From the Optimal Transport problem of Monge to Einstein's Gravitation through Euler's Hydrodynamics

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However, the present talk (based on Y.B. CRAS 22) seems unrelated to these works.

The (quadratic) Monge OT problem:

$$Monge_2(\rho_0, \rho_1)^2 = \inf \int_{\mathbb{R}^d} |T(x) - x|^2 \rho_0(x) dx,$$

for all Borel maps T for which $\rho_1(y)dy$ is the image by $y = T(x)$ of $\rho_0(x)dx$,

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which is convex in $(\rho, m = \rho v)$, Benamou-B. 2000.

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This was the prototype of the future field theories in Physics (Maxwell, Einstein, Schrödinger, Dirac)... and the first multidimensional evolution PDEs ever written!


PRO JUVENTUTE 1957
$e^{i\varphi} = \cos\varphi + i\sin\varphi$
LEONHARD EULER 1707-1783
5
+5 HELVETIA

XXI. Nous n'avons donc qu'à égaler ces forces accélératrices avec les accélerations actuelles que nous venons de trouver, & nous obtiendrons les trois équations suivantes :

$$P - \frac{1}{q}\left(\frac{dp}{dx}\right) = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right)$$

$$Q - \frac{1}{q}\left(\frac{dp}{dy}\right) = \left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right)$$

$$R - \frac{1}{q}\left(\frac{dp}{dz}\right) = \left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right)$$

Si nous ajoutons à ces trois équations premièrement celle, que nous a fournie la considération de la continuité du fluide :

$$\left(\frac{dq}{dt}\right)+\left(\frac{d.qu}{dx}\right)+\left(\frac{d.qv}{dy}\right)+\left(\frac{d.qw}{dz}\right)=0.$$

Si le fluide n'étoit pas compreſſible, la denſité q ſeroit la même en Z, & en Z′, & pour ce cas on auroit cette équation :

$$\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=0.$$

qui eſt auſſi celle ſur laquelle j'ai établi mon Mémoire latin allégué ci-deſſus.

The least action principle for the Euler equations

It amounts to looking for fields
$(t,x) \in \mathbb{R} \times \mathbb{R}^d \to (\rho, v)(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, critical points

$$\text{of} \quad \int \left(\frac{\rho |v|^2}{2} - \Phi(\rho)\right) dx dt \quad (\text{where} \;\; r\Phi''(r) = p'(r))$$

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Optimality equations read:

$$v = \nabla \phi, \quad \partial_t \phi + |\nabla \phi|^2/2 + \Phi(\rho) = 0$$

$$\Rightarrow \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) + \nabla(p(\rho)) = 0.$$

The pressureless case and the Monge problem

Without pressure, $p = 0$, the action principle reads

$$\text{crit}\ \left(\int \frac{\rho|v|^2}{2} dxdt\right), \quad \text{s.t.}\ \ \partial_t \rho + \nabla \cdot (\rho v) = 0$$

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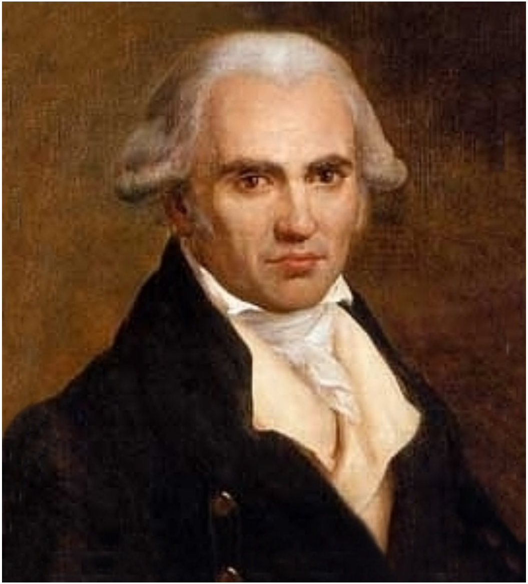
and leads to a *convex* minimization problem in $(\rho, \rho v)$ for suitable time-boundary conditions, which is nothing but the optimal transport problem of Monge in its Eulerian formulation!



A matrix-valued generalization

Find matrix-valued fields $(C, V)(x, \xi) \in \mathbb{R}^{4\times 4}$ over the tangent bundle $(x, \xi) \in (\mathbb{R}^4)^2$ of $\mathbb{R}^4$, critical points of

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subject to $\nabla_x \cdot C + \nabla_\xi \cdot (CV + VC) = 0$ and

$$C = \nabla_\xi A - \mathbb{I}_4 \nabla_\xi \cdot A, \quad V = \nabla_\xi W,$$

for some vector-potentials $A(x, \xi)$, $W(x, \xi) \in \mathbb{R}^4$.

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crit $\int C_j^k V_q^j V_k^q$ s.t. $\partial_{x^j} C_k^j + \partial_{\xi^j}(CV + VC)_k^j = 0$, $C_k^j = \partial_{\xi^k} A^j - \partial_{\xi^\gamma} A^\gamma \delta_k^j$, $V_k^j = \partial_{\xi^k} W^j$.

Theorem (Y.B. CRAS 2022) **in english on my webpage at LMO**

Let g be a smooth solution to the Einstein equations in vacuum $\mathbb{R}^4$ with Christoffel symbols $\Gamma =$"$g^{-1}\partial g$".

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Let g be a smooth solution to the Einstein equations in vacuum $\mathbb{R}^4$ with Christoffel symbols $\Gamma =$"$g^{-1}\partial g$". Then (C, V) solves the matrix-valued OT problem, where

$$C_k^j(x,\xi) = \partial_{\xi^k} A^j(x,\xi) - \partial_{\xi^q} A^q(x,\xi)\ \delta_k^j,$$

$$A^j(x,\xi) = \xi^j \, \mathrm{det} g(x)\ \cos(\frac{g_{\alpha\beta}(x)\xi^\alpha\xi^\beta}{2}),$$

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In addition:

$$\sqrt{-\det g(x)}\, g^{jk}(x) = cst \int (\xi^j A^k(x,\xi) + \xi^k A^j(x,\xi))d\xi.$$

A first comment: Einstein vs Monge

$$(x,\xi) \in \mathbb{R}^{4+4} \;\Leftarrow\; (t,x) \in \mathbb{R}^{1+d} \;;\; C_k^j(x,\xi) \;\Leftarrow\; \rho(t,x)$$

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CAVEAT! Our variational principle is designed only to get the right equations, not to get their solutions!!! (This is why we entirely ignored boundary conditions.)

A second comment. There is a (nearly convex!) formulation of this matrix-valued OT problem:

Find 4×4 matrix-valued fields $(C, V, M)(x, \xi)$ over the tangent bundle $(x, \xi) \in (\mathbb{R}^4)^2$, critical points of

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N.B. Here the optimization in V should be interpreted as the Legendre transform (but not Legendre-Fenchel!) of the non-convex function $V \in \mathbb{R}^{4\times 4} \to \mathrm{trace}(CV^2) \in \mathbb{R}$.

Einstein in OT form: some steps of the proof

Basic idea: view Γ as a collection of 4 vector fields over the tangent bundle $(x,\xi) \in \mathbb{R}^8$ which are linear in ξ: $V_k^j(x,\xi) = -\Gamma^j_{k\gamma}(x)\xi^\gamma$,

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$$R_{jkm}^n(x)\xi^m = \left((\partial_{x^k} + V_k^\gamma \partial_{\xi^\gamma})V_j^n - (\partial_{x^j} + V_j^\gamma \partial_{\xi^\gamma})V_k^n\right)(x,\xi)$$

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$$= \partial_{x^k} V_j^n + \partial_{\xi^j}(V_k^\gamma V_\gamma^n) - \partial_{x^j} V_k^n - \partial_{\xi^k}(V_i^\gamma V_\gamma^n),$$

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The zero-Ricci curvature 'tangent bundle' equation

$$\partial_{x^k} V_j^j + \partial_{\xi^j}(V_k^\gamma V_\gamma^j) - \partial_{x^j} V_k^j - \partial_{\xi^k}(V_j^\gamma V_\gamma^j) = 0$$

is going to play for Einstein (in vacuum) the role taken by the multiD Burgers equation $\partial_t v + \nabla(|v|^2/2) = 0$ for the quadratic OT problem in its hydrodynamical form.

A toy model : the multiD Burgers equation (1/4)

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Ignoring BC, let us look for critical points (A, V) of

$$\int \left(-\partial_t A \cdot V - \frac{(\nabla \cdot A)|V|^2}{2} \right) dxdt, \quad A = A(t,x) \in \mathbb{R}^d.$$

A toy model : the multiD Burgers equation (2/4)

Critical points (A, V) of

$$\mathcal{I}(A, V) = \int \left(-\partial_t A \cdot V - \frac{(\nabla \cdot A)|V|^2}{2} \right) dxdt.$$

$$\partial_A \mathcal{I}(A, V) = 0 \;\Rightarrow\; (1) \quad \partial_t V + \nabla(\frac{|V|^2}{2}) = 0$$

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$$\partial_V \mathcal{I}(A,V) = 0 \;\Rightarrow\; (2) \quad \partial_t A + V(\nabla \cdot A) = 0$$

(additional information that we are now going to use).

A toy model : the multiD Burgers equation (3/4)

We use (2) $\partial_t A + V(\nabla \cdot A) = 0$ to rewrite $\mathcal{I}(A, V)$ as:

$$\mathcal{I}_2(A, V) = \int \frac{(\nabla \cdot A)|V|^2}{2} dxdt.$$

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$\partial_B \mathcal{L}(A, V, B) = 0 \;\; \Rightarrow$ (2) (of course), $\partial_V \mathcal{L}(A, V, B) = 0 \;\; \Rightarrow \;\; (\nabla \cdot A)V - B(\nabla \cdot A) = 0$,
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Assuming that (A, V) is critical for $\mathcal{I}(A, V)$, we have

$\partial_t A + V(\nabla \cdot A) = 0$ and $\partial_t V + \nabla(|V|^2/2) = 0$.

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Assuming that (A, V) is critical for $\mathcal{I}(A, V)$, we have

$\partial_t A + V(\nabla \cdot A) = 0$ and $\partial_t V + \nabla(|V|^2/2) = 0$. Setting $B = V$, we are just in business!

A toy model : the multiD Burgers equation (4/4)

Let us now write everything in terms of $(\rho = \nabla \cdot A, V)$:

$$(2) \;\; \partial_t A + V(\nabla \cdot A) = 0 \;\Rightarrow\; \partial_t \rho + \nabla \cdot (\rho V) = 0,$$

$$\mathcal{I}_2(A, V) = \int \frac{(\nabla \cdot A)|V|^2}{2} dxdt \;\Rightarrow\; \int \frac{\rho |V|^2}{2} dxdt.$$

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$$\mathcal{I}_2(A, V) = \int \frac{(\nabla \cdot A)|V|^2}{2} dxdt \;\Rightarrow\; \int \frac{\rho |V|^2}{2} dxdt.$$

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THANKS!