On the Development of Shocks in Compressible Fluids

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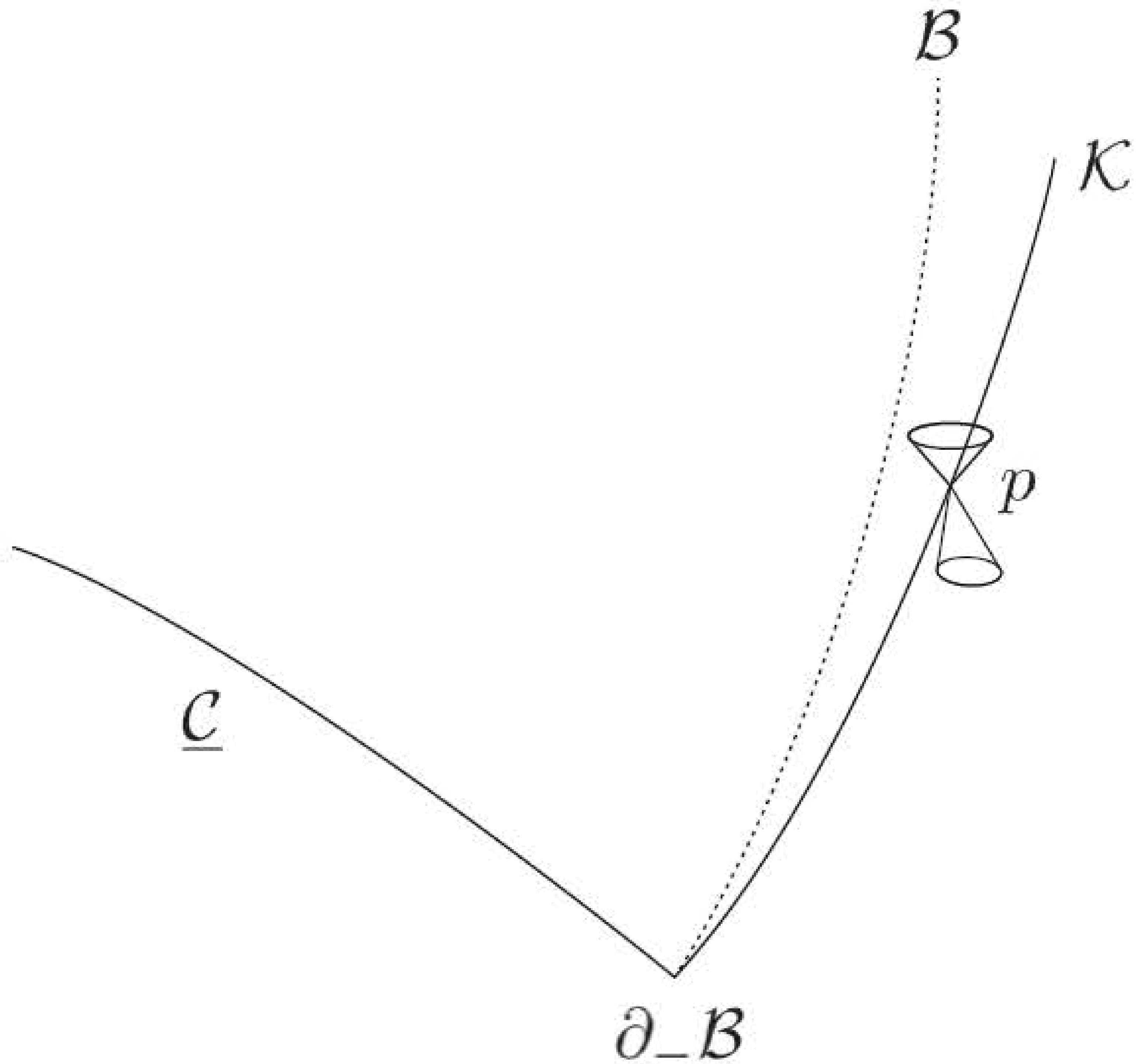
In this talk, I shall discuss the content of my recent monograph *The Shock Development Problem* (EMS Monographs in Mathematics, EMS Publishing House, 2019), which addresses the problem of the development of shocks in a compressible fluid past the point of their formation. This problem is formulated in the framework of the Eulerian equations of a compressible perfect fluid as completed by the laws of thermodynamics. These equations express the differential conservation laws of mass, momentum and energy and constitute a quasilinear hyperbolic 1st order system for the physical variables, that is the fluid velocity and the two positive quantities corresponding to a local thermodynamic equilibrium state. Smooth initial data for this system of equations leads to the formation of a surface in spacetime where the derivatives of the physical quantities with respect to the standard rectangular coordinates blow up.

Now, there is a mathematical notion of maximal development of initial data. As was first shown in my previous monograph on the topic of shocks in compressible fluids, the monograph *The Formation of Shocks in 3-Dimensional Fluids*, EMS Monographs in Mathematics, EMS Publishing House, 2007, this maximal development ends at a future boundary which consists of a regular part $\underline{\mathcal{C}}$ and a singular part $\mathcal{B}$ with a common past boundary $\partial_{-}\mathcal{B}$, the surface just mentioned. A solution of the Eulerian equations in a given spacetime domain defines a cone field on this domain, the sound cones. This defines a causal structure on the spacetime domain, equivalent to a conformal class of Lorentzian metrics, the *acoustical causal structure*. Relative to this structure $\partial_{-}\mathcal{B}$ is a spacelike surface, while $\underline{\mathcal{C}}$ is a null hypersurface.

Also $\mathcal{B}$ is in this sense a null hypersurface, however being singular, while its intrinsic geometry is that of a null hypersurface, its extrinsic geometry is that of a spacelike hypersurface, for, the past null geodesic cone in the spacetime manifold of a point on $\mathcal{B}$ does not intersect $\mathcal{B}$. The character of $\mathcal{B}$ and the behavior of the the solution at $\mathcal{B}$ were described in detail in the 2007 monograph by means of the introduction of a class of coordinates such that the rectangular coordinates as well as the physical variables are smooth functions of the new coordinates up to $\mathcal{B}$, but the Jacobian of the transformation to the new coordinates, while strictly negative in the past of $\mathcal{B}$, vanishes at $\mathcal{B}$ itself, a fact which characterizes $\mathcal{B}$. Now, the mathematical notion of maximal development of initial data, while physically correct up to $\underline{\mathcal{C}} \bigcup \partial_{-}\mathcal{B}$ is not physically correct up to $\mathcal{B}$.

The problem of the physical continuation of the solution is the *shock development problem*. In this problem one is required to construct a hypersurface of discontinuity $\mathcal{K}$, the shock hypersurface, lying in the past of $\mathcal{B}$ but having the same past boundary as the latter, namely $\partial_{-}\mathcal{B}$, and a solution of the Eulerian equations in the spacetime domain bounded in the past by $\underline{\mathcal{C}} \bigcup \mathcal{K}$, agreeing on $\underline{\mathcal{C}}$ with the data induced by the maximal development, while having jumps across $\mathcal{K}$ relative to the data induced on $\mathcal{K}$ by the maximal development, these jumps satisfying the jump conditions which follow from the integral form of the mass, momentum and energy conservation laws. Moreover, $\mathcal{K}$ is required to be a spacelike hypersurface relative to the acoustical structure corresponding to the prior solution and a timelike hypersurface relative to the acoustical structure corresponding to the new solution. Thus, the singular surface $\partial_{-}\mathcal{B}$ is the cause generating the shock hypersurface $\mathcal{K}$ [Figure].



$\mathcal{B}$
$\mathcal{K}$
p
$\underline{\mathcal{C}}$
$\partial_{-}\mathcal{B}$

The 2007 monograph actually considered the extension of the Eulerian equations to the framework of special relativity. In this framework the underlying geometric structure of the spacetime manifold is that of the Minkowski spacetime. On the other hand, the underlying spacetime structure of the original Eulerian equations, as of all of classical mechanics, is that of the Galilean spacetime. The monograph in collaboration with Miao *Compressible Flow and Euler's Equations*, Surveys of Modern Mathematics **9**, International Press & Higher Education Press, 2014, treated the same topics as the earlier monograph in the non-relativistic setting reaching similar results in a considerably simpler self-contained manner. The 2019 monograph treats the shock development problem in the non-relativistic as well as in the relativistic setting, showing how results in the former are deduced as limits of results in the latter. In this lecture I shall confine myself to the non-relativistic theory.

The Galilean structure has a distinguished family of hyperplanes, those of absolute simultaneity. By rectangular coordinates in the Galilean framework we mean a Galilei frame together with rectangular coordinates in Euclidean space and two such systems of coordinates are related by the Galilei group, which extends the Euclidean group. To achieve a fuller understanding of the mathematical structure we consider any number d of spatial dimensions greater than or equal to 2, the physical case being of course $d = 3$. We denote by $\mathbb{G}^d$ the corresponding Galilean spacetime.

The Lorentzian metric g on $\mathbb{G}^d$ given in rectangular coordinates by:

$$g = -\eta^2 dt \otimes dt + \sum_{i=1}^{d} (dx^i - v^i dt) \otimes (dx^i - v^i dt) := \sum_{\mu,\nu=0}^{d} g_{\mu\nu} dx^\mu \otimes dx^\nu \quad (x^0 = t) \tag{1}$$

is the simplest representative in the conformal class corresponding to the sound cone field. Here η is the sound speed, a thermodynamic function, and $v^i : i = 1, ..., d$ are the components of the fluid velocity. We call g the *acoustical metric*.

From the mathematical point of view the shock development problem is a free boundary problem, with nonlinear conditions at the free boundary $\mathcal{K}$, for a quasilinear hyperbolic 1st order system, with characteristic initial data on $\underline{\mathcal{C}}$ which are singular, in a prescribed manner, at $\partial_{-}\mathcal{B}$, the past boundary of $\underline{\mathcal{C}}$. It is shown that the singularity persists, not only as a discontinuity in the physical variables across $\mathcal{K}$, but also as a milder singularity propagating along $\underline{\mathcal{C}}$. While the physical variables and their 1st derivatives extend continuously across $\underline{\mathcal{C}}$, the 1st derivatives are only $C^{0,1/2}$ at $\underline{\mathcal{C}}$ from the point of view of the future solution.

What the 2019 monograph solves is not the general shock development problem but what we call the restricted shock development problem. One of the jump conditions, the Hugoniot relation, is a relation between thermodynamic quantities on the two sides of $\mathcal{K}$. According to this relation $\triangle s = O((\triangle p)^3)$, $\triangle s$ being the jump in entropy (per unit mass) and $\triangle p$ the jump in pressure. The restricted problem results if we neglect $\triangle s$. This simplification retains an essential difficulty of the general problem, namely the singular behavior as we approach $\partial_{-}\mathcal{B}$. The treatment is based on the spacetime 1-form β, defined by:

$$\beta = \left(h + \frac{1}{2}|v|^2\right) dt - \sum_{i=1}^{d} v^i dx^i := \sum_{\mu=0}^{d} \beta_\mu dx^\mu \tag{2}$$

where h is the enthalpy per unit mass.

In the general case this 1-form is not closed, but the 2-form $\omega = -d\beta$, the spacetime vorticity 2-form, satisfies the equation

$$i_u\omega = \theta ds \tag{3}$$

where

$$u = \frac{\partial}{\partial t} + \sum_{i=1}^{d} v^i \frac{\partial}{\partial x^i} := \sum_{\mu=0}^{d} u^\mu \frac{\partial}{\partial x^\mu} \tag{4}$$

is the spacetime fluid velocity and θ is the temperature. Equation 3 in fact expresses the whole content of the differential energy-momentum conservation laws.

It is complemented by the equation:

$$\sum_{\mu=0}^{d} \partial_\mu(\rho u^\mu) = 0 \tag{5}$$

which expresses the differential mass conservation law, ρ being the mass density. Taking h and s as our basic thermodynamic variables, the pressure p is expressed as a function of h and s, this expression constituting the *equation of state* encoding the mechanical properties of the fluid, and ρ and θ are defined by:

$$dp = \rho(dh - \theta ds) \tag{6}$$

while the sound speed η is given by:

$$\eta^2 = \left(\frac{dp}{d\rho}\right)_s \tag{7}$$

As a consequence of equation 3 ω satisfies the transport equation:

$$\mathcal{L}_u \omega = d\theta \wedge ds \tag{8}$$

where the right hand side, like ω itself, is of differential order 1, the physical variables themselves being of differential order 0. In the general case the vectorfield u defines a characteristic field complementing the sound cone field. The hypersurface generated by the integral curves of u initiating at $\partial_{-}\mathcal{B}$ divides the domain of the new solution into two subdomains, that bounded by $\underline{\mathcal{C}}$ and the hypersurface in question and that between the hypersurface and $\mathcal{K}$. In the former domain the equations coincide with those of the restricted problem, while in the latter domain the spacetime vorticity 2-form ω does not vanish being determined by $\triangle s$ and 8.

I shall now outline the mathematical methods of the 2019 monograph. A central role is played by the reformulation of the Eulerian equations in the domain $\mathcal{N}$ of the new solution. First a homeomorphism is defined of this domain onto

$$\mathcal{R}_{\delta,\delta} = R_{\delta,\delta} \times S^{d-1} \tag{9}$$

d being the spatial dimension and:

$$R_{\delta,\delta} = \{(\underline{u}, u) \in \mathbb{R}^2 \ : \ 0 \leq \underline{u} \leq u \leq \delta\} \tag{10}$$

being a domain in $\mathbb{R}^2$, which represents the range in $\mathcal{N}$ of two functions $\underline{u}$ and u the level sets of which are transversal acoustically null hypersurfaces denoted by $\underline{C}_{\underline{u}}$ and C_u respectively, with $\underline{C}_0 = \underline{\mathcal{C}}$.

In this representation the shock hypersurface boundary of $\mathcal{N}$ is:

$$\mathcal{K}^{\delta} = \{(\tau, \tau) \ : \ \tau \in [0, \delta]\} \times S^{d-1} \tag{11}$$

and $\partial_{-}\mathcal{B}$ is:

$$\partial_{-}\mathcal{B} = (0, 0) \times S^{d-1} = S_{0,0} \tag{12}$$

We denote by $S_{\underline{u},u}$ the surfaces:

$$S_{\underline{u},u} = \underline{C}_{\underline{u}} \bigcap C_u = (\underline{u}, u) \times S^{d-1} \tag{13}$$

In the following we use the summation convention according to which repeated upper and lower indices are understood to be summed over their range.

In the reformulation of the Eulerian equations the unknowns constitute a triplet $((x^\mu : \mu = 0, ..., d),\ b\ , (\beta_\mu : \mu = 0, ..., d))$, where the $(x^\mu : \mu = 0, ..., d)$ are functions on $R_{\delta,\delta} \times S^{d-1}$ representing rectangular coordinates in the corresponding domain in $\mathbb{G}^d$, and the $(\beta_\mu : \mu = 0, ..., d)$ are also functions on $R_{\delta,\delta} \times S^{d-1}$ and represent the rectangular components of the 1-form β. The unknown b is a mapping of $R_{\delta,\delta}$ into the space of vectorfields on S^{d-1}. The pair $((x^\mu : \mu = 0, ..., d),\ b)$ satisfies the *characteristic system*, a 1st order system of partial differential equations which for $d \geq 2$ is fully nonlinear. The $(\beta_\mu : \mu = 0, ..., d)$ satisfy the *wave system*, a quasilinear 1st order system of partial differential equations. The two systems are coupled through the $(g_{\mu\nu} : \mu, \nu = 0, ..., d)$, which represent the rectangular components of the acoustical metric and depend on the $(\beta_\mu : \mu = 0, ..., d)$.

More precisely, denoting by $\not{d}f$ the differential of a function f on the $S_{\underline{u},u}$, the coupling is through the functions $N^\mu : \mu = 0,...,d$ and $\underline{N}^\mu : \mu = 0,...,d$ defined in terms of the $\not{d}x^\mu : \mu = 0,...,d$ pointwise by the conditions:

$$g_{\mu\nu}N^\mu \not{d}x^\nu = 0, \quad g_{\mu\nu}N^\mu N^\nu = 0, \quad N^0 = 1 \tag{14}$$

and similarly for the $\underline{N}^\mu$. By reason of the quadratic nature of the 2nd of these conditions a unique pair $(N^\mu : \mu = 0,...,d),(\underline{N}^\mu : \mu = 0,...,d))$ up to exchange is pointwise defined by $(g_{\mu\nu} : \mu,\nu = 0,...,d)$ and $(\not{d}x^\mu : \mu = 0,...,d)$. The vectorfields N, $\underline{N}$ with rectangular components $(N^\mu : \mu = 0,...,d)$, $(\underline{N}^\mu : \mu = 0,...,d)$ are then null normal fields, relative to the acoustical metric g, to the surfaces $S_{\underline{u},u}$. The exchange ambiguity is removed by requiring N to be tangential to the C_u, $\underline{N}$ to be tangential to the $\underline{C}_{\underline{u}}$.

The function

$$c = -\frac{1}{2} g_{\mu\nu} N^{\mu} \underline{N}^{\nu} \tag{15}$$

is then bounded from below by a positive constant. Defining:

$$L = \frac{\partial}{\partial \underline{u}} - b, \quad \underline{L} = \frac{\partial}{\partial u} + b \tag{16}$$

and:

$$\rho = Lt, \quad \underline{\rho} = \underline{L}t \tag{17}$$

the characteristic system is simply:

$$Lx^{\mu} = \rho N^{\mu}, \quad \underline{L}x^{\mu} = \underline{\rho}\underline{N}^{\mu} \qquad : \; \mu = 0, ..., d \tag{18}$$

What is achieved by the reformulation just described is a regularization of the problem. That is, we are now seeking smooth functions on $R_{\delta,\delta} \times S^{d-1}$ satisfying the coupled system, the initial data themselves being represented by smooth functions on $\underline{C}_0$. The Jacobian of the transformation representing the mapping $((\underline{u}, u), \vartheta) \mapsto (x^\mu((\underline{u}, u), \vartheta) : \mu = 0, ..., d)$, $\vartheta \in S^{d-1}$, is of the form:

$$\frac{\partial(x^0, x^1, ..., x^d)}{\partial(\underline{u}, u, \vartheta^1, ..., \vartheta^{d-1})} = \rho\underline{\rho}e \tag{19}$$

$(\vartheta^A : A = 1, ..., d-1)$ being local coordinates on S^{d-1}.

Here ρ, $\underline{\rho}$, defined by 17, are non-negative functions, the inverse temporal density of the foliation of spacetime by the $\underline{C}_{\underline{u}}$ as measured along the generators of the C_u, the inverse temporal density of the foliation of spacetime by the C_u as measured along the generators of the $\underline{C}_{\underline{u}}$, respectively, and e is a function bounded from above by a negative constant. As a consequence, the Jacobian 19 vanishes where and only where one of ρ, $\underline{\rho}$ vanishes.

The function $\underline{\rho}$ is given on $\underline{C}_0$ by the initial data and while positive on $\underline{C}_0 \setminus S_{0,0}$, vanishes to 1st order at $S_{0,0}$, the last being a manifestation of the singular nature of the surface $\partial_- \mathcal{B}$. The function ρ on $\underline{C}_0$ represents 1st derived data on $\underline{C}_0$ and vanishes there to 0th order. It turns out that these are the only places where $\underline{\rho}$, ρ vanish in $R_{\delta,\delta} \times S^{d-1}$. A smooth solution of the coupled characteristic and wave systems once obtained then represents a solution of the original Eulerian equations in standard rectangular coordinates which is smooth in $\mathcal{N} \setminus \underline{\mathcal{C}}$ but singular at $\underline{\mathcal{C}}$ with the transversal derivatives of the β_μ being only Hölder continuous of exponent 1/2 at $\underline{\mathcal{C}}$ and, in addition, a stronger singularity at $\partial_- \mathcal{B}$, namely the blow up of the derivatives of the β_μ at $\partial_- \mathcal{B}$ in the direction tangential to $\underline{\mathcal{C}}$ but transversal to $\partial_- \mathcal{B}$.

In particular, the new solution is smooth at the shock hypersurface $\mathcal{K}$ except at its past boundary $\partial_{-}\mathcal{B}$, as is the prior solution which holds in the other side of $\mathcal{K}$, the past side, as $\mathcal{K} \setminus \partial_{-}\mathcal{B}$ lies in the interior of the domain of the maximal development.

The wave system consists of the equations

$$dx^{\mu} \wedge d\beta_{\mu} = 0, \quad g^{-1}(dx^{\mu}, d\beta_{\mu}) = 0 \tag{20}$$

expressed in terms of the representation on the domain 9.

The characteristic system for the pair $((x^\mu : \mu = 0, ..., d),\ b)$ together with the $(g_{\mu\nu} : \mu, \nu = 0, ..., d)$ which enter this system through the $(N^\mu : \mu = 0, ..., d)$ and the $(\underline{N}^\mu : \mu = 0, ..., d)$, manifest a new kind of differential geometric structure which involves the interaction of two geometric structures on the same underlying manifold, the first of these structures being the background Galilean structure and the other being the Lorentzian geometry deriving from the acoustical metric. As for the $(\beta_\mu : \mu = 0, ..., d)$ of the wave system, this is the set of functions obtained by evaluating the 1-form β on the set of translation fields $(\partial/\partial x^\mu : \mu = 0, ..., d)$ of the background structure.

If X is a vectorfield generating isometries of the background structure then:

$$\Box_{\tilde{g}}\beta(X) = 0 \tag{21}$$

where

$$\tilde{g} = \Omega g, \qquad \Omega = \left(\frac{\rho}{\eta}\right)^{2/(d-1)} \tag{22}$$

is a metric in the conformal class of the acoustical metric g. This fact, with a translation field substituted for X, plays a central role.

The derivatives of $((x^\mu : \mu = 0, ..., d),\ b)$ are controlled through the *acoustical structure equations*. These are differential consequences of the characteristic system, bringing out more fully the interaction of the two geometric structures. We have the induced metric $\not{g}$ on the surfaces $S_{\underline{u},u}$ and the functions:

$$\lambda = c\underline{\rho}, \quad \underline{\lambda} = c\rho \tag{23}$$

(see 17). While $\not{g}$ refers only to the acoustical structure, the functions λ, $\underline{\lambda}$, involve the interaction of the two geometric structures. These are acoustical quantities of 0th order. The quantities χ, $\underline{\chi}$, the two 2nd fundamental forms of $S_{\underline{u},u}$, give $\not{\mathcal{L}}_L \not{g}$, $\not{\mathcal{L}}_{\underline{L}} \not{g}$. The torsion forms η, $\underline{\eta}$ represent the connection in the normal bundle of $S_{\underline{u},u}$ in terms of the vectorfields L, $\underline{L}$ which along $S_{\underline{u},u}$ constitute basis sections of this bundle.

The commutator:

$$[L, \underline{L}] = \not{\zeta}_T b, \quad T = L + \underline{L} = \frac{\partial}{\partial \underline{u}} + \frac{\partial}{\partial u} \tag{24}$$

(see 16) is expressed in terms of η, $\underline{\eta}$. While the quantities χ, $\underline{\chi}$, η, $\underline{\eta}$ refer only to the acoustical structure, the structure equations assume a non-singular form only in terms of the quantities $\tilde{\chi}$, $\underline{\tilde{\chi}}$, $\tilde{\eta}$, $\underline{\tilde{\eta}}$ which involve both structures. The former are related to the latter as follows: up to order 1 remainders depending only on the $d\beta_\mu : \mu = 0, ..., d$, χ is equal to $\rho\tilde{\chi}$, $\underline{\chi}$ is equal to $\underline{\rho\tilde{\chi}}$, η is equal to $\rho\tilde{\eta}$, $\underline{\eta}$ is equal to $\underline{\rho\tilde{\eta}}$. Moreover $\tilde{\eta}$, $\underline{\tilde{\eta}}$, are expressed in terms of $\not{d}\lambda$, $\not{d}\underline{\lambda}$. Thus λ, $\underline{\lambda}$, $\tilde{\chi}$, $\underline{\tilde{\chi}}$ are the primary acoustical quantities, the first two being of 0th order, the second two of 1st order.

We come to the boundary conditions on $\mathcal{K}$. The jump $\triangle\beta_\mu$ is a function on $\mathcal{K}$, which at a given point on $\mathcal{K}$ represents the difference of β_μ, defined by the new solution which holds in the future of $\mathcal{K}$, at the point, from the corresponding quantity for the prior solution which holds in the past of $\mathcal{K}$, at the same point in the background Galilean spacetime. These jumps across $\mathcal{K}$ are subject to two conditions, one of which is linear and the other nonlinear. The linear jump condition decomposes into the two conditions:

$$\not{d}x^\mu \ \triangle\beta_\mu = 0, \qquad T^\mu \triangle\beta_\mu = 0 \tag{25}$$

As a consequence of the 1st of 25 $\triangle\beta_\mu$ can be expressed as a linear combination of the components:

$$\epsilon = N^\mu \triangle\beta_\mu, \quad \underline{\epsilon} = \underline{N}^\mu \triangle\beta_\mu \tag{26}$$

We denote by r the ratio:

$$r = -\frac{\epsilon}{\underline{\epsilon}} \tag{27}$$

In reference to the 2nd of 25, $T^\mu = Tx^\mu$ are the rectangular components of the vectorfield T and are given, in view of 18, 24 by:

$$T^\mu = \rho N^\mu + \underline{\rho}\underline{N}^\mu \tag{28}$$

Then in view of 23 the 2nd of 25 is equivalent to the following boundary condition for $\underline{\lambda}$:

$$r\underline{\lambda} = \lambda \quad : \text{ on } \mathcal{K} \tag{29}$$

Part of the acoustical structure equations are propagation equations along the integral curves of L and $\underline{L}$. The propagation equations for λ and for $\tilde{\chi}$ are supplemented by initial conditions on $\underline{C}_0$, while the propagation equations for $\underline{\lambda}$ and for $\underline{\tilde{\chi}}$ are supplemented by boundary conditions on $\mathcal{K}$. The boundary condition for $\underline{\tilde{\chi}}$ takes the form of a relation between $r\underline{\tilde{\chi}}$ and $\tilde{\chi}$ on $\mathcal{K}$ analogous to 29. The nonlinear jump condition takes the form of a relation between ϵ and $\underline{\epsilon}$ which, in the setting of the shock development problem, is shown to be equivalent to:

$$\epsilon = -j(\underline{\epsilon})\underline{\epsilon}^2, \quad \text{or} \quad r = j(\underline{\epsilon})\underline{\epsilon} \tag{30}$$

where j is a certain smooth function.

Let now X, Y be arbitrary vectorfields on $\mathcal{N}$. We define the *bi-variational stress* associated to the 1-form β and to the pair X, Y, to be the T^1_1 type tensorfield:

$$\dot{T} = \tilde{g}^{-1} \cdot \dot{T}_\flat \tag{31}$$

where $\dot{T}_\flat$ is the symmetric 2-covariant tensorfield:

$$\dot{T}_\flat = \frac{1}{2} \left(d\beta(X) \otimes d\beta(Y) + d\beta(Y) \otimes d\beta(X) - (d\beta(X), d\beta(Y))_g g \right) \tag{32}$$

which depends only on the conformal class of g. We then have the identity:

$$\mathrm{div}_{\tilde{g}} \dot{T} = \frac{1}{2} \Big(\Box_{\tilde{g}} \beta(X) \Big) d\beta(Y) + \frac{1}{2} \Big(\Box_{\tilde{g}} \beta(Y) \Big) d\beta(X) \tag{33}$$

In particular, if X, Y generate isometries of the background structure, then by 21:

$$\mathrm{div}_{\tilde{g}}\dot{T} = 0 \tag{34}$$

Setting X, Y to be the translation fields:

$$X = \frac{\partial}{\partial x^{\mu}}, \quad Y = \frac{\partial}{\partial x^{\nu}}$$

we have:

$$\beta(X) = \beta_{\mu}, \quad \beta(Y) = \beta_{\nu}$$

and we denote the corresponding bi-variational stress by $\dot{T}_{\mu\nu}$. The identity 33 takes in this case the form:

$$\mathrm{div}_{\tilde{g}}\dot{T}_{\mu\nu} = \frac{1}{2}(\Box_{\tilde{g}}\beta_{\mu})d\beta_{\nu} + \frac{1}{2}(\Box_{\tilde{g}}\beta_{\nu})d\beta_{\mu} \tag{35}$$

The usefulness of the concept of bi-variational stress in the context of a free boundary problem is in conjunction with the concept of variation fields. A *variation field* is here simply a vectorfield V on $\mathcal{N}$ which along $\mathcal{K}$ is tangential to $\mathcal{K}$. This can be expanded in terms of the translation fields $\partial/\partial x^\mu : \mu = 0, ..., d$:

$$V = V^\mu \frac{\partial}{\partial x^\mu} \tag{36}$$

The coefficients $V^\mu : \mu = 0, ..., d$ of the expansion are simply the rectangular components of V. To the variation field V we associate the column of 1-forms:

$${}^{(V)}\theta^\mu = dV^\mu \ : \ \mu = 0, ..., d \tag{37}$$

which we call *structure form* of V. Note that this depends on the background structure.

To a variation field V and to the row of functions $(\beta_\mu : \mu = 0, ..., d)$ we associate the 1-form:

$$ {}^{(V)}\xi = V^\mu d\beta_\mu \tag{38} $$

To the variation field V is associated the T^1_1 type tensorfield:

$$ {}^{(V)}S = V^\mu V^\nu \dot{T}_{\mu\nu} \tag{39} $$

In view of 31, 32, 38, we have:

$$ {}^{(V)}S = \tilde{g}^{-1} \cdot \ {}^{(V)}S_\flat \tag{40} $$

where ${}^{(V)}S_\flat$ is the symmetric 2-covariant tensorfield:

$$ {}^{(V)}S_\flat = {}^{(V)}\xi \otimes {}^{(V)}\xi - \frac{1}{2}({}^{(V)}\xi, {}^{(V)}\xi)_g g \tag{41} $$

which depends only on the conformal class of g.

The identity 35 together with the definition 37 implies the identity:

$$\begin{aligned}\mathrm{div}_{\tilde{g}}\ {}^{(V)}S = & ({}^{(V)}\xi,\ {}^{(V)}\theta^{\mu})_{\tilde{g}} d\beta_{\mu} - ({}^{(V)}\xi, d\beta_{\mu})_{\tilde{g}}\ {}^{(V)}\theta^{\mu} \\ & + ({}^{(V)}\theta^{\mu}, d\beta_{\mu})_{\tilde{g}}\ {}^{(V)}\xi + V^{\mu}(\Box_{\tilde{g}}\beta_{\mu})\ {}^{(V)}\xi \end{aligned} \tag{42}$$

A basic requirement on the set of variation fields V is that they span the tangent space to $\mathcal{K}$ at each point. The simplest way to achieve this is to choose one of the variation fields, which we denote by Y, to be at each point of $\mathcal{N}$ in the linear span of N and $\underline{N}$ and along $\mathcal{K}$ colinear to T, and to choose the other variation fields so that at each point of $\mathcal{N}$ they span the tangent space to the surface $S_{\underline{u},u}$ though that point.

We thus set:

$$Y = \gamma N + \overline{\gamma}\underline{N} \tag{43}$$

In view of 29, the requirement that Y is along $\mathcal{K}$ colinear to T reduces to:

$$\overline{\gamma} = r\gamma \ : \ \text{along } \mathcal{K} \tag{44}$$

The optimal choice is to set:

$$\gamma = 1 \tag{45}$$

in which case 44 reduces to:

$$\overline{\gamma} = r \ : \ \text{along } \mathcal{K} \tag{46}$$

and to extend $\overline{\gamma}$ to $\mathcal{N}$ by the requirement that it be constant along the integral curves of L:

$$L\overline{\gamma} = 0 \tag{47}$$

In 2 spatial dimensions there is an obvious choice of a variation field to complement Y, namely E, the unit tangent field of the curves $S_{\underline{u},u}$ (with the counterclockwise orientation). In higher dimensions, we complement Y with the $(E_{(\mu)} : \mu = 0, ..., d)$ which are the g-orthogonal projections to the surfaces $S_{\underline{u},u}$ of the translation fields $(\partial/\partial x^{\mu} : \mu = 0, ..., d)$ of the background structure. These are given by:

$$E_{(\mu)} = g_{\mu\nu}(\not{d}x^{\nu})^{\sharp} \tag{48}$$

We come to the fundamental energy identities. Given a vectorfield X, which we call *multiplier field*, we consider the vectorfield ${}^{(V)}P$ associated to X and to a given variation field V through ${}^{(V)}S$, defined by:

$$ {}^{(V)}P = -\,{}^{(V)}S \cdot X \tag{49} $$

We call ${}^{(V)}P$ the *energy current* associated to X and to V. Let us denote by ${}^{(V)}Q$ the divergence of ${}^{(V)}P$ with respect to the conformal acoustical metric $\tilde{g} = \Omega g$:

$$ \mathrm{div}_{\tilde{g}}\, {}^{(V)}P = {}^{(V)}Q \tag{50} $$

We have:

$$ {}^{(V)}Q = {}^{(V)}Q_1 + {}^{(V)}Q_2 + {}^{(V)}Q_3 \tag{51} $$

where:

$$ {}^{(V)}Q_1 = -\frac{1}{2}\, {}^{(V)}S^{\sharp} \cdot\, {}^{(X)}\tilde{\pi} \tag{52} $$

$$ \begin{aligned} {}^{(V)}Q_2 = &-({}^{(V)}\xi, {}^{(V)}\theta^{\mu})_{\tilde{g}} X\beta_{\mu} + ({}^{(V)}\xi, d\beta_{\mu})_{\tilde{g}}\, {}^{(V)}\theta^{\mu}(X) \\ &-({}^{(V)}\theta^{\mu}, d\beta_{\mu})_{\tilde{g}}\, {}^{(V)}\xi(X) \end{aligned} \tag{53} $$

$$ {}^{(V)}Q_3 = -\, {}^{(V)}\xi(X) V^{\mu} \Box_{\tilde{g}} \beta_{\mu} \tag{54} $$

In 52 ${}^{(V)}S^{\sharp}$ is the symmetric 2-contravariant tensorfield corresponding to ${}^{(V)}S$:

$$ {}^{(V)}S^{\sharp} = {}^{(V)}S \cdot \tilde{g}^{-1} \tag{55} $$

and

$$ {}^{(X)}\tilde{\pi} = \mathcal{L}_X \tilde{g} \tag{56} $$

is the *deformation tensor* of X, the rate of change of the conformal acoustical metric with respect to the flow generated by X.

Integrating 50 on a domain in $\mathcal{R}_{\delta,\delta}$ of the form:

$$\mathcal{R}_{\underline{u}_1,u_1} = R_{\underline{u}_1,u_1} \times S^{d-1} = \bigcup_{(\underline{u},u)\in R_{\underline{u}_1,u_1}} S_{\underline{u},u} \tag{57}$$

where, with $(\underline{u}_1, u_1) \in R_{\delta,\delta}$ we denote:

$$R_{\underline{u}_1,u_1} = \{(\underline{u}, u) \ : \ u \in [\underline{u}, u_1], \underline{u} \in [0, \underline{u}_1]\} \tag{58}$$

we obtain the *fundamental energy identity* corresponding to the variation field V and to the multiplier field X:

$$^{(V)}\mathcal{E}^{\underline{u}_1}(u_1) + {}^{(V)}\underline{\mathcal{E}}^{u_1}(\underline{u}_1) + {}^{(V)}\mathcal{F}^{\underline{u}_1} - {}^{(V)}\underline{\mathcal{E}}^{u_1}(0) = {}^{(V)}\mathcal{G}^{\underline{u}_1,u_1} \tag{59}$$

Here, ${}^{(V)}\mathcal{E}^{\underline{u}_1}(u_1)$ and ${}^{(V)}\underline{\mathcal{E}}^{u_1}(\underline{u}_1)$ are the *energies*:

$$
\begin{aligned}
{}^{(V)}\mathcal{E}^{\underline{u}_1}(u_1) &= \int_{C_{u_1}^{\underline{u}_1}} \Omega^{(d-1)/2}\ {}^{(V)}S_\flat(X,L) \\
{}^{(V)}\underline{\mathcal{E}}^{u_1}(\underline{u}_1) &= \int_{\underline{C}_{\underline{u}_1}^{u_1}} \Omega^{(d-1)/2}\ {}^{(V)}S_\flat(X,\underline{L})
\end{aligned}
\tag{60}
$$

and ${}^{(V)}\mathcal{F}^{\underline{u}_1}$ is the *flux*:

$$
{}^{(V)}\mathcal{F}^{\underline{u}_1} = \int_{\mathcal{K}^{\underline{u}_1}} \Omega^{(d-1)/2}\ {}^{(V)}S_\flat(X,M) \tag{61}
$$

where

$$
M = \underline{L} - L \tag{62}
$$

is a normal to $\mathcal{K}$ pointing to the interior relative to $\mathcal{N}$ or future of $\mathcal{K}$. In 60 we denote by $C_{u_1}^{\underline{u}_1}$ the part of C_{u_1} corresponding to $\underline{u} \leq \underline{u}_1$ and by $\underline{C}_{\underline{u}_1}^{u_1}$ the part of $\underline{C}_{\underline{u}_1}$ corresponding to $u \leq u_1$.

The right hand side of 59 is the *error integral*:

$$ {}^{(V)}\mathcal{G}^{\underline{u}_1,u_1} = \int_{\mathcal{R}_{\underline{u}_1,u_1}} 2a\Omega^{(d+1)/2}\ {}^{(V)}Q \tag{63} $$

which decomposes into:

$$ {}^{(V)}\mathcal{G}^{\underline{u}_1,u_1} = {}^{(V)}\mathcal{G}_1^{\underline{u}_1,u_1} + {}^{(V)}\mathcal{G}_2^{\underline{u}_1,u_1} + {}^{(V)}\mathcal{G}_3^{\underline{u}_1,u_1} \tag{64} $$

according to the decomposition 51 of ${}^{(V)}Q$.

The energies are positive semi-definite if the multiplier field X acoustically timelike future-directed. The conditions on X are found which make the flux coercive when ${}^{(V)}\xi$ satisfies on $\mathcal{K}$ the boundary condition which results by applying the variation field V to the nonlinear jump condition.

This boundary condition takes the form:

$$ {}^{(V)}\xi_+(A_+) = {}^{(V)}\xi_-(A_-) \tag{65} $$

where the subscripts $+$ and $-$ denote the future and past sides of $\mathcal{K}$ respectively and $A_\pm$ are the vectorfields:

$$ A_\pm = \frac{\triangle I}{\delta} - K_\pm \tag{66} $$

Here I is the mass current

$$ I = \rho u \tag{67} $$

The function δ is defined as follows. Let ζ to be the covectorfield along $\mathcal{K}$ such that at each $p \in \mathcal{K}$ the null space of ζ_p is $T_p\mathcal{K}$, $\zeta_p(U) > 0$ if the vector U points to the interior of $\mathcal{N}$, ζ being normalized by the condition that $\overline{\zeta}$, the restriction of ζ to the hyperplanes of absolute simultaneity, is of unit magnitude with respect to the Euclidean metric.

The nonlinear jump condition can then be stated in the form:

$$\zeta \cdot \triangle I = 0 \tag{68}$$

that is, the vectorfield $\triangle I$ along $\mathcal{K}$ is tangential to $\mathcal{K}$, while the linear jump conditions 25 take the form:

$$\triangle \beta = \delta \zeta \tag{69}$$

for some function δ on $\mathcal{K}$. This clarifies the meaning of the 1st term on the right in 66. As for the 2nd term, $K_{\pm}$ is a normal, relative to the acoustical metric, vectorfield along the future and past sides of $\mathcal{K}$ respectively. The 1st term on the right in 66, a vectorfield along $\mathcal{K}$ tangential to $\mathcal{K}$, is timelike future-directed with respect to the acoustical metric defined by the future solution. This fact plays a central role in the analysis of the coercivity of the flux integrant.

We show that under the boundary condition 65 the flux integrant in 61 is a coercive quadratic form in ${}^{(V)}\xi$ at a point of $\mathcal{K}$ if and only if the multiplier field X at the point belongs to the interior of a spheroidal cone contained in the positive outer half sound cone at the point corresponding to the future solution.

We then show that a suitable choice for the multiplier field is:

$$X = 3L + \underline{L} \quad : \text{ on } \mathcal{N} \tag{70}$$

With this choice there is a constant C' such that

$${}^{(V)}\mathcal{F}'^{\underline{u}_1} = {}^{(V)}\mathcal{F}^{\underline{u}_1} + 2C' \int_{\mathcal{K}^{\underline{u}_1}} \Omega^{(d-1)/2} ({}^{(V)}b)^2 \tag{71}$$

is positive-definite. Here ${}^{(V)}b$ is defined by the prior solution on the past side of $\mathcal{K}$.

Adding the 2nd term on the right in 71 to both sides of the energy identity 59, the last takes the form:

$$
\begin{aligned}
& {}^{(V)}\mathcal{E}^{\underline{u}_1}(u_1) + {}^{(V)}\underline{\mathcal{E}}^{u_1}(\underline{u}_1) + {}^{(V)}\mathcal{F}'^{\underline{u}_1} \\
& \quad = {}^{(V)}\underline{\mathcal{E}}^{u_1}(0) + {}^{(V)}\mathcal{G}^{\underline{u}_1,u_1} + 2C' \int_{\mathcal{K}^{\underline{u}_1}} \Omega^{(d-1)/2}({}^{(V)}b)^2
\end{aligned}
\tag{72}
$$

The *commutation fields* which are used to control the higher order analogues of the functions $\beta_\mu : \mu = 0, ..., d$ are then defined.

Denoting (see 11), for $\sigma \in [0, \delta]$,

$$\mathcal{K}_\sigma^\delta = \{(\tau, \sigma + \tau) \ : \ \tau \in [0, \delta - \sigma]\} \times S^{d-1} \tag{73}$$

(note that $\mathcal{K}_0^\delta = \mathcal{K}^\delta$) we require that at each point $q \in \mathcal{N}$, $q \in \mathcal{K}_\sigma^\delta$, the set of commutation fields C to span $T_q\mathcal{K}_\sigma^\delta$. As first of the commutation fields we take the vectorfield T. The remaining commutation fields are then required to span the tangent space to the $S_{\underline{u},u}$ at each point. In $d = 2$ spatial dimensions we choose E to complement T as a commutation field. For $d > 2$ we choose the $E_{(\mu)} : \mu = 0, ..., d$ (see 48) to complement T. Thus E for $d = 2$ and the $E_{(\mu)}$ for $d > 2$ play a dual role being commutation fields as well as variation fields. However, what characterizes the action of a variation field V is the corresponding structure form ${}^{(V)}\theta$, what characterizes the action of a commutation field C is the corresponding deformation tensor ${}^{(C)}\tilde{\pi} = \mathcal{L}_C \tilde{g}$.

The commutation fields generate higher order analogues of the functions $\beta_\mu : \mu = 0, ..., d$. In the following it is to be understood that in the case $d = 2$ the set of vectorfields $(E_{(\mu)} : \mu = 0, ..., d)$ is replaced by the single vectorfield E. At order $m + l$ we have:

$$ {}^{(m,\nu_1...\nu_l)}\beta_\mu = E_{(\nu_l)}...E_{(\nu_1)}T^m\beta_\mu \tag{74} $$

To these and to the variation field V there correspond higher order analogues of the 1-form ${}^{(V)}\xi$, namely:

$$ {}^{(V;m,\nu_1...\nu_l)}\xi = V^\mu d\ {}^{(m,\nu_1...\nu_l)}\beta_\mu \tag{75} $$

The preceding identities 35, 42, 49 - 54, 59, 64 which refer to β_μ and to ${}^{(V)}\xi$ all hold with these higher order analogues in the role of β_μ and ${}^{(V)}\xi$ respectively. However while for the original β_μ we have $\square_{\tilde{g}}\beta_\mu = 0$, hence the error term ${}^{(V)}Q_3$ (see 54) vanishes, this is no longer true for the higher order analogues.

Instead we have:

$$\Omega a \Box_{\tilde{g}} \ {}^{(m,\nu_1 \ldots \nu_l)}\beta_\mu = \ {}^{(m,\nu_1 \ldots \nu_l)}\tilde{\rho}_\mu \tag{76}$$

where

$$a = -\frac{1}{2} g(L, \underline{L}) = c\rho\underline{\rho} \tag{77}$$

(see 15, 18). The ${}^{(m,\nu_1 \ldots \nu_l)}\tilde{\rho}_\mu$, which we call *source functions*, obey certain recursion formulas which determine them for all m and l. The error terms at order $m + l$ which contain the acoustical quantities of highest order, $m + l + 1$, are contained in the error integral ${}^{(V;m,\nu_1 \ldots \nu_l)}\mathcal{G}_3^{\underline{u}_1,u_1}$, which by 63 and 54 is given by:

$$ {}^{(V;m,\nu_1 \ldots \nu_l)}\mathcal{G}_3^{\underline{u}_1,u_1} = -\int_{\mathcal{R}_{\underline{u}_1,u_1}} 2\Omega^{(d-1)/2} \ {}^{(V;m,\nu_1 \ldots \nu_l)}\xi(X) V^\mu \ {}^{(m,\nu_1 \ldots \nu_l)}\tilde{\rho}_\mu \tag{78}$$

The leading terms in ${}^{(m,\nu_1\ldots\nu_l)}\tilde{\rho}$ involving the acoustical quantities of highest order are:

$$\text{for } m=0: \quad \frac{1}{2}\rho(\underline{L}\beta_\mu)E_{(\nu_l)}\ldots E_{(\nu_1)}\text{tr}\tilde{\chi} + \frac{1}{2}\underline{\rho}(L\beta_\mu)E_{(\nu_l)}\ldots E_{(\nu_1)}\text{tr}\underline{\tilde{\chi}} \tag{79}$$

$$\text{for } m\geq 1: \quad \rho(\underline{L}\beta_\mu)E_{(\nu_l)}\ldots E_{(\nu_1)}T^{m-1}\not{\Delta}\lambda + \underline{\rho}(L\beta_\mu)E_{(\nu_l)}\ldots E_{(\nu_1)}T^{m-1}\not{\Delta}\underline{\lambda} \tag{80}$$

We now come to the main analytic method introduced in the 2019 monograph. To motivate the introduction of this method, we need to first discuss the difficulties encountered.

The difficulties arise in estimating the contribution of the terms involving the top order $m+l+1 = n+1$ acoustical quantities to the error integral ${}^{(V;m,\nu_1...\nu_l)}\mathcal{G}_{\underline{3}}^{u_1,u_1}$. In regard to 79 we must derive appropriate estimates for

$$\not{d}(E_{(\nu_{l-1})}...E_{(\nu_1)}\mathrm{tr}\tilde{\chi}),\ \not{d}(E_{(\nu_{l-1})}...E_{(\nu_1)}\mathrm{tr}\underline{\tilde{\chi}})\ :\ \nu_1,...,\nu_{l-1} = 0,...,d \quad (81)$$

In regard to 80 we must derive appropriate estimates for

$$E_{(\nu_l)}...E_{(\nu_1)}T^{m-1}\not{\Delta}\lambda,\ E_{(\nu_l)}...E_{\nu_1)}T^{m-1}\not{\Delta}\underline{\lambda}\ :\ \nu_1,...,\nu_l = 0,...,d. \quad (82)$$

Expressions for $L\mathrm{tr}\tilde{\chi}$ and $\underline{L}\mathrm{tr}\underline{\tilde{\chi}}$ are derived in terms of 2nd order quantities with vanishing 2nd order acoustical part. To be able to estimate $\mathrm{tr}\tilde{\chi}$, $\mathrm{tr}\underline{\tilde{\chi}}$ in terms of 1st order quantities, so that we can estimate 81 in terms of quantities of the top order $l+1 = n+1$, we must express the principal part of these expressions in the form $-L\hat{f}$ and $-\underline{L}\underline{\hat{f}}$ respectively, up to lower order terms, with $\hat{f}$ and $\underline{\hat{f}}$ being quantities of 1st order.

That this is possible follows from the fact that the quantities:

$$M = \frac{1}{2}\beta_N^2(a\not{\triangle}H - L(\underline{L}H)), \quad \underline{M} = \frac{1}{2}\beta_{\underline{N}}^2(a\not{\triangle}H - \underline{L}(LH)) \tag{83}$$

where a is defined by 77, are actually 1st order quantities. Here H a function of the $\beta_\mu : \mu = 0, ..., d$. This fact is a direct consequence of the equation $\Box_{\tilde{g}}\beta_\mu = 0$. The functions $\hat{f}$, $\underline{\hat{f}}$ each contain a singular term with coefficient λ^{-1}, $\underline{\lambda}^{-1}$ respectively. The functions $f = \lambda\hat{f}$, $\underline{f} = \underline{\lambda}\underline{\hat{f}}$ are then regular, and transferring the corresponding terms to the left hand side, we obtain propagation equations for the quantities:

$$\theta = \lambda \mathrm{tr}\tilde{\chi} + f, \quad \underline{\theta} = \underline{\lambda}\mathrm{tr}\underline{\tilde{\chi}} + \underline{f} \tag{84}$$

of the form:

$$L\theta = R, \quad \underline{L}\theta = \underline{R} \tag{85}$$

where R, $\underline{R}$ are again quantities of order 1.

To estimate 81 we introduce the quantities, of order $l+1=n+1$,

$$
\begin{aligned}
{}^{(\nu_1\ldots\nu_{l-1})}\theta_l &= \lambda \not{d}\Big(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}\mathrm{tr}\tilde{\chi}\Big)+\not{d}\Big(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}f\Big)\\
{}^{(\nu_1\ldots\nu_{l-1})}\underline{\theta}_l &= \underline{\lambda}\not{d}\Big(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}\mathrm{tr}\underline{\tilde{\chi}}\Big)+\not{d}\Big(E_{(\nu_{l-1})}\ldots E_{(\nu_1)}\underline{f}\Big)
\end{aligned}
\tag{86}
$$

and deduce from 85 the corresponding propagation equations. An analogous structure is found for $\not{\Delta}\lambda$ and $\not{\Delta}\underline{\lambda}$, which allows us to estimate 82.

The point is now that the optimal estimate for ${}^{(\nu_1 \dots \nu_{l-1})}\theta_l$ is a bound of $\| {}^{(\nu_1 \dots \nu_{l-1})}\theta_l \|_{L^2(\underline{C}^{u_1}_{\underline{u}_1})}$ by:

$$C \sum_{\nu_l} \underline{u}_1^{-3/2} \left\{ \int_0^{\underline{u}_1} {}^{(Y;0,\nu_1 \dots \nu_{l-1}\nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u}) d\underline{u} \right\}^{1/2} \tag{87}$$

This only allows the contribution of the 1st of 81 to the error integral ${}^{(Y;0,\nu_1 \dots \nu_l)}\mathcal{G}_3^{\underline{u}_1, u_1}$ to be bounded by:

$$C \sum_{\mu} \int_0^{\underline{u}_1} \left({}^{(Y;0,\nu_1 \dots \nu_l)}\underline{\mathcal{E}}^{u_1}(\underline{u}) \right)^{1/2} \left\{ \frac{1}{\underline{u}} \int_0^{\underline{u}} {}^{(Y;0,\nu_1 \dots \nu_{l-1}\mu)}\underline{\mathcal{E}}^{u_1}(\underline{u}') d\underline{u}' \right\}^{1/2} \frac{d\underline{u}}{\underline{u}} \tag{88}$$

a singular integral.

The greater difficulty is however that the optimal estimate for ${}^{(\nu_1 \ldots \nu_{l-1})}\underline{\theta}_l$ is a bound of $\|\underline{u}^2 \; {}^{(\nu_1 \ldots \nu_{l-1})}\underline{\theta}_l\|_{L^2(C_{u_1}^{\underline{u}_1})}$ the leading contribution to which is in terms of:

$$C \sum_{\nu_l} \left\{ \int_0^{\underline{u}_1} {}^{(Y;0,\nu_1 \ldots \nu_{l-1}\nu_l)}\underline{\mathcal{E}}^{\underline{u}_1}(\underline{u}) \frac{d\underline{u}}{\underline{u}^3} \right\}^{1/2} \tag{89}$$

a severely singular integral. Moreover, this only bounds $\|\underline{u}^2 \; {}^{(\nu_1 \ldots \nu_{l-1})}\underline{\theta}_l\|_{L^2(C_{u_1}^{\underline{u}_1})}$ rather than $\| \; {}^{(\nu_1 \ldots \nu_{l-1})}\underline{\theta}_l\|_{L^2(C_{u_1}^{\underline{u}_1})}$. The more severe singularity in connection with ${}^{(\nu_1 \ldots \nu_{l-1})}\underline{\theta}_l$ is due to the fact that the underlined quantities satisfy boundary conditions on $\mathcal{K}$ which are singular at $\partial_-\mathcal{K} = \partial_-\mathcal{B}$ due to the vanishing there of the factor r (see 29 and the text which follows). Nevertheless, from the point of view of scaling the two contributions to the error integrals are both borderline.

Similar results are obtained in regard to the error integral ${}^{(V;m,\nu_1...\nu_l)}\mathcal{G}_3^{\underline{u}_1,u_1}$ for $m \geq 1$, with $E_{(\nu_l)}...E_{(\nu_1)}T^{m-1}\not\Delta\lambda$, $E_{(\nu_l)}...E_{(\nu_1)}T^{m-1}\not\Delta\underline{\lambda}$ playing the roles of $\not d(E_{(\nu_{l-1})}...E_{(\nu_1)}\mathrm{tr}\tilde{\chi})$, $\not d(E_{(\nu_{l-1})}...E_{(\nu_1)}\mathrm{tr}\underline{\tilde{\chi}})$ respectively, and ${}^{(Y;m,\nu_1...\nu_l)}\underline{\mathcal{E}}^{\underline{u}_1}(\underline{u})$ playing the role of ${}^{(Y;0,\nu_1...\nu_l)}\underline{\mathcal{E}}^{\underline{u}_1}(\underline{u})$. (Compare 80 with 79.)

The new analytic method is designed to overcome the difficulties due to the appearance of the singular integrals. The starting point is the observation that, in view of the fact that these integrals are borderline from the point of view of scaling, they would become regular borderline integrals if the energies ${}^{(V;m,\nu_1....\nu_l)}\mathcal{E}^{\underline{u}}(u)$, ${}^{(V;m,\nu_1...\nu_l)}\underline{\mathcal{E}}^{u}(\underline{u})$ had a growth from the singularities at $\underline{u} = 0$ ($\underline{C}_0$) and $u = 0$ ($S_{0,0}$) like $\underline{u}^{2a}u^{2b}$ for some sufficiently large exponents a and b, and, moreover, the flux ${}^{(V;m,\nu_1...\nu_l)}\mathcal{F}'^{\tau}$ had a growth from the singularity at $\tau = 0$ ($S_{0,0}$) on $\mathcal{K}$ like τ^{2c}, where $c = a + b$.

This observation seems at first sight irrelevant since the required growth properties cannot hold for these quantities. However, given that the initial data of the problem are expressed as smooth functions of (u, ϑ), the derived data, that is the T derivatives on $\underline{C}_0$ of up to any desired order N of the unknowns $((x^\mu : \mu = 0, ..., n),\ \ b\ \ ,(\beta_\mu : \mu = 0, ..., n))$ in the characteristic and wave systems, are determined as smooth functions of (u, ϑ). Therefore we can define the Nth approximants $((x^\mu_N : \mu = 0, ..., n),\ \ b_N\ \ ,(\beta_{\mu,N} : \mu = 0, ..., n))$ as the corresponding Nth degree polynomials in $\tau = \underline{u}$ with coefficients which are known smooth functions of $(\sigma = u - \underline{u}, \vartheta)$. If the Nth approximants, so defined, are inserted into the equations of the characteristic and wave systems, these equations fail to be satisfied by errors which are known smooth functions of $(\underline{u}, u, \vartheta)$ and whose derivatives up to order n are bounded by a known constant times τ^{N-n+k} where k is a fixed integer depending on which equation we are considering.

Similarly, inserting the Nth approximants into the boundary conditions, these fail to be satisfied by errors which are known smooth functions of (τ, ϑ) and whose derivatives up to order n are likewise bounded by a known constant times τ^{N-n+k} where k is a fixed integer depending on which equation we are considering. Moreover, in connection with the equation $\square_{\tilde{g}} \beta_\mu = 0$ satisfied by an actual solution, the corresponding Nth approximant quantity $\square_{\tilde{g}_N} \beta_{\mu,N}$ is a known smooth function of $(\underline{u}, u, \vartheta)$ and whose derivatives up to order n are bounded by a known constant times τ^{N-n+k} where k is a fixed integer. For briefness, we shall confine the discussion to the case $d = 2$.

Note that in the above construction we have, in terms of $(\tau = \underline{u}, \sigma = u - \underline{u}, \vartheta)$ coordinates:

$$T_N = \frac{\partial}{\partial \tau} = T, \quad \Omega_N = \frac{\partial}{\partial \vartheta} = \Omega \tag{90}$$

independently of the approximation. On the other hand, L_N, $\underline{L}_N$, E_N depend on the approximation, the first two through b_N, and the last through $\not{g}_N$, where:

$$\not{g}_N = g_{\mu\nu,N}(\Omega x_N^\mu)(\Omega x_N^\nu) \tag{91}$$

We then define the difference quantities:

$$^{(m,l)}\breve{\beta}_\mu = E^l T^m \beta_\mu - E_N^l T^m \beta_{\mu,N} \tag{92}$$

$$^{(V;m,l)}\breve{\xi} = V^\mu d\ ^{(m,l)}\breve{\beta}_\mu \tag{93}$$

We also define ${}^{(V;m,l)}\breve{S}$ as in 40, 41 with ${}^{(V;m,l)}\breve{\xi}$ in the role of ${}^{(V)}\xi$. Defining then ${}^{(V;m,l)}\breve{P}$ in analogy with 49, we have:

$$\text{div}_{\tilde{g}}\, {}^{(V;m,l)}\breve{P} = {}^{(V;m,l)}\breve{Q} \tag{94}$$

where ${}^{(V;m,l)}\breve{Q}$ decomposes like 51 with the ${}^{(V;m,l)}\breve{Q}_i\ :\ i = 1, 2, 3$ given by formulas analogous to 52 - 54 with ${}^{(m,l)}\breve{\beta}_\mu$, ${}^{(V;m,l)}\breve{\xi}$ and ${}^{(V;m,l)}\breve{S}$ in the roles of β_μ, ${}^{(V)}\xi$ and ${}^{(V)}S$ respectively. Then a (m, l) *difference energy identity* follows, similar to 72, with the (m, l) difference quantities ${}^{(V;m,l)}\breve{\mathcal{E}}^{\underline{u}_1}(u_1)$, ${}^{(V;m,l)}\underline{\breve{\mathcal{E}}}^{u_1}(\underline{u}_1)$, ${}^{(V;m,l)}\breve{\mathcal{F}}'^{\underline{u}_1}$, ${}^{(V;m,l)}\breve{\mathcal{G}}^{\underline{u}_1,u_1}$ and ${}^{(V;m,l)}\breve{b}$ in the role of the corresponding original quantities in 72. Now, from the definition 92 and the preceding discussion it follows that for any solution of the problem the functions ${}^{(m,l)}\breve{\beta}_\mu$ vanish on $\underline{C}_0$ with all their T derivatives up to order n if we choose $N \geq n$.

We then have:

$$ {}^{(V;m,l)}\check{\underline{\mathcal{E}}}^{u_1}(0) = 0 \ \ : \ \text{for all } m = 0, ..., n \tag{95} $$

As a consequence, the (m,l) *difference energy identity* reads:

$$ \begin{aligned} &{}^{(V;m,l)}\check{\mathcal{E}}^{\underline{u}_1}(u_1) + {}^{(V;m,l)}\check{\underline{\mathcal{E}}}^{u_1}(\underline{u}_1) + {}^{(V;m,l)}\check{\mathcal{F}}'^{\underline{u}_1} \\ &= {}^{(V;m,l)}\check{\mathcal{G}}^{\underline{u}_1,u_1} + 2C' \int_{\mathcal{K}^{\underline{u}_1}} \Omega^{1/2}({}^{(V;m,l)}\check{b})^2 \end{aligned} \tag{96} $$

Expecting that the growth of ${}^{(V;m,l)}\check{\mathcal{E}}^{\underline{u}}(u)$, ${}^{(V;m,l)}\check{\underline{\mathcal{E}}}^{u}(\underline{u})$ is like $\underline{u}^{2a}u^{2b}$ and that the growth of ${}^{(V;m,l)}\check{\mathcal{F}}'^{\tau}$ is like τ^{2c}, we define the weighted quantities:

$$ {}^{(V;m,l)}\mathcal{B}(\underline{u}_1, u_1) = \sup_{(\underline{u},u)\in R_{\underline{u}_1,u_1}} \underline{u}^{-2a} u^{-2b} \; {}^{(V;m,l)}\breve{\mathcal{E}}^{\underline{u}}(u) \tag{97} $$

$$ {}^{(V;m,l)}\underline{\mathcal{B}}(\underline{u}_1, u_1) = \sup_{(\underline{u},u)\in R_{\underline{u}_1,u_1}} \underline{u}^{-2a} u^{-2b} \; {}^{(V;m,l)}\underline{\breve{\mathcal{E}}}^{u}(\underline{u}) \tag{98} $$

and:

$$ {}^{(V;m,l)}\mathcal{A}(\tau_1) = \sup_{\tau\in[0,\tau_1]} \tau^{-2(a+b)} \; {}^{(V;m,l)}\breve{\mathcal{F}}'^{\tau} \tag{99} $$

the exponents a, b being non-negative real numbers. Of course the above definitions do not make sense unless we already know that the quantities ${}^{(V;m,l)}\breve{\mathcal{E}}^{\underline{u}}(u)$, ${}^{(V;m,l)}\underline{\breve{\mathcal{E}}}^{u}(\underline{u})$, ${}^{(V;m,l)}\breve{\mathcal{F}}'^{\tau}$ have the appropriate growth properties. Making this assumption would introduce a vicious circle into the argument, so this is not what we do.

What we actually do is to first regularize the problem by giving initial data not on $\underline{C}_0$ but on $\underline{C}_{\tau_0}$ for $\tau_0 > 0$ but not exceeding a certain fixed positive number which is much smaller than δ. The initial data on $\underline{C}_{\tau_0}$ is modeled after the restriction to $\underline{C}_{\tau_0}$ of the Nth approximate solution, the difference being bounded by a fixed constant times τ_0^{N-1}. Similarly considering the mth derived data on $\underline{C}_{\tau_0}$, $m = 1, ..., n+1$ we show that the difference from the corresponding Nth approximants on $\underline{C}_{\tau_0}$ is bounded by a fixed constant times τ_0^{N-1-m}. The (m,l) difference energy identity now refers to the domain $\mathcal{R}_{\underline{u}_1,u_1,\tau_0}$ in $\mathcal{N}$ which corresponds to the domain:

$$R_{\underline{u}_1,u_1,\tau_0} = \{(\underline{u}, u) \ : \ u \in [\underline{u}, u_1], \ \underline{u} \in [\tau_0, \underline{u}_1]\} \tag{100}$$

in $\mathbb{R}^2$, so there is a 1st term on the right in this identity, which is:

$${}^{(V;m,l)}\underline{\check{\mathcal{E}}}^{u_1}(\tau_0) \leq C\tau_0^{2(N-1-m)} \tag{101}$$

Replacing the supremum over $R_{\underline{u}_1,u_1}$, the supremum over $[0,\tau_1]$, by the supremum over $\mathcal{R}_{\underline{u}_1,u_1,\tau_0}$, the supremum over $[\tau_0,\tau_1]$, respectively, in the definitions 97 - 99, and taking $N \geq m+1+c$, everything now makes sense. In fact, we take $N > m+\frac{5}{2}+c$, in which case the modifications in the resulting estimates for the quantities 97 - 99, tend to 0 as $\tau_0 \to 0$.

The argument relies on the derivation of energy estimates of the top order $m+l=n$ only. We derive estimates for the top order acoustical difference quantities corresponding to 86

$$
\begin{aligned}
{}^{(\nu_1,\ldots,\nu_{l-1})}\breve{\theta}_l &= {}^{(\nu_1\ldots\nu_{l-1})}\theta_l - {}^{(\nu_1\ldots\nu_{l-1})}\theta_{l,N} \\
{}^{(\nu_1,\ldots,\nu_{l-1})}\breve{\underline{\theta}}_l &= {}^{(\nu_1\ldots\nu_{l-1})}\underline{\theta}_l - {}^{(\nu_1\ldots\nu_{l-1})}\underline{\theta}_{l,N} \qquad (102)
\end{aligned}
$$

and to the quantities associated to $\not{\triangle}\lambda$, $\not{\triangle}\underline{\lambda}$, ignoring at first all but the top order terms.

We then derive the top order energy estimates, again ignoring all but the top order terms. The above estimates require taking the exponents a, b, c to be suitably large. This is in accordance with our preceding heuristic discussion. Moreover the estimates require that δ does not exceed a positive constant which is independent of m and n.

The preceding concern only the treatment of the principal terms, and these are estimated using only the fundamental bootstrap assumptions. The full treatment, which includes all the lower order terms, uses the complete set of bootstrap assumptions. Eventually, pointwise estimates are deduced and the bootstrap assumptions are recovered as strict inequalities. This recovery however requires a further smallness condition on δ.

There is a n_0 depending only on d, such that this further condition is associated to $n = n_0$. Given then any $n > n_0$, the nonlinear argument having closed at order n_0, the bootstrap assumptions are no longer needed, therefore no new smallness conditions on δ are required to proceed inductively to orders $n_0 + 1, ..., n$.

As already discussed, we first regularize the problem by giving the initial data on $\underline{C}_{\tau_0}$. We establish the existence of a solution to this regularized problem defined on the whole of $\mathcal{R}_{\delta,\delta,\tau_0}$ by applying a continuity argument. This argument relies at a local level on the work of Majda and Thomann (Comm. in Partial Differential Equations **12**(7), 777-828 (1987)) on the restricted local shock continuation problem, namely the problem of continuing locally in time a solution displaying a shock discontinuity initially.

After obtaining a solution on $\mathcal{R}_{\delta,\delta,\tau_0}$ satisfying the appropriate estimates we take τ_0 to be any member of a sequence $(\tau_{0,m} \ : \ m = M, M+1, M+2, \ . \ . \ .)$ converging to 0, and pass to the limit in a subsequence to obtain the solution to our problem.