Regularity and Turbulence

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$$E(k) = C\epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}, \quad \text{in inertial range: } k \leq k_d.$$

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Four-fifths law: homogeneous and isotropic turbulence, third order longitudinal moment: $\langle ((u(x+\ell) - u(x)) \cdot \frac{\ell}{|\ell|}))^3 \rangle = -\frac{4}{5}\epsilon|\ell|.$

Intermittency

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Structure functions:

$$\langle |u(x+\ell) - u(x)|^p \rangle \sim (\epsilon|\ell|)^{\frac{p}{3}} \left(\frac{|\ell|}{L}\right)^{-\alpha_p} = CU^p \left(\frac{|\ell|}{L}\right)^{\zeta_p}$$

with $\zeta_p = \frac{p}{3}$ for K'41. Nonzero $\alpha_p =$ intermittency corrections.

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and Reynolds number to infinity

$$Re = \frac{UL}{\nu}.$$

Mathematical considerations

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Forced NSE equations, smooth regime. Long time averages

$$-\fint \frac{dE(t)}{dt} dt = \nu \fint \|\nabla u(x,t)\|_{L^2}^2 dt - \fint (f(t)\cdot u(t))_{L^2} dt$$

What should ϵ be? Normally, with $\nu = 0$, $E(t)$ is unbounded. If $E(t)$ is bounded then the ϵ defined as the difference in the RHS, is zero.

Multiscale solutions

Theorem

Let Ω be an open set in $\mathbb{R}^3$. For any $\alpha \in (1,3)$, there exist families of smooth stationary solutions of the forced Navier-Stokes equations

$$u \cdot \nabla u + \nabla p = \nu \Delta u + F, \qquad \nabla \cdot u = 0,$$

with $u \in C_0^\infty(\Omega)$, $\nabla p \in C_0^\infty(\Omega)$, and such that $\nu \|\nabla u\|^2_{L^2(\Omega)}$ is bounded below, independently of ν as $\nu \to 0$. There is an inertial range of wave numbers $k \in [k_0, k_d]$ such that the dissipation wave number $k_d \sim \nu^{-\frac{1}{3-\alpha}}$ diverges with $\nu \to 0$ and the energy spectrum $E(k)$ obeys

$$E(k) \sim k^{-\alpha} \tag{1}$$

in the inertial range. $\alpha = \frac{5}{3}$, i.e. K41 is singled out as having the only scale independent prefactor. The force F is smooth, compactly supported, and bounded in $L^p(\Omega)$ for $p \in [1, \frac{6}{2+\alpha})$, $1 \leq \alpha \leq 2$ (and $p \in [1, \frac{6}{8-\alpha}]$, when $\alpha \in [2,3]$).

Compactly supported steady Euler solutions

Axisymmetric, steady, with swirl, compactly supported solutions found by Gavrilov. Constructed using the Grad-Shafranov equation by La, Vicol, -C:

$$u = \frac{\partial_z \psi}{r} e_r - \frac{\partial_r \psi}{r} e_z + \frac{F}{r} e_\phi$$

with $\psi = \psi(r, z)$ and $F = F(\psi)$, F arbitrary. Then $P = P(\psi)$ arbitrary, and ψ solves

$$\Delta^* \psi + FF' + \frac{P'}{r^2} = 0$$

where $' = \frac{d}{d\psi}$ and the Grad-Shafranov operator is

$$\Delta^* \psi = \partial_r^2 \psi - \frac{\partial_r \psi}{r} + \partial_z^2 \psi.$$

This set-up leads to steady solutions of the Euler equations with

$$P + p + \frac{|u|^2}{2} = constant.$$

Localizable, if and only if $u \cdot \nabla p = 0$, i.e. if and only if

$$u \cdot \nabla |u|^2 = 0.$$

Construction of multiscale solns NSE

$$u_B(x) \cdot \nabla u_B(x) + \nabla p_B(x) = 0, \quad \nabla \cdot u_B = 0$$

in the unit hollow annulus $A = \{x = (r, z) \mid \frac{1}{2} < |r - 1|^2 + |z|^2 < 1\}$ with

$$u_B \in (C_0^\infty(A))^3, \quad \nabla p_B \in C_0^\infty(A)$$

(constructed in La-Vicol-C after Gavrilov).

Take $\Omega \subset \mathbb{R}^3$, a sequence of points $x_j \in \Omega$, rotations $R_j \in O(3)$, and numbers $L > 0$, $T > 0$, $\ell > 0$ $\tau > 0$, with associated length scales and time scales

$$\ell_j = L2^{-\ell j}, \qquad \tau_j = T2^{-\tau j},$$

for $j = 1, 2, \ldots,$ such that functions

$$u_j(x) = \frac{L}{T} 2^{(\tau - \ell)j} R_j u_B \left(2^{\ell j} \frac{R_j^*(x - x_j)}{L} \right)$$

have disjoint supports in dilated, translated and tilted hollow annuli (a string of lifesavers, linked or not)

$$A_j = x_j + \ell_j R_j(A) \subset \Omega$$

Remark: choice of τ_j totally free, in particular they could be chosen differently and change sign with j.

Properties and choices

Supports of gradients of pressures ∇p_j also disjoint. Therefore, for any $N \geq 1$,

$$u(x) = \sum_{j=1}^{N} u_j(x).$$

solves the steady incompressible Euler equation. Note $u \in C_0^\infty(\Omega)^3$. Fix $\tau = a\ell$ with $a \in (\frac{3}{2}, \frac{5}{2})$. Keep L, T, ℓ free. Range of a: energy bounded, enstrophy diverging as $N \to \infty$.

$$k = L^{-1} 2^{\ell j},$$

The energy spectrum $E(k)$ is by definition the contribution of the kinetic energy at scale k, per unit mass and per scale:

$$E(k) = L^{-3} k^{-1} \|u_j\|^2_{L^2(\Omega)}$$

It follows that

$$E(k) = \frac{L^3}{T^2}(kL)^{-\alpha}, \qquad \alpha = 6 - 2a \in (1, 3).$$

Forced Navier-Stokes

$$\nu = \frac{L^2}{T} 2^{-(3-\alpha)N\ell}$$

$$\epsilon = \nu L^{-3} \|\nabla u\|^2_{L^2(\Omega)} = \frac{L^2}{T^3} C_1 .$$

$$E(k) = C_1^{-\frac{2}{3}} L^{\frac{5}{3}} \epsilon^{\frac{2}{3}} (kL)^{-\alpha}$$

K41: $\alpha = \frac{5}{3} \in (1, 3)$. ($a = \frac{13}{6}$).

$$E(k) = C_K \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

Only spectrum independent of L. The smallest scale $\ell_d = L2^{-N\ell}$ in terms of ϵ, ν:

$$\ell_d^{-1} = k_d = C_1^{-\frac{1}{3(3-\alpha)}} L^{\frac{4}{3(3-\alpha)}-1} \epsilon^{\frac{1}{3(3-\alpha)}} \nu^{-\frac{1}{3-\alpha}}$$

K41: $\alpha = \frac{5}{3}$ and familiar expression, independent of L:

$$k_d^{-1} = \ell_d = C \nu^{\frac{3}{4}} \epsilon^{-\frac{1}{4}} .$$

Incompressible Fluid Pressure

Momentum equation in homogeneous (constant density) incompressible Newtonian fluids

$$\partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p$$

with incompressibility constraint

$$\nabla \cdot u = 0$$

Gradient of pressure is the driver.

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Gradient of pressure is the driver. Pressure equation

$$-\Delta p = \nabla \cdot (u \cdot \nabla u)$$

In the absence of pressure: max principle, no NS singularity.

Intermittency and regularity

Known high derivatives a priori estimates:

$$\int_0^T \|u(t)\|_{H^m}^{\frac{2}{2m-1}} dt < \infty, \ m \geq 1 \quad \text{(Foias-Guillope-Temam '80)}$$

with consequences

$$\int_0^T \|u(t)\|_{L^\infty} dt < \infty, \ \text{(Tartar '80)} \quad \int_0^T \|\nabla u\|_{L^\infty}^{\frac{1}{2}} dt < \infty \ (C'90).$$

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By a different method, using vorticity:

$$\int_0^T \|\Delta(u(t))\|_{L^{\frac{4}{3},q}}^{\frac{4}{3}} dt < \infty \ \text{(C '90, Lions '96, Vasseur '10, Vasseur-Yang '21)}$$

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Grujic and collaborators: assuming sparse upper level sets of high derivatives, the gap between the known a priori estimates and conditions for global regularity vanishes asymptotically, as the number of derivatives tends to infinity. Cheskidov and Shvykoy: conditions based on Littlewood-Paley components: if intermittency "dimension" $< 3/2$.

Sufficient conditions for regularity, Navier-Stokes

Ladyzhenskaya-Prodi-Serrin:
$u \in L^p(dt; L^q(dx)),\ \frac{2}{p} + \frac{3}{q} = 1, \quad q > 3$. In particular,

$$\int_0^T \|u\|_{L^\infty}^2 dt < \infty \Rightarrow u \in C^\infty.$$

$q = 3$:

$$\sup_{t \le T} \|u\|_{L^3} < \infty \Rightarrow u \in C^\infty$$

(Escauriaza-Seregin-Sverak).
In terms of the pressure:

$$\int_0^T \|p\|_{L^3}^2 dt < \infty \Rightarrow u \in C^\infty$$

(Berselli, Galdi).

$$\inf_{x,t} p > -\infty \Rightarrow u \in C^\infty$$

(Seregin, Sverak).

Quantitative LPS

Theorem

Let $3 < q \leq \infty$. *If*

$$\int_0^T \|u(t)\|_{L^q}^{\frac{2q}{q-3}} dt \leq M_q < \infty,$$

then

$$\|u(t)\|_{\dot{H}^1}^2 \leq \|u(0)\|_{\dot{H}^1}^2 \exp\left[C\nu^{-\frac{q+3}{q-3}} \int_0^t \|u(s)\|_{L^q}^{\frac{2q}{q-3}} ds\right].$$

for $0 \leq t \leq T$. *Accessible.*

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Strong solutions: Given $u(T_0) \in V$ where V is the closure of C_0^∞ divergence-free fields in H^1, there exists a unique

$$u \in L^\infty(T_0, T_0 + \tau; V) \cap L^2(T_0, T_0 + \tau; H^2 \cap V).$$

Strong solutions are C^∞ smooth (unless boundary conditions or forcing obstruct).

Criterion in terms of pressure

Theorem

There exists an absolute constant C, such that, if $p = R_i R_j(u_i u_j)$ *satisfies the finite uniform integrability condition*

$$\exists\delta > 0, \forall t, \forall A \quad |A| \leq \delta \Rightarrow \int_A |p(x,t)|^{\frac{3}{2}} dx \leq \left(\frac{\nu}{C}\right)^3$$

for all $t \in [0, T]$*, then* $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$*.*

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for all $t \in [0,T]$, then $u \in L^\infty(0,T;L^3(\mathbb{R}^3))$. Moreover for $r \geq 4$,

$$\|u(\cdot,t)\|_{L^r} \leq \|u_0\|_{L^r} \exp\left(\frac{Ct\|u_0\|_{L^2}^2}{\nu\delta}\right).$$

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Accessible.

Condition is weaker than uniform integrability, because $\frac{\nu}{C}$ is fixed. It leads to explicit quantitative bounds on the enstrophy.

Ladyzhenskaya- Prodi-Serrin for Pressure

Theorem

Let $p = R_i R_j (u_i u_j)$. *Assume that there exists* $q > \frac{3}{2}$ *such that*

$$\int_0^T \|p(t)\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{2q-3}} dt < \infty$$

Then $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$ *and* u *is smooth on* $[0, T]$.

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Note that $q = \infty$ is allowed and the condition is

$$\int_0^T \|p(t)\|_{L^\infty} dt < \infty.$$

Structure function and regularity

Let

$$S_2(x,t)(r) = \int_{|y|\le 2r} \frac{|u(x+y)-u(x)|^2}{|y|^3}\,dy$$

Theorem

There exists an absolute constant C such that if

$$\exists\delta>0,\ \exists r(t),\ \int_0^T r^{-4}(t)dt<\infty,$$
$$|A|\le\delta \Rightarrow \int_A (S_2(x,t)(r))^{\frac{3}{2}}\,dx \le \left(\frac{\nu}{C}\right)^3$$

holds for on $[0,T]$*, then the solution of the Navier-Stokes equation is smooth on* $[0,T]$.

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holds for on $[0,T]$*, then the solution of the Navier-Stokes equation is smooth on* $[0,T]$*. Moreover, Accessible: for* $q \ge 4$

$$\|u(\cdot,t)\|_{L^q} \le \|u_0\|_{L^q} \exp\left(\frac{Ct\|u_0\|_{L^2}^2}{\nu\delta} + \nu^{-\frac{3}{2}}\|u_0\|_{L^2}^2\Gamma(t)\right)$$

where

$$\Gamma(t) = \sqrt{\int_0^t r^{-4}(s)ds}.$$

Nearly selfsimilar example

Let

$$u(x,t) = V + \text{smooth}$$

where the profile V satisfies

$$\|V(z+\cdot) - V(\cdot)\|_{L^3} \le UL\left(\frac{|z|}{L}\right)^s$$

for some $s > 0$. Let us note that this condition is satisfied by many functions with slow decay which are not in L^3 or even in L^2, such as $V(y) = (1+|y|)^{-\beta}$, $\beta > 0$. Of course, the condition is also satisfied on $B^s_{3,\infty}(\mathbb{R}^3)$.

We have

$$\int_A S_2(x,r)^{\frac{3}{2}} dx \le C_s (UL)^3 \left(\frac{r}{L}\right)^{3s}$$

We take

$$\frac{UL}{\nu} = Re(V) \le R$$

and see, by taking $\left(\frac{r}{L}\right)^{3s} \le C_s^{-1}(2CR)^{-3}$

$$\int_0^T L^{-4} dt < \infty,\ Re(V) \le R \quad \Rightarrow \text{ Regularity}$$

A Dini condition

Theorem

Assume that u satisfies

$$\|\delta_y u\|_{L^3} \le m(|y|)$$

where $\delta_y u(x,t) = u(x+y,t) - u(x,t)$, *and where* $0 \le m$ *is a time independent function satisfying*

$$\int_0^1 m^2(\rho) \frac{d\rho}{\rho} < \infty.$$

Then u is smooth on $[0, T]$

Clearly $m(r) \sim \log^{-\alpha}(r^{-1})$ with $\alpha > \frac{1}{2}$ is enough.

An L^q Dini Condition

Theorem

Let $3 < q \leq \infty$. Assume that u satisfies

$$\|\delta_y u\|_{L^q} \leq m(|y|, t)$$

where $\delta_y u(x,t) = u(x+y,t) - u(x,t)$, and where $0 \leq m$ is a time dependent function satisfying

$$\int_0^T \left[\int_0^1 m^2(\rho, t) \frac{d\rho}{\rho}\right]^{\frac{q}{q-3}} dt < \infty.$$

Then u is smooth on $[0, T]$

The case $q = \infty$ is allowed, and, in that case it is required that

$$\int_0^T \int_0^1 m^2(\rho, t) \frac{d\rho}{\rho} dt < \infty$$

Time dependent regions of interest

Theorem

Let $U(t)$, $G(t)$ and $r(t)$ be positive numbers such that

$$\int_0^T (r(t)^{-4} + U(t)^2 + G(t))dt < \infty.$$

Consider the set

$$B(t) = \{x \mid |u(x,t)| \geq U \text{ and } |\nabla u(x,t)| \geq G\}.$$

There exists an absolute constant C such that, if

$$\int_{|y| \leq r(t)} \left(\int_{B(t)} |\delta_y u(x,t)|^3 dx \right)^{\frac{2}{3}} \frac{dy}{|y|^3} \leq \left(\frac{\nu}{C} \right)^2$$

then the solution of Navier-Stokes equations is smooth and obeys explicit a priori bounds.

Multifractal scenario

We assume that the velocity increments

$$s_2(x,r) = \fint_{|y|=r} |u(x+y) - u(x)|^2 dS(y)$$

obey bounds

$$s_2(x,r) \le G^2 \left(\frac{r}{L}\right)^{2\alpha(x)}, \; S_2(x,r_0) \le CG^2 \frac{1}{\alpha(x)} \left(\frac{r_0}{L}\right)^{2\alpha(x)}$$

a.e. in $x \in B(t)$ with $0 < r < r_0 < L$.
In multifractal turbulent intermittent scenarios, it is assumed that there is a spectrum of near-singularities of Hölder exponent h and that these are achieved on sets Σ_h of dimension $d(h) \le 3$ which occur randomly with probability $d\mu(h)$.

Implementation

A region V_h around Σ_h, partition it in small disjoint cubes of size ρ with $\rho \leq r_0$. The multifractal assumption is that the number of such cubes of V_h is of the order $N_h(\rho) = \left(\frac{\rho}{L}\right)^{-d(h)}$. Assuming $\alpha(x) \geq h$ to hold on each such cube, we have

$$S_2(x, r_0) \leq CG^2 h^{-1} \left(\frac{r_0}{L}\right)^{2h}.$$

It follows that

$$\int_{B_U \cap V_h} S_2(x, r_0)^{\frac{3}{2}} dx \leq C(GL)^3 h^{-\frac{3}{2}} \left(\frac{r_0}{L}\right)^{3-d(h)+3h}.$$

Summing in h, remembering the frequency, we obtain

$$\int_{B_U \cap (\cup_h V_h)} S_2^{\frac{3}{2}}(x, r_0) dx \leq C(GL)^3 \int_0^1 h^{-\frac{3}{2}} \left(\frac{r_0}{L}\right)^{3-d(h)+3h} d\mu(h)$$

In the multifractal formalism, the structure function exponents are defined by

$$\zeta_p = \inf_h (3 - d(h) + ph).$$

$$\int_{B \cap (\cup_h V_h)} S_2(x, r_0)^{\frac{3}{2}} dx \leq C_\mu (GL)^3 \left(\frac{r_0}{L}\right)^{\zeta_3}$$

Euler equations

Notation:

$$a \mapsto X(a,t), \quad \partial_t X(a,t) = u(X(a,t),t).$$

$$D_t = \partial_t + u \cdot \nabla, \quad (D_t f) \circ X = \partial_t (f \circ X).$$

$$\nabla u = S + J, \quad S = \frac{1}{2}(\nabla u + (\nabla u)^T), \quad J = \frac{1}{2}(\nabla u - (\nabla u)^T)$$

$$Jv = \omega \times v$$

$$D_t \omega = S\omega, \qquad \text{“vorticity equation”}$$

$$D_t S + S^2 + J^2 + P = 0, \qquad \text{“S equation”}$$

$$J^2 = -\frac{|\omega|^2}{4}(I - \xi \otimes \xi)$$

$$P = \nabla\nabla p, \qquad \xi = \frac{\omega}{|\omega|}$$

$$D_t|\omega| = \alpha|\omega|, \qquad \alpha = PV \int D(\xi(x), \xi(x+y), y)|\omega(y)||y|^{-3} dy$$

Nonnegative Pressure Hessian blow up

If the matrix $P = \nabla\nabla p$ is nonnegative in a weak vorticity region carried by the flow, then there is blow up.

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Theorem

(C '95). (A simplified version). Let a be a marked point and assume that $T > 0$ is a (coherence) time with the properties that the vorticity is bounded and the pressure Hessian is nonnegative on the trajectory $x = X(a, t)$, $t \le T$:

$$\sup_{0 \le t \le T, x = X(a,t)} |\omega(x,t))| \le \Omega,$$

$$P(x,t)v \cdot v \ge 0, \quad \forall v \in \mathbb{R}^3, \ \text{for } x = X(a,t), \ t \le T.$$

Assume there exists a unit vector $v_0 \in \mathbb{S}^2$ such that

$$\sigma_0 := S(a,0)v_0 \cdot v_0 < -\frac{\Omega}{2}.$$

Then blow up occurs if the coherence time is long enough

$$T\Omega \ge \log\left(1 + \frac{2\Omega}{2|\sigma_0| - \Omega}\right).$$

Pressure Hessian component bound: no blow up

Theorem

(Chae-C, '21) Let $(u,p) \in C^1(\mathbb{R}^3 \times (0,T))$ *be a solution of the Euler equation with* $u \in C([0,T); W^{2,q}(\mathbb{R}^3))$, *for some* $q > 3$. *If*

$$\int_0^T \exp\left(\int_0^t ds \int_0^s \|[\zeta \cdot P\xi]_-(\tau)\|_{L^\infty} d\tau\right) dt < +\infty,$$

then $\limsup_{t\to T} \|u(t)\|_{W^{2,q}} < +\infty$.

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then $\limsup_{t\to T} \|u(t)\|_{W^{2,q}} < +\infty$. *In particular, if*

$$\limsup_{t\to T} (T - t)^2 \|[\zeta \cdot P\xi]_-(t)\|_{L^\infty} < 1$$

then $\limsup_{t\to T} \|u(t)\|_{W^{2,q}} < +\infty$.

Above: $[x]_- = \max\{-x, 0\}$,

$$\xi = \frac{\omega}{|\omega|}, \qquad \zeta = \frac{S\xi}{|S\xi|}.$$

Remarks

Because
$$[\zeta \cdot P\xi]_- \leq \|P\|$$
we have in particular that if
$$\int_0^T \exp\left(\int_0^t ds \int_0^s \|P(\tau)\|_{L^\infty} d\tau\right) dt < +\infty,$$
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Recall models (Vieillefosse, C'94) ($D_t \to \partial_t$) and the blow up for nonnegative P.

Localized version

Theorem

(Chae-C, '21) Let Let $(u,p) \in C^1(B(x_0,r)\times(T-r,T))$ *be a solution of the Euler equations with* $u \in C([T-r,T); W^{2,q}(B(x_0,r))) \cap L^\infty(T-r,T; L^2(B(x_0,r)))$ *for some* $q \in (3,\infty)$. *We suppose*

$$\int_{T-r}^{T} \|u(t)\|_{L^\infty(B(x_0,r))} dt < +\infty,$$

If

$$\int_{T-r}^{T} \exp\left(\int_0^t ds \int_0^s \|[\zeta\cdot P\xi]_-(\tau)\|_{L^\infty(B(x_0,r))} d\tau\right) dt < +\infty,$$

then for all $\epsilon \in (0,r)$ $\limsup_{t\nearrow T} \|u(t)\|_{W^{2,q}(B(x_0,\epsilon))} < +\infty$. *If*

$$\limsup_{t\to T} (T-t)^2 \|[\zeta\cdot P\xi]_-(t)\|_{L^\infty(B(x_0,r))} < 1,$$

then for all $\epsilon \in (0,r)$ $\limsup_{t\nearrow T} \|u(t)\|_{W^{2,q}(B(x_0,\epsilon))} < +\infty$.

Main idea of proof

We have

$$D_t^2\omega + P\omega = 0.$$

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Lagrangian variables:

$$\widetilde{\omega} = \omega\circ X, \qquad q = (P\xi,\zeta)\circ X$$

So, we have

$$\frac{d}{dt}\left|\frac{d}{dt}\widetilde{\omega}\right| + q|\widetilde{\omega}| = 0,$$

and, integrating in time,

Hessian bound implies no blow up, continued

$$\left|\frac{d}{dt}\widetilde{\omega}\right| = \gamma_0 - \int_0^t q|\widetilde{\omega}|d\tau$$

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Now

$$\frac{d}{dt}|\widetilde{\omega}| \leq \left|\frac{d}{dt}\widetilde{\omega}\right|$$

and so

$$|\widetilde{\omega}(t)| \leq |\widetilde{\omega}_0| + \gamma_0 t + \int_0^t ds \int_0^s q_-|\widetilde{\omega}|d\tau$$

holds pointwise, at fixed label a.

Hessian bound implies no blow up, continued

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$$|\widetilde{\omega}(t)| \leq |\widetilde{\omega}_0| + \gamma_0 t + \int_0^t ds \int_0^s q_-|\widetilde{\omega}|d\tau$$

holds pointwise, at fixed label a. We have a Gronwall Lemma:

Lemma

If $\alpha(t)$ is nondecreasing and $\beta(t) \geq 0$ then, for $y(t) \geq 0$ we deduce from

$$y(t) \leq \alpha(t) + \int_0^t ds \int_0^s \beta(\tau)y(\tau)d\tau$$

that

$$y(t) \leq \alpha(t)e^{\int_0^t ds \int_0^s \beta(\tau)d\tau}$$

Idea of proofs of conditional regularity NSE

All proofs based on a decomposition of the hydrodynamic pressure at scale r, with

$$\overline{f}(x,r) = \frac{1}{4\pi r^2}\int_{|x-y|=r} f(y)dS(y)$$

and the quantity $b(x,r)$ defined by

$$\boxed{b(x,r) = \overline{p}(x,r) + \frac{1}{4\pi r^2}\int_{|x-y|=r}\left(\frac{y-x}{|y-x|}\cdot u(y)\right)^2 dS(y)}$$

which obeys a local equation

Lemma

(C '13) Let Ω be an open set in $\mathbb{R}^3$, let $x \in \Omega$. Let $r < dist(x,\partial\Omega)$, and let u be a divergence-free vector field in $C^2(\Omega)^3$. Let $v \in \mathbb{R}^3$. Let p solve its equation in Ω. Then

$$r\partial_r b(x,r) + \frac{1}{4\pi}\int_{|\xi|=1}\sigma_{ij}(\xi)w_i(x+r\xi)w_j(x+r\xi)dS(\xi) = 0$$

where $\sigma_{ij}(\xi) = 3\xi_i\xi_j - \delta_{ij}$ and $w(x+r\xi) = u(x+r\xi) - v$.

Representation formulas

$$p(x) = \beta(x, r) + \pi(x, r)$$

This is valid for any $r > 0$.

- $\beta(x, r)$ is an explicit average of p at distance less than $2r$.

$$\beta(x, r) = \frac{1}{r}\int_r^{2r} \overline{p}(x, \rho)\, d\rho$$

- $p(x) - \beta(x, r)$ vanishes for harmonic functions.
- $\beta(x, r)$ obeys good bounds at fixed r.
- $\pi(x, r)$ has explicit integral representation in terms of squares of increments of velocity $u(x) - u(y)$ for $|x - y| \leq 2r$.
- $\pi(x, r)$ vanishes quadratically in r for almost all t.

Recap

- Beware of manufactured forces. Long time statistics are the main issue.
- Regularity conditions exist which permit multiscale, even multifractal scenarios.
- For Euler equations, a bound on one (well chosen) component of the Hessian of pressure implies regularity.