# Brakke’s mean curvature flow

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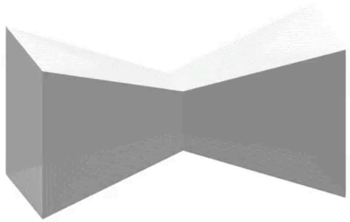
November 2, 2023

60th anniversary of CUHK

## 1. Introduction on the Brakke flow.

A family of smooth k-dimensional surfaces $M_t$ in $\mathbb{R}^n$ is mean curvature flow (MCF) if the normal velocity = the mean curvature vector. $\boxed{v = h}$

Even if the initial data is smooth, singularities can happen later.



==> a suitable weak solution for MCF : Brakke flow.

<u>How to formulate a weak notion of normal velocity:</u>

Suppose that $M_t$ is moving by the normal velocity $v = v(x,t)$ .

For any $\phi \in C_c^1(\mathbb{R}^n \times [0,\infty))$ ,

$$\frac{d}{dt}\Big(\int_{M_t} \phi(x,t)\, d\mathcal{H}^k(x)\Big) = \int_{M_t} \big(\phi_t + \nabla\phi \cdot v - \phi\, h \cdot v\big)\, d\mathcal{H}^k(x) = \int_{M_t} \{\phi_t + (\nabla\phi - \phi\, h)\cdot v\}\, d\mathcal{H}^k(x)$$

and conversely, if a smooth normal vector field $\tilde{v} = \tilde{v}(x,t)$ satisfies

$$\frac{d}{dt}\Big(\int_{M_t} \phi(x,t)\, d\mathcal{H}^k(x)\Big) = \int_{M_t} \{\phi_t + (\nabla\phi - \phi\, h)\cdot \tilde{v}\}\, d\mathcal{H}^k(x) \quad \text{for any } \phi \in C_c^1(\mathbb{R}^n \times [0,\infty)) ,$$

then $\tilde{v} \equiv v.$

Lemma If a normal vector field $\tilde{v}$ satisfies the inequality

$$\frac{d}{dt}\Big(\int_{M_t}\phi(x,t)\,d\mathcal{H}^k(x)\Big)\le\int_{M_t}\{\phi_t+(\nabla\phi-\phi\,h)\cdot\tilde{v}\}\,d\mathcal{H}^k(x)$$ for any non-negative $\phi\in C_c^1(\mathbb{R}^n\times[0,\infty))$

then we have $\tilde{v}\equiv v.$ So this inequality characterizes the normal velocity.

Proof The difference $w:=\tilde{v}-v$ satisfies $0\le\int_{M_t}(\nabla\phi-\phi\,h)\cdot w\,d\mathcal{H}^k(x)$

for any non-negative $\phi\in C_c^1(\mathbb{R}^n\times[0,\infty))$ . Suppose $x_0\in M_t$ .

For any $\lambda>0$ , let $\phi^\lambda(x,t):=\lambda^{-k+1}\phi(\lambda^{-1}(x-x_0),t)$ , plug this into above, change variables $z:=\lambda^{-1}(x-x_0)$ and let $\lambda\to 0+$ . Then we have

$$0\le\int_{T_{x_0}M_t}\nabla\phi(z,t)\cdot w(x_0,t)\,d\mathcal{H}^k(z)=\Big(\int_{T_{x_0}M_t}(\nabla\phi(z,t))^\perp\,d\mathcal{H}^k(z)\Big)\cdot w(x_0,t).$$ This implies $w(x_0,t)=0$ .

Lemma If a smooth family of k-dimensional surfaces $M_t$ satisfies

$$\frac{d}{dt}\Big(\int_{M_t}\phi(x,t)\,d\mathcal{H}^k(x)\Big)\le\int_{M_t}\{\phi_t+(\nabla\phi-\phi\,h)\cdot h\}\,d\mathcal{H}^k(x)$$ for any non-negative $\phi\in C_c^1(\mathbb{R}^n\times[0,\infty))$

then $M_t$ is a MCF.

Lemma If a smooth family of k-dimensional surfaces $M_t$ satisfies

$$\int_{M_{t_2}}\phi(x,t_2)\,d\mathcal{H}^k(x)-\int_{M_{t_1}}\phi(x,t_1)\,d\mathcal{H}^k(x)\le\int_{t_1}^{t_2}dt\int_{M_t}\{\phi_t+(\nabla\phi-\phi\,h)\cdot h\}\,d\mathcal{H}^k(x)$$ for any $0\le t_1<t_2<\infty$

and non-negative $\phi\in C_c^1(\mathbb{R}^n\times[0,\infty))$ then $M_t$ is a MCF.

Definition of Brakke flow A family of Radon measures $\mu_t$ is a Brakke flow if

1. For a.e. $t \geq 0$ there exist a countably k-rectifiable set $M_t$ and $\theta_t \in L^1_{loc}(M_t : \mathbb{N})$ such that $\mu_t = \theta_t \mathcal{H}^k\lfloor_{M_t}$ , in other word, $\int_{\mathbb{R}^n} \phi(x)\, d\mu_t = \int_{M_t} \phi(x)\theta_t(x)\, d\mathcal{H}^k(x)$ for any $\phi \in C_c(\mathbb{R}^n)$.
2. For a.e. $t \geq 0$ there exists a vector field $h \in L^2_{loc}(\mu_t)$ satisfying $\int_{M_t} \mathrm{div}_{T_x M_t} g\, d\mu_t = -\int_{M_t} h \cdot g\, d\mu_t$ for all $g \in C^1_c(\mathbb{R}^n : \mathbb{R}^n)$. Also it is locally square-integrable wrt space-time measure $d\mu := d\mu_t\, dt$. It is called the (generalized) mean curvature vector of $\mu_t$ . (cf. Brakke's perpendicularity theorem.)
3. For all $0 \leq t_1 < t_2 < \infty$ and non-negative $\phi \in C^1_c(\mathbb{R}^n \times [0,\infty))$, we have
$$\int \phi(x,t_2)\, d\mu_{t_2}(x) - \int \phi(x,t_1)\, d\mu_{t_1}(x) \leq \int_{t_1}^{t_2} dt \int \{\phi_t + (\nabla\phi - \phi\, h)\cdot h\}\, d\mu_t(x).$$

Observation If a smoothly moving k-dimensional $M_t$ has the property that $\mu_t := \mathcal{H}^k\lfloor_{M_t}$ is a Brakke flow, then $M_t$ is a MCF.

More generally If a family of Radon measures $\mu_t$ satisfies 1+2 and if there exists a vector field $u \in L^2_{loc}(\mu)$ such that

3*. for all $0 \leq t_1 < t_2 < \infty$ and non-negative $\phi \in C^1_c(\mathbb{R}^n \times [0,\infty))$ , we have
$$\int \phi(x,t_2)\, d\mu_{t_2}(x) - \int \phi(x,t_1)\, d\mu_{t_1}(x) \leq \int_{t_1}^{t_2} dt \int \{\phi_t + (\nabla\phi - \phi\, h)\cdot(h + u^\perp)\}\, d\mu_t(x),$$
we say that $\mu_t$ satisfies $\boxed{v = h + u^\perp}$ in the sense of Brakke.

## 2. $\varepsilon-$regularity theorem for the Brakke flow.

For stationary varifold (generalized minimal surface), we have Allard regularity theorem. The following is the precise parabolic analogue proved last year jointly with Stuvard, ``End-time regularity theorem for Brakke flows'' (preprint).

<u>Theorem</u>

There exist $\varepsilon = \varepsilon(n,k,p,q,E_1) \in (0,1)$ and $C = C(n,k,p,q,E_1) \in (1,\infty)$ with the following:

Suppose that $\mu_t$ ( $t \in (-3,0]$ ) is a family of Radon measures in $B_3^k \times B_1^{n-k} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ and satisfies $v = h + u^\perp$ (i.e. 1+2+3*) in the sense of Brakke in $(B_3^k \times B_1^{n-k}) \times (-3,0]$ with $\|u\|_{p,q} := \Big(\int_{-3}^0 \Big(\int_{B_3^k \times B_1^{n-k}} |u(x,t)|^p \, d\mu_t(x)\Big)^{\frac{q}{p}} dt\Big)^{\frac{1}{q}} < \varepsilon, \quad \alpha := 1 - \frac{k}{p} - \frac{2}{q} > 0,$

and further assume that $\sup_{t\in(-3,0], B_r(x) \subset B_3^k \times B_1^{n-k}} r^{-k} \mu_t(B_r(x)) \le E_1,$

$$L := \Big(\int_{-3}^0 \int_{B_3^k \times B_1^{n-k}} \mathrm{dist}\,(\mathbb{R}^k \times \{0\}, x)^2 \, d\mu_t(x) dt\Big)^{\frac{1}{2}} < \varepsilon, \qquad (B_2^k \times B_{1/2}^{n-k}) \cap \mathrm{spt}\, \mu_0 \neq \emptyset,$$

$\mu_{-2}(B_2^k \times B_1^{n-k}) \le 2^k \omega_k + \varepsilon$ (here, $\omega_k := \mathcal{H}^k(B_1^k)$), then, the support of $\mu_t$ in $B_1^k \times B_{1/2}^{n-k}$ for $\boxed{t \in [-1,0)}$ is represented as a graph of $C^{1,\alpha}$ function $f : B_1^k \times [-1,0) \to \mathbb{R}^{n-k}$ with

$$\|f\|_{C^{1,\alpha}(B_1^k \times [-1,0))} \le C(L + \|u\|_{p,q}).$$

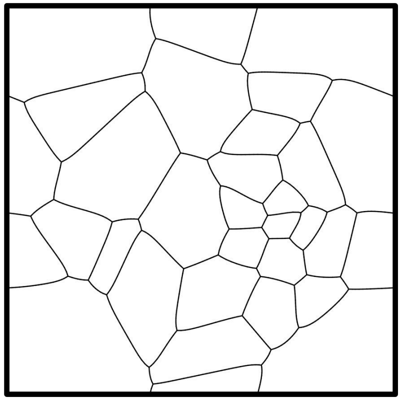
Remark If $u \in C^\beta$ then the estimate will be $\|f\|_{C^{2,\beta}(B_1^k\times[-1,0))} \le C(L + \|u\|_{C^\beta})$. Moreover, $v = h + u^\perp$ is satisfied classically. In the case of Brakke flow, the graph is smooth MCF in $(B_1^k \times B_{1/2}^{n-k}) \times (-1,0)$.

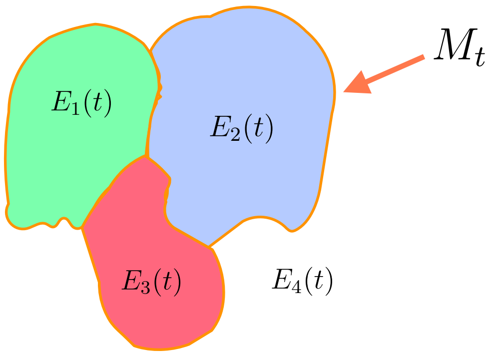
Remark The similar estimate but not up-to-the-end-time estimate was proved in Kasai-T. (`14), T. (`14). If the Brakke flow is a priori known to be obtained as a limit of smooth MCF, White (`05) proved the regularity theorem by a compactness argument for Brakke flow.

Remark Just like stationary varifold, one can prove that any Brakke flow is smooth on a ``dense set'' of the support using this theorem. But this does not say much about the size of the singularity set in general.

3. Co-dimension one existence theorem for Brakke flow.

Aim Given any ``n-dimensional set'' $M_0 \subset \mathbb{R}^{n+1}$, construct a Brakke flow starting from this set and existing for all time (or until it vanishes).





Assumptions 1 - 3 :

1. $N \geq 2$ and $E_{0,1}, \ldots, E_{0,N} \subset \mathbb{R}^{n+1}$ are non-empty mutually disjoint open sets.
2. $M_0 := \mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_{0,i}$ is countably n-rectifiable.
3. $\int_{M_0} \exp(-c_0|x|)\, d\mathcal{H}^n(x) < \infty$ for some $c_0 \geq 0$. (So $\cup_{i=1}^N \partial E_{0,i} = M_0$ .)

Conclusions (1-8):

There exist a Brakke flow $\{\mu_t\}_{t\geq 0}$ and $\{E_i(t)\}_{t\geq 0}$ for $i = 1, \ldots, N$ with the following.

1. $\mu_0 = \mathcal{H}^n\lfloor_{M_0},\ E_i(0) = E_{0,i}$ and $\lim_{t\to 0+} \mu_t = \mu_0$ if $\mathcal{H}^n(\cup_{i=1}^N \partial E_{0,i} \setminus \cup_{i=1}^N \partial^* E_{0,i}) = 0.$
2. Write $\Omega(x) := \exp(-c_0|x|)$. Then for all $t > 0$,

$$\int_{\mathbb{R}^{n+1}} \Omega(x)\, d\mu_t(x) + \int_0^t dt \int_{\mathbb{R}^{n+1}} \Omega(x)|h|^2\, d\mu_t(x) \leq \exp(c_0 t) \int_{M_0} \Omega(x)\, d\mathcal{H}^n(x).$$

   In particular, if $c_0 = 0$, (so $\mathcal{H}^n(M_0) < \infty$ ) $\mu_t(\mathbb{R}^{n+1}) + \int_0^t dt \int_{\mathbb{R}^{n+1}} |h|^2\, d\mu_t(x) \leq \mathcal{H}^n(M_0).$

3. $E_1(t), \ldots, E_N(t)$ are mutually disjoint open sets. Write $M_t := \mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_i(t).$

   Then for all t>0, $M_t = \cup_{i=1}^N \partial E_i(t) = \{x : (x,t) \in \mathrm{spt}\,(d\mu_t\, dt)\}.$

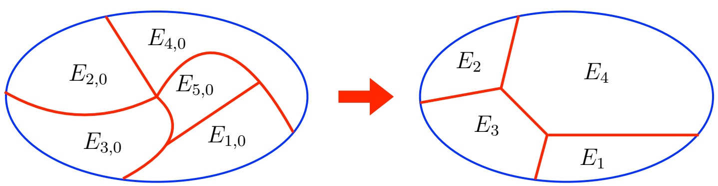
4. For all t>0, $\mathrm{spt}\,\mu_t \subset M_t$ and $\mathcal{H}^n(M_t \cap K) < \infty$ for any compact $K \subset \mathbb{R}^{n+1}.$
5. For a.e. t>0 and any $\delta > 0$ , $\mathcal{H}^{n-1+\delta}(M_t \setminus \mathrm{spt}\mu_t) = 0.$

6. For each $i=1,\ldots,N$, $\forall \xi \in C_c^1(\mathbb{R}^{n+1}\times(0,\infty);\mathbb{R})$, $\int_0^\infty \Big(\int_{E_i(t)} \partial_t \xi\, dx + \int_{\partial^* E_i(t)} (h\cdot\nu_{\partial^* E_i(t)})\,\xi\, d\mathcal{H}^n\Big) dt = 0.$

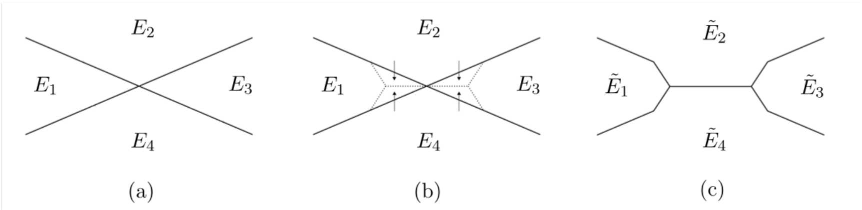
7. If for a.e. $t\in[0,T], \theta=1$ for $\mathcal{H}^n\lfloor_{M_t} - a.e.,$ then $\mu_t = \mathcal{H}^n\lfloor_{\cup_{i=1}^N \partial^* E_i(t)}$ for $a.e.\, t\in[0,T]$.

8. For n=1, for a.e. t>0, $\mathrm{spt}\,\mu_t$ consists of embedded $W^{2,2}\cap C^{1,\frac{1}{2}}$ curves with the endpoints meeting with angles of either 0°, 60° or 120°, and if N=2, only 0° (Kim-T. `20).

Remark We can also consider a fixed boundary setting in a strictly convex domain, in which case, as the time goes to infinity, the Brakke flow subsequentially coverges to a stationary varifold with possible multiplicity. In a sense, we can solve the Plateau problem by MCF using this result.




Remark The construction of solution is involved. First construct a time-discrete approximate flow. This involves smoothing of varifold and a restricted Lipschitz almost minimization, which gives a certain amount of regularity in a small length scale lost due to the smoothing.



## 4. Application: dynamic stability/instability of minimal surfaces

Question: Is there a minimal hypersurface with singularity from which there exists a non-trivial Brakke flow starting from it?

<u>Theorem</u> (Stuvard-T. preprint) Suppose $E_0 \subset \mathbb{R}^{n+1}$ is open.

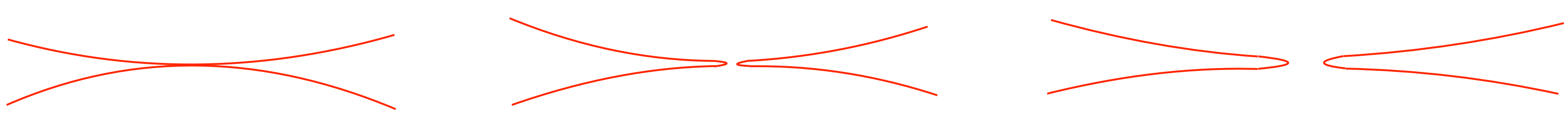
Assume that

1. $\mathcal{H}^n(\partial E_0 \setminus \partial^* E_0) = 0$ and define $M_0 := \partial E_0, \ \mu_0 = \mathcal{H}^n \lfloor_{M_0}.$
2. $\mu_0$ is stationary ( $h = 0$ ) in $B_1^{n+1}$ .
3. $\lim_{r \to 0+} \mathcal{H}^n \lfloor_{\frac{M_0}{r}} = q\, \mathcal{H}^n \lfloor_{\mathbb{R}^n \times \{0\}}, \quad \text{for some } q \in \{2, 3, 4, \ldots\}.$
4. There exist $\alpha > 1/2, \ r_0 > 0, \ C > 0$ such that

$$M_0 \cap \{(x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}, : \ |x'| \le r, \ |x_{n+1}| \le r_0\} \subset \{(x', x_{n+1}) \ : \ |x_{n+1}| \le \frac{Cr}{(\log(1/r))^\alpha}\}$$

for all $0 < r < r_0.$

Conclusion: Then there exists a Brakke flow $\mu_t$ (fixing boundary $\partial B_1^{n+1} \cap M_0$ ) such that $\lim_{t \to 0+} \mu_t = \mu_0 \text{ and } \mu_t(B_1^{n+1}) < \mu_0(B_1^{n+1}) \text{ for } t > 0.$



## 5. Further questions:

1. full characterization of Brakke flow w.r.t. Lipschitz minimization.
2. general almost everywhere regularity for 1-d (sheeting theorem).
3. analysis on the higher multiplicity part.
4. regularity of singular sets (1-d triple junction: T.-Wickramasekera `16).
5. how to de-singularize by restarting.
6. dynamically stable minimal surface — a bit like 2nd variation w.r.t. Lipschitz deformation.

References of today's talk:


- *On the existence of canonical multi-phase Brakke flows*, online first Advances in Calculus of Variations (with Salvatore Stuvard).
- *End-time regularity theorem for Brakke flows,* arXiv: 2212.07727 (with Salvatore Stuvard).
- *Dynamical instability of minimal surfaces at flat singular points*, arXiv:2008.13728 (with Salvatore Stuvard).
- *Existence and regularity theorems of one-dimensional Brakke flows*, Interfaces and Free Boundaries 22, (2020), 505–550 (with Lami Kim).
- *On the mean curvature flow of grain boundaries*, Annales de l'Institut Fourier (Grenoble) 67 (2017) no. 1, 43–142 (with Lami Kim).