Liouville type theorems in the Stationary Navier-Stokes and the related equations

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1. Introduction

Liouville's theorem for a harmonic function

Theorem 1 (Liouville, 1879)

If u is a smooth solution to the Laplace equation:

$$\Delta u = 0 \quad \text{on} \quad \mathbb{R}^n,$$

satisfying the *'boundary condition'* at infinity:

$$u(x) \to 0 \quad \text{as} \quad |x| \to +\infty,$$

then $u \equiv 0$.

- Similar Liouville type theorems also hold for many other "elliptic type" equations.
- More general version: "A harmonic function in $\mathbb{R}^n$ with sublinear growth growth at ininity is a constant."

- $\boxed{\text{Proof for } n = 1}$
  - If $u'' = 0$, then $u(x) = ax + b$.
  - The decay condition implies $a = b = 0$. ■

- $\boxed{\text{Proof for } n \geq 2 \text{ by the maximum principle(MP)}}$
  - If $u(x_0) > 0$ for some $x_0 \in \mathbb{R}^n$, then for sufficiently large $R > 0$, u has a (positive) interior maximum in $B(0, R)$, which is prohibited by MP.
  - Similarly, there exists no $x_1 \in \mathbb{R}^n$ such that $u(x_1) < 0$. Hence $u \equiv 0$ in $\mathbb{R}^n$. ■

- Similar Liouville type theorems also hold for many other PDEs.

  - $\boxed{\text{Example}}$ viscous vector Burgers' equations for $u(x) = (u_1, \cdots, u_n), x \in \mathbb{R}^n$.

$$u \cdot \nabla u = \Delta u, \quad \text{with} \quad u(x) \to 0 \quad \text{as} \quad |x| \to +\infty.$$

  - The maximum principle applied to each component u_j, $j = 1, \cdots, n$, implies $u \equiv 0$.

- Let $u = (u_1, u_2, u_3) = u(x)$ be a vector field, and $p = p(x)$ is a scalar function.

- Suppose the pair (u, p) be a smooth solution to the steady Navier-Stokes equations in $\mathbb{R}^3$.

$$(NS)\begin{cases} -\Delta u + (u\cdot\nabla)u = -\nabla p, \\ \nabla \cdot u = 0, \\ (BC): \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases}$$

$\boxed{\text{Question: } u = 0 \text{ is the only solution?}}$

- The question is an outstanding open problem.

- If we replace the domain $\mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$, then the answer is negative with a nontrivial solution $u(x) = x/|x|^3$.

Reduced Liouville type problem

- Since the above question appears to be too difficult, we usually consider the following reduced problem.

QUESTION: If we impose additionally $\int_{\mathbb{R}^3} |\nabla u|^2 dx < \infty$, then $u = 0$?

- The question was written explicitly in the book by Galdi(1994)
- The complete answer is still wide open.
- Many authors assume various (extra) sufficient conditions to prove the triviality of solution: "triviality criterion" $\Leftrightarrow$ "regularity criterion" for the time dependent (NS).

Galdi's $L^{9/2}$ condition

The first "triviality criterion" result appeared in Galdi's book(1994):

Theorem 2 (Galdi)

If $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$ is a smooth solution to (NS), then $u = 0$.

Proof :

- Let $1 < r < +\infty$ be arbitrarily chosen.
- Let $\eta \in C^\infty(\mathbb{R}^3)$ be a cut off function, which is radially non-increasing with $\eta = 1$ on $B(r)$ and $\eta = 0$ on $\mathbb{R}^3 \setminus B(2r)$ satisfying $|D^k \eta| \le cr^{-k}$.
- We multiply (NS) by $u\eta^2$ integrate over $B(2r)$ and apply integration by parts.

This yields

$$
\begin{aligned}
\int_{B(r)} |\nabla u|^2 dx \leq \frac{1}{2} \int_{B(2r)} |u|^2 \Delta \eta^2 dx + \\
&+ \frac{1}{2} \int_{B(2r)\setminus B(r)} |u|^2 u \cdot \nabla \eta^2 dx + \int_{B(2r)\setminus B(r)} pu \cdot \nabla \eta^2 dx \\
&\leq c \boxed{r^{-2} \int_{B(2r)\setminus B(r)} |u|^2 dx} + c \boxed{r^{-1} \int_{B(2r)} |u|^3 dx} \\
&+ c \boxed{r^{-1} \int_{B(2r)} |p||u| dx} \\
&= I + II + III.
\end{aligned}
$$

By Hölder's inequality we have, as $r \to +\infty$,

$$I \leq cr^{-\frac{1}{3}} \left(\int_{\mathbb{R}^3} |u|^{\frac{9}{2}} dx \right)^{\frac{4}{9}} \to 0,$$

$$II \leq c \left(\int_{B(2r)\setminus B(r)} |u|^{\frac{9}{2}} dx \right)^{\frac{2}{3}} \to 0,$$

$$\begin{aligned} III &\leq c \left(\int_{\mathbb{R}^3} |p|^{\frac{9}{4}} dx \right)^{\frac{4}{9}} \left(\int_{B(2r)\setminus B(r)} |u|^{\frac{9}{2}} dx \right)^{\frac{2}{9}} \\ &\leq \left(\int_{\mathbb{R}^3} |u|^{\frac{9}{2}} dx \right)^{\frac{4}{9}} \left(\int_{B(2r)\setminus B(r)} |u|^{\frac{9}{2}} dx \right)^{\frac{2}{9}} \\ &\to 0, \end{aligned}$$

where in III we used the Calderon-Zygmund inequality: $\|p\|_{L^q} \leq c\|u\|_{L^{2q}}^2, 1 < q < +\infty$.) $\square$

Remarks on the case with $\mathbb{R}^n$, $n \neq 3$

- Note that to control the nonlinear term(and the pressure term) we need to have estimate $|v|^2 v \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$, ($pv \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$,) which implies $v \in L^{\frac{3n}{n-1}}(\mathbb{R}^n)$.
- This can be obtained for $n \geq 4$ if $v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ (note $\frac{2n}{n-2} \leq \frac{3n}{n-1}$) which is implies by the condition $\int_{\mathbb{R}^n} |\nabla v|^2 dx < +\infty$. and the Sobolev inequality.
- In the case we recall the vorticity equation,

$$v \cdot \nabla \omega - \Delta \omega = 0, \quad \omega = \partial_1 v_2 - \partial_2 v_1.$$

- The maximum principle implies $\omega = 0$ if it decays to 0 at infinity.
- Hence, the Liouville problem is open only for $n = 3$.

Results around Galdi's one

- Although Galdi's result has a very simple proof, it looks difficult to improve it substantially.
- In order to consider space near $L^{9/2}$ we recall Lorentz space $L^{p,\ell}(\Omega)$:

$$\|f\|_{L^{p,\ell}(\Omega)} := \begin{cases} \left(\int_0^\infty \lambda^{\ell-1} m(\lambda)^{\frac{\ell}{p}} d\lambda \right)^{\frac{1}{\ell}} & \text{if} \quad \ell < +\infty \\ \sup_{\lambda>0} \lambda m(\lambda)^{\frac{1}{p}} & \text{if} \quad \ell = +\infty, \end{cases}$$

where $m(\lambda) = |\{x \in \Omega \,|\, |f(x)| > \lambda\}|$ is the distribution function of f.

Theorem 3 (Kozono, Terasawa, Wakasugi, '16)

Let u be a smooth solution to (NS) with bounded pressure. Then,

$$\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{3}} \leq C\|u\|_{L^{\frac{9}{2},\infty}}.$$

Therefore, if there exists small constant $\delta > 0$ such that

$$\|u\|_{L^{\frac{9}{2},\infty}} \leq \delta \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{3}}$$

Then, $u = 0$.

- Below we denote $M_{\gamma,q,\ell}(R) := R^{\gamma-\frac{3}{q}}\|u\|_{L^{q,\ell}(B_R\setminus B_{\frac{R}{2}})}$.

Theorem 4 (Seregin-Wang, '19)

Let u be a smooth solution to (NS).

(i) For $q > 3, 3 \leq \ell \leq \infty$, assume $\liminf_{R\to\infty} M_{\frac{2}{3},q,\ell}(R) < \infty$. Then,

$$\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{3}} \leq c \liminf_{R\to\infty} M_{\frac{2}{3},q,\ell}(R).$$

(ii) Therefore if there exists a 'small constant' $\delta > 0$ such that

$$\liminf_{R\to\infty} M_{\frac{2}{3},q,\ell}(R) \leq \delta \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{3}}$$

Then $u = 0$.

Remarks

- Notice that
$$M_{\frac{2}{3},\frac{9}{2},\frac{9}{2}}(R) = \|u\|_{L^{\frac{9}{2}}(B_R \setminus B_{\frac{R}{2}})}$$
and
$$M_{\frac{2}{3},\frac{9}{2},\infty}(R) = \|u\|_{L^{\frac{9}{2},\infty}}$$
- If $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$, then
$$\lim_{R\to\infty}\inf M_{\frac{2}{3},\frac{9}{2},\frac{9}{2}}(R) = \lim_{R\to\infty}\inf \|u\|_{L^{\frac{9}{2}}(B_R \setminus B_{\frac{R}{2}})} = 0.$$
Therefore (i) implies $\int_{\mathbb{R}^3} |\nabla u|^2 dx = 0$, and thus $u = 0$. Hence, the above theorem (i) implies Galdi's result.
- On the other hand, for the choice of $q = 9/2, \ell = +\infty$, since $M_{\frac{2}{3},\frac{9}{2},\infty} = \|u\|_{L^{\frac{9}{2},\infty}}$, we find Theorem 7 reduces to Theorem 6.

Logarithmic improvement of $L^{9/2}$ result

- The following is another result improving Galdi's result logarithmically.

Theorem 5 (DC, Wolf, '16)

Let u be a smooth solution to (NS) such that

$$\int_{\mathbb{R}^3} |u|^{\frac{9}{2}} \left\{ \log\left(2 + \frac{1}{|u|}\right) \right\}^{-1} dx < +\infty.$$

Then $u \equiv 0$.

- The proof uses Caccioppoli's inequality obtained directly from the stationary NS.

Theorem 6 (DC, '14)

Let u is a smooth solution to (NS) such that $\int_{\mathbb{R}^3} |\Delta u|^{\frac{6}{5}} dx < +\infty$, and $u \to 0$ at infinity, then $u \equiv 0$.

- Observing the Sobolev inequality in $\mathbb{R}^3$,

$$\|\nabla u\|_{L^2} \leq c\|\Delta u\|_{L^{\frac{6}{5}}},$$

  we see that the condition above has the same scaling as the Dirichlet integral condition.

- Since $\|u\|_{L^6} \leq c\|\Delta u\|_{L^{\frac{6}{5}}}$, we find that $u \cdot \Delta u \in L^1(\mathbb{R}^3)$.

Theorem 7 (DC, '20)

Let u is a smooth solution to (NS) such that $\int_{\mathbb{R}^3} |\nabla u|^2 dx < +\infty$ and $u \to 0$ at infinity. Suppose furthermore if at least one of the following belongs to $L^1(\mathbb{R}^3)$, then $u = 0$.

$$\boxed{u \cdot \Delta u, \quad \Delta |u|^2, \quad \Delta Q, \quad u \cdot \nabla Q}$$

where $Q = \frac{1}{2}|u|^2 + p$.

Outline of the proof: $\boxed{STEP\,1}$ We note from the vector calculus

$$\Delta \frac{|u|^2}{2} - u \cdot \Delta u = |\nabla u|^2 \in L^1(\mathbb{R}^3),$$

$$\Delta Q - \Delta \frac{|u|^2}{2} = \Delta p = -\sum_{j,k} \partial_j u_k \partial_k u_j \in L^1(\mathbb{R}^3).$$

$$\Delta Q - u \cdot \nabla Q = |\omega|^2 \in L^1(\mathbb{R}^3),$$

- Therefore, if one of the four belongs to $L^1(\mathbb{R}^3)$, then all of the others belong to $L^1(\mathbb{R}^3)$.

$\boxed{STEP\,2}$ (Show $\int_{\mathbb{R}^3} \Delta Q dx = 0$) Using the previous cut-off $\zeta = \zeta_r$, we find by DCT that

$$\int_{\mathbb{R}^3} \Delta Q dx = \lim_{r\to+\infty} \int_{\mathbb{R}^3} \Delta Q \zeta_r dx = \lim_{r\to+\infty} \int_{\mathbb{R}^3} Q\Delta\zeta_r dx = 0,$$

where we used the fact

$$\left| \int_{\mathbb{R}^3} Q\Delta\zeta_r dx \right| \leq \left(\int_{B(2r)\setminus B(r)} |Q|^3 dx \right)^{\frac{1}{3}} \to 0 \quad \text{as} \quad r \to +\infty,$$

since $\|Q\|_{L^3} \leq c\|u\|_{L^6}^2 \leq c\|\nabla u\|_{L^2}^2$, by the Calderon-Zygmund inequality.

$\boxed{STEP\ 3}$ (Show $\int_{\mathbb{R}^3} u \cdot \nabla Q dx = 0$) Applying the maximum principle to

$$\Delta Q - u \cdot \nabla Q = |\omega|^2 \geq 0,$$

we have $Q < 0$ on $\mathbb{R}^3$, and no local maximum for Q in $\mathbb{R}^3$, which implies $\{Q < \lambda\} \nearrow \mathbb{R}^3$ as $\lambda \nearrow 0$. Hence, by DCT

$$\int_{\mathbb{R}^3} u \cdot \nabla Q dx = \lim_{\lambda \nearrow 0} \int_{\{Q<\lambda\}} u \cdot \nabla Q dx = 0,$$

where we used the fact that for each $\lambda < 0$,

$$\begin{aligned} \int_{\{Q<\lambda\}} u \cdot \nabla Q dx &= \int_{\{Q=\lambda\}} u \cdot \nu Q dS \\ &= \lambda \int_{\{Q=\lambda\}} u \cdot \nu dS = \lambda \int_{\{Q<\lambda\}} \nabla \cdot u dx = 0. \end{aligned}$$

- We have shown

$$0 = \underbrace{\int_{\mathbb{R}^3} \Delta Q dx}_{\text{step2}} - \underbrace{\int_{\mathbb{R}^3} u \cdot \nabla Q dx}_{\text{step3}} = \int_{\mathbb{R}^3} |\omega|^2 dx. \quad \blacksquare$$

On the head pressure conditions

Theorem 8 (DC, Wolf, '16)

Let (v, p) be a smooth solution to (NS) and $Q = \frac{1}{2}|u|^2 + p$. We set $\|Q\|_{L^\infty} = M$. Then, we have the following inequality.

$$\int_{\mathbb{R}^3} \frac{|\nabla Q|^2}{|Q|} \left(\log \frac{eM}{|Q|} \right)^{-\alpha-1} dx \leq \frac{1}{\alpha} \int_{R^3} |\omega|^2 dx \quad \forall \alpha > 0.$$

If the the following holds,

$$\int_{\mathbb{R}^3} \frac{|\nabla Q|^2}{|Q|} \left(\log \frac{eM}{|Q|} \right)^{-\alpha-1} dx = o\left(\frac{1}{\alpha} \right) \quad \text{as} \quad \alpha \to 0,$$

then $u \equiv 0$.

- We introduce the following notations for iterated exponential/logarithmic functions.
- For $k \in \mathbb{N}$ let us define

$$\exp_k(x) = \overbrace{\exp(\exp(\cdots(\exp x)\cdots)}^{k-\text{times}},$$

$$\log_k(x) = \overbrace{\log(\log(\cdots(\log x)\cdots)}^{k-\text{times}}$$

and set $\exp_k(1) := e_k$, $e_0(x) = \log_0(x) := 1$ for all $x \in \mathbb{R}$.

Refinement/generalizations of Theorem 10

- We can prove for the asymptotic limit for the integrals inside the level sets defined by Q.

Theorem 9 (DC,'19)

Let (u,p) be a smooth solution of (NS), and $Q = \frac{1}{2}|u|^2 + p$ such that $\int_{\mathbb{R}^3} |\nabla u|^2 dx < +\infty$. We set $\inf_{x\in\mathbb{R}^3} Q(x) = m$. Then, for all $k \in \mathbb{N}\cup\{0\}$ we have

$$\lim_{\lambda\searrow 0} (\log_{k+1}(1/\lambda))^{-1} \int_{\{|Q|>\lambda\}} \frac{|\nabla Q|^2}{|Q| \prod_{j=0}^{k} \log_j\left(\frac{e_k|m|}{|Q|}\right)} dx = \int_{\mathbb{R}^3} |\omega|^2 dx$$

Therefore, if there exists $k \in \mathbb{N}\cup\{0\}$ such that

$$\int_{\{|Q|>\lambda\}} \frac{|\nabla Q|^2}{|Q| \prod_{j=0}^{k} \log_j\left(\frac{e_k|m|}{|Q|}\right)} dx = o\left(\log_{k+1}(1/\lambda)\right)$$

as $\lambda \searrow 0$, then $\int_{\mathbb{R}^3} |\omega|^2 dx = 0$, and thus $u = 0$ on $\mathbb{R}^3$.

For simplicity we consider the case $k = 0$.

Corollary 1

Under the assumptions of Theorem 8 it holds

$$\lim_{\lambda \searrow 0} (\log(1/\lambda))^{-1} \int_{\{|Q|>\lambda\}} \frac{|\nabla Q|^2}{|Q|} dx = \int_{\mathbb{R}^3} |\omega|^2 dx,$$

and therefore, if

$$\int_{\{|Q|>\lambda\}} \frac{|\nabla Q|^2}{|Q|} dx = o\left(\log(1/\lambda)\right),$$

as $\lambda \to 0$, then $u = 0$ on $\mathbb{R}^3$.

- The corollary implies that

$$\boxed{\text{If} \quad \int_{\mathbb{R}^3} \frac{|\nabla Q|^2}{|Q|} dx < +\infty, \quad \text{then} \quad u \equiv 0}$$

- Note that the integral $\int_{\mathbb{R}^3} \frac{|\nabla Q|^2}{|Q|} dx$ has the same scaling as the Dirichlet integral.

Oscillation growth conditions for the potential functions

- We say a matrix valued function $\boldsymbol{V}$ is potential function for vector field u if $\nabla \cdot \boldsymbol{V} = u$.

- The usual stream function ψ such that $\nabla \times \psi = u$ is the case where $\boldsymbol{V} = (V_{ij})$, $V_{ij} = -\epsilon_{ijk}\psi_k$ with ϵ_{ijk} being the standard skew-symmetric tensor.

- Note that for $\nabla \cdot \boldsymbol{V} = u$ we have

$$\boldsymbol{V} \in BMO(\mathbb{R}^3) \Leftrightarrow u \in BMO^{-1}(\mathbb{R}^3)$$

- Seregin firstly used sufficient conditions on potential function to obtain Liouville type result:

Theorem 10 (Seregin, '16)

If $u \in L^6(\mathbb{R}^3)$ is a smooth solution to (NS), and there exists a potential function $\boldsymbol{V} \in BMO(\mathbb{R}^3)$, then $u = 0$.

- In his later work Seregin removed L^6 condition, and substantially improved the above result.

Theorem 11 (Seregin, Analysis & Algebra, '18)

Let $3 < s < \infty$. If u is a smooth solution to (NS) such that there exists a vector field $\boldsymbol{V}$ with $u = \nabla \times \boldsymbol{V}$ satisfying

$$\left(\fint_{B(r)} |\boldsymbol{V} - \boldsymbol{V}_{B(r)}|^s dx \right)^{\frac{1}{s}} \leq C r^{\alpha(s)} \quad \forall 1 < r < +\infty,$$

where $\alpha(s) < \frac{s-3}{6(s-1)}$, then $u = 0$.

- In the above we used the notation

$$f_\Omega := \fint_\Omega f dx := \frac{1}{|\Omega|} \int_\Omega f dx.$$

- The above theorem, on the other hand, has been improved later as follows:

Theorem 12 (DC-Wolf, Cal. Var. PDE, '19)

Let $3 < s < \infty$. Let u be a smooth solution to (NS) and there exists a potential function $\boldsymbol{V}$ with such that

$$\left(\fint_{B(r)} |\boldsymbol{V} - \boldsymbol{V}_{B(r)}|^s dx \right)^{\frac{1}{s}} \leq C r^{\beta(s)} \quad \forall 1 < r < +\infty,$$

where $\beta(s) := \min\{\frac{s-3}{3s}, \frac{1}{6}\}$. Then, $u \equiv 0$.

- Since

$$\alpha(s) < \frac{s-3}{6(s-1)} < \min\left\{ \frac{s-3}{3s}, \frac{1}{6} \right\} = \beta(s)$$

for all $3 < s < +\infty$, our growth condition of oscillation is more relaxed than that of Theorem 14.

Extension to MHD system

- We first consider the stationary magnetohydrodynamics system(MHD) in $\mathbb{R}^3$.

$$(MHD)\begin{cases} -\Delta u + (u\cdot\nabla)u = -\nabla p + (B\cdot\nabla)B, \\ -\Delta B + (u\cdot\nabla)B - (B\cdot\nabla)u = 0, \\ \quad \nabla\cdot u = \nabla\cdot B = 0. \end{cases}$$

- Schulze['18] generalized Seregin's earlier result on the Navier-Stokes system to prove that $u, B \in BMO^{-1}(\mathbb{R}^3)\cap L^6(\mathbb{R}^3)$ implies $u = B = 0$.

- Z. Li and X. Pan['19] showed that in the axisymmetric MHD system in $\mathbb{R}^2\times\mathbb{T}$ with zero swirl and vanishing boundary condition with $\int_{\mathbb{R}^3}(|\nabla u|^2 + |\nabla B|^2)dx < +\infty$ implies $u = B = 0$.

- The following is our improvement of Schulze's result

Theorem 13 (DC, Wolf, '19)

Let (u, B) be a smooth solution to (MHD). Suppose there exist potential function $\mathbf{\Phi}, \mathbf{\Psi}$ for u, B respectively such that $\nabla \cdot \mathbf{\Phi} = u$, $\nabla \cdot \mathbf{\Psi} = B$, and

$$\fint_{B(r)} |\mathbf{\Phi} - \mathbf{\Phi}_{B(r)}|^6 dx + \fint_{B(r)} |\mathbf{\Psi} - \mathbf{\Psi}_{B(r)}|^6 dx \leq Cr \quad \forall 1 < r < +\infty.$$

Then, $u = B = 0$.

- The following is an immediate consequence of the above theorem.

Corollary 2

Let (u, B) be a smooth solution of (MHD) such that $u, B \in BMO^{-1}(\mathbb{R}^3)$(No need L^6 condition), then $u = B = 0$.

Extension to the Hall-magnetohydrodynamic equations

- We consider the Hall-MHD system in $\mathbb{R}^3$.

$$(HMHD)\begin{cases} -\Delta u + (u\cdot\nabla)u = -\nabla p + (B\cdot\nabla)B, \\ -\Delta B + (u\cdot\nabla)B - (B\cdot\nabla)u = \nabla\times((\nabla\times B)\times B), \\ \quad \nabla\cdot u = \nabla\cdot B = 0. \end{cases}$$

- This equations govern the the dynamics plasma flows of strong shear magnetic fields as in the solar flares, and there are many studies in the astrophysics community.

- The following is our Liouville type theorem for (HMHD).

Theorem 14 (DC, Wolf, JDE '19)

Let (u, B) be a smooth solution of (HMHD). Let us assume

$$r^{-8} \int\limits_{B(r)} |B - B_{B(r)}|^6 dx \to 0 \quad \text{as} \quad r \to +\infty,$$

and there exist $\mathbf{\Phi}, \mathbf{\Psi} \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3\times 3})$ such that

$$\fint_{B(r)} |\mathbf{\Phi} - \mathbf{\Phi}_{B(r)}|^6 dx + \fint_{B(r)} |\mathbf{\Psi} - \mathbf{\Psi}_{B(r)}|^6 dx \le Cr \quad \forall 1 < r < +\infty.$$

Then, $u = B = 0$.

Extension to non-Newtonian fluid equations

- We consider the following generalized version of the stationary Navier-Stokes equations in $\mathbb{R}^3$

$$(GNS)\begin{cases} -\boldsymbol{A}_p(u) + (u\cdot\nabla)u = -\nabla\pi \quad \text{in} \quad \mathbb{R}^3, \\ \nabla\cdot u = 0, \end{cases}$$

where $\boldsymbol{A}_p(u) = \nabla\cdot(|\boldsymbol{D}(u)|^{p-2}\boldsymbol{D}(u)), \quad 1 < p < +\infty$
with $\boldsymbol{D}(u) = \boldsymbol{D} = \frac{1}{2}(\nabla u + (\nabla u)^\top)$.

- The system is a power law fluid model of *non-Newtonian fluid*.

- For $p = 2$ it reduces to the usual stationary Navier-Stokes equations. For $1 < p < 2$ the fluid is called shear thinning, while in case $2 < p < +\infty$ the fluid is called shear thickening

- The following is our Liouville type theorem for the above non-Newtonian system.

Theorem 15 (DC, J. Wolf, J. Nonlinear Sci.'20)

(i) Let $\frac{3}{2} \leq p \leq \frac{9}{5}$: We suppose (u, π) is a smooth solution of (GNS). If

$$\int_{\mathbb{R}^3} |\nabla u|^p dx < +\infty, \quad \liminf_{R\to\infty} |u_{B(R)}| = 0$$

then $u \equiv 0$.

(ii) Let $\frac{9}{5} < p < 3$: Assume there exists a smooth $\boldsymbol{V}$ such that $\nabla \cdot \boldsymbol{V} = u$, and

$$\int_{B(r)} |\boldsymbol{V} - \boldsymbol{V}_{B(r)}|^{\frac{3p}{2p-3}} dx \leq C r^{\frac{9-4p}{2p-3}} \quad \forall 1 < r < +\infty.$$

Then, $u \equiv 0$.

- If we choose $p = 2$ in (ii) above, then we are reduced to the previous result for the usual Navier-Stokes equations(the case $s = 6$).

Corollary 3

Let u be a smooth solution of the stationary Navier-Stokes equations on $\mathbb{R}^3$. Suppose there exists $\boldsymbol{V} \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3\times 3})$ such that $\nabla \cdot \boldsymbol{V} = u$, and

$$\fint_{B(r)} |\boldsymbol{V} - \boldsymbol{V}_{B(r)}|^6 dx \le Cr \quad \forall 1 < r < +\infty.$$

Then, $u \equiv 0$.

p/Q condition

One can also impose the relative decay rate condition between the pressure and the head pressure to obtain a Liouville type theorem.

Theorem 16 (DC, JMFM '21)

Let (u, p) be a smooth decaying solution to (NS) having finite Dirichlet integral, and $Q = \frac{1}{2}|u|^2 + p$ be its head pressure. If either

$$\sup_{x\in\mathbb{R}^3} \frac{|u(x)|^2}{|Q(x)|} < +\infty,$$

or

$$\sup_{x\in\mathbb{R}^3} \frac{p(x)}{Q(x)} < +\infty.$$

Then, $u \equiv 0$.

- We observe that the ratios in the above conditions are scaling invariant.

Anisotropic conditions

- Below we use the notation.

$$\tilde{x}_1 := (x_2, x_3), \quad \tilde{x}_2 := (x_3, x_1), \quad \tilde{x}_3 := (x_1, x_2)$$

and for a domain $\Omega \subset \mathbb{R}^2$ we denote

$$\int_\Omega f\, d\tilde{x}_1 = \int_\Omega f\, dx_2 dx_3, \int_\Omega f\, d\tilde{x}_2 = \int_\Omega f\, dx_3 dx_1,$$

and

$$\int_\Omega f\, d\tilde{x}_3 = \int_\Omega f\, dx_1 dx_2.$$

- Given $0 < r, s \leq +\infty$, $i \in \{1, 2, 3\}$, we write $f \in L^r_{x_i} L^s_{\tilde{x}_i}(\mathbb{R} \times \mathbb{R}^2)$ if

$$\|f\|_{L^r_{x_i} L^s_{\tilde{x}_i}} := \left\{ \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |f|^s d\tilde{x}_i \right)^{\frac{r}{s}} dx_i \right\}^{\frac{1}{r}} < +\infty$$

for $0 < r, s < +\infty$ with obvious extensions to the case $r = +\infty$ or $s = +\infty$.

Then the following result holds.

Theorem 17 (DC, Appl. Math. Lett.'23)

Let v be a smooth solution to the stationary Navier-Stokes equations. Suppose

$$v \in L^6(\mathbb{R}^3) \cap L^q(\mathbb{R}^3),$$

and

$$v_i \in L_{x_i}^{\frac{q}{q-2}} L_{\tilde{x}_i}^{s}(\mathbb{R} \times \mathbb{R}^2), \quad \forall i = 1, 2, 3$$

with q, s satisfying

$$\frac{2}{q} + \frac{1}{s} \geq \frac{1}{2}, \quad \text{where} \quad 1 \leq s \leq \infty, \quad 2 < q < \infty.$$

Then, $v = 0$.

- In case is for $s = q = 6$ the above condition reduces to

$$v_i \in (L^{\frac{3}{2}}_{x_i} \cap L^6_{x_i}) L^6_{\tilde{x}_i}(\mathbb{R} \times \mathbb{R}^2) \quad \forall i = 1, 2, 3,$$

- This means that a mild decay in the planar direction $(\tilde{x}_i)$ combined with the faster decay in the direction orthogonal to the plane (x_i), implies the triviality of the solution.

Outline of the Proof

- Below we use the index sets

$$\mathcal{J}_1 = \{2,3\}, \quad \mathcal{J}_2 = \{1,3\}, \quad \mathcal{J}_3 = \{1,2\},$$

- Consider a smooth non-increasing real valued function $\psi : [0,+\infty) \to [0,1]$ defined by

$$\psi(s) = \begin{cases} 1, & \text{if} \quad 0 \leq s \leq 1, \\ 0, & \text{if} \quad s \geq 4. \end{cases}$$

- Given $R > 0$, we introduce cut-off functions

$$\varphi_R(x) = \prod_{j=1}^{3} \psi\left(\frac{x_j^2}{R^2}\right) \quad \text{and} \quad \tilde{\varphi}_{i,R}(x) = \prod_{j \in \mathcal{J}_i} \psi\left(\frac{x_j^2}{R^2}\right),$$

and the "slab domain"

$$D_i := \left\{\tilde{x}_i \in \mathbb{R}^2 \,|\, |x_j| < 2R, \quad \forall j \in \mathcal{J}_i\right\}, \quad i \in \{1,2,3\}.$$

- We multiply (NS) by $v\varphi_R$, and integrate it over $\mathbb{R}^3$. Then, after integration by part, we obtain

$$\begin{aligned}\int_{\{|x|<R\}} |\nabla v|^2 dx &\leq \int_{\mathbb{R}^3} \varphi_R(x)|\nabla v|^2 dx \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |v|^2 \Delta\varphi_R dx + \int_{\mathbb{R}^3} Qv \cdot \nabla\varphi_R dx \\ &:= I_1 + I_2, \end{aligned} \tag{1}$$

where $Q = \frac{1}{2}|v|^2 + p$ is the head pressure.

We estimate I_1 and I_2 as follows.

$$
\begin{aligned}
I_1 &= \frac{1}{2}\sum_{i=1}^{3}\int_{\{R\le|x_i|\le 2R\}}\int_{D_i}|v|^2\tilde{\varphi}_{i,R}\left\{\frac{2}{R^2}\psi'\left(\frac{x_i^2}{R^2}\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \frac{4x_i^2}{R^4}\psi''\left(\frac{x_i^2}{R^2}\right)\right\}d\tilde{x}_i dx_i \\
&\le \frac{C}{R^2}\sum_{i=1}^{3}\left(\int_{\{R\le|x_i|\le 2R\}}\int_{D_i}|v|^6 dx\right)^{\frac{1}{3}}\left(\int_{\{R\le|x_i|\le 2R\}}\int_{D_i}1\, d\tilde{x}_i dx_i\right)^{\frac{2}{3}} \\
&\le C\sum_{i=1}^{3}\left(\int_{\{R\le|x_i|\le 2R\}}\int_{\mathbb{R}^2}|v|^6 dx\right)^{\frac{1}{3}} \to 0
\end{aligned}
$$

as $R \to +\infty$.

- We use the anisotropic condition for the estimate of I_2.

$$I_2 \le \sum_{i=1}^{3} \frac{2}{R^2} \left| \int_{\mathbb{R}^3} x_i \tilde{\varphi}_{i,R} Q v_i \cdot \psi' \left(\frac{x_i^2}{R^2} \right) dx \right|$$

$$\le \text{(Hölder's inequalities, Calderon-Zygmund inequality)}$$

$$\le C R^{\frac{2(qs-q-2s)}{qs}-1} \left(\int_{\mathbb{R}^3} |v|^q dx \right)^{\frac{2}{q}} \times$$

$$\times \sum_{i=1}^{3} \left\{ \int_{\{R<|x_i|<2R\}} \left(\int_{\mathbb{R}^2} |v_i|^s d\tilde{x}_i \right)^{\frac{q}{s(q-2)}} dx_i \right\}^{\frac{q-2}{q}} .$$

- Our assumption of the theorem implies that

$$\frac{2(qs-q-2s)}{qs} - 1 \le 0.$$

- Therefore, $I_2 \to 0$ as $R \to +\infty$. ■

Extension to the n-dimensional MHD

We consider again the stationary MHD equations in $\mathbb{R}^n$,

$$\begin{aligned} -\Delta v + v \cdot \nabla v + \nabla p &= B \cdot \nabla B, \\ -\Delta B + v \cdot \nabla B - B \cdot \nabla v &= 0, \\ \nabla \cdot v = \nabla \cdot B &= 0, \end{aligned}$$

equipped with the uniform decay condition at spatial infinity,

$$|v(x)| + |B(x)| \to 0 \quad \text{as} \quad |x| \to +\infty.$$

Then we have the following extension of the previous result on the Navier-Stokes equations.

Theorem 18 (DC, '23)

Let v be a smooth solution to the n-D MHD system. Suppose

$$|v| + |B| \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n),$$

and

$$v_i \in L_{x_i}^{\frac{q}{q-2}} L_{\tilde{x}_i}^s(\mathbb{R} \times \mathbb{R}^{n-1}), \quad \forall i = 1, \cdots, n$$

with q, s satisfying

$$\frac{2}{q} + \frac{1}{s} \geq \frac{n-2}{n-1}, \quad \text{where} \quad 1 \leq s \leq \infty, \quad 2 < q < \infty.$$

Then, $v = 0$.

Thanks for your attention!