On the stability of homogeneous equilibria for the Vlasov-Poisson system on $\mathbb{R}^3$

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Introduction

We consider the Vlasov-Poisson system for a density function $f : \mathbb{R}^d_x \times \mathbb{R}^d_v \times \mathbb{R}_t \to \mathbb{R}_+$:

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f - q \nabla_x \phi \cdot \nabla_v f = 0, \quad q = \pm 1, \\ -\Delta_x \phi(x,t) = \int_{\mathbb{R}^3} f(x,v,t) dv, \\ f(0,x,v) = f_0(x,v). \end{cases} \tag{1}$$

This model is relevant in plasma physics (usually for $q = -1$) and in astrophysics (for $q = 1$). In dimension $d = 3$, solutions to (1) are global in time under rather mild assumptions (Lions-Perthame, Pfaffelmoser), but a complete understanding of their asymptotic behavior is still elusive.

Any smooth normalized function $M_0 : \mathbf{R}^d \to [0, \infty)$ satisfying $\int_{\mathbf{R}^d} M_0 \, dv = 1$ is a stationary solution of this equation, and we would like to study the stability of these solutions. Let

$$F = M_0(v) + f$$

so the perturbation f satisfies the equation

$$\begin{aligned} &(\partial_t + v \cdot \nabla_x) f + E \cdot \nabla_v M_0 + E \cdot \nabla_v f = 0, \\ &\rho(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\, dv, \qquad E := \nabla_x \Delta_x^{-1} \rho. \end{aligned} \tag{2}$$

Landau damping

In the periodic case $x \in \mathbb{T}^d$ the asymptotic stability result is known as nonlinear Landau damping (Mouhot-Villani, Bedrossian-Masmoudi-Mouhot, Grenier-Nguyen-Rodnianski).

The main assumptions are that the equilibrium M_0 is real-analytic and suitably decaying, and satisfies the key *Penrose stability condition*

$$\inf_{\xi \in \mathbb{Z}^d \setminus \{0\}, \Re z \geq 0} \left| 1 + \int_0^\infty r \widehat{M_0}(r\xi) e^{-zr} \, dr \right| > \kappa_0 > 0. \tag{3}$$

Under these assumptions nonlinear Landau damping occurs: the solution f corresponding to small and Gevrey smooth initial data $f_0 \in \mathcal{G}^{\gamma,\lambda}$, $\gamma > 1/3$ satisfying

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} f_0(x, v) \, dxdv = 0$$

converges rapidly to a final state $f_\infty(x, v)$, and the electric field E and the density charge ρ decay exponentially fast in $\langle t \rangle^\gamma$.

The Penrose criterion

The Penrose stability condition (3) on the equilibrium M_0 is critical for stability. Indeed, assume that f solves

$$(\partial_t + v\cdot\nabla_x)f + E\cdot\nabla_v M_0 = \mathcal{N},$$
$$\rho(x,t) = \int_{\mathbb{R}^d} f(t,x,v)\,dv, \quad E := \nabla_x\Delta_x^{-1}\rho$$

with initial data $f(0,x,v) = f_0(x,v)$. Taking Fourier transform in x we have and integrating in time we have

$$\widehat{f}(t,\xi,v) - e^{-itv\cdot\xi}\widehat{f_0}(\xi,v) + \int_0^t e^{-i(t-s)v\cdot\xi} i\partial_j M_0(v)\cdot(\xi_j/|\xi|^2)\widehat{\rho}(s,\xi)\,ds$$
$$= \int_0^t e^{-i(t-s)v\cdot\xi}\widehat{\mathcal{N}}(s,\xi,v)\,ds.$$

The Penrose criterion

Integrating now in v we derive the key equation

$$\widehat{\rho}(t,\xi) + \int_0^t (t-s)\widehat{M_0}((t-s)\xi)\widehat{\rho}(s,\xi)\,ds$$
$$= -\widetilde{\mu_0}(\xi, t\xi) - \int_0^t \widetilde{\mathcal{N}}(s,\xi,(t-s)\xi)\,ds,$$

for any $\xi \in \mathbb{Z}^d$, where $\widetilde{\cdot}$ denotes the Fourier transform in both variables x and v.

We can solve this equation using the Fourier transform in time.

For $\varepsilon > 0$ and $\tau \in \mathbb{R}$ let

$$Q_\varepsilon(\tau,\xi) := \int_0^\infty \widehat{\rho}(t,\xi)e^{-(\varepsilon+i\tau)t}\,dt,$$
$$K_\varepsilon(\tau,\xi) := \int_0^\infty r\widehat{M_0}(r\xi)e^{-(\varepsilon+i\tau)r}\,dr,$$

The Penrose criterion

The identity above gives

$$Q_\varepsilon(\tau,\xi)[1+K_\varepsilon(\tau,\xi)]$$
$$= -\int_0^\infty \widetilde{\mu_0}(\xi,t\xi)e^{-(\varepsilon+i\tau)t}\,dt - \int_0^\infty\int_0^\infty \widetilde{\mathcal{N}}(s,\xi,r\xi)e^{-(\varepsilon+i\tau)(s+r)}\,dsdr.$$

This can be inverted if

$$|1+K_\varepsilon(\tau,\xi)| > \kappa_0 > 0 \qquad \text{for any } \varepsilon\in(0,\infty),\ \tau\in\mathbb{R},\ \xi\in\mathbb{Z}^d,$$

which is the Penrose stability condition.

The key difficulty in the Euclidean case $x \in \mathbb{R}^d$ is that the Penrose condition

$$\inf_{\xi \in \mathbb{R}^d \setminus \{0\}, \varepsilon > 0, \tau \in \mathbb{R}} \left| 1 + \int_0^\infty r \widehat{M_0}(r\xi) e^{-(\varepsilon + i\tau)r} \, dr \right| > 0$$

cannot hold. This is because the left-hand side vanishes when $\xi = 0$, $\varepsilon = 0$, $\tau = \pm 1$,

$$1 + K_\varepsilon(\tau, 0) = 1 + \frac{1}{(\varepsilon + i\tau)^2} = \frac{[\varepsilon + i(\tau - 1)][\varepsilon + i(\tau + 1)]}{(\varepsilon + i\tau)^2}.$$

Because of this the density function ρ has only logarithmic decay in L^2 for the Maxwellian equilibrium M_0 (Glassey-Schaeffer), in sharp contrast with the periodic case $x \in \mathbb{T}^d$.

Asymptotic stability of the Poisson equilibrium

At the linearized level, the stability problem was addressed in recent work by Han-Kwan–Nguyen–Rousset and Bedrossian-Masmoudi-Mouhot. The conclusion is that for a suitable class of smooth equilibrium $M_0(v) = M_0'(|v|^2)$, the density ρ decomposes into a regular part and a singular part,

$$\rho(t,x) = \rho^R(t,x) + \rho^S_-(t,x) + \rho^S_+(t,x)$$

which have polynomial decay in L^∞,

$$\|\rho^R(t)\|_{L^\infty} \lesssim \frac{\log(2+t)}{\langle t\rangle^d}\big(\|f_0\|_{L^1_{x,v}} + \|f_0\|_{L^1_x L^\infty_v}\big),$$

and, for $k \in \{0,1\}$,

$$\|\rho^S_\pm(t)\|_{L^\infty} \lesssim \frac{\log(2+t)}{\langle t\rangle^{d/2+k-1}} \sum_{0\le l\le k} \big(\|\langle v\rangle^l \nabla_v^l f_0\|_{L^1_{x,v}} + \|\langle v\rangle^l \nabla_v^l f_0\|_{L^1_x L^\infty_v}\big),$$

In the same direction, the *screened Vlassov-Poisson* equation

$$(\partial_t + v \cdot \nabla_x)f + E \cdot \nabla_v M_0 + E \cdot \nabla_v f = 0,$$
$$\rho(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\,dv, \qquad E := \nabla_x(\Delta_x - 1)^{-1}\rho.$$

was analyzed recently by Bedrossian-Masmoudi-Mouhot in 3D. In this case the natural Penrose stability condition is

$$\inf_{\xi\in\mathbb{R}^d\setminus\{0\},\,\varepsilon>0,\tau\in\mathbb{R}} \left|1 + \int_0^\infty r\widehat{M_0}(r\xi)e^{-(\varepsilon+i\tau)r}\frac{|\xi|^2}{1+|\xi|^2}\,dr\right| \geq \kappa_0 > 0.$$

This condition is satisfied by a large family of equilibrium solutions M_0. Bedrossian-Masmoudi-Mouhot then prove a suitable analogue of the nonlinear Landau damping for the screened Vlassov-Poisson system in the Euclidean space $x \in \mathbb{R}^3$, at finite regularity.

Asymptotic stability of the Poisson equilibrium

We will be mainly interested in the Poisson equilibrium

$$M_0(v) := \frac{C_d}{(1+|v|^2)^{(d+1)/2}}, \qquad \widehat{M_0}(\xi) = e^{-|\xi|},$$

for a constant $C_d > 0$. We will also assume that $d = 3$.

Theorem 1. (I., Pausader, Wang, Widmayer) Assume that the initial perturbation $f_0 : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ of the Poisson equilibrium M_0 satisfies the smallness condition

$$\sum_{|\alpha|+|\beta|\leq 1} \|\langle v\rangle^5(\partial_x^\alpha \partial_v^\beta f_0)(x,v)\|_{L^\infty_x L^\infty_v} + \|\langle v\rangle^5(\partial_x^\alpha \partial_v^\beta f_0)(x,v)\|_{L^1_x L^\infty_v} \leq \varepsilon_0$$

where ϵ_0 is some sufficiently small absolute constant.

Then there is a unique global solution f of the Vlasov-Poisson system

$$(\partial_t + v \cdot \nabla_x) f + E \cdot \nabla_v M_0 + E \cdot \nabla_v f = 0,$$
$$\rho(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\, dv, \qquad E := \nabla_x \Delta_x^{-1} \rho.$$

The electric field and the density function decay at rates $\langle t \rangle^{-2+}$ and $\langle t \rangle^{-3+}$ over time respectively. The density decomposes as

$$\rho(t,x) = \rho^{stat}(t,x) + \rho^{osc}(t,x),$$

where the static part $\rho^{static}(t,x)$ decays faster at rate $\langle t \rangle^{-4+\delta}$ and the oscillatory part $\rho^{osc}(t,x)$ oscillates like e^{it} in time.

The full density f satisfies linear scattering, i.e.

$$\lim_{t \to \infty} f(t, x + tv, v) = f_\infty(x,v) \qquad \text{in } L^\infty_{x,v}.$$

Asymptotic stability of the Poisson equilibrium

Main ideas: Assume that $f : \mathbb{R}^3 \times \mathbb{R}^3 \times [0, T] \to \mathbb{R}$ is a regular solution on some time interval $[0, T]$. We consider the backwards characteristics defined by the functions $X, V : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{I}^2_T \to \mathbb{R}^3$ obtained by solving the ODE system

$$\partial_s X(x, v, s, t) = V(x, v, s, t), \qquad X(x, v, t, t) = x,$$
$$\partial_s V(x, v, s, t) = E(X(x, v, s, t), s), \qquad V(x, v, t, t) = v,$$

where $\mathcal{I}^2_T := \{(s, t) \in [0, T]^2 : \ s \leq t\}$. The main equation gives

$$\begin{aligned}\frac{d}{ds} & f(X(x, v, s, t), V(x, v, s, t), s) \\ &= [(\partial_s + v \cdot \nabla_x + E \cdot \nabla_v)f](X(x, v, s, t), V(x, v, s, t), s) \\ &= -E(X(x, v, s, t), s) \cdot \partial_v M_0(V(x, v, s, t)).\end{aligned}$$

Integrating over $s \in [0, t]$ and letting $M_0' := \nabla_v M_0$, we have

$$f(x, v, t) = f_0(X(x, v, 0, t), V(x, v, 0, t)) - \int_0^t E(X(x, v, s, t), s) \cdot M_0'(V(x, v, s, t))\, ds,$$

for any $(x, v, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times [0, T]$. Therefore

$$\rho(x, t) + \int_0^t \int_{\mathbb{R}^3} (t - s)\rho(x - (t - s)v, s) M_0(v)\, dvds = \mathcal{N}(x, t),$$

for any $(x, t) \in \mathbb{R}^3 \times [0, T]$, where

$$\mathcal{N}(x, t) := \mathcal{N}_1(x, t) + \mathcal{N}_2(x, t),$$

$$\mathcal{N}_1(x, t) := \int_{\mathbb{R}^3} f_0(X(x, v, 0, t), V(x, v, 0, t))\, dv,$$

$$\mathcal{N}_2(x, t) := \int_0^t \int_{\mathbb{R}^3} \{E(x - (t - s)v, s) \cdot M_0'(v) - E(X(x, v, s, t), s) \cdot M_0'(V(x, v, s, t))\}\, dvds.$$

In the case of the Poissin equilibrium we have the explicit identity

$$\frac{1}{1+K(\xi,\lambda)} = 1 - \frac{1}{2i}\Big[\frac{1}{|\xi|+i(\lambda-1)} - \frac{1}{|\xi|+i(\lambda+1)}\Big],$$

so we can solve the Volterra equation explicitly to see that

$$\widehat{\rho}(\xi,t) = \widehat{\mathcal{N}}(\xi,t) - \int_0^t \widehat{\mathcal{N}}(\xi,t-\tau)e^{-\tau|\xi|}\sin\tau\,d\tau,$$

for any $t \geq 0$. This is the main formula we use to prove bootstrap control of the solution.

Main bootstrap proposition: Assume that the density ρ decomposes as

$$\rho = \rho^{stat} + \Re\{e^{-it}\rho^{osc}\}$$

satisfying the bounds

$$\|\rho^{stat}\|_{Stat_\delta} + \|\rho^{osc}\|_{Osc_\delta} \leq \varepsilon_1,$$

where $\varepsilon_1 = \varepsilon_0^{2/3}$. Then there is an improved decomposition $\rho = \rho^{stat} + \Re\{e^{-it}\rho^{osc}\}$, such that the components ρ^{stat} and ρ^{osc} satisfy the improved bounds

$$\|\rho^{stat}\|_{Stat_\delta} + \|\rho^{osc}\|_{Osc_\delta} \lesssim \varepsilon_0.$$

Asymptotic stability of the Poisson equilibrium

The main norms are defined as follows: let B_T denote the space of continuous functions on $\mathbb{R}^3 \times [0, T]$ defined by the norm

$$\|f\|_{B_T} := \sup_{t\in[0,T]} \|f(t)\|_{B^0_t},$$
$$\|f(t)\|_{B^0_t} := \sup_{k\in\mathbb{Z}} \left\{\langle t\rangle^3 \|P_k f(t)\|_{L^\infty} + \|P_k f(t)\|_{L^1}\right\}.$$

Assume that $\delta \in [0, 1/100]$ is a small parameter and define the norms

$$\|f\|_{Stat_\delta} := \|\langle t\rangle^{1-2\delta} \langle \nabla_x\rangle f\|_{B_T},$$
$$\|f\|_{Osc_\delta} := \|\langle t\rangle^{-\delta} f\|_{B_T} + \|\langle t\rangle^{1-2\delta} \nabla_{x,t} f\|_{B_T}.$$

Finally, we define the Z-norm as

$$\|f\|_Z := \inf_{f=f^{stat}+\Re\{e^{-it} f^{osc}\}} \|f^{stat}\|_{Stat_\delta} + \|f^{osc}\|_{Osc_\delta},$$

and the main bootstrap proposition can be regarded as proving improved control on the density function ρ in the Z-norm.

A class of acceptable equilibria

Recall the basic formulas

$$\rho(x,t) = \int_0^t \int_{\mathbb{R}^3} G(x-y,t-s)\mathcal{N}(y,s)\, dyds,$$

where the Green's function G is defined by

$$K(\xi,\theta) := \int_0^\infty t\widehat{M}_0(t\xi)e^{-it\theta}dt,$$

$$\widehat{G}(\xi,\tau) := \delta_0(\tau) - 1_{[0,\infty)}(\tau)\frac{1}{2\pi}\int_{\mathbf{R}} \frac{K(\xi,\theta)}{1+K(\xi,\theta)}e^{i\theta\tau}\, d\theta.$$

and $1_{[0,\infty)}$ denoting the characteristic function of the interval $[0,\infty)$.

We consider a class of radially symmetric equilibria $M_0(v)$ such that $\int_{\mathbf{R}^3} M_0 dv = 1$. We define $m_0 : \mathbb{R} \to \mathbb{R}$ such that

$$m_0(r) = \int_{\mathbb{R}^2} M_0(r, y) dy, \tag{4}$$

where $r \in \mathbf{R}$, $y \in \mathbf{R}^2$, $(r, y) \in \mathbf{R}^3$. The reduced distribution m_0 is even, integrable and

$$\widehat{M_0}(\xi) = \widehat{m_0}(|\xi|).$$

For any $\vartheta \in (0, 1/2]$ we define the regions

$$\mathcal{D}_\vartheta := \{z \in \mathbb{C} : \, |\Im z| < \vartheta(1 + |\Re z|)\}.$$

A class of acceptable equilibria

We define the class $\mathcal{M}_{\vartheta,d}$, $\vartheta \in (0, \pi/4]$, $d > 1$, of "acceptable equilibria" as the set of radially symmetric functions $M_0 : \mathbf{R}^3 \to \mathbf{R}$ satisfying $\int_{\mathbf{R}^3} M_0 dv = 1$, and with the following properties:

(i) The function m_0 defined as in (4) is even, positive on $\mathbf{R}$, and extends to an analytic function $m_0 : \mathcal{D}_\vartheta \to \mathbb{C}$ satisfying the identies

$$m_0(z) = m_0(-z) = \overline{m_0(\overline{z})} \qquad \text{for any } z \in \mathcal{D}_\vartheta.$$

(ii) Moreover, $m_0'(r) < 0$ if $r \in (0, \infty)$ and

$$|m_0(z)| \lesssim (1 + |z|)^{-d} \qquad \text{for any } z \in \mathcal{D}_\vartheta.$$

Typical examples are polynomially decreasing or Gaussian equilibria, which lead to

$$m_0(r) = \frac{c_d}{[1 + r^2]^{\frac{d}{2}}}, \qquad m_0(r) = \pi^{-1/2} e^{-r^2}.$$

Theorem 2. (I., Pausader, Wang, Widmayer) Assume that $d > 1$, $M_0 \in \mathcal{M}_{\vartheta,d}$ is an acceptable equilibrium, and recall

$$K(\xi,\theta) := \int_0^\infty t\widehat{M}_0(t\xi)e^{-it\theta}\,dt,$$

$$\widehat{G}(\xi,\tau) := \delta_0(\tau) - \mathbb{1}_{[0,\infty)}(\tau)\frac{1}{2\pi}\int_{\mathbf{R}} \frac{K(\xi,\theta)}{1+K(\xi,\theta)}e^{i\theta\tau}\,d\theta.$$

Then there exists $r_0 > 0$ and $\gamma_0 = \gamma_0(\vartheta, d, r_0) \in (0,\infty)$ such that the following claims hold:

(i) At high frequency, we have a perturbation of a kinetic density: if $|\xi| > r_0/2$, then we can write $\widehat{G}$ in the form

$$\widehat{G}(\xi,\tau) = \delta_0(\tau) + e^{-\gamma_0\tau|\xi|}\mathcal{E}_h(|\xi|,\tau),$$

where, for any $a \in \{0,\ldots,10\}$, $\tau \geq 0$, and $|r| > r_0/2$,

$$r^a|\partial_r^a\mathcal{E}_h(r,\tau)| \lesssim r^{-1}.$$

A class of acceptable equilibria

(ii) At low frequencies, we have an additional oscillatory component: if $|r| < 2r_0$ then we can write $\widehat{G}$ in the form

$$\widehat{G}(\xi,\tau) = \delta_0(\tau) + \Re\big\{i[1+\mathfrak{m}_l(|\xi|)]e^{i\tau\omega(|\xi|)}\big\} + e^{-\gamma_0\tau|\xi|}\mathcal{E}_l(|\xi|,\tau)$$

where, for any $a \in \{0,\dots,10\}$, $\tau \geq 0$, and $r < 2r_0$,

$$r^a|\partial_r^a \mathfrak{m}_l(r)| \lesssim r^{d-1} + r^2\log(1/r),$$
$$r^a|\partial_r^a \mathcal{E}_l(r,\tau)| \lesssim r^{d-1} + r^2\log(1/r).$$

The dispersion relation $\omega_1 := \Re\omega$ and the dissipation coefficient $\omega_2 := \Im\omega \geq 0$ satisfy the bounds

$$r^a|\partial_r^a(\omega_1(r)-1)| + r^a|\partial_r^a(\omega_2(r))| \lesssim r^{d-1} + r^2\log(1/r),$$
$$\omega_2(r) \in \Big[\frac{-\pi m_0'(\omega_1(r)/r)}{4r^2}, \frac{-\pi m_0'(\omega_1(r)/r)}{r^2}\Big],$$

for any $a \in \{0,\dots,10\}$, $\tau \geq 0$, and $r < 2r_0$.