Direct Sampling Methods for General Nonlinear Inverse Problems

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OUTLINE

- Motivation of sampling-type methods
- General framework of direct sampling methods
- DSMs for Inverse acoustic/EM scattering, EIT, DOT
- DSMs for moving inhomogeneous media
- Optimal control approach for Sobolev scale
Most Popular Approach for Inverse Problems

Most IPs: parameter identifications in PDEs, e.g.,

EIT, DOT, Inverse Scattering, Seismic Tomography, …

Stationary PDE:

\[ L_q(u) = 0 \]

or time-dependent PDE:

\[ D_t^\alpha u - L_q(u) = 0 \]

Inverse problem is to solve

\[ u(q) = u^\delta \quad \text{on} \quad \Gamma \]

Mostly the solution parameter \( q \) tells us information

geometric shape/location & distributional values
Least-squares formulation with regularization

IPs are mostly ill-posed: \( u(q) = u^\delta \) on \( \Gamma \)

transform to a nearby “well-posed” problem:

\[
\min_{q \in K} J(q) = \text{data fitting} \ (u(q) - u^\delta) + \beta \Phi(q)
\]

A most crucial mathematical issue:

choose \( K, \Phi, \beta \) s.t. it is stable wrt data
Output LS Tikhonov regularization:

$$\min_{q \in K} J(q) = \text{data fitting} \ (u(q) - u^\delta) + \beta \Phi(q)$$

1st approach: coupled optimality PDE system

- Forward PDE;
- Adjoint PDE;
- Variational Inequality

Singularities: parameters mostly disconts, unknown

e.g., conductivity in EIT, refractive index in inverse medium
Iterative Solvers for Nonlinear Optim Systems

Least-squares minimization:

$$\min_{q \in K} J(q) = \text{data fitting } (u(q) - u^\delta) + \beta \Phi(q)$$

2nd approach: iterative

Most popular iterative, e.g., Newton type:

- need to choose $\beta$, $h$, $\Delta t$,
- need good initial guess of $q$,
- repeated forward solutions,
- need the derivatives of $u(q)$ wrt changes of $q$,
- often very sensitive to noise,

Often very expensive & challenge to solve

Is it always worthwhile or necessary to do so?
Alternative Solvers

Indeed not worthwhile or necessary to do

$$\min_{q \in K} J(q) = \text{data fitting } (u(q) - u^\delta) + \beta \Phi(q)$$  \hspace{1cm} (1)

1\textsuperscript{st}: if noise not small, we can see from

$$q - q^{\delta,h} = (q - q^\delta) + (q^\delta - q^{\delta,h}) = O(\delta^\gamma) + O(h^\alpha)$$

2\textsuperscript{nd}: no good accuracy needed for the concerned applications

Alternative solvers, overcoming technical barriers:
no need good initial guess of \(q\),
no repeated forward solutions,
no need the derivatives of \(u(q)\) wrt changes of \(q\)
Can we reconstruct Shape & Location, without Physics?

Inverse acoustic, EM, elastic wave

EIT, DOT, MRI, … …
Linear Sampling Method

Colton-Kirsch 96: a truly revolutionary algorithm!

Inverse acoustic scattering: \[ \Delta u + k^2 n^2(x) u = 0 \]

Consider the far-field operator \[ F : L^2(S^{N-1}) \rightarrow L^2(S^{N-1}) \]

\[ (Fg)(\hat{x}) = \int u_\infty(\hat{x}, d) g(d) \, ds(d), \quad \hat{x} \in S^{N-1} \]

and the far-field equation for \( g \):

\[ Fg = \Phi_\infty(\cdot, z) \quad \forall z \in \mathbb{R}^N \]

Solve for \( g \) at each \( z \), and look at its energy

\[ \| g(\cdot, z) \|_{L^2(S)} \]
Algorithm of LSM

- Turns inverse scattering into solving integral equations

- Algorithm of LSM: select a numerical cut-off value $c$

1. Select a grid $T_h$ of sampling points, covering $D$
2. At each $z$, solve the far-field equation for $g(\cdot, z)$
3. Determine

$$z \in D \text{ if } \|g(\cdot, z)\| \leq c;$$

$$z \notin D \text{ if } \|g(\cdot, z)\| > c$$
No effective strategies to choose numerical cut-off values.

Huge computational efforts:
need to solve the far-field equation for each sampling point, e.g.,

for an \( n \times n \times n \) grid, need to solve \( n^3 \) ill-posed equations

The grid should be very fine to get a fine reconstruction
New Variants of LSM

- Li-Liu-Zou, SISC 09:
  Multilevel Linear Sampling Method,
  reduce computational complexity from $O(n^3)$ to $O(n^2)$

- Li-Liu-Zou, SISC 10:
  Strengthened LSM with a Reference Obstacle,
  provide a deterministic technique to select feasible numerical cut-off values
Multilevel Linear Sampling Method

MLSM : get rid of remote and inner cells
Numerical Example I
Numerical Example I
Numerical Example I
Numerical Example II
Sampling-type Methods

- Linear sampling method (Colton-Kirsch 96);
- Factorization method (Kirsch 98);
- Point source & multipole method (Potthast 98);
- Probe method (Potthast 01);
- Reciprocity Gap Sampling Method (Colton-Haddar, 05)
- Subspace-based optimization method (Chen 08)

... ... 

- Monographs:
  - Potthast, Chapman & Hall, 01;
  - Kirsch, Grinberg, Oxford 07;
  - Cakoni, Colton, Monk: SIAM 11;
  - Cakoni, Colton, Springer 14;
  - Nakamura, Potthast, IOP, 15;
  - X Chen, Wiley, 2018; ... ...
But

when we apply these methods, we may still encounter several common difficulties
(I) Cut-off Values & Noise

6 incidents & 30 receivers

inaccurate cut-off values
(II) Large Data for LSMs
And LSMs

derived only for wave-type inverse problems
Find methods for more realistic cases

- Apply even with data from a single incident field or a single set of Cauchy data
- Insensitive to data noise
- Involve no solutions of ill-posed & well-posed linear or nonlinear systems
- Apply to general inverse problems

Clearly, hard to have efficient methods for all these

Let us try what we can do
DSMs for General Inverse Problems

- Inverse acoustic medium scattering, Ito-Jin-Zou 12;
- Inverse EM medium scattering, Ito-Jin-Zou 13;

- Non-wave type IPs:
  - Electric impedance tomography, Chow-Ito-Zou 14;
  - Diffusive optical tomography, Chow-Ito-Liu-Zou 14;
  - Moving objects, Chow-Ito-Zou 16;
  - Several other important applications, Chow-Han-Zou 20
Define a Sobolev dual product on $\Gamma$ with index $\gamma$:

$$\langle \chi, \phi \rangle_{\gamma, \Gamma} \quad \forall \chi \in Y, \phi \in Z$$

Select probing & testing funcs $\{\eta_x\}, \{\mu_x\}$ based on PDEs

(1) nearly orthogonal wrt $\langle \cdot, \cdot \rangle_{\gamma}$, i.e., $\forall x \in \Omega, y \in D$,

$$K(x, y) = \frac{\langle \eta_x, \mu_y \rangle_{\gamma}}{|\eta_x|_Y}$$

kernel like a Gaussian

(2) family of testing funcs is fundamental over testing points:

$$u - u_0 \approx \sum_k a_k \mu_{x_k} \quad \text{on} \quad \Gamma$$
General Index functions for DSMs

We define the index function

\[ I(x) := \frac{\langle \eta_x, u - u_0 \rangle_{\gamma, \Gamma}}{\eta_x \mid Y} \quad \forall \, x \in \Omega \]

Then the index provides a probability:

\[ I(x) \approx \sum_k a_k \frac{\langle \eta_x, \mu x_k \rangle_{\gamma, \Gamma}}{\eta_x \mid Y} \]
Acoustic wave, TM or TE mode:

\[ \Delta u + k^2 n^2(x)u = 0 \]

Fundamental solution \( G \):

\[ \Delta G + k^2 G = \delta(x - y) \]

By Lippmann-Schwinger representation:

\[ u^s(x) = \int_{\Omega} G(x, y) I(y) dy \approx \sum w_j G(x, y_j) \]

From the above:

\[ \int \bar{u}^s(x) \bar{G}(x, x_p) ds \approx k^{-1} \sum w_j \text{Im}(G(y_j, x_p)) \]
Direct Sampling Algorithm
(Ito-Jin-Zou 2011)

Index func for probability of sampling point:

\[ \Phi(x_p) = \frac{|\langle u^s, G(\cdot,x_p) \rangle_\Gamma|}{\| u^s \| \| G(\cdot,x_p) \|} \]
Numerical Examples I

Two incidents: 20% noise
DSM for Inverse EM Media Scattering
(Ito-Jin-Zou 2013)

Time harmonic EM system:
\[
i\omega E + \nabla \times H = 0 \quad \text{in} \quad \mathbb{R}^d
- i\omega \mu H + \nabla \times E = 0 \quad \text{in} \quad \mathbb{R}^d
\]

Fundamental solution \( G \):
\[
( -\Delta - k^2 ) G(x, y) = \delta(x - y)
\]

Maxwell fundamental soln:
\[
\Phi(x, y) = k^2 G(x, y) I + D^2 G(x, y)
\]
Direct Sampling Algorithm
(Ito-Jin-Zou 2013)

Nearly orthogonality:

\[ \int_{\Gamma} (\Phi(x, x_p)p, \overline{\Phi}(x, x_q)q) ds \approx k^{-1}(p, \Im(\Phi(x_p, x_q))q) \quad \forall p \in \mathbb{C}^d, \; q \in \mathbb{R}^d \]

By Lippmann-Schwinger representation:

\[ E^s(x) = \int_{\Omega} \Phi(x, y)J(y)dy \approx \sum_j \Phi(x, y_j)J(y_j) |\tau_j|, \]

\[ \langle E^s, \Phi(\cdot, x_p)q \rangle_{L^2(\Gamma)} \approx k^{-1} \sum_j |\tau_j|(J(y_j), \Im(\Phi(x_p, y_j))q) \]

Index func for probability of sampling point:

\[ \Psi(x_p; q) = \frac{|\langle E^s, \Phi(\cdot, x_p)q \rangle_{\Gamma}|}{\|E^s\|_{\Gamma} \|\Phi(\cdot, x_p)q\|_{\Gamma}} \]

polarization q: basically quite arbitrary; incident polarization works well
Numerical Examples I

Two incidents, same polarizations \( p \) & \( q \): 20% noise
Numerical Examples II

Two incidents, same polarizations p & q: 20% noise
Electrical Impedance Tomography:

\[ \nabla \cdot (\sigma \nabla u) = 0 \quad \text{in} \quad \Omega \]

Inject current on \( \Gamma \):

\[ g = \sigma \frac{\partial u}{\partial \nu} \]

Measure potential on \( \Gamma \):

\[ f = u \]

EIT:

given \((f, g)\), recover electrical conductivity \(\sigma(x)\)
Choice of Probing & Testing Spaces/Funcs

Define on the measurement surface $\Gamma$:

$$\langle \chi, \phi \rangle_{\gamma, \Gamma} := \langle (\Delta_{\Gamma})^{\gamma} \chi, \phi \rangle \quad \forall \chi \in H^{2\gamma}(\Gamma), \phi \in L^2(\Gamma)$$

Select probing & testing funcs $\{\eta_x\}, \{\mu_y\}$ s.t.

1. Nearly orthogonal wrt $\langle \cdot, \cdot \rangle_{\gamma}$, i.e., $\forall x \in \Omega, y \in D$,

$$K(x, y) = \frac{\langle \eta_x, \mu_y \rangle_{\gamma, \Gamma}}{|\eta_x|_Y}$$

like a Gaussian

2. The testing family is fundamental:

$$u - u_0 \approx \sum_k a_k \mu y_k$$ on $\Gamma$
Choose of Probing Functions

Defining

$$-\Delta w_{x,d} = -d \cdot \nabla \delta_x \quad \text{in} \quad \Omega; \quad \frac{\partial w_{x,d}}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega$$

Dipole potential:

$$D_{x,d}(\xi) := c_n \frac{(x-\xi) \cdot d}{|x-\xi|^n}, \quad \xi \in \mathbb{R}^n$$

Set

$$\varphi_{x,d} = D_{x,d} - w_{x,d} :$$

$$-\Delta \varphi_{x,d} = 0 \quad \text{in} \quad \Omega; \quad \frac{\partial \varphi_{x,d}}{\partial \nu} = \frac{\partial D_{x,d}}{\partial \nu} \quad \text{on} \quad \partial \Omega$$

Probing functions as dir. derivative of Green func:

$$\eta_{x,d}(\xi) := w_{x,d}(\xi) = -d \cdot \nabla G_x(\xi) \quad \forall \xi \in \Gamma$$
For 3D spheric measurement surface:

$$\eta_{x,d}(\xi) = \frac{d \cdot \xi - \frac{(x-\xi) \cdot d}{|x-\xi|}}{\sqrt{4\pi(1-|x-\xi|-x \cdot \xi + 1)}}$$

For 2D circular measurement curve:

$$\eta_{x,d}(\xi) = \frac{1}{\pi} \frac{(\xi-x) \cdot d}{|x-\xi|^2}$$
Verification of Fundamental Properties

For p.w. constant inclusions $\Omega_1, \Omega_2, \cdots, \Omega_k$:

$$(u - u_0)(\xi) = - \sum_i \int_{\partial \Omega_i} [\eta] \frac{\partial G}{\partial \nu} d\nu \approx \sum_k a_k \eta x_k, d_k(\xi)$$

so testing funcs take the same as probing, with $d_k = \nu(x_k)$

Similarly for p.w. smooth inclusions
Verification of Orthogonality

For circular measurement curve:

\[
\langle \eta x, d_x, \mu y, d_y \rangle_{\gamma, S^1} = 2 \text{Re} \left( \frac{e^{i(\theta d_x - \theta d_y - \theta x + \theta y)}}{r_x r_y} G^{2\gamma}(r_x r_y e^{i(\theta x - \theta y)}) \right)
\]

with a complex polynomial

\[
G^\beta(z) := \left( z \frac{\partial}{\partial z} \right)^\beta \left( \frac{1}{1 - z} \right) = \sum_{m=0}^{\infty} m^\beta z^m
\]

so when \( y \approx x \) and \( d_y \approx d_x \), like a Gaussian

Similarly for spherical measurement surface, but much more technical
Index Function

Index function for EIT:

\[ K(x, y) = \frac{\langle \eta_x, G_y \rangle \gamma}{|\eta_x|_Y} = \frac{\langle (-\Delta \Gamma)^\gamma \chi, \phi \rangle}{|\eta_x|_Y} \]

With the Sobolev index

\[ \gamma = 2 \]
Numerical Experiments

Four separated square objects

5% noise

Thin square ring object:
DSM for DOT

Diffusive optical tomography in absorption medium $\Omega$ with absorption coeff $\mu$ & photon density $u$:

$$-\Delta u + \mu uu = 0 \quad \text{in} \quad \Omega$$

Inject current on $\Gamma$:

$$g = \frac{\partial u}{\partial n}$$

Measure density on $\Gamma$:

$$f = u$$

DOT:

given $(f, g)$, recover the absorption coeff $\mu$
General Principle of DSM

Define on the measurement surface $\Gamma$:

$$\langle \chi, \phi \rangle_{\gamma, \Gamma} := \langle (-\Delta_{\Gamma})^{\gamma} \chi, \phi \rangle \quad \forall \chi \in H^{2\gamma}(\Gamma), \phi \in L^2(\Gamma)$$

Select a set of probing & testing funcs $\{\eta_x\}$ & $\{\mu_x\}$:

1. nearly orthogonal wrt $\langle \cdot, \cdot \rangle_{\gamma}$, i.e., $\forall x \in \Omega, y \in D$,

$$K(x, y) = \frac{\langle \eta_x, \mu_y \rangle_{\gamma, \Gamma}}{|\eta_x|_Y}$$ like a Gaussian

2. family probing funcs is fundamental:

$$(u - u_0)(\xi) \approx \sum_k a_k \mu_{y_k}(\xi) \quad \forall \xi \in \Gamma$$
Choice of Testing Functions

- Green function:
  \[-\Delta G_x + \mu_0 G_x = \delta_x \text{ in } \Omega; \quad \frac{\partial G_x}{\partial \nu} = 0 \text{ on } \partial\Omega\]

- Scattered potential: \(u - u_0\) on \(\Gamma\):
  \((u - u_0)(\xi) = \int_D G_y(\xi)(\mu_0 - \mu(y))u(y) \, dy \quad \forall \xi \in \Gamma\)

- Fundamental representation:
  \((u - u_0)(\xi) \approx \sum_k a_k G_{y_k}(\xi) \quad \forall \xi \in \Gamma\)

- Green functions: good candidates for testing funcs
Choice of Probing Functions

Green function:

\[-\Delta w_x + \mu_0 w_x = \delta_x \text{ in } \Omega; \quad w_x = 0 \text{ on } \Gamma; \quad \frac{\partial w_x}{\partial \nu} = 0 \text{ on } \partial\Omega \setminus \Gamma\]

Fundamental solution in the whole space \(\Phi_x\)

Define \(\psi_x\):

\[-\Delta \psi_x + \mu_0 \psi_x = 0 \text{ in } \Omega; \quad \psi_x = \Phi_x \text{ on } \Gamma; \quad \frac{\partial \psi_x}{\partial \nu} = \frac{\partial \Phi_x}{\partial \nu} \text{ on } \partial\Omega \setminus \Gamma\]

Probing functions, \(w_x = \Phi_x - \psi_x\):

\[\eta_x(\xi) := \frac{\partial w_x}{\partial \nu}(\xi) \quad \forall \xi \in \Gamma\]
Probing functions for special geometries

For 2D circular measurement curve:

\[ \eta_x(y) = \frac{1 - |x|^2}{2\pi|x-y|^2} \quad \forall \ y \in S^1 \]

Orthogonality or Gaussian like behaviour:

\[ K(x, z) = \frac{\langle \eta_x, G_z \rangle_1}{\frac{1}{2} |\eta_x|^2} = C(x) \left\{ \frac{r_z r_x \cos(\theta_x - \theta_z)(1+r_z^2 r_x^2)-2r_z^2 r_x^2}{(1-2r_z r_x \cos(\theta_x - \theta_z)+r_z^2 r_x^2)^2} \right\} \]
Recall the kernel functions for DOT:

\[ K(x, y) = \frac{\langle \eta_x, G_y \rangle_\gamma}{|\eta_x|_Y} = \frac{\langle (-\Delta \Gamma)^\gamma \chi, \phi \rangle}{|\eta_x|_Y} \]

with the Sobolev index

\[ \gamma = 1 \]
Example I (5% noise)

Severely ill-posed, 4 inclusions close to each other & to the boundary, but reconstructions quite satisfactory:
5% noise, only one Cauchy data, data far away from inclusions
General Principle of time-dependent DSM

Define a Sobolev dual product on \( \Gamma \times (\tau_0, T) \):

\[ \langle \chi, \phi \rangle_{\gamma, \Gamma \times (\tau_0, T)} \quad \forall \chi \in Y, \phi \in Z \]

Select probing & testing funcs \( \{\eta_{x,t}\}, \{\mu_{y,s}\} \) based on PDEs

(1) nearly orthogonal wrt \( \langle \cdot, \cdot \rangle_{\gamma} \), i.e., \( \forall y \in D, s \in (\tau_0, T), \)

Kernel func:

\[
K(x, t; y, s) = \frac{\langle \eta_{x,t}, \mu_{y,s} \rangle_{\gamma, \Gamma \times (\tau_0, T)}}{|\eta_{x,t}|_Y} \quad \text{Gaussian}
\]

(2) testing funcs are fundamental over set of testing points:

\[
u - u_0 \approx \sum_{k,j} a_{k,j} \mu_{y_k,s_j} \quad \text{on} \quad \Gamma \times (\tau_0, T)
\]
We define the index function

\[ I(x, t) := \frac{\langle \eta_{x,t}, u - u_0 \rangle_{\gamma, \Gamma \times (\tau_0, T)} }{|\eta_{x,t}|_Y} \quad \forall x \in \Omega, t \in (0, T) \]

Then the index provides a probability:

\[ I(x, t) \approx \sum_k a_{k,j} \frac{\langle \eta_{x,t}, \mu_{y_k,s_j} \rangle_{\gamma, \Gamma \times (\tau_0, T)} }{|\eta_{x,t}|_Y} \]
DSM for Moving Potential

Heat conduction/moving DOT:

\[
\frac{\partial u}{\partial t} = a \Delta u - q(x, t)u
\]

Measure heat intensity on \(\Gamma: u\)
corresponding to one single \(u_0\)

Inverse Problem:

given \(u\), recover \(q(x, t)\)
Testing Functions

\( U_0 \): heat intensity with background potential \( Q_0 \)

\[
\frac{\partial (u-u_0)}{\partial t} - a \Delta (u - u_0) = -q_0 (u - u_0) - (q - q_0)u
\]

we have

\[
(u - u_0)(x, t) = - \int_0^T \int_{D(t)} \Phi(x - y, t - s) c(y, s) \, dy \, ds
\]

with fundamental solution

\[
\Phi(x, t) = \frac{1}{4 \pi at} \exp \left( - \frac{|x|^2}{4at} \right)
\]

Therefore

\[
(u - u_0)(x, t) \approx \sum_{k,j} c_{kj} \Phi(x - y_k, t - s_j) \quad \forall (x, t) \in \Gamma \times (0, T)
\]

Probing functions:

\[
\eta_{x,t} := \Phi_{x,t}(y, s) \equiv \Phi(x - y, t - s) \chi_+(t - s - \delta)
\]
Index Function

Define on the measurement surface:

\[ \langle \chi, \phi \rangle_{\alpha, \Gamma \times (0,t)} := \langle \Delta_x^\alpha \chi, \phi \rangle \quad \forall \chi \in H^{2\alpha}(\Gamma), \phi \in L^2(\Gamma) \]

DSM index functions:

\[ I_0^\alpha(x, t) = \frac{\langle \eta_x, t, u-u_0 \rangle_{\alpha, \Gamma \times (0,t)} \right|_{\omega_x^\alpha, t, Y} |w_x^\alpha, t|_Y \]

Normalization:

\[ \hat{I}_0^\alpha(x, t) = \frac{|I_0^\alpha(x, t)|}{\max |I_0^\alpha(x, t)|} \]
DSM index functions: real time reconstruction

\[ I_0^\alpha (x, t) = \frac{\langle \eta_{x,t}, u - u_0 \rangle_\alpha, \Gamma \times (0,t) }{|\omega_{x,t}^\alpha | Y} \]

no any data after time t needed
Behavior of index for point source \( q = \delta_{(0.5,0)}(x)\delta_1(t) \)

zeroth order index \( I_0^\alpha \)

\( \alpha = 2 \)

max at \( t=0.5 \), far from \( t=1 \), further away
From the behaviour of $I_0^\alpha$, we see big drop in spatial maximum with time as time goes on from $t=1$, so a rate of change of $I_0^\alpha$ may capture the inclusion more effectively:

$$I_{\gamma}^\alpha = \frac{\partial^\gamma}{\partial t^\gamma} I_0^\alpha$$
Behavior of index for point source \( q = \delta_{(0.5,0)}(x)\delta_1(t) \)

1st order index \( I_1^\alpha \)

\( \alpha = 2 \)

max well reached at \( t=1 \)
Verification of Index Function

Consider the kernel

\[ K_0^\alpha(x, t; y, s) = \frac{\int_0^{t-\delta} \int_\Gamma \Phi(y-z, s-k) \Delta_z^\alpha \Phi(x-z, t-k) d\sigma_z dk}{\sqrt{\int_0^{t-\delta} \int_\Gamma (\Delta_z^\alpha \Phi(x-z, t-k))^2 d\sigma_z dk}} \]

and its derivatives

\[ K_\gamma^\alpha(x, y, t, s) := \partial_t^\gamma K_0^\alpha(x, y, t, s) \]

For some \( t > t_0 \),

\[ \partial_t \left( \sqrt{\int_0^{t-\delta} \int_\Gamma (\Delta_z^\alpha \Phi(x-z, t-k))^2 d\sigma_z dk} \right) = O(t_0^{-2\alpha-2}) \]

Explicit relation for \( \alpha, \gamma \in \mathbb{N}, t > t_0 \):

\[ K_\gamma^\alpha(x, t; y, s) = \frac{\partial_t^\gamma \left( \int_0^{t-\delta} \int_\Gamma \Phi(y-z, s-k) \Delta_z^\alpha \Phi(x-z, t-k) d\sigma_z dk \right)}{\sqrt{\int_0^{t-\delta} \int_\Gamma (\Delta_z^\alpha \Phi(x-z, t-k))^2 d\sigma_z dk}} + O(t_0^{-2\alpha-2}) \]

Max at \( y = x, s = t \), but shift away more if \( \alpha - \gamma \) bigger
Numerical Experiments I

\[ \Gamma(t) = \left( \frac{t}{8T} + 0.25 \right) \left( \cos \left( \frac{t\pi}{3} \right), \sin \left( \frac{t\pi}{3} \right) \right), \quad t \in (0, 5) \]

there is a time lag, reconstructed inclusion follows closely exact inclusion

5% noise

green *
yellow :
centre of inclusion mass
Numerical Experiments II

\[ \Gamma(t) = \left( -\frac{t}{5} + \frac{1}{2} + \frac{1}{40} \cos(t\pi), -\frac{t}{20} + \frac{2}{3} - \frac{1}{5} \cos(t\pi) \right), \quad t \in (0, 5) \]

Initially, for \( t < 2 \), the reconstructed inclusion tries to find the exact inclusion. Once it succeeds to approach the exact inclusion for \( t > 2 \), it starts to follow the exact path. The recovered trajectory can even follow very fine turnings as exact one from \( t > 4 \) onwards.
Numerical Experiments III

\[ \Gamma_1(t) = \left( \frac{2}{3} \cos \left( \frac{t \pi}{10} \right), \frac{1}{2} \sin \left( \frac{t \pi}{10} \right) \right), \quad \Gamma_2(t) = \left( -\frac{2}{3} \cos \left( \frac{2t \pi}{15} \right), -\frac{1}{2} \sin \left( \frac{2t \pi}{15} \right) \right) \]

Initially, strongly coupling, gradually, clearly seen 2 objects, with different speeds, highly ill-posed

Initially, strongly coupling,

Gradually, clearly seen 2 objects,
Numerical Experiments III

\[ \Gamma_1(t) = \left( \frac{2}{3} \cos \left( \frac{t\pi}{10} \right), \frac{1}{2} \sin \left( \frac{t\pi}{10} \right) \right), \quad \Gamma_2(t) = \left( -\frac{2}{3} \cos \left( \frac{2t\pi}{15} \right), -\frac{1}{2} \sin \left( \frac{2t\pi}{15} \right) \right) \]

after long time, signal to noise weak, less stable, more oscillatory

2 objects, with one set of data, very challenging task

still tracing 2 objects reasonably well
An Optimal Control framework

- **Forward equation:**
  \[ L_q(u) = 0, \quad Bu = f \]

- **Background equation:**
  \[ L_{q_0}(u_0) = 0, \quad B u_0 = f \]

- **Parameter-to-solution:**
  \[ L_{q_0}(u - u_0) = -(L_q - L_{q_0})(u) := J(q - q_0) \]
  \[ u - u_0 = G \left[ J(q - q_0) \right] \]
An Optimal Control Framework

Parameter-to-solution:

\[ u - u_0 = G \left[ J(q - q_0) \right] \]

\[ E : u - u_0 \rightarrow (u - u_0)|_{\Gamma} \]

Index function:

\[ I = (\Phi \circ W_{X,\gamma}(\eta) \circ E \circ G) \left[ J(q - q_0) \right] \]

Hope \( I \) provides an estimate of support of \( J(c - c_0) \)

\[ \Phi \circ W_{X,\gamma}(\eta) \circ E \circ G \approx \text{id} : X \rightarrow X^* \]

\[ \min_{\Phi,\gamma} \| \Phi \circ W_{X,\gamma}(\eta) \circ E \circ G - \text{id} \|_{X \rightarrow X^*}^2 \]
Recall the kernel function:

\[ K(x, y) = \frac{\langle \eta_x, G_y \rangle^\gamma}{|\eta_x|_Y} = \frac{\langle ( - \Delta \Gamma )^\gamma \chi, \phi \rangle}{|\eta_x|_Y} \]

Sobolev index:

- Wave-type: \( \gamma = 0 \)
- EIT: \( \gamma = 2 \)
- DOT: \( \gamma = 1 \)
Features of DSMs

- Computationally very cheap, completely parallel
- Stability: straightforward
- Works for a single measurement data
- Robust against noise in data, due to orthogonality:
  
  high frequency components in data orthogonal to fundamental solutions on measurement surface
Other related sampling methods

R Potthast 2010, inverse obstacle scattering

ZM Chen et al. (since 2013): reverse time migration, inverse obstacle acoustic & EM scattering

H Ammari, et al.

WK Park, et al.

HY Liu, XD Liu, JZ Li, YK Guo, … …

… …

Mostly for inverse wave scattering
THANK YOU!