Gauss curvature type flows: convergence, stability and applications

Pengfei Guan
(McGill)

Recent Advances in Nonlinear PDEs and their Applications
CUHK, November 2, 2023

The Gauss curvature flow

$\Omega \subset \mathbb{R}^{n+1}$ bounded convex domain, $X : \partial\Omega \to \mathbf{R^{n+1}}$ position vector.
$K(x)$ Gauss curvature.

Gauss curvature flow:

$$\frac{\partial X(x,t)}{\partial t} = -K(x,t)\nu \tag{0.1}$$

Introduced by *Firey* to study *Shapes of worn stones.*

Mod (**PDE**), Firey reasoned flow contracts to a point after finite time.

Under symmetry condition, he proved that the point is round.

$$\text{Firey introduced entropy:} \quad \mathscr{E}_{\Omega(t)} = \int_{\mathbb{S}^n} \log u,$$

he proved it's monotone decreasing along normalized flow,

$$\frac{\partial X(x,t)}{\partial t} = -(K(x,t) - u)\nu.$$

PDE: Existence, contracts to a point after finite time. (*K.S. Chou*).

Monge-Ampere type equation.
Cheng-Yau, Pogorelov, Caffarelli-Nirenberg-Spruck, Krylov...

Flow by power of Gauss curvature,

$$\frac{\partial X(x,t)}{\partial t} = -K^{\alpha}(x,t)\nu \tag{0.2}$$

also contract to a point at finite time (*Andrews*).

The normalized flow of (0.2) with $|\Omega(t)| = |B_1|$:

$$\frac{\partial X(x,t)}{\partial t} = \left(-\frac{K^{\alpha}(x,t)}{\int_{\mathbb{S}^n} K^{\alpha-1}} + u\right)\nu. \tag{0.3}$$

(0.3) converges to sphere if

1. $\alpha = \frac{1}{n}$ (*Chow*), or
2. $n = 2, \alpha = 1$ (*Andrews*).

For $n \geq 3, \alpha > \frac{1}{n+2}$,

- Convergent to a soliton $K^\alpha = u$, (*Guan-Ni, Andrews-Guan-Ni*).
- Soliton is the unit sphere (*Brendle-Choi-Daskopolous*).

$\alpha = \frac{1}{n+2}$, Affine flow, converges to ellipsoid (*Andrews*).

$$X_t = -f^{\alpha}(\nu)K^{\alpha}\nu, \ \alpha > 0, \ \ 0 < f \in C^2(\mathbb{S}^n). \tag{0.4}$$

Finite time contraction (*Andrews*). The normalized flow

$$X_t = -\frac{f^{\alpha}(\nu)K^{\alpha}}{\fint_{\mathbb{S}^n} f^{\alpha}K^{\alpha-1}}\nu + X. \tag{0.5}$$

Soliton of flow (0.5)

$$\sigma_n(u_{ij} + u\delta_{ij}) = fu^{-p} \text{ on } \mathbb{S}^n, \quad p = \frac{1}{\alpha}. \tag{0.6}$$

Lutwak's L^p-Minkowski problem. The classical Minkowski problem corresponds to $p = 0$ in (0.6).

Entropy

$$\mathscr{E}_{\alpha,f}(\Omega) := \sup_{z_0 \in \Omega} \mathscr{E}_\alpha(\Omega, z_0),$$

$$\text{where } \mathscr{E}_\alpha(\Omega, z_0) = \frac{\alpha}{\alpha - 1} \log \left(\fint_{\mathbb{S}^n} u_{z_0}(x)^{1-\frac{1}{\alpha}} f(x) \right).$$

A variational problem:

$$\text{Minimize } \mathscr{E}_\alpha(\Omega) \text{ under constraint } |\Omega| = c. \tag{0.7}$$

A critical point of entropy functional is a solution to (0.6).

Find a path to the minimizer of the constraint problem (0.7).

Candidate: Flow (0.5).

Convergence of flow (0.5)?

In the case $f = 1$,

1. Entropy monotone along flow (0.5),
2. Entropy controls diameter,
3. there is entropy point estimate

$$\exists! z_e,\ \mathscr{E}_\alpha(\Omega) = \mathscr{E}_\alpha(\Omega, z_e), \quad dist(z_e, \partial\Omega) \geq \delta(d(\Omega), n, |\Omega|). \tag{0.8}$$

General f, *monotonicity* still holds

$$\mathscr{E}_{\alpha,f}(\Omega_{t_2}, z) - \mathscr{E}_{\alpha,f}(\Omega_{t_1}, z) = -\int_{t_1}^{t_2} \left(\frac{\fint_{\mathbb{S}^n} h^{\alpha+1}(x,t)\, d\sigma_t}{\fint_{\mathbb{S}^n} h(x,t)\, d\sigma_t \cdot \fint_{\mathbb{S}^n} h^{\alpha}(x,t)\, d\sigma_t} - 1 \right) dt,$$

$$h(x,t) \doteqdot f(x) u_z^{-\frac{1}{\alpha}}(x,t) K(x,t),\ d\sigma_t(x) = \frac{u_z(x,t)}{K(x,t)} d\theta(x).$$

Entropy point estimate (0.8) fails for $\mathscr{E}_{\alpha,f}$ in general $\forall \alpha > \frac{1}{n}$.

Weak convergence of flow (0.5) yields:

Theorem 1

For $\alpha > \frac{1}{n+2}$ and finite non-trivial Borel measure μ on $\mathbb{S}^n$, $n \geq 1$, there exists a weak solution of (0.6) provided:

(i) $\alpha > 1$ *and μ is not concentrated onto any great subsphere* $x^{\perp} \cap \mathbb{S}^n$, $x \in \mathbb{S}^n$.

(ii) $\alpha = 1$ *and μ satisfies that $\forall \ell$-subspace $L \subset \mathbb{R}^{n+1}$, $1 \leq \ell \leq n$,*

  (a) $\mu(L \cap \mathbb{S}^n) \leq \dfrac{\ell}{n+1} \cdot \mu(\mathbb{S}^n)$;

  (b) *"=" in (a) for a $L \subset \mathbb{R}^{n+1}$ implies $\exists$ a complementary linear $(n+1-\ell)$-subspace $\widetilde{L} \subset \mathbb{R}^{n+1}$ such that* $\operatorname{supp} \mu \subset L \cup \widetilde{L}$.

(iii) $\frac{1}{n+2} < \alpha < 1$ *and $d\mu = f\, d\theta$ for non-negative* $f \in L^{\frac{n+1}{n+2-\frac{1}{\alpha}}}(\mathbb{S}^n)$.

If f is bounded, it's a result of *Chou-Wang*.

1. $\alpha > 1$, *Chen-Li-Zhu*.
2. $\alpha = 1$, *Böröczky-Lutwak-Yang-Zhang* even case, *Chen-Li-Zhu* general case.
3. $\frac{1}{n+2} < \alpha < 1$, *Bianchi-Böröczky-Colesanti-Yang*.
4. For $\alpha < \frac{1}{n+2}$, there is a recent work of *Guang-Li-Wang*.

Conditions in Theorem 1 is for the **control diameter by entropy**.

Anisotropic approach was discussed in *Andrews-Böröczky-Guan-Ni* under symmetry assumptions.

Weak convergence of (0.5) proved by *Böröczky-Guan*.

(0.4) contract to a point z, assume it's the origin.

Lemma 2

Along (0.5),

(a). *The entropy* $\mathcal{E}_{\alpha,f}(\Omega_t)$ *is monotonically decreasing,*

$$\mathcal{E}_{\alpha,f}(\Omega_{t_2}) \le \mathcal{E}_{\alpha,f}(\Omega_{t_1}), \quad \forall t_1 \le t_2 \in [0,\infty). \tag{0.9}$$

(b). $\forall t_0 \in [0,\infty)$,

$$\mathcal{E}_{\alpha,f}(\Omega_{t_0},0) \ge \mathcal{E}_{\alpha,f,\infty} + \int_{t_0}^{\infty} \left(\frac{⨏_{\mathbb{S}^n} h^{\alpha+1}(x,t)\, d\sigma_t}{⨏_{\mathbb{S}^n} h(x,t)\, d\sigma_t \cdot ⨏_{\mathbb{S}^n} h^{\alpha}(x,t)\, d\sigma_t} - 1 \right) dt, \tag{0.10}$$

where

$$h(x,t) \doteqdot f(x) u_0^{-\frac{1}{\alpha}}(x,t) K(x,t), \ \mathcal{E}_{\alpha,f,\infty} \doteqdot \lim_{t\to\infty} \mathcal{E}_{\alpha,f}(\Omega_t).$$

Monotonicity ensures entropy bound. Conditions in Theorem 1 yields

$$D(\Omega(t)) \leq C. \tag{0.11}$$

$|\Omega(t)| = |B(1)|$, non-collapsing estimate

$$\frac{\rho_+(\Omega(t))}{\rho_-(\Omega(t))} \leq C,$$

where ρ_+ and ρ_- are the outer and inner radii of the convex body.

Set

$$\eta(t) = \oint_{\mathbb{S}^n} h(x,t)\, d\sigma_t. \tag{0.12}$$

As $\oint_{\mathbb{S}^n} h(x,t)\, d\sigma_t$ is monotone and bounded from below and above by diameter estimates,

$$C \geq \lim_{t\to\infty} \oint_{\mathbb{S}^n} h(x,t) = \eta \geq \frac{1}{C} \tag{0.13}$$

(0.10) implies that $\exists\{t_k\}_{k=1}^{\infty}$, $\lim_{k\to\infty} u(x,t_k) = u_\infty(x)$ and

$$\lim_{k\to\infty} \frac{\oint_{\mathbb{S}^n} h^{\alpha+1}(x,t_k)\,d\sigma_{t_k}}{\oint_{\mathbb{S}^n} h(x,t_k)\,d\sigma_{t_k} \cdot \oint_{\mathbb{S}^n} h^{\alpha}(x,t_k)\,d\sigma_{t_k}} = 1. \tag{0.14}$$

Holder Room: $\frac{1}{p}+\frac{1}{q}=1$, set $\beta = \min\{\frac{1}{p},\frac{1}{q}\}$, $\forall F \in L^p,\ G \in L^q$,

$$\frac{\int_M |FG|d\mu}{\|F\|_{L^p}\|G\|_{L^q}} - 1 \le -\beta \int_M \left(\frac{|F|^{\frac{p}{2}}}{(\int_M |F|^p d\mu)^{\frac{1}{2}}} - \frac{|G|^{\frac{q}{2}}}{(\int_M |G|^q d\mu)^{\frac{1}{2}}} \right)^2. \tag{0.15}$$

The extra room is crucial to deduce weak convergence of flow (0.5).

Denote $\sigma_{n,t_k}(x) = \sigma_n(u_{ij}(x,t_k) + u(x,t_k)\delta_{ij})$. Then

$$\lim_{k\to\infty} \oint_{\mathbb{S}^n} |u^{\frac{1}{\alpha}}(x,t_k)\sigma_{n,k}(x) - \frac{f(x)}{\eta}| = 0.$$

L^p Christoffel-Minkowski problem

$$\sigma_k(u_{ij}+u\delta_{ij}) = u^{-p}f, \quad \text{on } \mathbb{S}^n.$$

Soliton of

$$X_t = -f^{\alpha}\left(\frac{\sigma_n(\kappa)}{\sigma_{n-k}(\kappa)}\right)^{\alpha}\nu. \tag{0.16}$$

Open Problem: condition on f, (0.16) would contract to a point?

A form of flows for general F (not necessary homogeneous)

$$X_t = -f(\nu, X)F(\kappa)\vec{\nu}. \tag{0.17}$$

There is a recent work by *Guan-Huang-Liu*, but (0.16) does not satisfy structural condition.

Difficulty: upper bound of curvature, or lower bound of principal radii.

$$X_t = -f(K)\nu, \tag{0.18}$$

$$f(K) = K^{\alpha} + g(K),\ \exists \delta > 0,\ \lim_{s \to +\infty} \left|\frac{g(s)}{s^{\alpha-\delta}}\right| = 0.$$

Chen-Guan-Huang:

Theorem 3

X_0 strictly convex smooth hypersurface in $\mathbb{R}^{n+1}$, for $\alpha > \frac{1}{n+1}$, flow (0.18) converges to a round sphere in $\mathbb{R}^{n+1}$ in the C^{∞}-topology after re-scaling.

Open Problem: convergence for $\frac{1}{n+1} \geq \alpha > \frac{1}{n+2}$.

Normalized flow

$$X_t = X - \frac{f(e^{nt}K)}{\fint_{\mathbb{S}^n} \frac{f(e^{nt}K)}{K}} \nu, \tag{0.19}$$

Recall entropy

$$\mathscr{E}_\alpha(\Omega) := \sup_{z_0 \in \Omega} \frac{\alpha}{\alpha - 1} \log \fint_{\mathbb{S}^n} u_{z_0}^{1-\frac{1}{\alpha}} \tag{0.20}$$

Lemma 4

$\alpha > \frac{1}{n+1}$, along flow (0.19), $\exists C > 0,\ \gamma > 0,\ \mathscr{E}_\alpha(\Omega_t) + Ce^{-\gamma t}$ is non-increasing.

Thus entropy is bounded, so is the diameter.

Proof of monotonicity in easy case $\alpha = 1$, u satisfies

$$u_t = u - \frac{K + e^{-nt} g(e^{nt}K)}{1 + \fint_{\mathbb{S}^n} e^{-nt} g(e^{nt}K)}.$$

$$
\begin{aligned}
\frac{d\mathscr{E}_\alpha(\Omega_t)}{dt} &\le \frac{\fint_{\mathbb{S}^n}(1+\frac{g(e^{nt}K)}{e^{nt}K}) - \fint_{\mathbb{S}^n} u_{e(t)}^{-1}(K+e^{-nt}g(e^{nt}K))}{1+\fint_{\mathbb{S}^n}\frac{g(e^{nt}K)}{e^{nt}K}} \\
&\le \frac{\fint_{\mathbb{S}^n}(1+\frac{g(e^{nt}K)}{e^{nt}K}) - \fint_{\mathbb{S}^n}(-\frac{u}{K}+2\sqrt{1+\frac{g(e^{nt}K)}{e^{nt}K}})}{1+\fint_{\mathbb{S}^n}\frac{g(e^{nt}K)}{e^{nt}K}} \\
&\le \frac{C\fint_{\mathbb{S}^n}\frac{g(e^{nt}K)}{e^{nt}K}}{1+\fint_{\mathbb{S}^n}\frac{g(e^{nt}K)}{e^{nt}K}} \le \frac{Ce^{-n\delta t}\fint_{\mathbb{S}^n}K^{-\delta}}{1+Ce^{-n\delta t}\fint_{\mathbb{S}^n}K^{-\delta}}.
\end{aligned}
$$

Blaschke-Santaló inequality and Hölder inequality, $\fint_{\mathbb{S}^n} u_s K^{-1} = 1$

$$
\fint_{\mathbb{S}^n} K^{-\delta} d \le \left(\fint_{\mathbb{S}^n} u_s^{\frac{-\delta}{1-\delta}}\right)^{1-\delta} \le \left(\fint_{\mathbb{S}^n} u_s^{-(n+1)}\right)^{\frac{\delta}{n+1}} = 1.
$$

Convergent to sphere: *Andrew-Guan-Ni*, *Brendle-Choi-Daskopolous*.

Gauss curvature flow in space forms

$N^{n+1}(\kappa)$ $(\kappa = \pm 1)$ space form, flow by power of Gauss curvature

$$X_t = -K^{\alpha}\nu \tag{0.21}$$

$\mathbb{S}^{n+1}_{+} \to \mathbb{R}^{n+1}$: Projecting $\mathbb{S}^{n+1}_{+}$ to its tangent space at north pole.

$\mathbb{H}^{n+1} \to \mathbb{R}^{n+1}$: Beltrami transformation.

$(N^{n+1}, \bar{g})$ warped product space,

$$\bar{g} = d\rho \cdot d\rho + \phi^2(\rho) g_{\mathbb{S}^n}.$$

2nd FF:

$$h_{ij} = \sigma(\sqrt{\phi^2 + |\nabla\rho|^2})^{-1}(-\phi\rho_{ij} + 2\phi'\rho_i\rho_j + \phi^2\phi'\delta_{ij}).$$

Set $r = \frac{\phi}{\phi'}$,

$$h_{ij} = q(r, \nabla r)(-r r_{ij} + 2 r_i r_j + r^2\delta_{ij}).$$

$$K_{N^{n+1}} = Q(r, \nabla r) K_{\mathbb{R}^{n+1}}.$$

Flow (0.1) in $N^{n+1}(\kappa)$ is converted to flow in $\mathbb{R}^{n+1}$

$$\frac{\partial X(x,t)}{\partial t} = -\psi(\|X\|, X\cdot\nu)K^{\alpha}(x,t)\nu, \quad \text{in } \mathbb{R}^{n+1}, \tag{0.22}$$

$$\psi = (1+\kappa\|X\|^2)^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa(X\cdot\nu)^2)^{-\frac{n+2}{2}\alpha+\frac{1}{2}}.$$

It's a special case of (0.17).

Work of *Chen-Huang*,

Theorem 5

$\forall\alpha > 0$, flow (0.21) contracts a point at a finite time T. $\forall\alpha > \frac{1}{n+2}$, it converges to a round geodesic ball in $\mathbb{N}^{n+1}(\kappa)$ in the C^∞-topology after re-scaling.

Normalization:

$$X_t = X - \frac{\psi K^\alpha}{\int_{\mathbb{S}^n} \psi K^{\alpha-1}} \nu, \tag{0.23}$$

$$\psi = (1+\kappa\frac{|X|^2}{e^{2t}})^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\frac{u^2}{e^{2t}})^{-\frac{n+2}{2}\alpha+\frac{1}{2}}.$$

Almost monotonicity.

Lemma 6

$\exists C, t^*$, under the normalized flow (0.23), $\mathscr{E}_\alpha(\Omega_t) + Ce^{-\frac{2(n+1)}{2n+1}t}$ is non-increasing when $t \geq t^*$ sufficiently large. Furthermore, $\forall \alpha \geq \frac{1}{n+2}$.

$$\lim_{t\to\infty}\left(\mathscr{E}(\Omega_t) + Ce^{-\frac{2(n+1)}{2n+1}t}\right) = \mathscr{E}_\alpha^\infty \in \mathbb{R}.$$

With entropy under control, the convergence to sphere follows from works of *Andrews-Guan-Ni* and *Brendle-Choi-Daskopolous*.

Thank You