Spiral strategies for blocking fire

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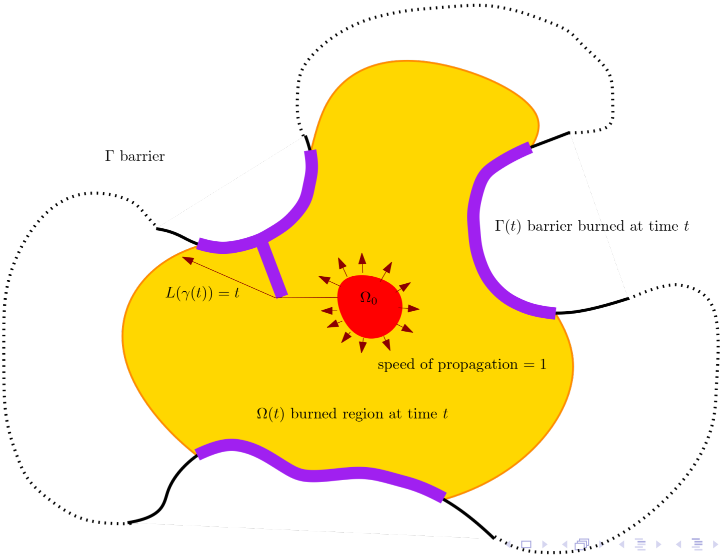
Fire blocking problem



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- The barrier is *admissible* if the part $\Gamma(t) = \Gamma \cap \mathrm{clos}(\Omega(t))$ reached by the fire has length $\leq \sigma t$,

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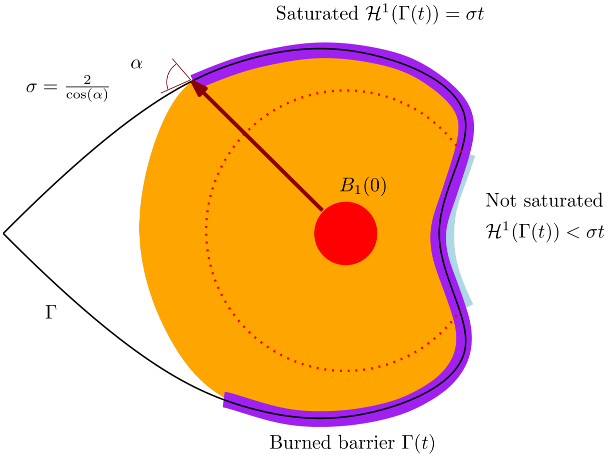
The *fire blocking problem* is whether there exists an admissible barrier (strategy) encircling the fire in finite time.

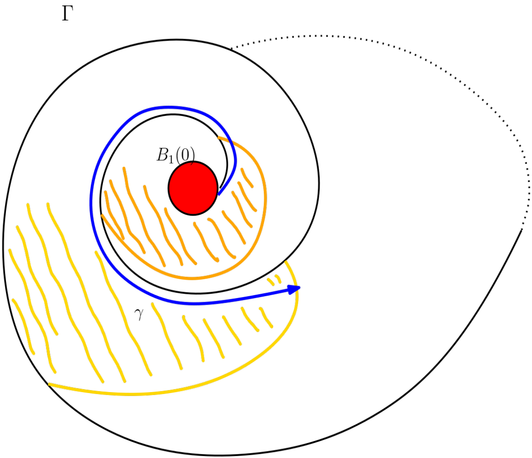
Simple closed curve



$\sigma > 2$ OK

[Bressan (..., Chiri), optimal strategies Bressan-De Lellis (Robyr)]

Spiral-like barrier



$\sigma > 2.61\ldots$ OK
[Bressan (...), Klein-Langetepe-Levcopoulos]

If $\sigma \leq 2$ then the fire cannot be blocked

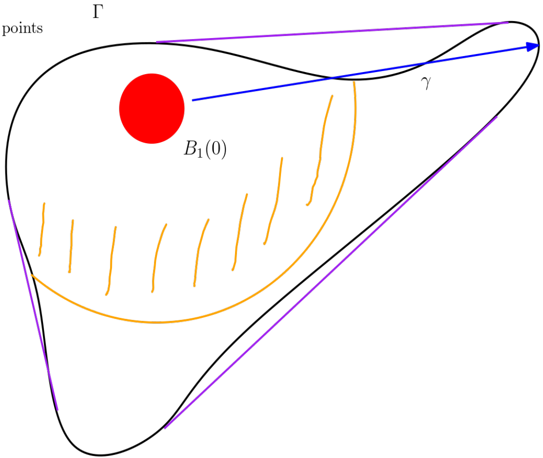
[Bressan:] if $\sigma \leq 1$ then the fire cannot be blocked, if $\sigma \leq 2$ it cannot be blocked with a simple closed curve

1) The barrier burns at least 2 points

2) Hence it not saturated

3) By a homotopy argoment
deform Γ into
a convex closed curve

4) The length of Γ is
stringtly greater than $L(\gamma)$



If $\sigma \leq 2.61...$ then the fire cannot be blocked by a spiral

[Klein-Langetepe-Levcopoulos:] if $\sigma \leq \frac{1+\sqrt{5}}{2}$ then a spiral cannot block the fire

Assume that when the barrier arrives to P_i it has already overtaken P_{i-1}.

The next point P_{i+1} must satisfy

$$1 + \frac{|P_i - P_{i-1}|}{\sigma} + \frac{|P_{i+1} - P_i|}{\sigma} \leq |P_i| \leq |P_i - P_{i-1}|$$

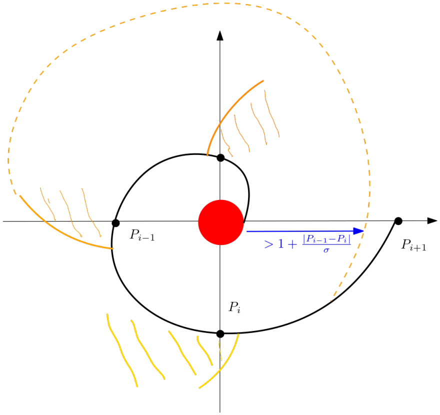
Hence

$$\frac{|P_{i+1} - P_1|}{\sigma} \geq \frac{1}{\sigma - 1} + \frac{|P_i - P_{i-1}|}{\sigma(\sigma - 1)}$$

If $\sigma \leq \frac{1+\sqrt{5}}{2}$, then

$$\frac{|P_{i+1} - P_i|}{\sigma} > 1 + |P_i - P_{i-1}|$$

so that the series is diverging.



Our result

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Theorem

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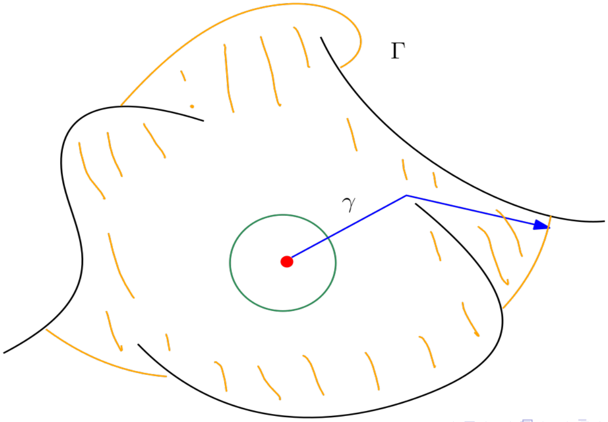
Some remarks:

1. it is possible to tune the method to obtain $\sigma \leq 2.5$ (with additional technicalities)
2. the critical case $\sigma \leq 2.61\dots$ is very delicate: any perturbation could lead to a closing spiral.

Some simplifications

- The fire can be assumed to start in $(0,0)$, i.e. u is the distance from the origin avoiding Γ
- The barrier is outside the unit ball $B_1(0)$



Some notation

Let

$$\mathcal{A}(t[x]) = \underbrace{t[x]}_{\text{time taken by the fire}} - \frac{1}{\sigma} \underbrace{L(t[x])}_{\text{length of burned barrier}}$$

be the *admissibility functional*: an admissible barrier satisfies $\mathcal{A} \geq 0$ by definition.

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Saturated set $S = \{x \in \Gamma : \mathcal{A}(t[x]) = 0\}$

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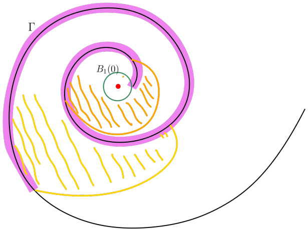
In the following an *optimal barrier* is the barrier with blocks the fire with minimal length.

Definition of spiral

Definition

A *spiral barrier* Γ is a simple curve $\gamma(s)$, $s \in [0, L)$, such that

- $\gamma(0) = (1, 0)$, $\dot\gamma(0) = e^{i\theta}$, $\theta \in [0, \pi/2]$,
- $\dot\gamma \wedge \ddot\gamma(s) \geq 0$,
- $\{u = t\} \cap \Gamma$ is a connected arc.

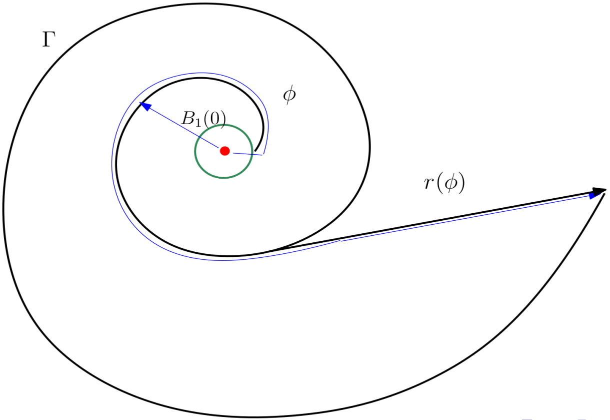


For optimal barriers some of the above assumptions are redundant.

Definition of spiral

The barrier can be described by means of the coordinates:

- the rotation angle ϕ of the optimal ray,
- the length $r(\phi)$ of the free part of the optimal ray.



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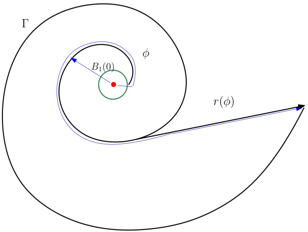
The equations is the delayed DE

$$\frac{dr(\phi)}{d\phi} = \cot(\beta(\phi))r(\phi) - R(\phi'),$$

with $R(\phi')$ radius of curvature, $\phi = 2\pi + \phi' + \beta(\phi')$, and $\beta \in [0, \pi/2]$ is a control parameter.



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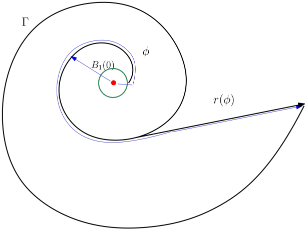


The *admissibility functional* is

$$\mathcal{A}(\phi) = 1 + L(\phi') + r(\phi) - \cos(\alpha)L(\phi) \geq 0, \quad \cos(\alpha) = \frac{1}{\sigma}.$$

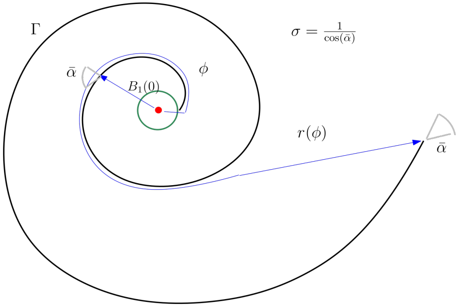
A different optimality condition

Definition

We say that the admissible Γ is optimal at the angle $\bar{\phi}$ if $r(\bar{\phi})$ is minimal.

In particular, if $r(\bar{\phi}) < 0$ then the barrier blocks the fire before $\bar{\phi}$.

An example of solution is for $\beta = \bar{\alpha}$,



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We say that the admissible Γ is optimal at the rotation $\bar{\phi}$ if $r(\bar{\phi})$ is minimal.

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An example of solution is for $\beta = \alpha$,

$$\frac{dr(\phi)}{d\phi} = \cot(\alpha) r(\phi) - \frac{r(\phi - 2\pi - \alpha)}{\sin(\alpha)},$$

$$r(\phi) = \begin{cases} e^{\cot(\alpha)\phi} & \phi \in [0, 2\pi), \\ e^{\cot(\alpha)(\phi - 2\pi)}(e^{\cot(\alpha)2\pi} - 1) & \phi \in [2\pi, 2\pi + \alpha). \end{cases}$$

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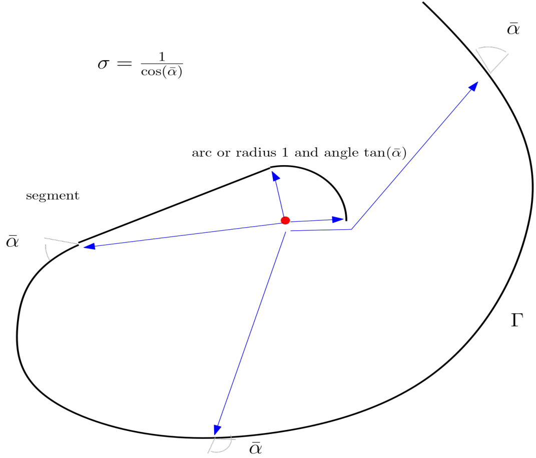
One can prove that this $r(\phi)$ blows up if there is a real solution to

$$\lambda = \cot(\alpha) - \frac{e^{-\lambda(2\pi+\alpha)}}{\sin(\alpha)},$$

and this happens only if $\alpha \leq \bar{\alpha} = 1.17\ldots$, $\sigma = 1/\cos(\bar{\alpha}) = 2.61\ldots$

An example of (locally) optimal barrier



Lemma

The above barrier is optimal for angles $[0, 2\pi]$.

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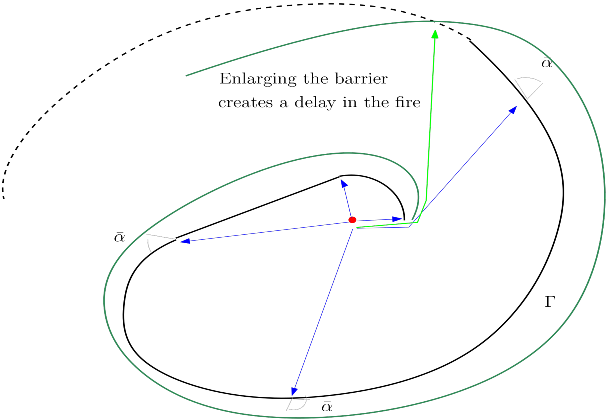
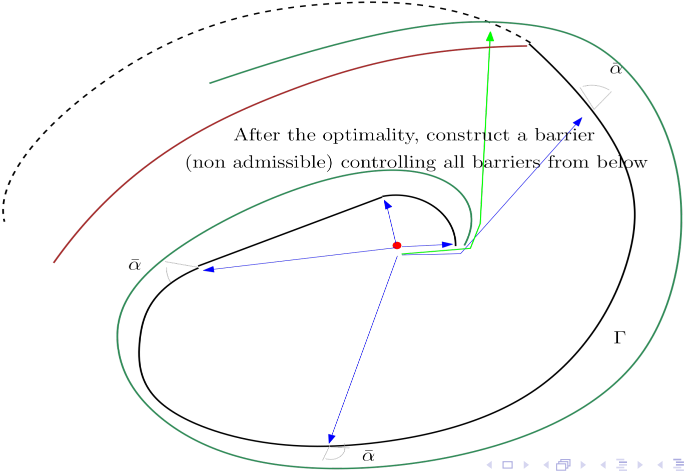


Corollary

It is not optimal for $\phi > 2\pi$, and the optimal barrier depends on $\bar{\phi}$.

The (almost) fastest closing spiral

Problem: given a spiral barrier $r(\phi)$, replace the part $\phi \geq \phi_0$ with a faster closing spiral.



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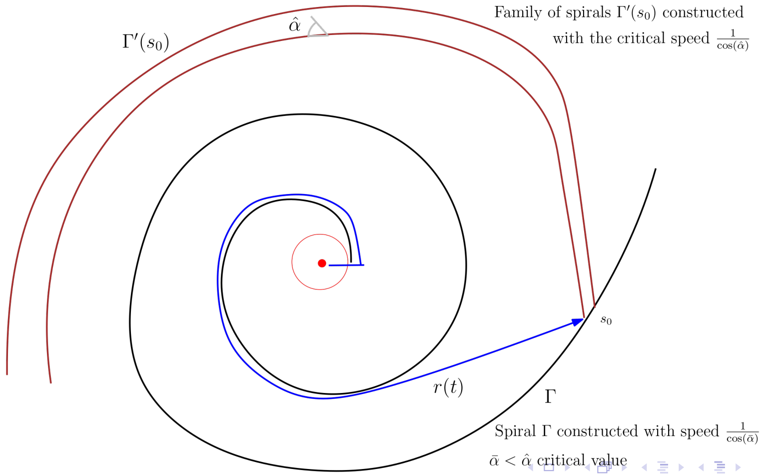

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Conclude the proof by showing that:

- the saturated spiral with $\hat{\alpha}$ equal to the critical angle $\frac{1}{\cos(\hat{\alpha})} = 2.61\ldots$, does not close: actually it growth exponentially
- the family of perturbed spirals $\Gamma'(s_0)$ is increasing w.r.t. s_0 if $\bar{\alpha} \leq 2.3$ (even something better if you like technicalities)
- deduce that the narrowest spiral in the family is the one starting from the initial $(1, 0)$, and thus all spirals are exponentially diverging
- conclude by

$$r(\phi) = \Gamma'(\phi, 0) + \int_0^\phi \frac{d\Gamma'(\phi, s_0)}{ds_0} ds_0 \geq \Gamma'(\phi, 0).$$

Thanks for the attention!