X_2 : M^n \to \bar{M}^{\bar{n}}$ which is bounded by $\bar{C}$. Combining (2.4.9) (2.4.10) and (2.4.11), we have

$$|(\Delta_1 - \Delta_2)\bar{X}_2|^2 \le C g_1^{\delta\gamma} \langle \bar{X}_{1*}(\frac{\partial}{\partial y^\delta}) - P_\sigma^{-1}(\bar{X}_{2*}(\frac{\partial}{\partial y^\delta})), \bar{X}_{1*}(\frac{\partial}{\partial y^\gamma}) - P_\sigma^{-1}(\bar{X}_{2*}(\frac{\partial}{\partial y^\gamma}))\rangle_{\bar{g}}. \tag{2.4.12}$$

By choosing an orthonormal frame at y so that $g_{1\alpha\beta} = \delta_{\alpha\beta}$, then we have

$$Hess(\psi)(Z_\alpha, Z_\beta)g_1^{\alpha\beta} = \sum_{\alpha=1}^{n} Hess(\psi)(Z_\alpha, Z_\alpha).$$

Note that

$$Z_\alpha = Z_{\alpha 1} + Z_{\alpha 2}, \qquad \text{for} \quad \alpha = 1, 2, \cdots, n,$$

where $Z_{\alpha 1} = \frac{\partial \bar{X}_1^{\bar{\xi}}}{\partial y^\alpha} \frac{\partial}{\partial \bar{z}_1^{\xi}} = \bar{X}_{1*}(\frac{\partial}{\partial y^\alpha})$ and $Z_{\alpha 2} = \frac{\partial \bar{X}_2^{\bar{\zeta}}}{\partial y^\alpha} \frac{\partial}{\partial \bar{z}_2^{\zeta}} = \bar{X}_{2*}(\frac{\partial}{\partial y^\alpha})$.

Recall that by Theorem 2.1.2 (ii), there is a constant C such that if $d_{\bar{M}}(\bar{z}_1, \bar{z}_2) \leq \min\{\frac{1}{4\sqrt{C}}, \frac{\bar{\delta}}{2}\}$, we have

$$(\nabla^2 d^2)(Z, Z) \geq 2|Z_1 - P_\sigma^{-1} Z_2|^2 - C|Z|^2 d^2 \quad \text{for all } Z \in T_{(\bar{z}_1, \bar{z}_2)} \bar{M}^{\bar{n}} \times \bar{M}^{\bar{n}},$$

where $Z = Z_1 + Z_2$, $Z_1 \in T_{\bar{z}_1} \bar{M}^{\bar{n}}$, $Z_2 \in T_{\bar{z}_2} \bar{M}^{\bar{n}}$. Hence if $d_{\bar{M}}(\bar{X}_1, \bar{X}_2) \leq \min\{\frac{1}{4\sqrt{C}}, \frac{\bar{\delta}}{2}\}$, then

$$\sum_{\alpha=1}^{n} Hess(\psi)(Z_\alpha, Z_\alpha) \geq \sum_{\alpha=1}^{n} 2|\bar{X}_{1*}(\frac{\partial}{\partial y^\alpha}) - P_\sigma^{-1} \bar{X}_{2*}(\frac{\partial}{\partial y^\alpha})|^2 - C d_{\bar{M}}(\bar{X}_1, \bar{X}_2)^2 \tag{2.4.13}$$

since $|Z_\alpha| \leq C$.

Combining (2.4.8), (2.4.12) and (2.4.13), if $u^{\frac{1}{2}} \leq \min\{\frac{1}{4\sqrt{C}}, \frac{\bar{\delta}}{2}\}$, then we have

$$\begin{aligned}(\frac{\partial}{\partial t} - g_1^{\alpha\beta} \hat{\nabla}_\alpha \hat{\nabla}_\beta) u(y,t) &\leq C d_{\bar{M}}(\bar{X}_1, \bar{X}_2) \sum_{\alpha=1}^{n} 2|\bar{X}_{1*}(\frac{\partial}{\partial y^\alpha}) - P_\sigma^{-1} \bar{X}_{2*}(\frac{\partial}{\partial y^\alpha})| \\ &\quad - 2\sum_{\alpha=1}^{n} |\bar{X}_{1*}(\frac{\partial}{\partial y^\alpha}) - P_\sigma^{-1} \bar{X}_{2*}(\frac{\partial}{\partial y^\alpha})|^2 + C d_{\bar{M}}^2(\bar{X}_1, \bar{X}_2) \\ &\leq Cu. \end{aligned} \tag{2.4.14}$$

Now we show that $u^{\frac{1}{2}} \leq \min\{\frac{1}{4\sqrt{C}}, \frac{\bar{\delta}}{2}\}$ on some time interval $[0, T_5]$.

For any $(y,t)\in\hat{M}\times[0,T_4]$, we have

$$\begin{aligned}u^{\frac{1}{2}}(y,t)\le& d_{\bar{M}}(X_1\circ F_1^{-1}(y,t),X_1\circ F_1^{-1}(y,0))+d_{\bar{M}}(X_1\circ F_1^{-1}(y,0),X_2\circ F_2^{-1}(y,0))\\&+d_{\bar{M}}(X_2\circ F_2^{-1}(y,t),X_2\circ F_2^{-1}(y,0))\\ \triangleq& I_1+I_2+I_3.\end{aligned}\tag{2.4.15}$$

By the mean curvature flow equation (1.1), we know

$$I_2\le d_{\bar{M}}(X_1(y,t),X_1(y,0))+d_{\bar{M}}(X_2(y,t),X_2(y,0))\le 2\sqrt{n}\bar{C}t.$$

By (2.3.4) (2.3.23), for any $x_1,x_2\in M^n$, we get

$$\frac{\partial}{\partial t}\hat{d}(F_1(x_1,t),F_1(x_2,t))\ge -C,$$

this implies

$$\hat{d}(x_1,x_2)\le\hat{d}(F_1(x_1,t),F_1(x_2,t))+Ct.\tag{2.4.16}$$

By (2.4.16) and Lemma 2.3.2, it follows

$$\begin{aligned}I_1=&d_{\bar{M}}(X_1\circ F_1^{-1}(y,t),X_1\circ F_1^{-1}(y,0))\\ \le&d_{(M,g^1(\cdot,t))}(F_1^{-1}(y,t),y)\\ \le&C\hat{d}(F_1^{-1}(y,t),y)\\ \le&Ct+C\hat{d}(y,F_1(y,t))\\ \le&C\sqrt{t}.\end{aligned}$$

The estimate of I_3 is similar. Therefore, we have

$$u^{\frac{1}{2}}(y,t)\le C\sqrt{t}\tag{2.4.17}$$

for some constant C depending only on $\bar{C}$, $\bar{\delta}$, T and $\bar{n}$.

Although $g_1^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_\beta$ is not the standard Laplacian, the maximum principle is still applicable. For completeness, we include the proof in the following.

Since the curvature of $(M,\hat{g})$ is bounded, it is well-known that there is a function φ such that

$$\begin{aligned}\frac{1}{C}(1+d_{\hat{g}}(y_0,y)) \le \varphi(y) \le C(1+d_{\hat{g}}(y_0,y))\\ |\hat{\nabla}\varphi|+|\hat{\nabla}^2\varphi| \le C.\end{aligned}$$

Note g_1 is equivalent to $\hat{g}$. For any small $\varepsilon>0$ and big $A>0$, we have

$$(\frac{\partial}{\partial t}-g_1^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_\beta)(e^{-Ct}u(y,t)-\varepsilon e^{At}\varphi) \le -\frac{\varepsilon A}{2}e^{At}\varphi<0.$$

Then the classical maximum principle implies that for any fixed t_0 the maximal value of $(e^{-Ct}u(y,t)-\varepsilon e^{At}\varphi)$ on $M\times[0,t_0]$ can not be achieved for any point (y,t) with $0<t\le t_0$. Hence $e^{-Ct}u(y,t)-\varepsilon e^{At}\varphi\le 0$ for any $t\in[0,T_5]$ for some $T_5>0$. Let $\varepsilon\to 0$, we conclude that $u\equiv 0$ on $[0,T_5]$. This implies $\bar{X}_1=\bar{X}_2$, on $M\times[0,T_5]$. We complete the proof of Proposition 2.4.1. $\square$

2.5 Proof of the uniqueness theorem 1.1

Now we are ready to prove Theorem 1.1. Let $X_1(x,t)$ and $X_2(x,t)$ be two solutions to the mean curvature flow with bounded second fundamental form and with the same initial data. We solve the corresponding harmonic map flow (2.4.4) (2.4.5) (with the same target $(M,\hat{g})$) respectively to obtained two solutions $F_1(x,t)$ and $F_2(x,t)$ on some common time interval. Then $\bar{X}_1=X_1\circ F_1^{-1}$ and $\bar{X}_2=X_2\circ F_2^{-1}$ are two solutions to the mean curvature flow in harmonic map gauge with the same initial value. By Proposition 2.4.1 we know $\bar{X}_1\equiv\bar{X}_2$ on $[0,T_5]$. So in order to prove $X_1(x,t)\equiv X_2(x,t)$, we only need to show $F_1\equiv F_2$.

We know

$$\begin{aligned}\Delta_1 F_1^\alpha &= g_1^{\beta\gamma}(\hat{\Gamma}^\alpha_{\beta\gamma}-\Gamma^\alpha_{1\beta\gamma})\circ F_1,\\ \Delta_2 F_2^\alpha &= g_2^{\beta\gamma}(\hat{\Gamma}^\alpha_{\beta\gamma}-\Gamma^\alpha_{2\beta\gamma})\circ F_2.\end{aligned}$$

Since $\bar{X}_1 \equiv \bar{X}_2$, we know $g_{1\alpha\beta}(y,t) = g_{2\alpha\beta}(y,t)$ on $[0, T_5]$, and the vector fields $V_1 \equiv V_2$ on the target, where

$$\begin{aligned} V_1^\alpha &= g_1^{\beta\gamma}(\hat{\Gamma}^\alpha_{\beta\gamma} - \Gamma^\alpha_{1\beta\gamma}), \\ V_2^\alpha &= g_2^{\beta\gamma}(\hat{\Gamma}^\alpha_{\beta\gamma} - \Gamma^\alpha_{2\beta\gamma}). \end{aligned}$$

Therefore, the two families of maps F_1 and F_2 satisfy the same ODE with the same initial value:

$$\begin{cases} \dfrac{\partial}{\partial t} F_1 = V \circ F_1 \\ F_1(\cdot, 0) = Identity, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t} F_2 = V \circ F_2 \\ F_2(\cdot, 0) = Identity. \end{cases}$$

So for any $x \in M^n$, letting γ be a shortest geodesic(parametrized by arc length) on the target with $\gamma(0) = F_1(x,t)$ and $\gamma(l) = F_2(x,t)$, we have

$$\begin{aligned} \frac{\partial}{\partial t}\hat{d}(F_1(x,t), F_2(x,t)) &= \langle V, \gamma'(l)\rangle - \langle V, \gamma'(0)\rangle \\ &= \langle P_\gamma^{-1} V - V, \gamma'(0)\rangle \\ &\leq \sup_{y\in\gamma} |\hat{\nabla} V|(y,t)\hat{d}(F_1(x,t), F_2(x,t)), \end{aligned}$$

where $P_\gamma^{-1}V$ is the parallel transport of $V(F_2(x,t),t)$ along the geodesic γ back to the tangent space of the point $F_1(x,t)$. We have seen in the proof of Corollary 2.3.6 that $\sup\limits_{y\in\gamma} |\hat{\nabla} V|(y,t) \leq \frac{C}{\sqrt{t}}$ for some C depending on x but independent of t. Since $\hat{d}(F_1(x,0), F_2(x,0)) \equiv 0$, we conclude that

$$F_1(x,t) \equiv F_2(x,t).$$

So we have proved $X_1(x,t) = X_2(x,t)$, for all $x \in M$ and $t \in [0, T_5]$. Clearly, we can extend the interval $[0, T_5]$ to the whole $[0, T]$ by applying the same argument on $[T_5, T]$.

The proof of Theorem 1.1 is completed. □

Corollary 1.2 is a direct consequence of Theorem 1.1. Indeed, let $\bar{\sigma}$ and σ be two isometries of $(\bar{M}^{\bar{n}}, \bar{g})$ and (M^n, g) respectively such that $(\bar{\sigma} \circ X_0)(x) = (X_0 \circ \sigma)(x)$ for any $x \in M^n$. Simple computation shows that $\bar{\sigma} \circ X_t$ and $X_t \circ \sigma$ are two solutions to the mean curvature flow (1.1) with bounded second fundamental form on $M^n \times [0, T]$ and with the same initial value, then by Theorem 1.1, we have

$$(\bar{\sigma} \circ X_t)(x) = (X_t \circ \sigma)(x)$$

for any $x \in M^n$ and $t \in [0, T]$. The proof of the Corollary 1.2 is completed. □

Chapter 3

Pseudolocality Theorem

In this chapter, we establish the pseudolcality theorems 1.4, 1.5 for the mean curvature flow. As an application, the strong uniqueness theorem 1.3 of the mean curvature flow is proved.

We begin with a few terminologies for the sake of convenience. An n-dimensional submanifold $M \subset \bar{M}$ is said to be a local δ- Lipschitz graph of radius r_0 at $P \in M$, if there is a normal coordinate system $(y^1 \cdots y^{\bar{n}})$ of $\bar{M}$ around P with $T_P M$ =span$\{\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}\}$, a vector valued function $F : \{y' = (y^1, \cdots, y^n) \mid (y^1)^2 + \cdots + (y^n)^2 < r_0^2\} \to \mathbb{R}^{\bar{n}-n}$ with $F(0) = 0$, $|DF|(0) = 0$ such that $M \cap \{|y'| < r_0\} = \{(y', F(y')) \mid |y'| < r_0\}$ and $|DF|^2(y') = \sum_{i,\beta} \frac{\partial F^\beta}{\partial y^i} \frac{\partial F^\beta}{\partial y^i} < \delta^2$. The submanifold M_0 is said to be graphic in the ball $B_{\bar{M}}(x_0, r_0)$, if the above holds for $\delta = \infty$.

We say a submanifold $M \subset \bar{M}$ is properly embedded in a ball $B_{\bar{M}}(x_0, r_0)$ if either M is closed or ∂M has distance $\geq r_0$ from x_0. We say a submanifold $M \subset \bar{M}$ is properly embedded in $\bar{M}$ if either M is closed or there is an $x_0 \in \bar{M}$ such that M is properly embedded in $B_{\bar{M}}(x_0, r_0)$ for any $r_0 > 0$. It is clear that if $\bar{M}$ is complete and M is properly embedded in $\bar{M}$, then M is complete. A properly embedded submanifold M is said to be uniform graphic with radius r_0 if for any $x_0 \in M$ it is graphic in the ball $B_{\bar{M}}(x_0, r_0)$.

The following lemma says that if the second fundamental form is controlled, then (a piece of) the sub-manifold is a local δ-Lipschitz graph of suitable radius.

Lemma 3.1 *Let $\bar{M}$ be an $\bar{n}-$dimensional complete Riemannian manifold satisfying*

$$|\bar{R}m| + |\bar{\nabla}\bar{R}m|(x) \leq \bar{C}, \qquad inj(\bar{M}) \geq i_0 > 0.$$

There exists a constant $C_1 > 0$ with the following property. Let $\{x^1, \cdots, x^{\bar{n}}\}$ be normal coordinates of $\bar{M}$ of radius r_0 around x_0 with $T_{x_0}M = span\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$, where M is an n-dimensional submanifold properly embedded in $B_{\bar{M}}(x_0, r_0)$, $x_0 \in M$, $r_0 \leq \frac{1}{C_1}$, and the second fundamental form $|A| \leq \frac{1}{r_0}$. Then there exists a map $F : \{(x^1, \cdots, x^n) \mid ({x^1}^2 + \cdots + {x^n}^2)^{\frac{1}{2}} < \frac{r_0}{96}\} \to \mathbb{R}^{\bar{n}-n}$ with $F(0) = 0$, $|DF|(0) = 0$ such that the connected component containing x_0 of $M \cap \{(x^1, \cdots, x^{\bar{n}}) \mid ({x^1}^2 + \cdots + {x^n}^2)^{\frac{1}{2}} < \frac{r_0}{96}\}$ can be written as a graph $\{(x', F(x')) \mid |x'| = ({x^1}^2 + \cdots + {x^n}^2)^{\frac{1}{2}} < \frac{r_0}{96}\}$ and

$$|DF|(x') \leq \frac{36}{r_0}|x'|, \tag{3.1}$$

$x' = (x^1, \cdots, x^n) \in B_{\mathbb{R}^n}(0, \frac{r_0}{96})$, where $|DF|(x')^2 = \sum_{i=1}^{n} \sum_{\alpha=n+1}^{\bar{n}} \frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\alpha}{\partial x^i}(x')$.

Proof. Let $X = (X^1, \cdots, X^{\bar{n}}) = (x', F(x')), x' = (x^1, \cdots, x^n)$, be a graph representation of the local isometric embedding of the connected component containing x_0 of $M \cap \{(x^1, \cdots, x^{\bar{n}}) \mid ({x^1}^2 + \cdots + {x^n}^2)^{\frac{1}{2}} < r_1\}$(for some $r_1 \leq \frac{r_0}{96}$) into $\bar{M}$ under the exponential map.

Define

$$|\nabla F|^2 = \sum_{i,j=1}^{n} \sum_{\alpha=n+1}^{\bar{n}} \frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\alpha}{\partial x^j} g^{ij}, |DF|^2 = \sum_{i=1}^{n} \sum_{\alpha=n+1}^{\bar{n}} \frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\alpha}{\partial x^i}.$$

By choosing C_1 large, we have

$$\frac{1}{2}\delta_{\alpha\beta} \leq \bar{g}_{\alpha\beta} \leq 2\delta_{\alpha\beta}, \quad |\bar{\Gamma}^\gamma_{\alpha\beta}| \leq 1, \quad \frac{1}{2}\delta_{ij} \leq g_{ij} \leq 2(1 + |DF|^2)\delta_{ij}.$$

For $\alpha \geq n + 1$, $i, j \leq n$, recall the coefficients of the second fundamental form is given by

$$A^\alpha_{ij} = \frac{\partial X^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial X^\alpha}{\partial x^k} + \bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial X^\beta}{\partial x^i}\frac{\partial X^\gamma}{\partial x^j} = \nabla^2_{ij}F^\alpha + \bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial X^\beta}{\partial x^i}\frac{\partial X^\gamma}{\partial x^j}.$$

Note that

$$|\bar{\Gamma}^{\alpha}_{\beta\gamma}\frac{\partial X^{\beta}}{\partial x^i}\frac{\partial X^{\gamma}}{\partial x^j}|^2 = \bar{\Gamma}^{\alpha'}_{\beta'\gamma'}\frac{\partial X^{\beta'}}{\partial x^i}\frac{\partial X^{\gamma'}}{\partial x^j}\bar{\Gamma}^{\alpha}_{\beta\gamma}\frac{\partial X^{\beta}}{\partial x^k}\frac{\partial X^{\gamma}}{\partial x^l}g^{ik}g^{jl}\bar{g}_{\alpha\alpha'} \le C(\bar{n}),$$

$$\begin{aligned}|\nabla^2 F|^2 &= \sum_{\alpha,\beta\ge n+1; i,j,k,l\le n}\nabla^2_{ij}F^{\alpha}\nabla^2_{kl}F^{\beta}\delta_{\alpha\beta}g^{ik}g^{jl}\\ &\le 4(|A|^2 + C(\bar{n}))\\ &\le 4r_0^{-2} + C(\bar{n}),\end{aligned}$$

and

$$|\nabla|\nabla F|| \le |\nabla^2 F|.$$

This implies

$$|\nabla F|(\cdot) \le 3r_0^{-1}d_M(x_0,\cdot). \tag{3.2}$$

Since $g_{ij} \le 2(\delta_{ij} + \frac{\partial F^{\alpha}}{\partial x^i}\frac{\partial F^{\alpha}}{\partial x^j}) \le 2(1+|DF|^2)\delta_{ij}$, it follows that

$$|\nabla F|^2 \ge \frac{1}{4}\frac{|DF|^2}{1+|DF|^2}$$

and

$$|DF|^2 \le \frac{4|\nabla F|^2}{1-4|\nabla F|^2}. \tag{3.3}$$

Combining (3.2)and (3.3), it follows that

$$|DF|(\cdot) \le 9r_0^{-1}d_M(x_0,\cdot) \qquad \text{on} \quad B_M(x_0,\frac{r_0}{24}).$$

Since $d_M(x_0,\cdot) \le 2d_{\bar{M}}(x_0,\cdot)$ by (2.1.5), we have

$$|DF|(\cdot) \le 18r_0^{-1}\sup_{B_M(0,\frac{r_0}{24})}(1+|DF|)|x'| \le 36r_0^{-1}|x'|,$$

and we conclude that

$$|DF|(x') \le 36r_0^{-1}|x'|, \qquad \text{whenever} \quad |x'| \le \frac{r_0}{96}.$$

The above argument shows that there is $C_1 > 0$ such that under the exponential map, once the connected component of M can be expressed as a graph $(x', F(x'))$

on $B_{\mathbb{R}^n}(0, r_1)$, for $r_1 \le \frac{r_0}{96}$, then the estimate (3.1) holds. Hence the connected component of M can be expressed as a graph on the ball $B_{\mathbb{R}^n}(0, \frac{r_0}{96})$. $\square$

For future applications in pseudolocality theorem, we need a local graph representation for mean curvature flow.

Lemma 3.2 *Fix $k \ge 1$. Let $\bar{M}$ be an $\bar{n}-$dimensional complete manifold satisfying*

$$\sum_{i=0}^{k+1} |\bar{\nabla}^i \bar{R}m|(x) \le \bar{C}, \qquad inj(\bar{M}) \ge i_0 > 0.$$

There exists a constant $C_1 > 0$ with the following property. Suppose M_s, $s \in [-r_0^2, 0]$ is a solution of MCF properly embedded in $B_{\bar{M}}(x_0, r_0)$, $x_0 \in M_0$, $r_0 \le \frac{1}{C_1}$, with $\sum\limits_{i=0}^{k} |\nabla^i A| r_0^{i+1} \le 1$ on $B_{\bar{M}}(x_0, r_0)$. Denote by $x_0^s \in M_s$ the orbit of x_0. Let $\{x^1, \cdots, x^{\bar{n}}\}$ be normal coordinates of $\bar{M}$ of radius r_0 around x_0 with $T_{x_0}M_0 = span\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$. Then there exist a family of smooth maps $F_s : \{(x^1, \cdots, x^n) \mid ({x^1}^2 + \cdots + {x^n}^2)^{\frac{1}{2}} < \frac{r_0}{C_1}\} \to \mathbb{R}^{\bar{n}-n}$ with $F_0(0) = 0$, $|D_0 F|(0) = 0$, $e\bar{x}p_{x_0}((0, F_s(0))) = x_0^s$ such that the connected component of $M_s \cap \{(x^1, \cdots, x^{\bar{n}}) \mid ({x^1}^2 + \cdots + {x^n}^2)^{\frac{1}{2}} < \frac{r_0}{C_1}\}$ (under the exponential map $e\bar{x}p_{x_0}$) containing x_0^s can be written as a graph $\{(x', F_s(x')) \mid |x'| = ({x^1}^2 + \cdots + {x^n}^2)^{\frac{1}{2}} < \frac{r_0}{C_1}\}$; moreover we have $\sum\limits_{i=1}^{k+2} r_0^{i+1} |D^i F_s| \le C_1$.

Proof. Actually, by the mean curvature flow equation $\frac{\partial}{\partial s} X = \Delta X$, where $X = (x', F_s(x'))$ is the graph representations on $B(0, r_1)$ for some $r_1 < \frac{r_0}{C_1}$, we have information on $|\frac{\partial}{\partial s} F_s| r_0 + |\frac{\partial}{\partial s} DF_s| r_0^2 \le C_1$. This gives $|F_s(0)| \le Csr_0^{-1}$ and $|DF_s|(0) \le Csr_0^{-2}$. Similarly, by integrating $|\nabla |\nabla F|| \le |\nabla^2 F|$, we know the graph representation holds in a ball of uniform radius $\frac{r_1}{C_1}$. The higher derivative $D^i F$ can be estimated by $\sum_{j \le i} |\nabla^j F|$ by definitions. $\square$

Now we state the pseudolocality theorem for the mean curvature flow.

Theorem 3.3 *(Pseudolocality) Let $\bar{M}$ be an $\bar{n}$-dimensional complete manifold satisfying $\sum\limits_{i=0}^{3} |\bar{\nabla}^i \bar{R}m| \le c_0^2$ and $inj(\bar{M}) \ge i_0 > 0$. Then for every $\alpha > 0$ there exist*

$\varepsilon > 0$, $\delta > 0$ with the following property. Suppose we have a smooth solution to the mean curvature flow $M_t \subset \bar{M}$ properly embedded in $B_{\bar{M}}(x_0, r_0)$ for $t \in [0, T]$ with $0 < T \leq \varepsilon^2 r_0^2$, and assume that at time zero, M_0 is a local δ-Lipschitz graph of radius r_0 at $x_0 \in M_0$ with $r_0 \leq \frac{i_0}{2}$. Then we have an estimate of the second fundamental form

$$|A|(x,t)^2 \leq \frac{\alpha}{t} + (\varepsilon r_0)^{-2} \tag{3.4}$$

on $B_{\bar{M}}(x_0, \varepsilon r_0) \cap M_t$, for any $t \in [0, T]$.

Proof. We argue by contradiction. By scaling we may assume $r_0 = 1$. Suppose there exist fixed $c_0 > 0$, $i_0 > 0$, $\alpha > 0$, and a sequence of $\varepsilon, \delta \to 0$ and smooth solutions to the mean curvature flow $M_t \subset \bar{M}$ for $t \in [0, T] \subseteq [0, \varepsilon^2]$ such that at time zero, M_0 is a local δ- Lipschitz graph of radius 1 at $x_0 \in M$. But there is some (x_1, t_1) satisfying $0 \leq t_1 \leq T$ and $x_1 \in B_{\bar{M}}(x_0, \varepsilon)$ such that

$$|A|(x_1, t_1)^2 > \frac{\alpha}{t_1} + \varepsilon^{-2}.$$

Denote by E_α the set of points (x, t) satisfying $|A|(x,t)^2 \geq \frac{\alpha}{t}$. Now we use the Perelman's point-picking technique [26] to choose another point which controls nearby points in its scale.

Lemma 3.4 *For any $K > 0$ with $K\varepsilon < \frac{1}{100n}$, let M_t be assumed as in the theorem, suppose $|A|(x_1, t_1)^2 \geq \frac{\alpha}{t_1} + \varepsilon^{-2}$ for some (x_1, t_1) satisfying $0 \leq t_1 \leq T \leq \varepsilon^2$ and $x_1 \in B_{\bar{M}}(x_0, \varepsilon)$, then one can find $(\bar{x}, \bar{t}) \in E_\alpha$ with $0 < \bar{t} \leq T$, $d_{\bar{M}}(x_0, \bar{x}) \leq (2K+1)\varepsilon$ such that*

$$|A|(x,t) \leq 4Q \tag{3.5}$$

whenever $\bar{t} - \frac{3}{4}\alpha Q^{-2} \leq t \leq \bar{t}$, $d_{\bar{M}}(x, \bar{x}) \leq KQ^{-1}$, where $Q = |A|(\bar{x}, \bar{t})$.

Proof. Firstly, we claim that there exists $(\bar{x}, \bar{t}) \in E_\alpha$ with $0 < \bar{t} \leq T$, $d_{\bar{M}}(x_0, \bar{x}) \leq (2K+1)\varepsilon$ such that

$$|A|(x,t) \leq 4|A|(\bar{x}, \bar{t})$$

whenever $(x,t) \in E_\alpha$, $0 \leq t \leq \bar{t}$, $d_{\bar{M}}(x_0, x) \leq d_{\bar{M}}(x_0, \bar{x}) + K|A|(\bar{x}, \bar{t})^{-1}$.

The argument is by contradiction. If (x_1, t_1) can not be chosen for $(\bar{x}, \bar{t})$, one can find $(x_2, t_2) \in E_\alpha$ with $0 \leq t_2 \leq t_1$, $d_{\bar{M}}(x_0, x_2) \leq d_{\bar{M}}(x_0, x_1) + K|A|(x_1, t_1)^{-1}$, $|A|(x_2, t_2) > 4|A|(x_1, t_1)$. Inductively, we have a sequence of $(x_k, t_k) \in E_\alpha$ with $0 \leq t_k \leq t_{k-1}$, $d_{\bar{M}}(x_0, x_k) \leq d_{\bar{M}}(x_0, x_{k-1}) + K|A|(x_{k-1}, t_{k-1})^{-1}$, $|A|(x_k, t_k) > 4|A|(x_{k-1}, t_{k-1})$. Therefore we have

$$|A|(x_k, t_k) > 4^{k-1}|A|(x_1, t_1) \geq 4^{k-1}\varepsilon^{-1}$$

and $d_{\bar{M}}(x_0, x_k) \leq d_{\bar{M}}(x_0, x_1) + K\sum_{i=1}^{\infty}(4^{i-1}|A|(x_1, t_1))^{-1} \leq (2K+1)\varepsilon < \frac{1}{2}$. Since the solution is smooth, we get a contradiction as k large enough.

For the chosen $(\bar{x}, \bar{t})$, if $(x, t) \notin E_\alpha$, $\bar{t} - \frac{3}{4}\alpha Q^{-2} \leq t \leq \bar{t}$, then

$$|A|^2(x, t) \leq \frac{\alpha}{t} \leq \frac{\alpha}{\bar{t} - \frac{3}{4}\alpha Q^{-2}} \leq 4Q^2.$$

If $(x, t) \in E_\alpha$ and $d_{\bar{M}}(x, \bar{x}) \leq K|A|(\bar{x}, \bar{t})^{-1}$, by above claim we still have the estimate. The lemma is proved. $\square$

Continuing the proof of Theorem 3.3.

Choose $K = \frac{1}{\sqrt{\varepsilon}}$. Let $(\bar{x}, \bar{t})$ be the point obtained in Lemma 3.4. Consider the auxiliary functions

$$\varphi(x, t) = (4\pi(\bar{t} - t))^{-\frac{n}{2}} e^{-(1+\frac{1}{\varepsilon}(t-\bar{t}))\frac{d^2_{\bar{M}}(\bar{x}, x)}{4(\bar{t}-t)} - \frac{n}{2\varepsilon}t}, \psi(x, t) = (1 - \frac{d_{\bar{M}}(\bar{x}, x)^2 + 3nt}{\rho^2})^3_+$$

on $\bar{M} \times [0, \bar{t}]$, where $\rho = \min\{\frac{1}{2}, \frac{1}{c_0\sqrt{e}}, i_0, \sqrt{\varepsilon}\}$. They are also functions on M by composing the inclusion maps. We will compute their equations on M. Since the sectional curvature of $\bar{M}$ satisfies $-c_0^2 \leq sec \leq c_0^2$, by comparison theorem and mean curvature flow equation, we have

$$\begin{aligned}
(\frac{\partial}{\partial t}+\Delta)d_{\bar{M}}(\bar{x},\cdot)^2 &= 4d\bar{\nabla}d_{\bar{M}}\cdot H+tr(Hess(d^2_{\bar{M}}(\bar{x},\cdot))\mid_{TM}) \\
&\geq 4d\bar{\nabla}d_{\bar{M}}\cdot H+2n\frac{c_0d_{\bar{M}}(\bar{x},\cdot)}{\tan c_0d_{\bar{M}}(\bar{x},\cdot)} \\
&\geq 4d\bar{\nabla}d_{\bar{M}}\cdot H+2n(1-\frac{1}{2}c_0^2d^2_{\bar{M}}(\bar{x},\cdot)), \\
(\frac{\partial}{\partial t}-\Delta)d_{\bar{M}}(\bar{x},\cdot)^2 &= -tr(Hess(d^2_{\bar{M}}(\bar{x},\cdot))\mid_{TM}) \\
&\geq -2nc_0d_{\bar{M}}(\bar{x},\cdot)\coth(c_0d_{\bar{M}}(\bar{x},\cdot))\geq -3n
\end{aligned}$$

whenever $d_{\bar{M}}(\bar{x},\cdot)^2<\min\{\frac{1}{c_0^2e},i_0^2\}$, $t\in[0,\bar{t}]$. Hence we have

$$(\frac{\partial}{\partial t}-\Delta)\psi\leq 0 \tag{3.6}$$

and

$$\begin{aligned}
(\frac{\partial}{\partial t}+\Delta-|H|^2)\varphi =&\varphi[\frac{n}{2(\bar{t}-t)}-\frac{1+\frac{1}{\varepsilon}(t-\bar{t})}{4(\bar{t}-t)}(\frac{\partial}{\partial t}+\Delta)d_{\bar{M}}(\bar{x},\cdot)^2 \\
&-\frac{(1+\frac{1}{\varepsilon}(t-\bar{t}))d_{\bar{M}}(\bar{x},\cdot)^2}{4(\bar{t}-t)^2}+\frac{(1+\frac{1}{\varepsilon}(t-\bar{t}))^2|\nabla d_{\bar{M}}(\bar{x},\cdot)^2|^2}{16(\bar{t}-t)^2} \\
&-\frac{\frac{1}{\varepsilon}d_{\bar{M}}(\bar{x},\cdot)^2}{4(\bar{t}-t)}-\frac{n}{2\varepsilon}-|H|^2] \\
\leq&\varphi[-\frac{1+\frac{1}{\varepsilon}(t-\bar{t})}{(\bar{t}-t)}d_{\bar{M}}\bar{\nabla}d_{\bar{M}}\cdot H-\frac{(1+\frac{1}{\varepsilon}(t-\bar{t}))d_{\bar{M}}(\bar{x},\cdot)^2}{4(\bar{t}-t)^2} \\
&+\frac{(1+\frac{1}{\varepsilon}(t-\bar{t}))^2|\nabla d_{\bar{M}}(\bar{x},\cdot)^2|^2}{16(\bar{t}-t)^2} \\
&-\frac{[\frac{1}{\varepsilon}-(1+\frac{1}{\varepsilon}(t-\bar{t}))nc_0^2]d_{\bar{M}}(\bar{x},\cdot)^2}{4(\bar{t}-t)}-|H|^2] \\
\leq&-|H+(1+\frac{1}{\varepsilon}(t-\bar{t}))\frac{d_{\bar{M}}(\bar{x},\cdot)\bar{\nabla}^{\perp}d_{\bar{M}}(\bar{x},\cdot)}{2(\bar{t}-t)}|^2\varphi
\end{aligned} \tag{3.7}$$

whenever $d_{\bar{M}}(\bar{x},\cdot) < \rho$, $t \in [0,\bar{t}]$. We used $0 < 1 + \frac{1}{\varepsilon}(t-\bar{t}) \le 1$. In the above and following argument, we regard the mean curvature flow M_t is a smooth family of $F_t : M \to \bar{M}$, $(\varphi\psi)\circ F_t$ is a C^2 function on $M \times [0,\bar{t}]$ with compact support in M. So $\int_{M_t} \varphi\psi = \int_M \varphi\psi dv_t$ is a C^2 function in t. Combining (3.6) and (3.7), we get the monotonicity formula (which generalizes Huiken's Monotonicity Formula[23])

$$\frac{d}{dt}\int_{M_t} \varphi\psi \le -\int_{M_t} |H + (1+\frac{1}{\varepsilon}(t-\bar{t}))\frac{d_{\bar{M}}(\bar{x},\cdot)\bar{\nabla}^{\perp} d_{\bar{M}}(\bar{x},\cdot)}{2(\bar{t}-t)}|^2\varphi\psi \tag{3.8}$$

on $[0,\bar{t}]$. This implies

$$\begin{aligned}&\int_{\bar{t}-\frac{1}{2}\alpha Q^{-2}}^{\bar{t}} [\int_{M_t} |H + (1+\frac{1}{\varepsilon}(t-\bar{t}))\frac{d_{\bar{M}}(\bar{x},\cdot)\bar{\nabla}^{\perp} d_{\bar{M}}(\bar{x},\cdot)}{2(\bar{t}-t)}|^2\varphi\psi]dt\\ &\le \int_{M_{\bar{t}-\frac{1}{2}\alpha Q^{-2}}} \varphi\psi - \int_{M_{\bar{t}}} \varphi\psi.\end{aligned} \tag{3.9}$$

Since the solution is smooth and properly embedded, ψ is compactly supported, we have $\lim_{t\to\bar{t}_-}\int_{M_t}\varphi\psi = e^{-\frac{n}{2\varepsilon}\bar{t}}(1-\frac{3n\bar{t}}{\rho^2})^3$. Now we claim that there is $\beta > 0$ such that as $\varepsilon, \delta \to 0$, we have

$$\int_{M_{\bar{t}-\frac{1}{2}\alpha Q^{-2}}} \varphi\psi \ge (1+\beta)e^{-\frac{n}{2\varepsilon}\bar{t}}(1-\frac{3n\bar{t}}{\rho^2})^3. \tag{3.10}$$

We still argue by contradiction. Suppose not, then there is a subsequence of $\varepsilon, \delta \to 0$ and

$$\int_{\bar{t}-\frac{1}{2}\alpha Q^{-2}}^{\bar{t}} [\int_{M_t} |H + (1+\frac{1}{\varepsilon}(t-\bar{t}))\frac{d_{\bar{M}}(\bar{x},\cdot)\bar{\nabla}^{\perp} d_{\bar{M}}(\bar{x},\cdot)}{2(\bar{t}-t)}|^2\varphi\psi dv]dt \le \beta \to 0. \tag{3.11}$$

Parabolic scaling the solution around $(\bar{x},\bar{t})$ with the factor Q and shifting the $\bar{t}$ to 0 and $\bar{x}$ to the origin O, i.e. let $(\tilde{M},\tilde{g}) = (\bar{M}, Q^2\bar{g})$ be the new target manifold, $\tilde{M}_s = M_{\bar{t}+Q^{-2}s}$, $-\frac{3}{4}\alpha \le s \le 0$ be the new family of submanifolds, which is still solution of the mean curvature flow. By (3.5), the normalized second fundamental form satisfies $|\tilde{A}| \le 4$ on $B_{\tilde{M}}(\bar{x},K)$, $-\frac{3}{4}\alpha \le s \le 0$. By Theorem 2.2.2, we have $|\tilde{\nabla}\tilde{A}| \le Const.$ on $B_{\tilde{M}}(\bar{x},\frac{K}{2})$, $-\frac{5}{8}\alpha \le s \le 0$. Note that $K \to \infty$.

Now we are going to consider the convergence of the mean curvature flow on changing target manifolds. We clarify the meaning of the convergence in the following.

Denote the orbit of $\bar{x}$ under mean curvature flow by $\bar{x}^s \in \tilde{M}_s$ such that $\bar{x}^0 = \bar{x}$. Note the injectivity radius of the new target $(\tilde{M}, \tilde{g})$ tends to infinity as $\varepsilon \to 0$. Let $\{x^1, \cdots, x^{\bar{n}}\}$ be normal coordinates of $\tilde{M}$ of radius $\gg 1$ around $\bar{x}$ with $T_{\bar{x}}\tilde{M}_0 = span\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$, and $\tilde{g}_{\alpha\beta}$ be the metric coefficients of $\tilde{M}$ in this coordinates. By [20], we have $|\tilde{g}_{\alpha\beta} - \delta_{\alpha\beta}|(x) \leq CQ^{-2}|x|^2$ and $|\partial\tilde{g}_{\alpha\beta}| + |\partial^2\tilde{g}_{\alpha\beta}| \leq C$. By Arzela-Ascoli theorem, after taking a subsequence of $\varepsilon \to 0$, $\tilde{g}_{\alpha\beta}$ tends to $\delta_{\alpha\beta}$ in $C^{2-\gamma}$ topology for any $0 < \gamma < 1$.

By Lemma 3.2, there exist a family of maps $F_s : \{(x^1, \cdots, x^n) \mid ({x^1}^2 + \cdots + {x^n}^2)^{\frac{1}{2}} < 1\} \to \mathbb{R}^{\bar{n}-n}$ with $F_0(0) = 0$, $|DF_0|(0) = 0$, such that the connected component containing $\bar{x}^s$ of $\tilde{M}_s \cap \{(x^1, \cdots, x^{\bar{n}}) \mid ({x^1}^2 + \cdots + {x^n}^2)^{\frac{1}{2}} < 1\}$ can be written as a graph $\{(x', F_s(x')) \mid |x'| = ({x^1}^2 + \cdots + {x^n}^2)^{\frac{1}{2}} < 1\}$. Moreover, we can show

$$\sum_{i=1}^{4} |D^i F| + \sum_{i=1}^{2} (|\frac{\partial^i}{\partial s^i} F| + |D^i \frac{\partial F}{\partial s}|) \leq C,$$

where D and the norm are the natural differential and norm in Euclidean ordinates of $N \subset \mathbb{R}^n$ and the garget $\mathbb{R}^{\bar{n}}$. By Arzela-Ascoli theorem, $F(x', s)$ will converge to $F^\infty(x', s)$ in the topology of $\mathcal{C}^{\frac{3}{2}}(\overline{B(0, \frac{1}{2})} \times [-\frac{5\alpha}{8}, 0], \mathbb{R}^{\bar{n}}) \cap \mathcal{C}^3(\overline{B(0, \frac{1}{2})}, \mathbb{R}^{\bar{n}})$.

If we set $X = (x', F(x'))$ being the map from $N := B(0, 1)$ to $\tilde{M}$, then the mean curvature flow equation can be written as

$$\frac{\partial X}{\partial s} = \Delta X,$$

where Δ is the harmonic Laplacian defined by using the induced metric $X^*\tilde{g}$ and the target metric $\tilde{g}$. Since $X^*\tilde{g}$ is defined by DF and $\tilde{g}$, after taking a subsequence of $\varepsilon \to 0$, we know $X^*\tilde{g}$ converges in $\mathcal{C}^{1-\gamma}(\overline{B(0, \frac{1}{2})} \times [-\frac{5\alpha}{8}, 0])$ topology.

Denote by $\hat{M}_s = \tilde{M}_s \cap \exp_{\bar{x}}\{|x'| < 1\}$, and $\hat{M} = \cup_{s \in [-\frac{\alpha}{2}, 0]} \hat{M}_s$. By summing up the above discussion, the piece $\hat{M}$ of $\tilde{M}$ containing $(\bar{x}, 0)$ will converge to a solution of the mean curvature flow (in the classical sense) which is embedded in the Euclidean space $\mathbb{R}^{\bar{n}}$ with $|\hat{A}_\infty|(O, 0) = 1$ and $|\hat{A}_\infty|(\cdot, s) \leq 4$ on $[-\frac{\alpha}{2}, 0]$.

On the other hand, let $\tilde{\varphi} = Q^{-n}\varphi = (4\pi(-s))^{-\frac{n}{2}} e^{-(1+\frac{s}{Q^2\varepsilon})\frac{d^2_{\tilde{M}}(\bar{x},\cdot)}{4(-s)} - \frac{n}{2\varepsilon}(\bar{t}+Q^{-2}s)}$,

note that

$$|H+(1+\frac{1}{\varepsilon}(t-\bar{t}))\frac{d_{\bar{M}}(\bar{x},\cdot)\bar{\nabla}^{\perp}d_{\bar{M}}(\bar{x},\cdot)}{2(\bar{t}-t)}|_{\bar{g}}^{2}Q^{-2}=|\tilde{H}-(1+\frac{s}{Q^{2}\varepsilon})\frac{d_{\tilde{M}}(\bar{x},\cdot)\tilde{\nabla}^{\perp}d_{\tilde{M}}(\bar{x},\cdot)}{2s}|_{\tilde{g}}^{2},$$

$$\psi=(1-\frac{Q^{-2}d_{\tilde{M}}(\bar{x},\cdot)^{2}+3n\bar{t}+3nQ^{-2}s}{\rho^{2}})_{+}^{3}\to 1,$$

$$\tilde{\varphi}\to(4\pi(-s))^{-\frac{n}{2}}e^{-\frac{|\cdot|^{2}}{4(-s)}}\quad\text{ and }\quad\varphi\psi dv=\tilde{\varphi}\psi d\tilde{v}.$$

Since $\hat{M}_s\subset\tilde{M}_s$, by passing (3.11) to limit, we have

$$\int_{-\frac{1}{2}\alpha}^{0}[\int_{\hat{M}_{s}^{\infty}}|\hat{H}_{\infty}-\frac{x^{\perp}}{2s}|^{2}(4\pi(-s))^{-\frac{n}{2}}e^{-\frac{|x|^{2}}{4(-s)}}]ds=0,$$

where we denote the limit of $\hat{M}_s$ by $\hat{M}_s^{\infty}$, $\hat{H}_{\infty}$ the mean curvature on the limit. This implies

$$\hat{H}_{\infty}=\frac{x^{\perp}}{2s}\qquad\text{for }\ s\in[-\frac{\alpha}{2},0].$$

The boundedness of the second fundamental form on $\hat{M}_0^{\infty}$ implies $x^{\perp}\equiv 0$ on $\hat{M}_0^{\infty}$. Since the second fundamental form and its twice covariant derivative of $\hat{M}_s^{\infty}$ are bounded for $s\in[-\frac{\alpha}{2},0]$, $\hat{M}_s^{\infty}$ are $C^{4-\gamma}$ submanifolds for any $\gamma>0$. Moreover by the higher derivative estimates in Theorem 2.2.2 (in Euclidean space), $\hat{M}_0^{\infty}$ is smooth.

Note $0\in\hat{M}_0^{\infty}$, after a orthogonal transformation, we may assume $T_0\hat{M}_0^{\infty}=\{(x_1,x_2,\cdots,x_n,0,\cdots,0)\}$. Clearly we still have the condition $x^{\perp}\equiv 0$ on $\hat{M}_0^{\infty}$. We may write $\hat{M}_0^{\infty}$ as a graph (at least locally near 0) $(x',f_1(x'),\cdots,f_{\bar{n}-n}(x'))$ where $x'=(x_1,\cdots,x_n)$. Now $x^{\perp}=(x',f_1(x'),\cdots,f_{\bar{n}-n}(x'))^{\perp}\equiv 0$ implies $\sum\limits_{p=1}^{n}\frac{\partial f_i}{\partial x_p}x_p=f_i(x')$. So f_i is homogenous of degree 1. Since $Df_i(0)=0$, we conclude $f_i\equiv 0$. Hence we know $\hat{M}_0^{\infty}$ is an n-dimensional linear subspace $\mathbb{R}^n$ of $\mathbb{R}^{\bar{n}}$.

This contradicts $|\hat{A}_{\infty}|(O,0)=1$ and we complete the proof of (3.10).

Note that $B_{\bar{M}}(\bar{x},\rho)\subseteq B_{\bar{M}}(x_0,\rho+(2K+1)\varepsilon)\subseteq B_{\bar{M}}(x_0,4\sqrt{\varepsilon})$. Combining (3.10) and monotonicity formula (3.8), we know

$$\int_{M_0\cap B_{\bar{M}}(x_0,4\sqrt{\varepsilon})}(4\pi\bar{t})^{-\frac{n}{2}}e^{-(1-\frac{\bar{t}}{\varepsilon})\frac{d_{\bar{M}}^{2}(\bar{x},x)}{4\bar{t}}}dv\geq\int_{M_t}\varphi\psi dv\,|_{t=\bar{t}-\frac{1}{2}\alpha Q^{-2}}\geq(1+\beta)e^{-\frac{n}{2\varepsilon}\bar{t}}(1-\frac{3n\bar{t}}{\rho^{2}})^{3}.\tag{3.12}$$

By assumption, there is a normal coordinate system $(y^1 \cdots y^{\bar{n}})$ of $\bar{M}$ around x_0 with $T_{x_0}M_0$ =span$\{\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}\}$ and a vector valued function $F : \{y' = (y^1, \cdots, y^n) \mid (y^1)^2 + \cdots + (y^n)^2 < 1\} \to \mathbb{R}^{\bar{n}-n}$ with $F(0) = 0$, $|DF|(0) = 0$, $|DF|^2(y') = \sum_{i,\gamma} \frac{\partial F^\gamma}{\partial y^i} \frac{\partial F^\gamma}{\partial y^i} \leq \delta^2$ such that $M_0 \cap \{|y'| < 1\} = \{(y', F(y')) \mid |y'| < 1\}$. Let $P : \mathbb{R}^{\bar{n}} \to \mathbb{R}^n$ be the orthogonal projection into the first n-components. Let $exp_{x_0}(\bar{y}) = \bar{x}$ and $\bar{y}' = P\bar{y}$. For $x \in B_{\bar{M}}(x_0, 4\sqrt{\varepsilon})$, let $exp_{x_0}(y) = x$ and $y' = Py$. Since the curvature of $\bar{M}$ is bounded by c_0^2, by comparison theorem on the ball $B_{T_{x_0}\bar{M}}(o, 4\sqrt{\varepsilon})$, we have

$$d_{\bar{M}}(\bar{x}, x) \geq \frac{\sin(4c_0\sqrt{\varepsilon})}{4c_0\sqrt{\varepsilon}}|\bar{y} - y| \geq (1 - 3c_0^2\varepsilon)|\bar{y} - y| \geq (1 - 3c_0^2\varepsilon)|\bar{y}' - y'|. \quad (3.13)$$

On the other hand, also by comparison theorem, the Riemannian volume element dv of M_0 satisfies

$$\exp_{x_0}^* dv \leq [\frac{\sinh(c_0 d_{\bar{M}}(x_0, \cdot))}{c_0 d_{\bar{M}}(x_0, \cdot)}]^n dv_{\exp_{x_0}^{-1} M_0} \leq [1 + 16c_0^2\varepsilon]^n dv_{\exp_{x_0}^{-1} M_0} \quad (3.14)$$

whenever $x \in M_0 \cap B_{\bar{M}}(x_0, 4\sqrt{\varepsilon})$. By definition, it is clear that

$$dv_{\exp_{x_0}^{-1} M_0} \leq (1 + |DF|^2)^{\frac{n}{2}} dy^1 \cdots dy^n \leq (1 + \delta^2)^{\frac{n}{2}} dy^1 \cdots dy^n. \quad (3.15)$$

Combining (3.13),(3.14) and (3.15), we have

$$\int_{M_0 \cap B_{\bar{M}}(x_0, 4\sqrt{\varepsilon})} (4\pi\bar{t})^{-\frac{n}{2}} e^{-(1-\frac{\bar{t}}{\varepsilon})\frac{d^2_{\bar{M}}(\bar{x},x)}{4\bar{t}}} dv$$

$$\leq (1 + \delta^2)^{\frac{n}{2}}(1 + 16c_0^2\varepsilon)^n(1 - \varepsilon)^{-\frac{n}{2}}(1 - 3c_0^2\varepsilon)^{-n}$$

$$\times \int_{(|y^1|^2 + \cdots + |y^n|^2)^{\frac{1}{2}} < 4\sqrt{\varepsilon}} [\frac{4\pi\bar{t}}{(1 - \varepsilon)(1 - 3c_0^2\varepsilon)^2}]^{-\frac{n}{2}} e^{-\frac{\frac{|\bar{y}' - y'|^2}{4\bar{t}}}{(1-\varepsilon)(1-3c_0^2\varepsilon)^2}} dy^1 \cdots dy^n$$

$$\leq (1 + \delta^2)^{\frac{n}{2}}(1 + 16c_0^2\varepsilon)^n(1 - \varepsilon)^{-\frac{n}{2}}(1 - 3c_0^2\varepsilon)^{-n}.$$

By (3.12) and the fact $\bar{t} \leq \varepsilon^2$, we conclude that

$$(1 + \delta^2)^{\frac{n}{2}}(1 + 16c_0^2\varepsilon)^n(1 - \varepsilon)^{-\frac{n}{2}}(1 - 3c_0^2\varepsilon)^{-n}(1 - 3n\varepsilon)^{-3}e^{\frac{n\varepsilon}{2}} \geq (1 + \beta),$$

which is a contradiction as $\varepsilon, \delta \to 0$. We complete the proof of the Theorem. □

Theorem 3.5 *Let $\bar{M}$ be an $\bar{n}$-dimensional manifold satisfying $\sum_{i=0}^{3}|\bar{\nabla}^i \bar{R}m| \leq c_0^2$ and $inj(\bar{M}) \geq i_0 > 0$. Then there is $\varepsilon > 0$ with the following property. Suppose we have a smooth solution $M_t \subset \bar{M}$ to the mean curvature flow properly embedded in $B_{\bar{M}}(x_0, r_0)$ for $t \in [0,T]$ where $r_0 < \frac{i_0}{2}$, $0 < T \leq \varepsilon^2 r_0^2$. We assume that at time zero, $x_0 \in M_0$, and the second fundamental form satisfies $|A|(x) \leq r_0^{-1}$ on $M_0 \cap B_{\bar{M}}(x_0, r_0)$ and assume M_0 is graphic in the ball $B_{\bar{M}}(x_0, r_0)$. Then we have*

$$|A|(x,t) \leq (\varepsilon r_0)^{-1} \tag{3.16}$$

for any $x \in B_{\bar{M}}(x_0, \varepsilon r_0) \cap M_t$, $t \in [0,T]$.

Proof. By scaling we may assume $r_0 = 1$. By Lemma 3.1, for any $\delta > 0$, there is $0 < r_\delta < 1$ such that the connected component of $M_0 \cap B_{\bar{M}}(x_0, \frac{1}{96})$ containing x_0 contains a δ-Lipschitz graph of radius $2r_\delta$ at x_0. By our graphic assumption, we conclude that $M_0 \cap B_{\bar{M}}(x_0, r_\delta)$ is a δ-Lipschitz graph. So Theorem 3.3 is applicable with radius r_δ.

Consequently, for any $\alpha > 0$, there exists an $\varepsilon_\alpha > 0$ such that

$$|A|(x,t)^2 \leq \frac{\alpha}{t} + \varepsilon_\alpha^{-2} \tag{3.17}$$

whenever $x \in M_t \cap B_{\bar{M}}(x_0, \varepsilon_\alpha)$, $t \in [0, \varepsilon_\alpha^2] \cap [0,T]$. Let α be a fixed small constant to be determined later. It turns out that we only need to choose $\alpha = \alpha(c_0, \bar{n}, n)$ finally. Choose $\varepsilon = \min\{\sqrt{\alpha}\varepsilon_\alpha, 10^{-1}\}$. Then by (3.17) we have

$$|A|(x,t)^2 \leq \frac{2\alpha}{t} \tag{3.18}$$

whenever $x \in M_t \cap B_{\bar{M}}(x_0, \varepsilon_\alpha)$, $t \in [0, \varepsilon^2] \cap [0,T]$.

Claim $|A|(x,t) \leq \varepsilon^{-1}$ *holds on $M_t \cap B_{\bar{M}}(x_0, \varepsilon)$, $t \in [0, \varepsilon^2] \cap [0,T]$.*

Suppose $|A|(x_1, t_1) > \varepsilon^{-1}$ holds for some point (x_1, t_1), $x_1 \in M_{t_1} \cap B_{\bar{M}}(x_0, \varepsilon)$, $t_1 \in [0, \varepsilon^2] \cap [0,T]$. We can choose another point $(\bar{x}, \bar{t})$, $\bar{x} \in M_{\bar{t}} \cap B_{\bar{M}}(x_0, 4\varepsilon)$, $\bar{t} \in [0, \varepsilon^2] \cap [0,T]$ such that $Q = |A|(\bar{x}, \bar{t}) \geq \varepsilon^{-1}$ and

$$|A|(x,t) \leq 4Q \tag{3.19}$$

whenever $x \in M_t$, $d_{\bar{M}}(\bar{x}, x) \leq Q^{-1}$, $0 \leq t \leq \bar{t}$.

Actually $(\bar{x}, \bar{t})$ can be constructed as the limit of a finite sequence (x_i, t_i) satisfying $0 \leq t_k \leq t_{k-1}$, $d_{\bar{M}}(x_0, x_k) \leq d_{\bar{M}}(x_0, x_{k-1}) + |A|(x_{k-1}, t_{k-1})^{-1}$, $|A|(x_k, t_k) \geq 4|A|(x_{k-1}, t_{k-1})$. Since

$$|A|(x_k, t_k) \geq 4^{k-1}|A|(x_1, t_1) \geq 4^{k-1}\varepsilon^{-1},$$

$d_{\bar{M}}(x_0, x_k) \leq d_{\bar{M}}(x_0, x_1) + \sum_{i=1}^{\infty}(4^{i-1}|A|(x_1, t_1))^{-1} \leq 3\varepsilon < \frac{1}{2}$, and the solution is smooth, the sequence must be finite and the last element fits.

Note that $3n\bar{t}Q^2 \leq 6n\alpha \leq \frac{1}{2}$ by choosing $\alpha \leq \frac{1}{12n}$. Let $\psi = (1 - \frac{d^2_{\bar{M}}(\bar{x}, \cdot) + 3nt}{Q^{-2}})^3_+$, then we have

$$(\frac{\partial}{\partial t} - \Delta)\psi \leq 0$$

whenever $d_{\bar{M}}(\bar{x}, \cdot)^2 < \min\{\frac{1}{c_0^2 e}, i_0^2\}$, $t \in [0, \bar{t}]$. On the other hand, by (2.2.2), the second fundamental form satisfies

$$(\frac{\partial}{\partial t} - \Delta)|A|^2 \leq -|\nabla A|^2 + C(\bar{n})|A|^4 + C(\bar{n})(1 + c_0^2)(|A|^2 + |A|).$$

Hence

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)(\psi|A|^2) &\leq -|\nabla A|^2\psi + C(\bar{n})|A|^4\psi + C(\bar{n})(1 + c_0^2)(|A|^2 + |A|)\psi + 4|\nabla A||A||\nabla\psi| \\ &\leq C(\bar{n})|A|^4\psi + C(\bar{n})(1 + c_0^2)(|A|^2 + |A|)\psi + 4\frac{|\nabla\psi|^2}{\psi}|A|^2 \\ &\leq C(\bar{n})|A|^4\psi + C(\bar{n})(1 + c_0^2)(|A|^2 + |A|)\psi + 144Q^2|A|^2\psi^{\frac{1}{3}}\end{aligned} \tag{3.20}$$

on $[0, \bar{t}]$. By (3.19)(3.20), we have

$$(\frac{\partial}{\partial t} - \Delta)(\psi|A|^2) \leq C(\bar{n})Q^4 + C(\bar{n})(1 + c_0^2)(Q + Q^2).$$

From the maximum principle, it follows

$$\begin{aligned}(\psi|A|^2)_{\max}|_{t=\bar{t}} &\leq 1 + C(\bar{n})Q^4\bar{t} + C(\bar{n})(1 + c_0^2)(Q + Q^2)\bar{t} \\ &\leq 1 + 2\alpha C(\bar{n})Q^2 + C(\bar{n})(1 + c_0^2)(\sqrt{2\alpha\bar{t}} + 2\alpha).\end{aligned}$$

Note that

$$(\psi|A|^2)_{\max}\,|_{t=\bar{t}} \geq \psi|A|^2(\bar{x},\bar{t}) \geq (1-3nQ^2\bar{t})^3 Q^2 \geq (1-18n\alpha)Q^2,$$

hence we have

$$(1-18n\alpha)Q^2 \leq 1+2\alpha C(\bar{n})Q^2 + C(\bar{n})(1+c_0^2)(\sqrt{2\alpha\bar{t}}+2\alpha).$$

This implies

$$Q^2 \leq \frac{1+C(\bar{n})(1+c_0^2)(\sqrt{2\alpha}+2\alpha)}{1-(18n+2C(\bar{n}))\alpha}.$$

Choosing suitable small $\alpha = \alpha(c_0,\bar{n},n)$, we have $Q^2 \leq 2$, which is a contradiction with $Q^2 > \varepsilon^{-2}$. So the Claim is proved. □

Corollary 3.6 *Let $\bar{M}$ be an $\bar{n}$-dimensional complete manifold satisfying $\sum_{i=0}^{3}|\bar{\nabla}^i \bar{R}m| \leq c_0^2$ and $inj(\bar{M}) \geq i_0 > 0$. Let $X_0 : M \to \bar{M}$ be an n-dimensional isometrically properly embedded submanifold with bounded second fundamental form $|A| \leq c_0$ in $\bar{M}$. We assume $M_0 = X_0(M)$ is uniform graphic with some radius $r > 0$. Suppose $X(x,t)$ is a smooth solution to the mean curvature flow (1.1) on $M \times [0,T_0]$ properly embedded in $\bar{M}$ with X_0 as initial data. Then there is $0 < T_1 \leq T_0$ depending upon c_0, i_0, r and the dimension $\bar{n}$ such that*

$$|A|(x,t) \leq 2c_0$$

for all $x \in M$, $0 \leq t \leq T_1$.

Proof. By Theorem 3.5, there is $\epsilon > 0$ such that for any $x_0 \in M$, we have

$$|A|(x,t) \leq \epsilon^{-1}$$

on $B_{\bar{M}}(x_0,\epsilon)$, $t \in [0,\epsilon^2] \cap [0,T]$. Let $[0,\gamma) \subset [0,\epsilon^2] \cap [0,T]$ be the maximal time interval so that the orbit of x_0, $x_0^t \in B_{\bar{M}}(x_0,\epsilon)$ for $t \in [0,\gamma]$. Then by the mean curvature flow equation, we know

$$\frac{d}{dt} d_{\bar{M}}(x_0,x_0^t) \leq C\epsilon^{-1},$$

for any $t \in [0, \gamma]$. This implies $\gamma \geq \frac{\varepsilon^2}{C}$ for some $C = C(n, \bar{n})$. Choosing $\varepsilon = \frac{\epsilon}{\sqrt{C}}$, $T = \min\{T_0, \varepsilon^2\}$, we conclude that the second fundamental forms are uniformly bounded by the constant ε^{-1} on $M \times [0, T]$. Once the second fundamental form is bounded, since we assumed $\sum_{i=0}^{3} |\bar{\nabla}^i \bar{R}m| \leq c_0^2$, we have gradient estimate $|\nabla A| \leq \frac{C}{\sqrt{t}}$, and hence suitable linear growth function with bounded first and second derivatives can be constructed. Therefore we can apply the maximum principle to the equation of $|A|$ to conclude a uniform estimate $|A| \leq 2c_0$, for any $t \in [0, \frac{1}{C(\bar{n})c_0^2}]$. Set $T_1 = \min\{T, \frac{1}{C(\bar{n})c_0^2}\}$. The proof is completed. □

Theorem 1.3 follows as a corollary of Theorem 1.1 and Corollary 3.6.

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