Bounded Morse index solutions of Allen-Cahn equation on Riemann surfaces

Juncheng Wei

University of British Columbia

Joint work with Yong Liu, Frank Pacard

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In celebration of 60th birthday of CUHK

Aim of My Talk

Minimal submanifolds in Riemann manifold

$$\Sigma^k \subset M^n$$

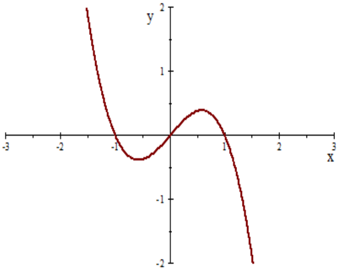
Simplest case:

$$k = 1, n = 2$$

Σ^k: minimal sub-manifolds arising from Allen-Cahn

1. Introduction: The Allen-Cahn equation

$$-\Delta u = u - u^3 \text{ in } \mathbb{R}^N, \quad |u| < 1.$$



Double-well Potential

- Energy functional

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- Function $W(u) = \frac{1}{4}(1 - u^2)^2$ has two minima ($u = \pm 1$) of equal depth ($J(\pm 1) = 0$) (double well potential).
- Most of the results are true for any double-well $W(u)$; but some double-well is better than others.

ε version

$$-\varepsilon \Delta u_\varepsilon = \frac{1}{\varepsilon}(u_\varepsilon - u_\varepsilon^3) \text{ in } \Omega.$$

Deep connections with minimal surface theory.

- u_ε local minimizer, then the interface $\{u_\varepsilon = 0\} \to$ minimal hypersurfaces.
  Modica-Mortola (Γ-Convergence), Kohn-Sternberg, Caffarelli-Cordoba ($C^{1,\alpha}$ estimates), Hutchison-Tonegawa (Quantization), Tonegawa-Wickramasekera ($C^{1,\alpha}$ estimates for stable solutions), Wang-Wei ($C^{2,\alpha}$ estimates)
- Recently there is a renewed interest in building min-max theory of Allen-Cahn and the corresponding in minimal surfaces

- Chodosh-Mantoulidis (Annals Math 2019): Using Allen-Cahn to prove Multiplicity One Conjecture of Minimal Surfaces by Marques-Neves in $\mathbb{R}^3$.
- Dey (2022): Equivalence of Almgren-Pitts theory of min-max embedded minimal surfaces theory of Marques-Neves (2015) and Ljusternik-Schnirlman min-max theory of Allen-Cahn
- Chodosh-Mantoulidis (Publ.IHES 2023): Using Allen-Cahn to compute the $p-$width of S^2 $(\omega_p(S^2))$.

$$\omega_p(M) = \lim_{\epsilon\to 0} c_p^\epsilon = \lim_{\epsilon\to 0} \inf_{\gamma(A)\geq p} \max_A [\int_M [\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{4\epsilon}W(u)]]$$

Hutchinson-Tonegawa's Quantitization Result

General Theory: Let u_ϵ be a sequence of Allen-Cahn equation

$$\epsilon \Delta u_\epsilon + \frac{1}{\epsilon}(1 - u_\epsilon^2) u_\epsilon = 0 \text{ on } (M^N, g)$$

with bounded energy

$$E_\epsilon(u_\epsilon) = \int \epsilon |\nabla u_\epsilon|^2 + \frac{1}{4\epsilon}(1 - u_\epsilon^2)^2 \lesssim 1.$$

Then the following holds (Hutchinson-Tonegawa (2000)): for the nodal sets $\{u_\epsilon = 0\}$ there is a naturally associated limiting stationary $(N-1)-$varifold V with integer density m_V. Roughly speaking, one can define generalized mean curvature H_V and stationary varifold implies

$$H_V = 0$$

- In two dimensions Tonegawa (2014)): V is smooth geodesics away from isolated points.
- For stable solutions, Tonegawa-Wickramasekera (2017): V is stable and a smooth stable minimal hypersurface (outside of a codimension 7 singular set).

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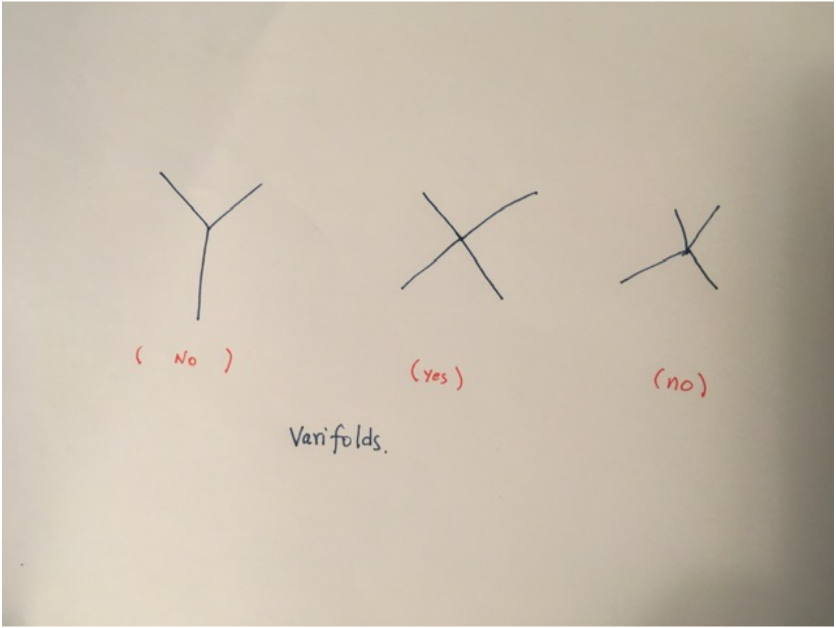
In general stationary varifolds can be very complicated, even on surfaces.


(No)
(yes)
(no)
Varifolds.

Chodosh-Mantoulidis Result

Theorem (Chodosh-Mantoulidis (preprint 2021,Pub.IHES 2023)):
Let (M^2, g) be a closed Riemann 2-manifold. Fix the Sine-Gordon double-well potential

$$W(u) = \frac{1 + \cos \pi u}{\pi^2}$$

Let u_ϵ be a sequence of solutions to the sine-Gordon equation

$$\epsilon^2 \Delta u_\epsilon + \sin \pi u_\epsilon = 0 \text{ on } (M^2, g)$$

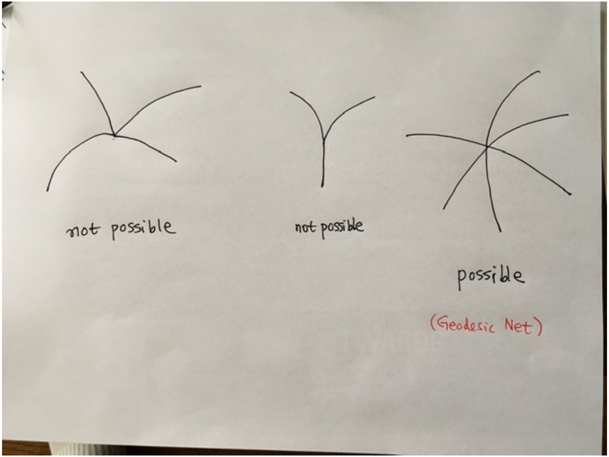
with bounded Morse index and energy

$$m(u_\epsilon) + E_\epsilon(u_\epsilon) \lesssim C$$

Then along a subsequence the "ϵ_i-phase transition 1-varifolds $V[u_\epsilon]$ converge to a stationary integral 1-varifold V such that

$$V = \sum_{j=1}^{N} v(\sigma_j, \mathbf{1}_{\sigma_j})$$

for $\sigma_1, ..., \sigma_N$ (possibly repeated) are primitive closed geodesics in $(M; g)$.

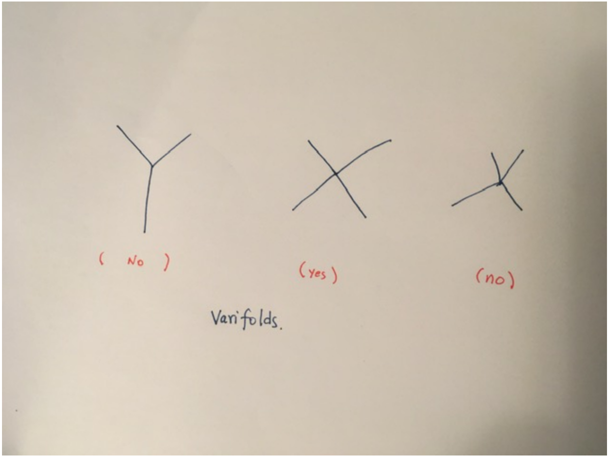

not possible
not possible
possible
(Geodesic Net)

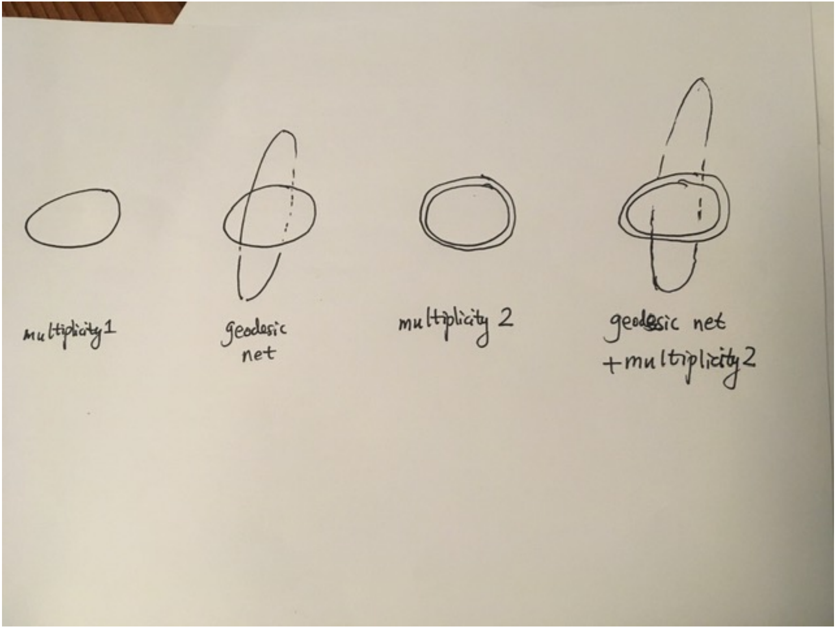
Consequences of Chodosh-Mantoulidis' Theorem

The stationary 1d varifold arising from sine-Gordon double-well potential can only have two possibilities

1) Geodesics Networks (Crossing)

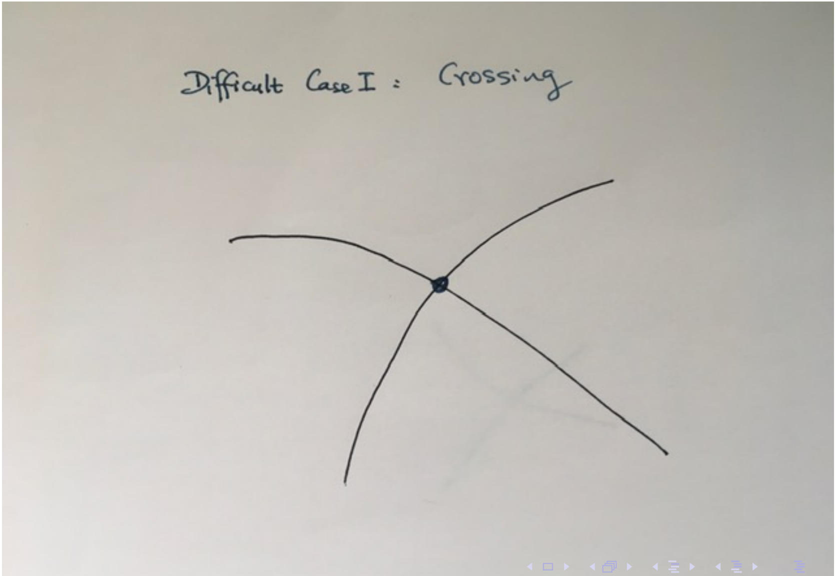
2) Higher multiplicities (Collapsing)




multiplicity 1
geodesic net
multiplicity 2
geodesic net +multiplicity 2

Proof of Chodosh-Mantoulidis Theorem

Geodesics nets



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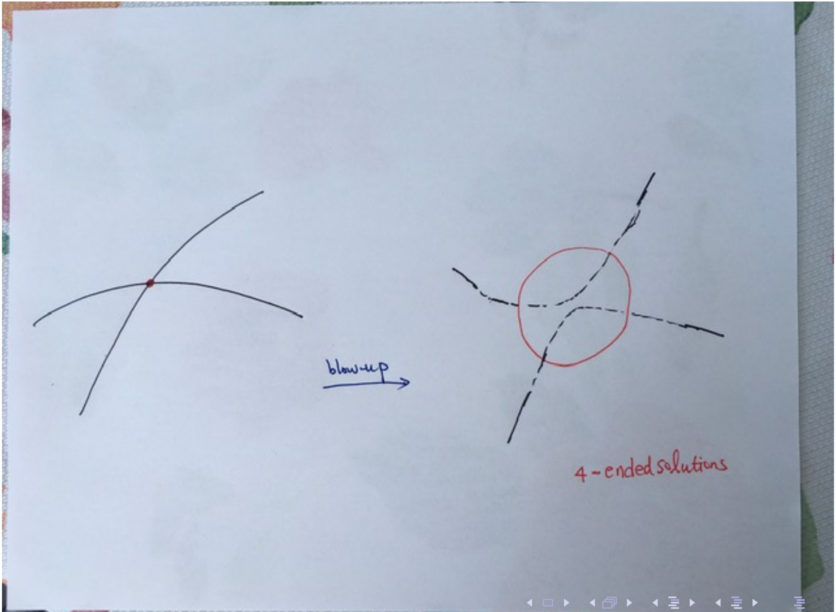

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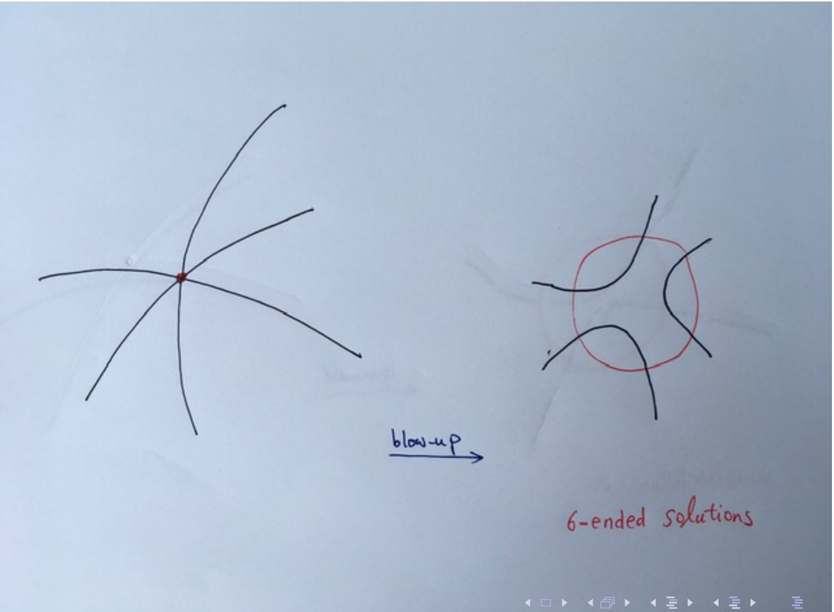

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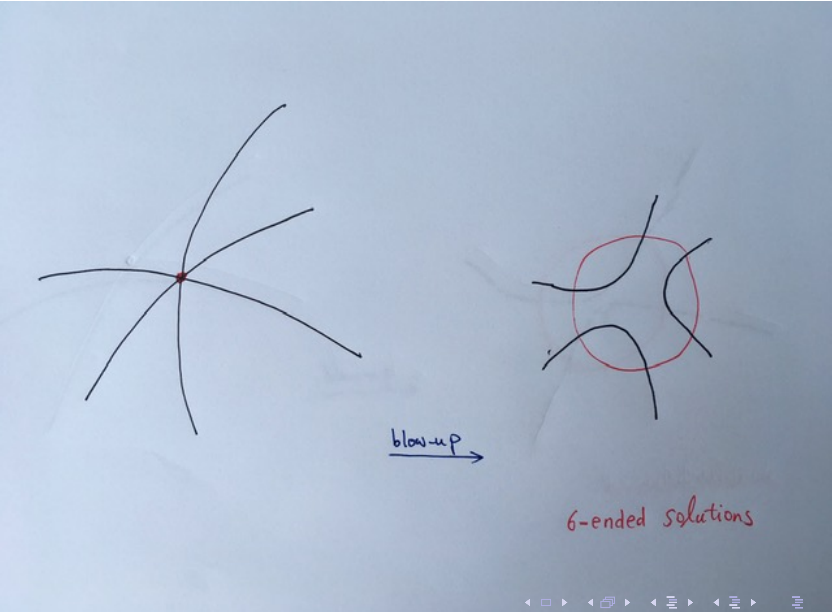

Classification of Finite Morse index Solutions in $\mathbb{R}^2$

(AC) $$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^2$$

Finite Morse Index: there exists a compact set K such that

$$\int |\nabla \phi|^2 + (3u^2 - 1)\phi^2 \geq 0, \ \forall \phi \in C_0^\infty(\mathbb{R}^2 \backslash K)$$

Wang-Wei (2019): Finite Morse Index $\longleftrightarrow$ Finite Ends
The set of $2n-$-ended solutions is called $\mathcal{M}_{2n}$.

Question: Complete Classification of $\mathcal{M}_{2n}$?

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All these questions are completely open for Allen-Cahn equation. (Gui (2009),del Pino-Kowalczyk-Pacard-Wei (2010), del Pino-Kowalczyk-Pacard (2012), Kowalczyk-Liu-Pacard (2014), Gui-Liu-Wei (2017),... for $\mathcal{M}_4$.)

A Complete Classification of $\mathcal{M}_{2n}$?

But for elliptic sine-Gordon equation all these questions are completely answered.
Using the fact that finite Morse index implies finite ends, and also "integrable system theory", we can explicitly write down all the solutions of multiple-ended solutions to another double-well potential—the elliptic Sine-Gordon equation

$$-\Delta u = \sin(\pi u), |u| < 1 \text{ in } \mathbb{R}^2$$

$$\text{double-well potential } W(u) = \frac{1 + \cos(\pi u)}{\pi^2}$$

$4-$ended saddle-solution:

$$4\arctan\left(\frac{\cosh(\frac{y}{\sqrt{2}})}{\cosh(\frac{x}{\sqrt{2}})}\right) - 1$$

$2m-$ended Solutions

Let $p_j, q_j, j = 1, ..., n$, be *real* numbers satisfying $p_j^2 + q_j^2 = 1$.

$$\alpha(j,k) := \frac{(p_j - p_k)^2 + (q_j - q_k)^2}{(p_j + p_k)^2 + (q_j + q_k)^2}.$$

$$a(j_1, ..., j_n) := 1, \text{ if } n = 0,1,$$

$$a(j_1, ..., j_n) := \prod_{k<l\leq m} \alpha(j_k, j_l), \text{ if } n \geq 2.$$

$$\eta_j = p_j x + q_j y + \eta_j^0$$

Then $U_n := 4\arctan\frac{g_n}{f_n} - \pi$ where

$$f_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\sum_{\{n,2k\}} \left[a\left(j_1, ..., j_{2k}\right) \exp\left(\eta_{j_1} + ... + \eta_{j_{2k}}\right) \right] \right),$$

$$g_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \left(\sum_{\{n,2k+1\}} \left[a\left(j_1, ..., j_{2k+1}\right) \exp\left(\eta_{j_1} + ... + \eta_{j_{2k+1}}\right) \right] \right).$$

is a $2n-$ended solution. Conversely,

A Complete Classification of $\mathcal{M}_{2n}$

Theorem [Y. Liu-J.Wei (2021, 79 pages)] Suppose ϕ is a $2n$-end solution of the equation $-\Delta\phi = \sin\phi$. Then there exist parameters $p_j, q_j, \eta_j^0, j = 1, ..., n$, such that $\phi = U_n$, where U_n is defined before. As a consequence

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Main idea: Inverse Scattering transform.

Asymptotic behavior at ∞

The $2n$ ends are given by half lines:

$$(x, y) \cdot e_j^{\perp} = r_j.$$

The direction of the j-th line is the unit vector e_j.

A consequence of the explicit formula of all solutions in $\mathcal{M}_{2n}$ is the following: up to relabelling, the ends satisfy

$$e_{2k+1} = -e_{2k}$$

From this fact, Chodosh-Mantoulidis (Pub.IHES 2023) proved geodesic net theorem.

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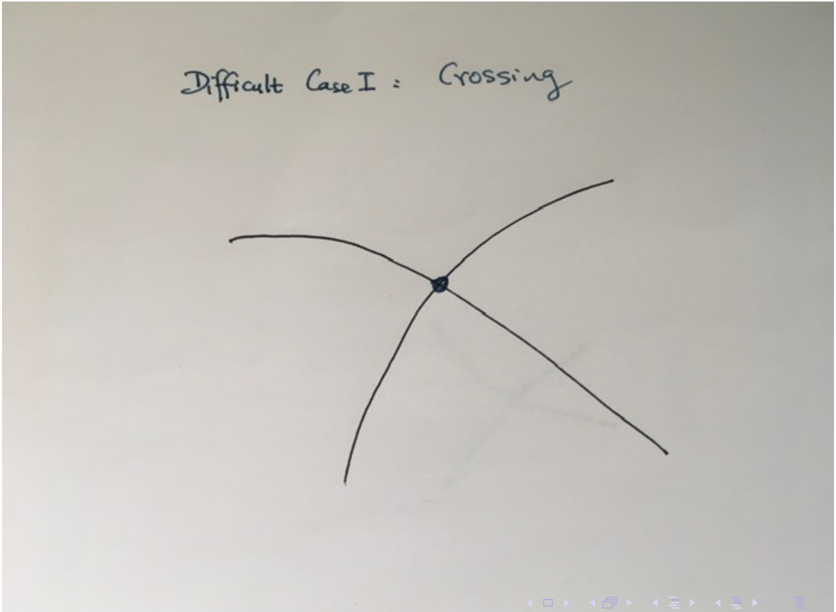
- 5. Computation of $p-$width of Riemann surfaces $\omega_p(M, g)$
  Chodosh-Mantoulidis (Publ.IHES 2023): computation of $\omega_p(S^2, g_0)$ ($p-$width of S^2).

2.Geodesic Networks



Construction and Morse Index of solutions on Geodesic Networks

$$\epsilon^2 \Delta u + \sin(\pi u) = 0 \ \text{ on } (M^2, g)$$

Theorem

(Liu-Pacard-Wei (2022)) Let $L_1, \ldots, L_n$ be a geodesic net, with each geodesic being embedded and nondegenerate. Suppose they intersect at k distinct points, and at each intersection points k_i lines intersect. Then for each $\varepsilon > 0$ small enough, there exists a solution u_ε to the elliptic sine-Gordon equation

$$\epsilon^2 \Delta u_\epsilon + \sin(\pi u_\epsilon) = 0 \text{ on } (M^2, g)$$

whose zero set is close to $L_1 \cup \ldots \cup L_n$. Moreover, the Morse index of u_ε is equal to

$$\sum_{i=1}^{k} \frac{k_i(k_i-1)}{2} + \sum_{j=1}^{n} Ind\,(L_j).$$

Construction

WLOG, $k_i = 2, i = 1, ..., k$.
The construction relies on two steps. In the first step, we construct an approximate solution, by putting four-end solutions around each intersection point.
In the second step, we perturb the approximate solution into a true one. Here we need to translate and rotate the four-end solutions near each intersection point. By adjusting these parameters, we should be able to define the adjusted approximate solution. This relies on the nondegeneracy of the geodesic net.

Compute the Morse index–From eigenfunction of linearized AC operator to eigenfunction of the Jacobi operator

Let $g = g_\varepsilon$ be a sequence of eigenfunctions of the operator $\mathcal{L}$ with negative eigenvalue:

$$\mathcal{L}g = -\varepsilon^2 \Delta g - \cos\left(u_\varepsilon\right) g = -\beta^2 g.$$

We analyze the asymptotic behavior of these functions.

Lemma

Assume $\beta = \beta_\varepsilon \to 0$ as $\varepsilon \to 0$. There exists a constant C independent of ε such that $|\beta| \leq C\varepsilon$.

This provides an estimate of those negative eigenvalues close to 0. They are expected to be related with the Jacobi operator.

Sketch of the proof

Assume to the contrary that there was a sequence β such that $\varepsilon^{-1}\beta$ tends to infinity.
Away from the intersection point $p_{i,j}$, the projection η of the function g onto $H'(t)$ satisfies approximately

$$-\Delta_L \eta - Ric(\nu,\nu)\eta = \varepsilon^{-2}\beta^2\eta.$$

This implies that away from the intersection point, the function η essentially behaves like $e^{\pm\varepsilon^{-1}\beta t}$. This together with the assumption that the L^∞ norm of the sequence of eigenfunctions g is uniformly bounded imply that there exist intersection point p_j, constants m and c, such that

$$\|g\|_{L^\infty(B_{m\varepsilon}(p_j))} \geq c > 0.$$

Then as ε tends to 0, the function $g(\varepsilon z)$ converges to a bounded function η_0 of the equation

$$-\Delta\eta_0 - \cos(U_\alpha)\eta_0 = 0 \text{ in } R^2.$$

Hence $\eta_0 = c_1 \partial_x U_\alpha + c_2 \partial_y U_\alpha$ for some constants c_1, c_2, where U_α denotes the four-end solutions.

The spectrum of the Jacobi operator J_i on $H^1(L_i)$ has the form $\mu_1 < \mu_2 \leq \mu_3 \leq ...$ Let k_1 be the eigenfunction corresponding to μ_1, normalized such that $\|k_1\|_{L^\infty} = 1$. We can assume k_1 is always positive.

The operator $\mathcal{L}$ has an eigenfunction w_1 which is close in the local Fermi coordinate to the function $k_1(s)\mathbf{U}_{i,j}(\varepsilon^{-1}t)$ with eigenvalue $\bar{\mu}_1\varepsilon^2$, where $\bar{\mu}_1 - \mu_1 = O(\varepsilon)$:

$$-\varepsilon^2 \Delta w_1 - \cos(u) w_1 = \bar{\mu}_1 \varepsilon^2 w_1.$$

Rescaling the functions by setting

$$W(s,t) = w_1(\varepsilon s, \varepsilon t), \text{ and } G(s,t) = g(\varepsilon s, \varepsilon t),$$

we get

$$-\Delta W - \cos(U)W = \bar{\mu}_1 \varepsilon^2 W,$$
$$-\Delta G - \cos(U)G = -\beta^2 G.$$

Let Ω be the ball of radius $\varepsilon^{-1} r_0$ centered at the intersection point in the rescaled domain, where r_0 is a fixed small constant.

$$\left(\beta^2 + \bar{\mu}^2 \varepsilon^2\right) \int_\Omega (GW) = \int_{\partial\Omega} (\partial_\nu WG - \partial_\nu GW).$$

Estimating the boundary integral appeared in the right hand side, we get the desired estimate of β.

Fix a large constant m. Let (s, t) be the local Fermi coordinate around p_j. Define the projection of g onto H' as

$$q(s) := \int_{-\delta}^{\delta} g(s, t) H'\left(\varepsilon^{-1} t\right) dt.$$

For $s < -m\varepsilon$, the solution u is close to $H\left(\varepsilon^{-1} t\right)$. Write

$$g(s, t) = q(s) H'\left(\varepsilon^{-1} t\right) + \phi.$$

Using the exponential decay of ϕ, we deduce

$$-\Delta_L q - Ric(\nu, \nu) q = \varepsilon^{-2} \beta^2 q + O(\varepsilon), \text{ if } s < -10\varepsilon |\ln \varepsilon|.$$

Introduce function $\gamma(s)$, which depends on ε and solves

$$\begin{cases} -\Delta_L \gamma - Ric(\nu,\nu)\gamma = \varepsilon^{-2}\beta^2\gamma, s > -\delta. \\ \gamma(\delta) = q(\delta), \gamma'(\delta) = q'(\delta). \end{cases}$$

Construct an eigenfunction ξ of the linearized Allen-Cahn operator of the form $\xi = \tilde{\gamma}(s)\mathbf{U}_{i,j} + \psi$ in the region where $s \in (-2\delta, 2\delta)$, with

$$\|\tilde{\gamma} - \gamma\|_{L^\infty([-\delta,\delta])} = O(\varepsilon), \text{ and } |\psi| \le Ce^{-\delta\varepsilon^{-1}|s|},$$

and ξ satisfying

$$-\varepsilon^2\Delta\xi - \cos(u)\xi = \beta^2\xi.$$

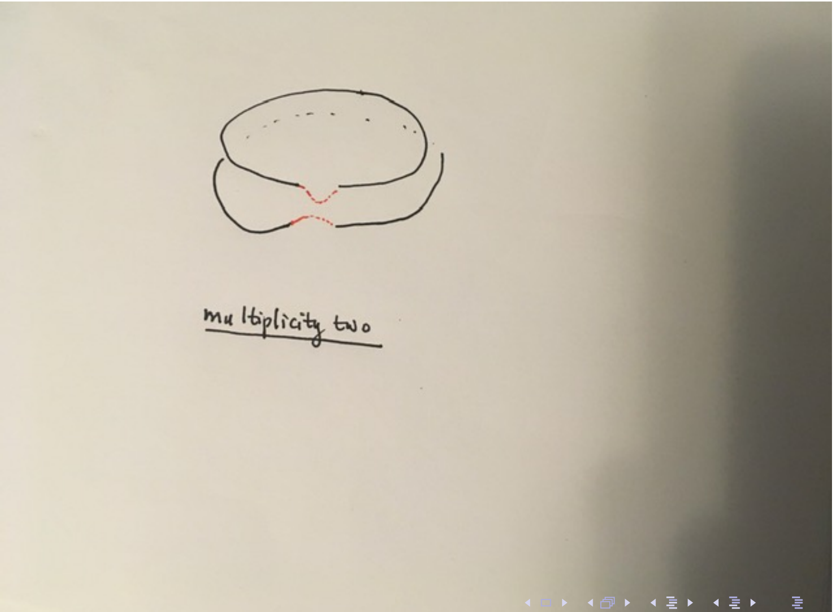
There holds

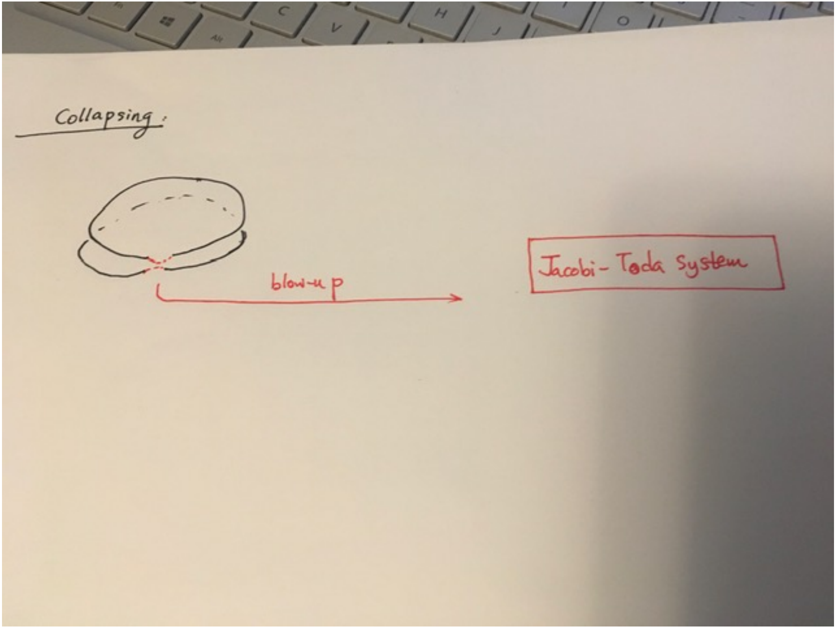
$$\int_{\partial\Omega} (\partial_\nu \xi g - \partial_\nu g \xi) = 0.$$

This implies that approximately, g is a multiple of ξ.

Repeat this argument and do the analysis across all the intersection point on the geodesic L_j.
In particular, in the region where $s \in [-10\delta, -\delta]$, q is approximately a multiple of γ.
This together with q is continuous imply that as ε tends to 0, γ will converge to an eigenfunction of the Jacobi operator, defined on the closed geodesic L_j.
Hence this type of eigenvalues correspond to that of the Jacobi operator.

3. Multiplicity two interfaces




Collapsing:
blow-up
Jacobi-Toda system

Interaction between different interfaces has the form

$$\triangle_\Gamma f_k + (|A|^2 + Ric) f_k = \frac{A}{\varepsilon}\left[e^{-\frac{\sqrt{2}}{\varepsilon}(f_{k,\varepsilon} - f_{k-1,\varepsilon})} - e^{-\frac{\sqrt{2}}{\varepsilon}(f_{k+1,\varepsilon} - f_{k,\varepsilon})} \right] + h.o.t.$$

Jacobi-Toda Systems

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A major consequence of the existence of multiplicity two interfaces with bounded Morse index is the existence of bouncing Jacobi fields.

Bouncing Jacobi fields

Let $R = Ric(\nu,\nu)(= K_g)$ be the Gaussian curvature along the geodesic L in the normal direction. The Jacobi operator J has the form

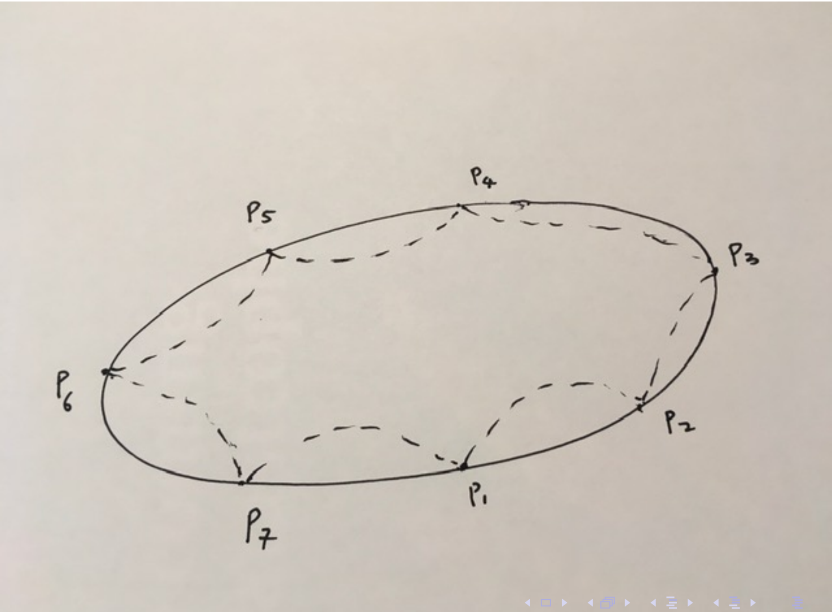
$$u \to u'' + Ru.$$

Definition

A continuous function ϕ defined on the geodesic L is called a bouncing Jacobi field with k minimums of the Jacobi operator J, if ϕ satisfies the following conditions: There exist k distinct points $p_1, ..., p_k \in L$, such that

(A1) $J\phi = 0$, in $L \setminus \{p_1, ..., p_k\}$.
(A2) $\phi(p_i) = 1$, for $i = 1, ..., k$.
(A3) $\phi'_-(p_i) = -\phi'_+(p_i) \neq 0$, for $i = 1, ..., k$.
(A4) $\phi(s) \geq 1$, for all $s \in L$.

Bouncing Jacobi Field



Necessary condition for multiplicity two: bouncing Jacobi field

Theorem

(Liu-Pacard-Wei-Ye (2022)) Let u_ε be a sequence of multiplicity two solutions of the Allen-Cahn equation with uniformly bounded Morse index. Suppose the upper part of the nodal set of u_ε is represented by the graph of f_ε defined on the geodesic L. Assume L is nondegenerated. Then as $\varepsilon \to 0$, the function $\frac{f_\varepsilon}{\|f_\varepsilon\|_{L^\infty}}$ tends to $a\phi$, where $a > 0$ is constant and ϕ is a bouncing Jacobi field of L.

The case with higher multiplicity is more delicate, which in principle could lead to combination of geodesic network and more complicated bouncing solutions.

Sketch of the proof

Suppose the lower and upper part of the nodal set of u_ε is represented by the graph of $f_{\varepsilon,1}, f_{\varepsilon,2}$. By arguments of Chodosh-Mantoulidis, the function $\phi_\varepsilon = \frac{f_{\varepsilon,2}-f_{\varepsilon,1}}{\|f_{\varepsilon,2}-f_{\varepsilon,1}\|_{L^\infty}}$ converges, away from finitely many points $p_1, ..., p_k$ to solution of the equation $J\phi = 0$, in $\tilde{L} := L \setminus \{p_1, ..., p_k\}$.

Case 1. $\inf_{s\in\tilde{L}} \phi(s) > 0$.
In this case, if $\frac{\|f_{\varepsilon,1}-f_{\varepsilon,2}\|_{L^\infty}}{\varepsilon|\ln\varepsilon|} \leq C < +\infty$, then near the point p_i, the function $g_{\varepsilon,i} := \varepsilon^{-1} f_{\varepsilon,i}$ and $g_\varepsilon = g_{\varepsilon,2} - g_{\varepsilon,1}$ satisfies

$$g''_{\varepsilon,2} + Rg_{\varepsilon,2} - \bar{c}\varepsilon^{-2} e^{-g_{\varepsilon,2}+g_{\varepsilon,1}} \sim 0.$$
$$g''_{\varepsilon,1} + Rg_{\varepsilon,1} + \bar{c}\varepsilon^{-2} e^{-g_{\varepsilon,2}+g_{\varepsilon,1}} \sim 0.$$
$$g''_\varepsilon + Rg_\varepsilon - 2\bar{c}\varepsilon^{-2} e^{-g_\varepsilon} \sim 0.$$

Using $|g_\varepsilon'| < C |\ln \varepsilon|$ in $L \backslash \{p_1, ..., p_k\}$, we get

$$\min_s g_\varepsilon(s) > 2|\ln \varepsilon| - 2\ln|\ln \varepsilon| - O(1).$$

It follows that $g_\varepsilon'' + Rg_\varepsilon - 2\bar{c}\varepsilon^{-2}e^{-g_\varepsilon} = O(\varepsilon^\sigma)$ for some $\sigma > 0$. This regime is exactly what have been analyzed in the construction step. Around p_i, g_ε is essentially governed by the Toda equation. The function ϕ is then continuous and the function $\xi_\varepsilon := g_{\varepsilon,1} + g_{\varepsilon,2}$ satisfies

$$\xi_\varepsilon'' + R\xi_\varepsilon = O(\varepsilon^\sigma).$$

Nondegeneracy of the geodesic L implies $\xi_\varepsilon \to 0$, leading to $\phi_+'(p_i) = -\phi_-'(p_i)$. This means that ϕ is a bouncing Jacobi field.

If $\frac{\|f_{\varepsilon,1}-f_{\varepsilon,2}\|_{L^\infty}}{\varepsilon|\ln\varepsilon|} \to +\infty$, then $\min_s g_\varepsilon(s) > 3|\ln\varepsilon|$ for ε large. That is, the distance between two layers is large.
This implies ϕ_ε converges to a smooth kernel of the Jacobi operator, contrary to the nondegeneracy of L.

Case 2. $\inf_{s\in L}\phi(s) = 0$.
Assume $\phi_\varepsilon(p) \to 0$ for some $p \in L$.
Similar arguments as before tells us that at the point p,

$$\phi'_+(p) = -\phi'_-(p).$$

Hence ϕ and $-\phi$ patch together to form a nontrivial kernel of the Jacobi operator, contrary to the nondegeneracy of L again.

Existence of solutions for the Allen-Cahn equation with multiplicity two

Theorem

(Liu-Pacard-Wei (2022)) Let L be a geodesic with total length 2π embedded in the two dimensional surface M. Let $n \geq 1$ be a fixed integer. Suppose L has a bouncing Jacobi field ϕ with n minimums and with index k. Then for each ε small, the Allen-Cahn equation has a solution u_ε with energy close to $4\pi\mathbf{e}$ and Morse index $n + k$, provided that ϕ is nondegenerated in the sense described below. Here $\mathbf{e}$ is the energy of the heteroclinic solution of the Allen-Cahn equation. Moreover, the transition layer of u_ε has multiplicity two.

Remark: Under suitable conditions, we can show the existence of bouncing Jacobi fields with some additional information on their Morse index(this is not the index of L). In particular, we can prove the existence of Morse index $2n$ solutions for the AC equation. We can also show the existence with higher multiplicity under certain assumptions.

Existence of bouncing Jacobi fields

Lemma

Assume $R > 0$. For each $n \in \mathbb{N}$ satisfying

$$n > 2\sqrt{\|R\|_{L^\infty}}, \tag{1}$$

there exists at least one bouncing Jacobi fields with exactly n minimums. Moreover, if $2\sqrt{\|R\|_{L^\infty}} < 1$, there are at least two distinct bouncing solutions with one local minimum.

The existence follows from the method used by Qian-Torres-(SIMA 2005), using the Poincare-Birkhoff theorem about the existence of fixed points for area preserving maps.
We provided a variational proof, which also captures more information about the Morse index of the bouncing solutions.
If R changes sign, bouncing Jacobi fields may still exist.
Intuitively, this result tells us that there is some relation between n and the index of the geodesic.

Sketch of the proof

First consider the case of $n \geq 2$ bouncing points.
For each pair $s_1, s_2 \in [0, 2\pi]$ with $s_1 < s_2$, identified as two distinct points on L, define

$$H\left(s_1, s_2\right) := \inf_{\phi - 1 \in H_0^1([s_1, s_2])} \int_{s_1}^{s_2} \left(\phi'^2 - R\phi^2\right) ds.$$

If $H\left(s_1, s_2\right) > -\infty$, then the infimum is achieved at some function ϕ^*, and $\phi^{*\prime\prime} + R\phi^* = 0$. Since R is positive, there holds $H\left(s_1, s_2\right) = 2\phi^{*\prime}\left(s_2\right) - 2\phi^{*\prime}\left(s_1\right) < 0$, and

$$\phi^*\left(s\right) > 1, \text{ for } s \in \left(s_1, s_2\right).$$

Also define

$$H\left(s_2, s_1\right) := \inf_{\phi - 1 \in H_0^1([s_2, s_1 + 2\pi])} \int_{s_2}^{s_1 + 2\pi} \left(\phi'^2 - R\phi^2\right) ds.$$

Let

$$\Omega_0 := \{(s_1, ..., s_n) : 0 < s_1 < ... < s_n \leq 2\pi\}.$$

Let $s_{n+1} := s_1 + 2\pi$ and define

$$\mathcal{H}(s_1, ..., s_n) = \sum_{i=1}^{n} H(s_i, s_{i+1}).$$

A *critical point* of $\mathcal{H}$ corresponds to a bouncing Jacobi field.
$\mathcal{H} \leq 0$. To find a maximum for $\mathcal{H}$, consider

$$\Omega_1 := \{(s_1, ..., s_n) \in \Omega_0 : H(s_i, s_{i+1}) > -\infty \text{ for } i = 1, ..., n\}.$$

The assumption $n > 2\sqrt{\|R\|_{L^\infty}}$ ensures that this set is nonempty.
Indeed, the n-tuple $(\bar{s}_1, ..., \bar{s}_n)$ with $\bar{s}_j = \frac{2j\pi}{n}$ is in this set:

$$\begin{aligned} H(\bar{s}_i, \bar{s}_{i+1}) &= \inf_{\phi \in H_0^1(\bar{s}_i, \bar{s}_{i+1})} \int_{\bar{s}_i}^{\bar{s}_{i+1}} \left[\phi'^2 - R(\phi+1)^2\right] ds \\ &\geq \inf_{\phi \in H_0^1(\bar{s}_i, \bar{s}_{i+1})} \int_{\bar{s}_i}^{\bar{s}_{i+1}} \left[\left(\frac{n^2}{4} - R\right)\phi^2 - 2R\phi - R\right] ds \\ &> -\infty. \end{aligned}$$

Define

$$\mathcal{M} := \sup_{z\in\Omega_1} \mathcal{H}(z).$$

Then $\mathcal{M}$ is achieved in the interior of Ω_1. Indeed, if for a pair $0 < s < t \leq 2\pi$, $H(s,t) > -\infty$, then we have

$$H(s,t) < H\left(s, \frac{s+t}{2}\right) + H\left(\frac{s+t}{2}, t\right).$$

This implies that for a sequence $\{z_k\}$ with $z_k \to \partial\Omega_1$ as $k \to +\infty$, there holds

$$\lim \sup_{k\to+\infty} \mathcal{H}(z_k) < \mathcal{M}.$$

In the case of $n = 1$, consider the function

$$\mathcal{H}(s) := H(s, s+2\pi), \text{ for } s \in [0, 2\pi].$$

It is defined on L and satisfies $\mathcal{H} \leq 0$. Under the assumption that $2\sqrt{\|R\|_{L^\infty}} < 1$, the function $\mathcal{H}$ is also bounded from below.

Nondegeneracy of bouncing Jacobi field

Definition

A bouncing Jacobi field ϕ with k minimums is nondegenerate, if the following problem only has the trivial solution $\eta = 0$:

$$\begin{cases} J\eta = 0, \text{ in } L \setminus \{p_1, ..., p_k\}, \\ \eta_+(p_i) + \eta_-(p_i) = 0, i = 1, ..., k, \\ \eta'_+(p_i) + \eta'_-(p_i) = -\frac{2R(p_i)\phi(p_i)\eta_+(p_i)}{\phi'_+(p_i)}, i = 1, ..., k. \end{cases} \tag{2}$$

This definition can be derived from the second variation of the energy functional

$$\int_0^{2\pi} \left(\Phi'^2 - R\Phi^2\right) ds.$$

where $\Phi(q_i) = 1$, for some q_i close to $p_i, i = 1, ..., k$.

From bouncing Jacobi field to solutions of Jacobi-Toda equation

We would like to construct 2π-periodic solutions for the following Jacobi-Toda equation

$$u'' + Ru - \bar{c}\varepsilon^{-2}e^{-u} = 0,\ \ s \in [0, 2\pi],$$

where $\bar{c}$ is a positive constant. This equation is closely related to AC equation.

Lemma

Suppose ϕ is a nondegenerate bouncing Jacobi field. Then for $\varepsilon > 0$ small enough, the above Jacobi-Toda equation has a positive C^2 solution u_ε defined on L, with $\|u\|_{L^\infty} = O(|\ln \varepsilon|)$.

Sketch of the proof

First step. Construct solutions away from the bouncing points: Assume the bouncing Jacobi field ϕ only has two minimums, at p_1, p_2, with $0 < p_1 < p_2 \leq 2\pi$.
Let $\delta_1, \delta_2 > 0$ be sufficiently small, with $\delta_i = O\left(\frac{\ln|\ln \varepsilon|}{|\ln \varepsilon|}\right)$.
Assume p_1^*, p_2^* are close to p_1, p_2, to be determined later on. Let $M := 2\left|\ln \varepsilon\right| + 2\ln\left|\ln \varepsilon\right|$.
Consider the following boundary value problems:

$$\begin{cases} \mathbf{u}'' + R\mathbf{u} - \bar{c}\varepsilon^{-2}e^{-\mathbf{u}} = 0, \text{ in } \left(p_2^* + \delta_2, p_1^* - \delta_1 + 2\pi\right), \\ \mathbf{u}\left(p_2^* + \delta_2\right) = \mathbf{u}\left(p_1^* - \delta_1 + 2\pi\right) = M, \end{cases}$$

and

$$\begin{cases} \mathbf{v}'' + R\mathbf{v} - \bar{c}\varepsilon^{-2}e^{-\mathbf{v}} = 0, \text{ in } \left(p_1^* + \delta_1, p_2^* - \delta_2\right), \\ \mathbf{v}\left(p_1^* + \delta_1\right) = \mathbf{v}\left(p_2^* - \delta_2\right) = M. \end{cases}$$

Write $\mathbf{u} = \bar{\phi}_1 + \eta_1$ and $\mathbf{v} = \bar{\phi}_2 + \eta_2$. Here $\bar{\phi}_1, \bar{\phi}_2$ satisfy

$$\begin{cases} \bar{\phi}_1'' + R\bar{\phi}_1 = 0, \text{in } \left(p_2^* + \delta_2, p_1^* - \delta_1 + 2\pi\right) \\ \bar{\phi}_1\left(p_2^* + \delta_2\right) = \bar{\phi}_1\left(p_1^* - \delta_1 + 2\pi\right) = M, \end{cases}$$

and

$$\begin{cases} \bar{\phi}_2'' + R\bar{\phi}_2 = 0, \text{ in } \left(p_1^* + \delta_1, p_2^* - \delta_2\right), \\ \bar{\phi}_2\left(p_1^* + \delta_1\right) = \bar{\phi}_2\left(p_2^* - \delta_2\right) = M. \end{cases}$$

We are lead to

$$\begin{cases} \eta_1'' + R\eta_1 = \bar{c}\varepsilon^{-2}e^{-\left(\bar{\phi}_1 + \eta_1\right)}, \text{ in } \left(p_2^* + \delta_2, p_1^* - \delta_1 + 2\pi\right), \\ \eta_1\left(p_2^* + \delta_2\right) = \eta_1\left(p_1^* - \delta_1 + 2\pi\right) = 0. \end{cases}$$

$\bar{c}\varepsilon^{-2}e^{-\left(2\bar{\phi}_1 + \eta_1\right)}$ can be regarded as a perturbation term. In the interval $\left[p_2^* + \delta_2, p_1^* - \delta_1 + 2\pi\right]$, there holds

$$\varepsilon^{-2}e^{-\bar{\phi}_1} \le C\left|\ln\varepsilon\right|^{-1}.$$

The existence of a solution $\bar{\eta}_1 = o(1)$ follows from standard perturbation argument. Similar approach yields the solution $\mathbf{v}$.

For fixed δ_1, δ_2, positive and small, define the map

$$\begin{aligned} \mathcal{G} : (p_1^*, p_2^*) \to \\ & (\mathbf{u}'(p_2^* + \delta_2) + \mathbf{v}'(p_2^* - \delta_2), \\ & \mathbf{u}'(p_1^* - \delta_1 + 2\pi) + \mathbf{v}'(p_1^* + \delta_1)). \end{aligned}$$

Since ϕ is a nondegenerated bouncing Jacobi field, the linearization of this map at the point (p_1, p_2) is invertible. Applying the implicit function theorem, we obtain $(\bar{p}_1^*, \bar{p}_2^*)$, depending on δ_1, δ_2, such that the corresponding solutions $\mathbf{u}, \mathbf{v}$ satisfy $\mathcal{G}(\bar{p}_1^*, \bar{p}_2^*) = 0$. That is, the slopes of $\mathbf{u}, \mathbf{v}$ match with each other.

Step 2. Adjust the parameters δ_1 and δ_2.
We assume that R is constant in a neighbourhood of p_i. The general case follows from a perturbation argument.
There exists a solution $\mathbf{w} \leq M$, solving

$$\begin{cases} \mathbf{w}'' + R\mathbf{w} - \bar{c}\varepsilon^{-2}e^{-\mathbf{w}} = 0, \text{ in } \left(\bar{p}_1^* - \delta_1, \bar{p}_1^* + \delta_1\right), \\ \mathbf{w}\left(\bar{p}_1^* - \delta_1\right) = \mathbf{w}\left(\bar{p}_1^* + \delta_1\right) = M. \end{cases}$$

The relation between $\mathbf{w}'\left(\bar{p}_1^* - \delta_1\right)$ and δ_1 is given by

$$\delta_1 = \frac{\ln\left|\ln\varepsilon\right|}{\mathbf{w}'\left(\bar{p}_1^* - \delta_1\right)}\left(1 + o\left(1\right)\right).$$

The slope of the solution $\mathbf{u}$ has the form

$$\mathbf{u}'\left(\bar{p}_1^* - \delta_1\right) \sim \bar{\phi}_1'\left(\left(\bar{p}_1^* - \delta_1\right)\right) \sim M\phi'\left(\bar{p}_1^* - \delta_1\right).$$

We need to solve an equation for δ_1 of the form:

$$\delta_1 \sim \frac{\ln|\ln\varepsilon|}{M\left|\phi'_-\left(p_1\right)\right|} \sim \frac{\ln|\ln\varepsilon|}{M\phi'\left(p_1\right)}.$$

Then we can find $\delta_1 \in \left[\frac{\ln|\ln\varepsilon|}{3M\left|\phi'_-(p_1)\right|}, \frac{\ln|\ln\varepsilon|}{M\left|\phi'_-(p_1)\right|}\right]$ such that

$$\mathbf{w}'\left(\bar{p}_1^* - \bar{\delta}_1\right) = \mathbf{u}'\left(\bar{p}_1^* - \bar{\delta}_1\right).$$

Similar arguments apply to δ_2. Patching these solutions together, we then obtain a smooth solution u of the Jacobi-Toda equation. From the construction, we see that the L^∞ norm of u is of the order $O\left(|\ln\varepsilon|\right)$.

The Morse index of solutions to the Jacobi-Toda equation

The linearized Jacobi-Toda operator around the solution u is

$$J^{*}\eta := -\eta'' - R\eta - \bar{c}\varepsilon^{-2}e^{-u}\eta,\ \eta \in H^{1,2}(L).$$

Since $\bar{c}\varepsilon^{-2}e^{-u} > 0$, the Morse index of the operator J^{*} is bounded from below by the Morse index of the geodesic L.

Lemma

Suppose that the bouncing Jabobi field ϕ is nondegenerated. Then the spectrum of the operator J^{*} is away from 0, uniformly in ε.

Sketch of the proof

Assume to the contrary that there was a sequence of eigenfunctions η_ε and eigenvalues λ_ε, such that

$$J^* \eta = \lambda_\varepsilon \eta$$

with $\lambda_\varepsilon \to 0, \|\eta_\varepsilon\|_{L^\infty} = 1$.
The solution u has local minimum at the points $\bar{p}_j^*$. There exists a universal constant $c > 0$, such that for fixed small constant δ independent of ε, we have the estimate

$$\varepsilon^{-2} e^{-u} \leq \varepsilon^{c\delta}, \text{ in } L \setminus \cup_j \left(\bar{p}_j^* - \delta, \bar{p}_j^* + \delta \right).$$

It follows that away from these points $\bar{p}_j^*$, η_ε converges to a solution η^* of $J\eta = 0$, where J is the Jacobi operator of the geodesic.

The behaviour of the solution near each $\bar{p}_i^*$ is more delicate. Let ξ_1 be solution of the equation

$$\gamma'' + \bar{c}\varepsilon^{-2}e^{-u}\gamma = 0$$

with $\xi_1(0) = 0$ and odd. Let ξ_2 be the even solution of this equation with $\xi_2(0) = 1$. That is, kernels of the linearized Toda equation. Then near each $\bar{p}_i^*$, $\eta \sim a\xi_1 + b\xi_2$.
Since $\|\eta\|_{L^\infty}$ norm is assumed to be uniformly bounded with respect to ε, we find that $|a|$ is also uniformly bounded, and $|b| \to 0$ as $\varepsilon \to 0$. Since $|\eta'(\bar{p}_1^* - \delta_1)|$ is also uniformly bounded,

$$|b| \ln |\ln \varepsilon| \to 0, \text{ as } \varepsilon \to 0.$$

This together with the fact that ξ_1 is even then implies

$$\eta_+^*(p_i) = -\eta_-^*(p_i).$$

Next we would like to prove

$$\eta_{+}^{*\prime}(p_i) + \eta_{-}^{*\prime}(p_i) = -\frac{2R(p_i)\,\phi(p_i)\,\eta_{+}^{*}(p_i)}{\phi_{+}'(p_i)}.$$

Fix an index i, and assume $\bar{p}_i^* = 0$. Consider the function

$$\omega(s) := \eta(s) - \eta(-s).$$

Since R is constant around 0, the function ω still satisfies the linear equation $J^*\omega = 0$ in this interval. Let q be the point close $\bar{p}_i^*$ where $u = m \sim 2|\ln \varepsilon|$. We fix any $\sigma > 0$ small. There holds

$$\begin{aligned}\omega'(q) - \omega'(\bar{p}_i - \sigma) &= \int_{\bar{p}_i-\sigma}^{q} \omega''(s)\,ds \\ &= \int_{\bar{p}_i-\sigma}^{q} \left(-R\omega - \bar{c}\varepsilon^{-2}e^{-u}\omega\right) ds.\end{aligned}$$

At $\bar{p}_1^* - \delta_1$, $\varepsilon^{-2} e^{-u} = m >> R$, and around $\bar{p}_1^* - \delta_1$,

$$u \sim m \left[1 + \phi' \left(p_i \right) \left(s - \left(\bar{p}_1^* - \delta_1 \right) \right) \right].$$

As a consequence,

$$\begin{aligned} \omega' \left(q \right) - \omega' \left(\bar{p}_i - \sigma \right) &= - \int_{p_i - \sigma}^{q} \varepsilon^{-2} e^{-u} \omega ds + O \left(\sigma \right) \\ &= \frac{2R \left(p_i \right) \phi \left(p_i \right) \omega_- \left(p_i \right)}{\phi'_- \left(p_i \right)} + O \left(\sigma \right). \end{aligned}$$

This implies

$$\eta_+^{*\prime} \left(p_i \right) + \eta_-^{*\prime} \left(p_i \right) = - \frac{2R \left(p_i \right) \phi \left(p_i \right) \eta_+^* \left(p_i \right)}{\phi'_+ \left(p_i \right)}.$$

Eigenvalue problem associated to bouncing Jacobi fields

Associated with the nondegenerated bouncing Jacobi field, we have the following eigenvalue problem(EVP)

$$\begin{cases} J\eta = -\lambda\eta, \text{ in } L\backslash\left\{p_1, ..., p_k\right\}, \\ \eta_+\left(p_i\right) = -\eta_-\left(p_i\right), i = 1, ..., k. \\ \eta'_+\left(p_i\right) + \eta'_-\left(p_i\right) = -\frac{2R(p_i)\phi(p_i)\eta_+(p_i)}{\phi'_+(p_i)}, i = 1, ..., k. \end{cases}$$

Theorem

Let u be the solution of the Jacobi-Toda equation obtained from bouncing Jacobi field with k minimums, which arises from maximizing the functional $\mathcal{H}$. Then the Morse index of the operator J^* is equal to $2k$.

Sketch of the proof

Consider the sequence of eigenfunctions η_ε such that

$$J^*\eta = -\lambda_\varepsilon \eta,$$

with $\lambda_\varepsilon > 0$. We normalize η_ε such that $\|\eta_\varepsilon\|_{L^\infty} = 1$.
Case 1. $\frac{\lambda_\varepsilon}{|\ln \varepsilon|^2} \to 0$, as $\varepsilon \to 0$.
We first of all analyze solutions of the equation

$$-\xi'' - \bar{c}\varepsilon^{-2}e^{-u}\xi = -\lambda\xi. \tag{3}$$

Define the rescaled the function $\mu = \xi\left(\frac{\varepsilon e^{\frac{\alpha}{2}} s}{\sqrt{\bar{c}}}\right)$ for suitable α. Let Γ be the even solution of the standard Toda equation with $\Gamma(0) = 0$. Then μ will satisfy the following normalized equation:

$$-\mu'' - e^{-\Gamma}\mu = -\frac{\lambda\varepsilon^2 e^\alpha}{\bar{c}}\mu.$$

By assumption, $\lambda\varepsilon^2 e^\alpha \to 0$ as $\varepsilon \to 0$. Now the behaviour of η around $\bar{p}_i^*$ is essentially determined by (3) . Hence using the same arguments as before, we find that η_ε converge to an eigenfunction of the problem (EVP).

Case 2. $\frac{\lambda_\varepsilon}{|\ln \varepsilon|^2} > \sigma > 0$, for some σ independent of ε.
In the interval $L \setminus \cup (p_i - \delta_i, p_i + \delta_i)$, the equation $J^* \eta = 0$ has the form

$$-\eta'' \sim -O\left(|\ln \varepsilon|^2\right) \eta.$$

This implies that

$$|\eta| \to 0, \text{ in } L \setminus \cup (p_i - \delta_i, p_i + \delta_i).$$

Hence the L^∞ norm of η is achieved around p_i. This implies that

$$\frac{\lambda \varepsilon^2 e^\alpha}{\bar{c}} \to \lambda^*,$$

where $-\lambda^*$ is the unique negative eigenvalue of the operator L :

$$\phi \to -\phi'' - e^{-\Gamma} \phi.$$

Moreover, the rescaled function μ converges to the eigenfunction of the operator L.

Now we want to show the number of negative eigenvalues of the problem (EVP) is equal to k. This corresponds to the fact that the Morse index of the critical point $(p_1, ..., p_k)$ of the function $\mathcal{H}$ is equal to k.
We first claim that the Morse index of the eigenvalue problem (EVP) is at most k. Suppose to the contrary that there were $k+1$ negative eigenvalues of (EVP) with corresponding normalized eigenfunctions $\gamma_1, ..., \gamma_{k+1}$, with

$$\int_L (\gamma_i \gamma_j) = \delta_{i,j},$$

where δ_{ij} is the Kronecher symbol. We can find a linear combination $\Gamma := c_1\gamma_1 + ... + c_{k+1}\gamma_{k+1}$, with $|c_i| \leq 1$, such that Γ equals zero at $p_1, ..., p_k$.

By the minimal property of each ϕ_i, for σ small,

$$I\left(\phi+\sigma\Gamma\right)\geq 0.$$

Here I is the energy functional $\int_L \left(\Phi'^2 - R\Phi^2\right) ds$. On the other hand, by the definition of negative eigenvalues and the computation of the second variation of the energy functional, we have $I\left(\phi+\sigma\Gamma\right) < 0$. This is a contradiction.

Hence the Morse index of the eigenvalue problem (EVP) is at most k.

This also implies that the Morse index of J^* is at most $2k$.

Next we prove that the Morse index of the eigenvalue problem (EVP) is at least k.
Since $(p_1, ..., p_k)$ is a maximizer of $\mathcal{H}$ and is nondegenerated,

$$\mathcal{H}\left(p_1 + e_1, ..., p_k + e_k\right) - \mathcal{H}\left(p_1, ..., p_k\right) \leq -\sigma\left(e_1^2 + ... + e_k^2\right).$$

Hence there are k linearly independent functions $\eta_1, ..., \eta_k$ such that

$$I\left(\phi + \sigma v\right) \leq 0,$$

for $v \in \mathrm{Span}\left\{\eta_1, ..., \eta_k\right\}$ and $|\sigma|$ small. This implies that the Morse index of the eigenvalue problem (EVP) is at least k.
Then we can show that the corresponding solutions of the Allen-Cahn equation has Morse index $2k$.

Solutions with higher multiplicity

The Jacobi-Toda system with $k(k > 2)$ components:

$$\phi_i'' + R\phi_i + \bar{c}\varepsilon^{-2}\left(e^{\phi_i - \phi_{i+1}} - e^{\phi_{i-1} - \phi_i}\right) = 0, \;\; i = 1, ..., k, \;\; \text{in } L.$$

To find smooth solution to this system with bounded Morse indices, we consider the corresponding Jacobi system

$$\phi_i'' + R\phi_i = 0, \;\; i = 1, ..., k. \tag{4}$$

The notion of bouncing Jacobi fields discussed before can be generalized to this system. The most general case is quite complicated. Here we consider the simplest nontrivial case for this system.

Multiple-component bouncing Jacobi fields

We are interested in multiple-component bouncing Jacobi fields of this Jacobi system satisfying the following properties:
(B1) There exist points $p_2, ..., p_k$ with $p_i \neq p_{i+1}$, such that ϕ_j is smooth in $L \setminus \{p_j, p_{j-1}\}$ for all j.
(B2) For $j = 2, ..., k$, ϕ_j is continuous and not of C^1 at p_j, p_{j+1}. Moreover,

$$\phi'_{j-1}(p_{j,+}) = \phi_j(p_{j,-}) \neq 0, \text{ and } \phi'_{j-1}(p_{j,-}) = \phi'_j(p_{j,+}) \neq 0.$$

(B3) For $j = 2, ...k-1$, $\phi_j - \phi_{j-1} \geq 1$, and

$$\phi_j(p_j) - \phi_{j-1}(p_j) = 1.$$

Existence of multiple-component bouncing Jacobi fields

Theorem

Suppose $R > 0$ and $\|R\|_{L^\infty} < \frac{1}{2k}$. Then the Jacobi system has a multiple-component bouncing Jacobi field.

Each component has two bouncing points, except the first and last components.

The proof of this result is a generalization in the case of one equation, and is of variational nature.

Sketch of the proof

For any point $p_k \in L$ and $h > 0$, under the assumption that $\|R\|_{L^\infty} < \frac{1}{2k}$, the minimization problem

$$\min_{\phi(p_k)=h} \int_L \left(\phi'^2 - R\phi^2\right) ds$$

has a unique solution ϕ_k. With ϕ_k at hand, we consider the initial value problems:

$$\begin{cases} \phi'' + R\phi = 0, \text{ for } s > p_k, \\ \phi\left(p_k\right) = h - 1, \phi'\left(p_k\right) = \phi_k'\left(p_{k,-}\right). \end{cases}$$

This ODE has a unique solution ϕ_{k-1}^+. Similarly, the problem

$$\begin{cases} \phi'' + R\phi = 0, \text{ for } s < p_k, \\ \phi\left(p_k\right) = h - 1, \phi'\left(p_k\right) = \phi_k'\left(p_{k,+}\right) \end{cases}$$

has a unique solution ϕ_{k-1}^-. They can be regarded as functions defined on L.

There is a point $p_{k-1} \in L$ such that

$$\phi_{k-1}^{+}\left(p_{k-1}\right) = \phi_{k-1}^{-}\left(p_{k-1}\right).$$

The functions ϕ_{k-1}^{+} patches with ϕ_{k-1}^{-} yielding a function ϕ_{k-1} defined on L. We then consider the initial value problems

$$\begin{cases} \phi'' + R\phi = 0, \text{ for } s > p_{k-1}, \\ \phi\left(p_{k-1}\right) = \phi_{k-1}\left(p_{k-1}\right) - 1, \phi'\left(p_{k-1}\right) = \phi_{k-1}'\left(p_{k-1,-}\right), \end{cases}$$

and

$$\begin{cases} \phi'' + R\phi = 0, \text{ for } s < p_{k-1}, \\ \phi\left(p_{k-1}\right) = \phi_{k-1}\left(p_{k-1}\right) - 1, \phi'\left(p_{k-1}\right) = \phi_{k-1}'\left(p_{k-1,+}\right). \end{cases}$$

We can again find $p_{k-2} \in L$ such that solutions of these two problems equal each other. Then these two solutions patch together and yield a function ϕ_{k-2} defined on L. Repeat this procedure and obtain functions ϕ_j and points p_j, with $j = k-1, ..., 2$. The assumption $\|R\|_{L^\infty} < \frac{1}{2k}$ ensures that this procedure is well defined.

Let ϕ_1 be solution of the minimization problem

$$\min_{\phi(p_2)=\phi_2(p_2)-1} \int_L \left(\phi'^2 - R\phi^2\right) ds.$$

Define the energy functional

$$G\left(p_k, h\right) = \sum_{j=1}^{k} \int_L \left(\phi_j'^2 - R\phi_j^2\right) ds.$$

If $\left(p_k, h\right)$ is a critical point of this function, then the corresponding functions $\phi_1, ..., \phi_k$ is the desired solution. Let

$$t_0 := \sup_{p_k \in L, h \in R} G\left(p_k, h\right).$$

As $|h| \to +\infty$, $G\left(p_k, h\right) \to -\infty$. Hence $t_0 < +\infty$. It then follows that t_0 is the maximum of G and it is achieved.