Free Interface Problems for the Incompressible Inviscid Resistive MHD

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Outline

I Introduction

- Formation of the plasma-vacuum problem in Eulerian coordinates;
- Physical Energy-Dissipation Law;
- Review of literatures.

II Main Results

- Reformulation in flattening coordinates;
- Main results on a prior estimates and global well-posedness.

III Ideas of the analysis

- Local well-posdeness;
- A priori estimates and the regularizing effects of the magnetic fields.

IV Results on plasma-plasma problem

I. Introduction

Aim: Consider the plasma-vacuum and plasma-plasma interface problems in a horizontal periodic slab in $\mathbb{R}^3$ impressed by a uniform non-horizontal magnetic field.

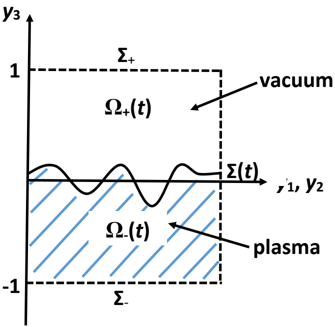


Fig. 1. Plasma-vacuum interface

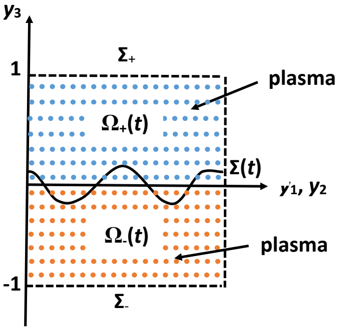


Fig. 2. Plasma-Plasma interface

§1.1 Formulation of the plasma-vacuum interface in Eulerian coordinates.

Consider the plasma-vacuum interface problem in $\Omega = \mathbb{T}^2 \times [-1, 1]$ impressed by a uniform transversal magnetic field $\bar{B}$ with $\bar{B}_3 \neq 0$, such that

Plasma region:

$$\Omega_-(t) = \{(y_h, y_3) \triangleq (y_1, y_2, y_3) \in \mathbb{T}^2 \times \mathbb{R} \mid -1 < y_3 < \eta(t, y_h)\} \quad (1.1)$$

Vacuum region:

$$\Omega_+(t) = \{y \in \mathbb{T}^2 \times \mathbb{R} \mid \eta(t, y_h) < y_3 < 1\} \quad (1.2)$$

I. Introduction

P-V interface:

$$\Sigma(t) \triangleq \{y \in \mathbb{T}^2 \times \mathbb{R} \mid y_3 = \eta(t, y_h)\} \tag{1.3}$$

$$\eta : \mathbb{R}^+ \times \mathbb{T}^2 \to \mathbb{R} \quad \text{is unknown;} \tag{1.4}$$

Upper and lower fixed boundaries are $\Sigma_\pm \triangleq \mathbb{T}^2 \times \{\pm 1\}$.
In the plasma region $\Omega_-(t)$, the flow is given by the incompressible, inviscid and resistive magnetohydrodynamics equation (MHD)

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \text{curl } B \times B \\ \operatorname{div} u = 0 \\ \partial_t B = \text{curl } E, \quad E = u \times B \quad - k \text{ curl } B \\ \operatorname{div} B = 0 \end{cases} \tag{1.5}$$

where

u :	velocity field
B :	magnetic field
p :	pressure
E :	the electric field of the plasma
$k > 0$:	the magnetic diffusion coefficient

In the vacuum region $\Omega_+(t)$, the magnetic field $\hat{B}$ and the electric field $\hat{E}$ are assumed to satisfy the pre-Maxwell equations:

$$\begin{cases} \text{curl } \hat{B} = 0, & \mathrm{div}\hat{B} = 0 \quad \text{in } \Omega_+(t) \\ \partial_t \hat{B} = \text{curl } \hat{E}, & \mathrm{div}\hat{E} = 0 \quad \text{in } \Omega_+(t) \end{cases} \tag{1.6}$$

I. Introduction

The free interface satisfies the kinematic boundary condition

$$\partial_t \eta = u \cdot \mathcal{N} \quad \text{on } \Sigma(t) \tag{1.7}$$

with $\mathcal{N} = (-\nabla_h \eta, 1) \triangleq (-\partial_1 \eta, -\partial_2 \eta, 1)$ begin the upperward normal vector of $\Sigma(t)$.

Furthermore, across the $\Sigma(t)$, the balance of normal stress and classical jump conditions for the magnetic and electric fields should be satisfied.

Balance of Normal Stress:

$$(pI + \frac{1}{2}|B|^2 I - B \otimes B)\mathcal{N} = \left(\frac{1}{2}|\hat{B}|^2 I - \hat{B} \otimes \hat{B}\right)\mathcal{N} - \sigma H \mathcal{N} \text{ on } \Sigma(t) \tag{1.8}$$

with I being the 3×3 Identity matrix, $\sigma > 0$ surface tension, H: the mean curvature of Σ

$$H = \mathrm{div}_h \left(\frac{\nabla_h \eta}{\sqrt{1 + |\nabla_h \eta|^2}} \right).$$

Classical jump conditions of magnetic and electric fields:

$$B \cdot \mathcal{N} = \hat{B} \cdot \mathcal{N}, \ (E - \hat{E}) \times \mathcal{N} = u \cdot \mathcal{N}(B - \hat{B}) \text{ on } \Sigma(t) \quad (1.9)$$

Under the consideration that B is close to $\bar{B}$ so that $B \cdot \mathcal{N} = \hat{B} \cdot \mathcal{N} \neq 0$, then (1.7) and (1.8) are equivalent to

$$p = -\sigma H, \quad B = \hat{B}, \quad E \times \mathcal{N} = \hat{E} \times \mathcal{N} \qquad (1.10)$$

(B.C.): The upper wall Σ_+ is assumed to be perfectly insulating:

$$\hat{B} \times e_3 = \bar{B} \times e_3, \quad \hat{E} \cdot e_3 = 0 \quad \text{on} \quad \Sigma_+; \tag{1.11}$$

while the lower wall Σ_- is assumed to be impermeable and perfectly conducting:

$$u \cdot e_3 = 0, \quad B \cdot e_3 = \bar{B} \cdot e_3, \quad E \times e_3 = 0 \quad \text{on } \Sigma_- \tag{1.12}$$

with $e_3 = (0, 0, 1)$.

(I.C.): Given initial surface $\Sigma(0)$ as the graph of $\eta(0) = \eta_0 : \mathbb{T}^2 \to \mathbb{R}$, which yield $\Omega_-(0)$ and $\Omega_+(0)$. We also specify $u(0) = u_0 : \Omega_-(0) \to \mathbb{R}^3$, and $B(0) = B_0 : \Omega_-(0) \to \mathbb{R}^3$.

Thus the plasma-vacuum interface problem is to look for $(u, B, p, \eta, \hat{B}, \hat{E})$ satisfying (1.5), (1.6), (1.7), (1.10), (1.11), (1.12) and (I.C.).

Remark (1.1)

Mathematically, as Ladyzenskaya-Solonnikov, one may regard the electric field $\hat{E}$ in vacuum as a secondary variable. Indeed, set

$$b = B - \bar{B}, \quad \hat{b} = \hat{B} - \bar{B}. \tag{1.13}$$

Then (1.5)-(1.7), (1.10)-(1.12) imply the following problem

$$
\begin{cases}
\partial_t u + u\cdot\nabla u + \nabla p = \text{curl } b \times (\bar{B} + b) & \Omega_-(t)\\
\mathrm{div} u = 0 & \Omega_-(t)\\
\partial_t b = \text{curl } E, E = u \times (\bar{B} + b) - k \text{ curl } b & \Omega_-(t)\\
\mathrm{div} b = 0 & \Omega_-(t)\\
\text{curl}\hat{b} = 0, \mathrm{div}\hat{b} = 0 & \Omega_+(t)\\
\partial_t \eta = u\cdot\mathcal{N} & \Sigma(t)\\
p = -\sigma H, b = \hat{b} & \Sigma(t)\\
\hat{b}\times e_3 = 0 & \Sigma_+\\
u_3 = 0, b_3 = 0, E\times e_3 = 0 & \Sigma_-\\
\eta|_{t=0} = \eta_0, b|_{t=0} = b_0, u|_{t=0} = u_0 & \Omega_-(0)
\end{cases}
\tag{1.14}
$$

Remark (1.2)

Once (1.14) is solved, then $\hat{E}$ can be recovered by solving the following elliptic system,

$$\begin{cases} curl\ \hat{E} = \partial_t \hat{b}, \quad \mathrm{div}\hat{E} = 0 & in\ \Omega_+(t) \\ \hat{E} \times \mathcal{N} = E \times \mathcal{N} & on\ \Sigma(t) \\ \hat{E}_3 = 0 & on\ \Sigma_+(t) \end{cases} \tag{1.15}$$

Remark (1.3)

Formally, the magnetic field in vacuum, $\hat{b}$, can be suppressed in (1.14) too. Indeed, $\hat{b}$ can be determined by $b \cdot \mathcal{N}$ on $\Sigma(t)$ through the following problem:

$$\begin{cases} \textit{curl}\ \hat{b} = 0, \operatorname{div} \hat{b} = 0 & \textit{in}\ \Omega_+(t) \\ \hat{b} \cdot \mathcal{N} = b \cdot \mathcal{N} & \textit{on}\ \Sigma(t) \\ \hat{b} \times e_3 = 0 & \textit{on}\ \Sigma_+ \end{cases} \tag{1.16}$$

This implies that the jump condition $b = \hat{b}$ on $\Sigma(t)$ in (1.14) could be regarded as a nonlocal boundary condition for b:

$$b \times \mathcal{N} = B^t(b \cdot \mathcal{N}) \times \mathcal{N} \quad \text{on } \Sigma(t) \tag{1.17}$$

where $B^t(b \cdot \mathcal{N})$ is the solution to (1.16).

§1.2 Physical Energy-Dissipation Law

Key fact: The classical solution to the problem (1.14) admits the following energy identify:

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega_-(t)}(|u|^2+|b|^2)dy+\int_{\Omega_+(t)}|\hat{b}|^2dy + \int_{\mathbb{T}^2}2\sigma(\sqrt{1+|\nabla_h\eta|^2}-1)dy_h\right)+k\int_{\Omega(t)}|\nabla\times b|^2dy=0 \tag{1.18}$$

which can be derived by using energy estimates and making use of the structure (1.15) satisfied by the electric field $\hat{E}$ in vacuum.

(1.18) will be the basis of the energy method to analyze the problem (1.14).

Remark (1.4)

The fact (1.18) can also be derived by introducing the so called virtual magnetic field in $\Omega_{-}(t)$ as by Ladyzenskaya-Solonnikov for the viscous an resistive MHD.

§1.3 Review of Literature

(1) Local well-posedness (LWP):

- There are huge amount of studies on free surface Euler equations:
  - Water waves for the irrotational Euler equations:
    - Nalimov, '74; Yosihara, '82; Carig, '85; ...
    - S. J. Wu, '97, '99; Lanes, '95; Ambrouse-Masoudi, '05, '09; ...
  - Water waves for the general Euler equations, under Taylor sign condition or surface tension:
    - Christodoulou-Lindblad, '00; Lindblad, '05; Coutand-Shkoller, '07; Shatah-Zeng, '08; Zhang-Zhang, '08;
    - Masmoudi-Rousset, '17; Wang-Xin, '15.
  - Vortex Sheets, with surface tension:
    - Ambrosae-Masmoudi, '03, '07; Cheng-Coutand-Shkoller, '08; Shatah-Zeng, '08, '11.

I. Introduction

- Compared with the pure fluids, there are only recent studies on the free interface problems for the ideal (inviscid and non-resistive) MHD and viscous and resistive MHD:
  - Plasma-Vacuum interface problem; under the assumption: $B \cdot \mathcal{N} = \hat{B} \cdot \mathcal{N} \equiv 0$ on $\Sigma(t)$;
    1. Magnetic stability condition ($\Leftrightarrow$ Non-Collinearity Condition): $|B \times \hat{B}| > 0$ on $\Sigma(t)$;
       - Morando-Trakhinin-Trebesdi, '14; linear problem;
       - Sun-Wang-Zhang, '19: Nonlinear local well-posedness!
    2. Hydrodynamic stability; Taylor sign condition: $-\nabla(p + \frac{1}{2}|B|^2 - \frac{1}{2}|\hat{B}|^2) \cdot \mathcal{N} > 0$ on $\Sigma(t)$
       - Hao-Luo, '14: $\hat{B} \equiv 0$, a priori estimates;
       - Gu-Wang, '19: $\hat{B} = 0$, well-posedness.

I. Introduction

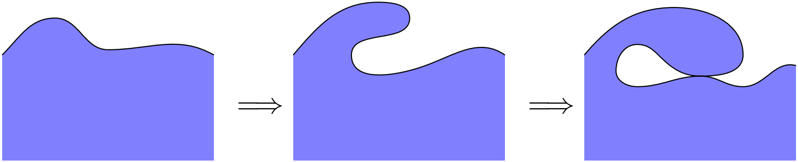
- Plasma-Plasma interface problem (Current-Vortex sheets):
  Syrovatskij stability condition:
  $|[u] \times B_+|^2 + |[u] \times B_-|^2 < 2|B_+ \times B_-|^2$ on $\Sigma(t)$:
  - Coulombel-Morando-Secchi-Trebeschi '12; A priori estimates (under stronger condition);
  - Sun-Wang-Zhang, '18: well-posedness;
  - Compressible case: Chen-Wang '2008, Trakhinin '2009.

- Plasma-Vacuum interface problem for viscous and resistive MHD:
  - Padula-Solonnikov, '10; Solonnikov, '12, '16.

Remark (1.5)

The Non-Collinearity condition and the Syrovatskij condition show the stabilizing effects of the magnetic field on the local well-posedness of interface problems in inviscid fluids since either the Taylor-sign condition or non-zero surface tension is necessary for the local well-posed of the one-phase problem, and the non-zero surface tension is necessary for the local well-posedness of the vortex sheets problem. However, it requires $B \cdot N = 0$ *on* Σ.

(2) Finite time singularities: Development in finite time of splash/splat singularities for free boundary problems for some large initial data:



- Inviscid flows:
  Castro-Córadoba-Fefferman-Gancedo-Gómez-Serrano, '13; Coutand-Shkoller, '14; Coutand, '19.

I. Introduction

- Viscous flows:
  Castro-Córadoba-Fefferman-Gancedo-Gómez-Serrano, '19; Coutand-Shkoller, '15 arXiv.
- The two-phase interface problem:
  Fefferman-Ionescu-Lie, '16; Coutand-Shkoller, '16; Coutand, '19.

(3) Global well-posedness:

- Irrotational Euler flows: horizontally non-periodic setting with "small" data: Wu, '09, '11; Germain-Masmoudi-Shatah, '12, '15; Ionescu-Pusateri, '15, '17; Alazard-Delort, '15; Deng-Ionescu-Pausader-Pusateri, '17; ...

I. Introduction

- Navier-Stokes flows: Solonnikov, '77, '88; Beale, '81, '83; Nishida-Teramoto-Yoshihara, '04; Hataya, '09; Guo-Tice, '13; Wang-Tice-Kim, '14; Tan-Wang, '14; ...
- Viscous and resistive MHD: "small" data around the zero magnetic field:
  Solonnikov-Frolova, '13; Solonnikov, '16;
- Viscous and non-resistive MHD:
  Y. Wang, '19; global existence plasma-plasma interface problem around a transversal uniform magnetic field.

(4) Motivations:

- It is still open whether the free surface incompressible Euler equations for general small initial data admits a global unique solution or not, except the case of irrotational flows where certain dispersive effects can be used to establish global well-posedness. This is even so for 2D!
- Some global well-posedness of free surface problems for "general small" initial data have been established for viscous fluids (either Navier-Stokes, or viscous MHD). These results rely heavily on the dissipation and regularization effects of the viscosity for the velocity field. It is quite open for inviscid fluids!

- In the absence of the viscosity for the velocity field, the magnetic field may provide some stabilizing effects for the local well-posedness of some free interface problem for the inviscid MHD. However, there is no any global well-posedness results for the inviscid MHD. In the free surface problems in a horizontally slab impressed by a uniform non-horizontal magnetic field, even the local well-posedness of either plasma-vacuum or plasma-plasma interface problem is highly non-trivial. In this talk, I will present some global well-posedness results for the free interface problems for the inviscid and resistive MHD. Note that this is a subtle and difficult issue since the free surface is transported by the fluid velocity, and the global existence of classical solutions to the Cauchy problem in 2D is unknown. Our results reveal strong stabilizing effect of the magnetic field based on an induced damping structure for the fluid vorticity due to the resistivity and the transversal magnetic field.

§2.1 Reformulation in flattening coordinates

Flattening coordinates

- The equilibrium domains:

$$\Omega_- := \mathbb{T}^2 \times (-1, 0), \quad \Omega_+ := \mathbb{T}^2 \times (0, 1) \tag{2.1}$$

and their interface

$$\Sigma := \mathbb{T}^2 \times \{0\}. \tag{2.2}$$

- The physical domains can be flattened via the mapping

$$\Omega_\pm \ni x \mapsto (x_h, \varphi(t, x) := x_3 + \bar{\eta}(t, x)) =: \Phi(t, x) = y \in \Omega_\pm(t) \tag{2.3}$$

where $\bar{\eta} = \chi(x_3) P\eta : \chi(0) = 1,\ \chi(\pm 1) = 0,\ P\eta$ is the harmonic extension of η onto $\mathbb{R}^3$.

- Set

$$\partial_i^\varphi = \partial_i - \partial_i\bar{\eta}\partial_3^\varphi, \quad i = t, 1, 2, \quad \partial_3^\varphi = \frac{1}{\partial_3\varphi}\partial_3 \tag{2.4}$$

$$\begin{gathered}(\nabla^\varphi)_i = \partial_i^\varphi, \quad i = 1, 2, 3, \quad \mathrm{div}^\varphi = \nabla^\varphi\cdot, \\ \mathrm{curl}^\varphi = \nabla^\varphi\times, \quad \Delta^\varphi = \mathrm{div}^\varphi\nabla^\varphi\end{gathered} \tag{2.5}$$

$$[b] = \hat{b}|_\Sigma - b|_\Sigma \tag{2.6}$$

II. Main Results

Reformulation:

- In flattening coordinates, the Problem (1.4) is equivalent to:

$$
\begin{cases}
\partial_t^{\varphi} u + u \cdot \nabla^{\varphi} u + \nabla^{\varphi} p = \mathrm{curl}^{\varphi} b \times (\bar{B} + b) & \Omega_- \\
\mathrm{div}^{\varphi} u = 0 & \Omega_- \\
\partial_t^{\varphi} b = \mathrm{curl}^{\varphi} E, \ \ E = u \times (\bar{B} + b) - k \ \mathrm{curl}^{\varphi} b & \Omega_- \\
\mathrm{div}^{\varphi} b = 0 & \Omega_- \\
\mathrm{curl}^{\varphi} \hat{b} = 0, \quad \mathrm{div}^{\varphi} \hat{b} = 0 & \Omega_+ \\
\partial_t \eta = u \cdot \mathcal{N} & \text{on } \Sigma \\
p = -\sigma H, \quad [b] = 0 & \text{on } \Sigma \\
\hat{b} \times e_3 = 0 & \text{on } \Sigma_+ \\
u_3 = 0, \quad b_3 = 0, \quad E \times e_3 = 0 & \text{on } \Sigma_- \\
(u, b, \eta)|_{t=0} = (u_0, b_0, \eta_0)
\end{cases}
\tag{2.7}
$$

- Then the energy-dissipation law (1.18) becomes

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{\Omega_-}(|u|^2+|b|^2)d\nu_t+\int_{\Omega_+}|\hat{b}|^2 d\nu_t \\ +\int_{\mathbb{T}^2}2\sigma(\sqrt{1+|\nabla_h\eta|^2}-1)\Big)+k\int_{\Omega_-}|\mathrm{curl}^{\varphi}\, b|^2 d\nu_t=0 \tag{2.8}$$

where $d\nu_t := \partial_3\varphi\, dx$ is the volume elements.

§2.2 Statement of the Main Results

Assumptions on Initial Data

- Zero-average condition:

$$\int_{\mathbb{T}^2}\eta_0=0 \tag{2.9}$$

- $2N$-th order compatibility condition for (u_0, b_0, η_0):

$$
\begin{cases}
\mathrm{div}^{\varphi_0} u_0 = \mathrm{div}^{\varphi} b_0 = 0 \text{ on } \Omega_-; \ u_{0,3} = b_{0,3} = 0 \text{ on } \Sigma_-; \\
\left[\partial_t^j b(0)\right] \times \mathcal{N}_0 = 0 \text{ on } \Sigma, \ \partial_t^j E(0) \times e_3 = 0 \text{ on } \Sigma_-, \\
\quad j = 0, \cdots, 2N-1.
\end{cases}
\tag{2.10}
$$

Remark (2.1)

It can be verified easily that (2.9) implies

$$
\int_{\mathbb{T}^2} \eta(x,t) = 0 \quad \textit{for all} \quad t \geq 0
$$

Remark (2.2)

The $2N$-th order compatibility conditions are necessary for local well-posedness theory in the high order regularity contest. However, due to the non-local and nonlinear nature of the problem (2.7), the construction of initial data satisfying the $2N$-th order compatibility conditions is highly technical and non-trivial. We can achieve this by using the implicit function theorem.

Energy and Dissipation Functionals

- Sobolev Norm:

$$||f||_m := ||f||_{H^m(\Omega_\pm)}, \text{ and } |f|_s := ||f||_{H^s(\mathbb{T}^2)}, k \geq 0, s \in \mathbb{R}$$

Anisotropic norm:

$$||f||_{k,l} := \sum_{\alpha \in N^2, |\alpha| \leq l} ||\sigma^\alpha f||_k$$

- For $N \geq 4$, the high-order energy is defined as

$$\begin{aligned} E_{2N} = &\sum_{j=0}^{2N} ||\partial_t^j u||_{2N-j}^2 + \sum_{j=0}^{2N-1} ||\partial_t^j b||_{2N-j+1}^2 + ||\partial_t^{2N} b||_0^2 \\ &+ \sum_{j=0}^{2N-1} ||\partial_t^j \hat{b}||_{2N-j+1}^2 + ||\partial_t^{2N} \hat{b}||_0^2 + \sum_{j=0}^{2N-1} ||\partial_t p||_{2N-j}^2 \\ &+ \sum_{j=0}^{2N-1} |\partial_t^j \eta|_{2N-j+\frac{3}{2}}^2 + |\partial_t^{2N} \eta|_1^2 + |\partial_t^{2N+1} \eta|_{-\frac{1}{2}}^2 . \end{aligned} \tag{2.11}$$

Remark (2.3)

One of the key parts in proving the global well-posedness of (2.7) is to show that $E_{2N}(t)$ for $N \geq 8$ is bounded for all $t \geq 0$. To this end, one needs to derive a sufficiently fast time-decay of certain lower-order Sobolev norms of the solution, which will be achieved by some dissipation estimates.

- Dissipation functional: For $N+4 \leq n \leq 2N$,

$$
\begin{aligned}
D_n := & \sum_{j=0}^{n-1} ||\partial_t^j u||^2_{n-j-1} + \sum_{j=0}^{n-2} ||\partial_t^j b||^2_{n-j} + \sum_{j=0}^{n} ||\partial_t^j b||^2_{1,n-j} \\
& + \sum_{j=0}^{n} ||\partial_t^j \hat{b}||^2_{n-j+1} + \sum_{j=0}^{n-2} ||\partial_t^j p||^2_{n-j-1} \qquad (2.12) \\
& + \sum_{j=0}^{n-2} |\partial_t^j \eta|^2_{n-j+1/2} + |\partial_t^{n-1}\eta|^2_1 + |\partial_t^n \eta|^2_0
\end{aligned}
$$

- Note that the dissipation functional D_{2N} cannot control E_{2N}. Furthermore, in the derivation of the dissipation estimates for D_n, the following lower-order energy functional is involved:

$$\begin{aligned}\mathcal{E}_n := \quad & ||u||^2_{n-1} + ||u||^2_{0,n} + \sum_{j=1}^{n} ||\partial_t^j u||^2_{n-j} + ||b||^2_n \\ & + \sum_{j=1}^{n-1} ||\partial_t^j b||^2_{n-j+1} + ||\partial_t^n b||^2_0 + ||\hat{b}||^2_n \\ & + \sum_{j=1}^{n-1} ||\partial_t^j \hat{b}||^2_{n-j+1} + ||\partial_t^n \hat{b}||^2_0 + \sum_{j=0}^{n-1} ||\partial_t^j p||^2_{n-j} \\ & + \sum_{j=0}^{n-1} |\partial_t^j \eta|^2_{n-j+3/2} + |\partial_t^n \eta|^2_1 + |\partial_t^{n+1} \eta|^2_{-1/2}\end{aligned} \tag{2.13}$$

In fact, it is $\mathcal{E}_n$ that would decay, but not E_n.

Main Results:

Theorem (Plasma-Vacuum Interface)

Let $k > 0$, $\bar{B}_3 \neq 0$, $\sigma > 0$, and $N \geq 8$ (an integer) be fixed. Assume that the initial (u_0, b_0, η_0) is given such that

(i) $u_0 \in H^{2N}(\Omega_-)$, $b_0 \in H^{2N+1}(\Omega_-)$, $\eta_0 \in H^{2N+\frac{3}{2}}(\Sigma)$, $E_{2N}(0) < +\infty$

(ii) *(2.9) and (2.10) are satisfied.*

Then $\exists$ universal constant $\varepsilon_0 > 0$ such that if $E_{2N}(0) \leq \varepsilon_0$, then $\exists|$ global solution $(u, p, \eta, b, \hat{b})$ to the plasma-vacuum interface problem (2.7). Moreover, for all $t \geq 0$, it holds that

Theorem (Plasma-Vacuum Interface) (continued)

$$E_{2N}(t) + \int_0^t D_{2N}(s)ds \leq cE_{2N}(0) \tag{2.14}$$

and

$$\sum_{j=1}^{N-5}(1+t)^{N-5-j}\mathcal{E}_{N+4+j}(t) + \sum_{j=0}^{N-6}\int_0^t (1+s)^{N-5-j}D_{N+4+j}(s)ds \leq cE_{2N}(0) \tag{2.15}$$

Remark (2.4)

The theorem implies in particular that $\sqrt{\mathcal{E}_{N+4}(t)} \leq c(1+t)^{-\frac{N-5}{2}}$*, which is integrable in time for* $N \geq 8$*. This decay result can be regarded as "almost exponential" decay rate. Since* η *is such that the mapping* $\Phi(t,\cdot)$*, defined in (2.3), is a diffeomorphism for each* $t \geq 0$*, one may change coordinates to* $y \in \Omega_{\pm}(t)$ *to obtain a global in time decay solution to (1.14).*

Remark (2.5)

The theorem provides the first results for the global well-posedness of free surface problems without viscosity for the general incomperessible rotational flows. This is due to the strong coupling between the fluid and the diffusive transversal magnetic field. In contrast to the earlier works on the local well-posedness of free inteface problems for ideal MHD, where the tangential magnetic field play the important role, here the global well-posedness depends crucially on the transversally of the magnetic field. Indeed, our analysis fails for the case $\bar{B}$ *being horizontal. For example, for* $B = \hat{B} = \bar{B} = e_1 = (1, 0, 0)$. *Take* $u_1 \equiv 0$, $\Rightarrow$ *2D Euler!*

Remark (2.6)

The surface tension is important for the theory here ($\sigma > 0$)*. Indeed, to solve (2.7) with the desired regularities of b (and $\hat{b}$) in (2.11), even locally in time, one needs* $\eta \in H^{2N+\frac{1}{2}}$ *due to the magnetic diffusion term* $curl^{\varphi}\, curl^{\varphi}$*. In the case* $\sigma = 0$*, it seems that only* H^{2N} *regularity for* η *is available. Hence* $\sigma > 0$ *is necessary here even for local well-posedness! This is different from the viscous case where the viscosity has a regularizing effect of* $\frac{1}{2}$ *order for* η *and so* $\sigma > 0$ *is unnecessary!*

Remark (2.7)

It should be noted even the local well-posedness of the interface problem (2.7) is unknown and non-trivial, which is of independent interests. Indeed, note that it is difficult to apply the ideas for previous local well-posedness of interface problems for ideal MHD (see Gu-Wang, Morando-Trakhinin-Trebeschi, etc.) where the parallelness of the magnetic field to the interface is important! Even though the magnetic diffusion has a regularizing effect for the magnetic field, one of the main difficulties in constructing solutions to (2.7) lies in solving the magnetic system due to the non-local boundary conditions for the magnetic field. For the viscous and resistive MHD, Padula-Solonnikov solved the magnetic system in the framework of full parabolic regularity theory, which unfortunately <u>cannot</u> be applied to the inviscid problem due to the less regularity of the velocity.

Remark (2.7) (continued)

Out strategy is to solve the magnetic system in the framework of energy method, which is naturally consistent with the Euler equations, so that the solution can be constructed as the limit of the approximate solutions to an elaborate chosen regularization.

Remark (2.8)

The main ideas and strategies for the plasma-vacuum interface problem can be modified to study the plasma-plasma interface problem to obtain its global well-posedness.

Strategy:
LWP + Global a priori estimates + Continuity Argument $\Rightarrow$ GWP

§3.1 Local well-posedness (LWP)

- Since the Lorentz force is of lower-order regularity compared with magnetic diffusion,

  Main Strategy: Decompose (2.7) $\approx$ Hydrodynamic part on $\Omega_- \oplus$ Magnetic part on $\Omega \oplus$ iteration scheme

- Hydrodynamic part on Ω_-: For $F = \mathrm{curl}^{\tilde{\varphi}} b \times (\bar{B} + b)$ with given $\tilde{\varphi}$ and b, solve the following free surface incompressible Euler equations with surface tension:

$$
\begin{cases}
\partial_t^{\varphi} u + u \cdot \nabla^{\varphi} u + \nabla^{\varphi} p = F & \text{in } \Omega_- \\
\mathrm{div}^{\varphi} u = 0 & \text{in } \Omega_- \\
\partial_t \eta = u \cdot \mathcal{N}, \ \ p = -\sigma H & \text{on } \Sigma \\
u_3 = 0 & \text{on } \Sigma_- \\
(u, \eta)|_{t=0} = (u_0, \eta_0)
\end{cases}
\tag{3.1}
$$

Remark (3.1)

The hydrodynamic part (3.1) can be solved is a similar way as Coutand-Shkoller '07.

- Magnetic part on Ω: For $G = u \times (\bar{B} + \tilde{b})$ with u, $\tilde{b}$, and η given, solve the following fixed initial boundary value problem for the magnetic field $(b, \hat{b})$:

$$\begin{cases} \partial_t^{\varphi} b + k\, \mathrm{curl}^{\varphi}\mathrm{curl}^{\varphi} b = \mathrm{curl}^{\varphi} G & \text{in } \Omega_- \\ \mathrm{div}^{\varphi} b = 0 & \text{in } \Omega_- \\ \mathrm{curl}^{\varphi} \hat{b} = 0, \ \mathrm{div}^{\varphi} \hat{b} = 0 & \text{in } \Omega_+ \\ [b] = 0 & \text{on } \Sigma \\ \hat{b} \times e_3 = 0 & \text{on } \Sigma_+ \\ b_3 = 0, \ k\mathrm{curl}^{\varphi} b \times e_3 = G \times e_3 & \text{on } \Sigma_- \\ b|_{t=0} = b_0 \end{cases} \tag{3.2}$$

Remark (3.2)

This is the major difficult part of LWP due to nonlocal boundary condition on Σ. However, in the more regular case (i.e. u satisfies NS equation). (3.2) was solved by Padula-Solonnikov ('10) with η being a small perturbation of flat case ($\eta = 0$) by employing the full parabolic regularity. However, such a full parabolic regularity of solving (3.2) is not consistent in the iteration scheme to construct solutions to (2.7) since the hyperbolic Euler equations could not provide such higher regularity for u and η.

Now Approach: We solve (3.2) in the functional framework based on the energy structure (2.8).

Step 3.1: Consider the following regularized problem:

$$\begin{cases} \partial_t^{\varphi^\varepsilon} b^\varepsilon + k\ \mathrm{curl}^{\varphi^\varepsilon}\mathrm{curl}^{\varphi^\varepsilon} b^\varepsilon = \mathrm{curl}^{\varphi^\varepsilon}(G^\varepsilon - \Psi^\varepsilon) & \text{in } \Omega_- \\ \mathrm{div}^{\varphi^\varepsilon} b^\varepsilon = 0 & \text{in } \Omega_- \\ \mathrm{curl}^{\varphi^\varepsilon} \hat{b}^\varepsilon = 0,\ \mathrm{div}^{\varphi^\varepsilon} \hat{b}^\varepsilon = 0 & \text{in } \Omega_+ \\ [b^\varepsilon] = 0 & \text{on } \Sigma \\ \hat{b}^\varepsilon \times e_3 = 0 & \text{on } \Sigma_+ \\ b^\varepsilon_\varepsilon = 0,\ k\mathrm{curl}^{\varphi^\varepsilon} b^\varepsilon \times e_3 = G^\varepsilon \times e_3 & \text{on } \Sigma_- \end{cases} \tag{3.3}$$

where $\varepsilon > 0$: smoothing parameter; $\varphi^\varepsilon = \varphi(\eta^\varepsilon)$; η^ε and G^ε are smooth regularizations of η and G; Ψ^ε: corrector to be constructed, which are crucial to satisfy the compatibility condition for (3.3).

Step 3.2: Solve (3.3) in the higher order regularity context by modifying the arguments due to Padula-Solonnikov ('10).

Step 3.3: To derive the uniform estimates (independent of $\varepsilon > 0$) for the solution to (3.3) with the desired regularity in our functional framework. To this end, we make important use of the following regularizing electric field in vacuum, $\hat{E}^\varepsilon$, which solves

$$\begin{cases} \mathrm{curl}^{\varphi^\varepsilon} \hat{E}^\varepsilon = \partial_t^{\varphi^\varepsilon} \hat{b}^\varepsilon, \ \mathrm{div}^{\varphi^\varepsilon} \hat{b}^\varepsilon = 0 & \text{in } \Omega_+ \\ \hat{E}^\varepsilon \times N^{-\varepsilon} = (-k\mathrm{curl}^{\varphi^\varepsilon} b^\varepsilon + G^\varepsilon - \Psi^\varepsilon) \times N^\varepsilon & \text{on } \Sigma \\ \hat{E}_3^\varepsilon = 0 & \text{on } \Sigma_+ \end{cases} \tag{3.4}$$

whose solvability is classical (see Cheng-Shkoller ('17)).

Step 3.4: The solution to (3.2) is then obtained as the limit of solutions to (3.3) as $\varepsilon \to 0^+$ after deriving the uniform estimates on the approximate solutions on a time interval independent of ε by a variant of the derivation of the estimates for (2.7) to be sketched below.

Finally, we can construct the local solution to (2.7) by the method of successive approximations based on the solvability of (3.1) and (3.2).$\square$

§3.2 A Priori Energy Estimates

Our derivation of a priori estimates for the solutions to (2.7) is based on the physical energy-dissipation structure (2.8), and involves the vacuum electric field $\hat{E}$ which solves:

$$\begin{cases} \mathrm{curl}^{\varphi}\hat{E} = \partial_t^{\varphi}\hat{b},\ \mathrm{div}^{\varphi}\hat{E} = 0 & \text{in } \Omega_+ \\ \tilde{E}\times\mathcal{N} = E\times\mathcal{N} & \text{on } \Sigma \\ \hat{E}_3 = 0 & \text{on } \Sigma_+ \end{cases} \tag{3.5}$$

and the estimates of $\hat{E}$ in terms E_{2N}, $\mathcal{E}_{2N}$, D_{2N} can be obtained easily by the Hodge theory.

Tangential energy estimates:

- Applying the energy-dissipation structure law (2.8) to the high order temporal and horizontal spatial derivatives ∂^α for $\alpha \in \mathbb{N}^{1+2}$ with $|\alpha| \le 2N$ yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega_-}(|\partial^\alpha u|^2+|\partial^\alpha b|^2)d\nu_t+\int_{\Omega_+}|\partial^\alpha \hat{b}|^2 d\nu_t \right.\\ &\left.+\int_\Sigma \sigma|\nabla\partial^\alpha\eta|^2\right)+k\int_{\Omega_+}|\mathrm{curl}^\varphi\partial^\alpha b|^2 d\nu_t \\ =\ & -\int_{\Omega_-}\partial^\alpha p\,[\partial^\alpha,\mathrm{div}]\,u d\nu_t-\int_\Sigma \sigma\partial^\alpha H\,[\partial^\alpha,\mathcal{N}]\,u \\ &-\int_{\Omega_+}\partial^\alpha\hat{E}\cdot[\partial^\alpha,\mathrm{curl}^\varphi]\,\hat{b}d\nu_t+\Sigma_R, \end{aligned} \tag{3.6}$$

where Σ_R denotes nonlinear terms which can be controlled by the energies!

- When $\alpha_0 \leq 2N-1$, the first three terms on the right hand side of (3.6) can be shown to be also of Σ_R.
- When $\alpha_0 = 2N$, the difficulty is that $\partial_t^{2N} p$, $\partial_t^{2N} H$ and $\partial_t^{2N} \hat{E}$ seem to be out of control. However, integrating by parts in times shows the third term is of Σ_R, so it remains to estimate the first two terms. As we observed earlier, integrating by parts in both time and space in an appropriate order and then employing a crucial cancellation between $\partial_t^{2N} p$ and $\sigma \partial_t^{2N} H$ on Σ by using the dynamical boundary condition, one can show that the first two terms are of Σ_R too!

- The above arguments lead to the following tangential energy evolution estimate:

$$\begin{aligned} & \bar{E}_{2N}(t) + \int_0^t \bar{D}_{2N}(s)ds \\ \leq \quad & E_{2N}(0) + E_{2N}^{\frac{3}{2}}(t) + \int_0^t \sqrt{\mathcal{E}_{N+4}(s)}(E_{2N}(s) + D_{2N}(s))ds \end{aligned} \tag{3.7}$$

where the tangential energy and dissipation functionals are defined by

$$\begin{aligned} \bar{E}_n := \quad & \sum_{j=0}^{n} ||\partial_t^j u||_{0,n-j}^2 + \sum_{j=0}^{n} ||\partial_t^j b||_{0,n-j}^2 \\ & + \sum_{j=0}^{n} ||\partial_t^j \hat{b}||_{0,n=j}^2 + \sum_{j=0}^{n} |\partial_t^j \eta|_{n-j+1}^2, \end{aligned} \tag{3.8}$$

$$\bar{D}_n := \sum_{j=0}^{n} ||\text{curl}\, \partial_t^j b||_{0,n-j}^2 \tag{3.9}$$

- To show that $\mathcal{E}_{N+4}(t)$ decays sufficiently fast so that $\sqrt{\varepsilon_{N+4}(t)}$ is integrable in time (since the energy cannot be dominated by the dissipation), we can derive the following set of tangential energy evolution estimates different from (3.7):

$$\frac{d}{dt}(\bar{E}_n + B_n) + \bar{D}_n \lesssim \sqrt{E_{2N}} D_n, \;\; n = N+4, \cdots, 2N-2, \tag{3.10}$$

with B_n satisfying $|B_n| \lesssim \sqrt{E_{2N}} \mathcal{E}_n$.

- Improved tangential dissipation estimates: Note that the tangential dissipation $\bar{D}_n$ contains only curl-estimate of b. We can improve this as follows. Set

$$\bar{\mathcal{D}}_n := \sum_{j=0}^{n} ||\partial_t^j b||_{1,n-j}^2 + \sum_{j=0}^{n} ||\partial_t^j \hat{b}||_{n-j+1}^2 \qquad (3.11)$$

(1) H^1-dissipation estimates of b and full dissipation estimates on $\hat{b}$:

$$\bar{\mathcal{D}}_{2N} \lesssim \bar{D}_{2N} + \mathcal{E}_{N+4}(E_{2N} + D_{2N}), \qquad (3.12)$$

$$\bar{\mathcal{D}}_n \lesssim \bar{D}_n + D_{N+4}E_{2N}, \ \ n = N+4, \cdots, 2N-1, \qquad (3.13)$$

which follows from Hodge-type estimates.

(2) Tangential dissipation estimates for u: (due to the coupling, $\bar{B}_3 \neq 0$)

- $\bar{B} \cdot \nabla$-dissipation estimates on u:

$$\sum_{j=0}^{n-1} ||\bar{B} \cdot \nabla \partial_t^j u_3||_{0,n-j-1}^2 + \sum_{j=0}^{n-1} ||\bar{B} \cdot \nabla \partial_t^j (k\partial_3 b_h + \bar{B}_3 u_h)||_{0,n-j-1}^2 \lesssim \bar{\mathcal{D}}_n + D_{N+4} E_{2N} \quad (3.14)$$

which follows by projecting the magnetic equations onto the vertical and horizontal components respectively. Thus using Poincare-type inequality related to $\bar{B} \cdot \nabla$ together with boundary conditions on $\Sigma_- \Rightarrow$.

- Tangential dissipation estimates for u:

$$\sum_{j=0}^{n-1}(||\partial_t^j u||_{0,n-j-1}^2 + |\partial_t^j u|_{n-j-1}^2) \lesssim \bar{\mathcal{D}}_n + D_{N+4}E_{2N} \quad (3.15)$$

where $\bar{B}_3 \neq 0$ and $k > 0$ are all used crucially!

Normal Derivative Estmates:

The heart of the analysis is to derive the estimates involving the normal derivatives of u and b. The key of this is the observation of the damping structure for the fluid vorticity field induced by the magnetic field.

- Induced damping structure for the vorticity:

  The fluid vorticity $\mathrm{curl}^{\varphi} u$ satisfy

$$\partial_t^{\varphi}(\mathrm{curl}^{\varphi} u) + u \cdot \nabla^{\varphi}(\mathrm{curl}^{\varphi} u) = \bar{B} \cdot \nabla^{\varphi}(\mathrm{curl}^{\varphi} b) + \cdots \quad (3.16)$$

  with $+\cdots$ being some nonlinear terms. Note that $\mathrm{div}^{\varphi} b = 0 \Rightarrow$

$$\bar{B}_3 \partial_3^{\varphi}(\mathrm{curl}^{\varphi} b)_1 = \bar{B}_3 \partial_1^{\varphi}(\mathrm{curl}^{\varphi} b)_3 + \bar{B}_3(\mathrm{curl}^{\varphi}\mathrm{curl}^{\varphi} b)_2,$$

$$\bar{B} \cdot \nabla^{\varphi} u_2 = \bar{B}_h \cdot \nabla_h^{\varphi} u_2 - \bar{B}_3(\mathrm{curl}^{\varphi} u)_1 + \bar{B}_3 \partial_2^{\varphi} u_3.$$

  Thus $\partial_t^{\varphi} b = \mathrm{curl}^{\varphi} E$ implies that

$$
\begin{aligned}
& \bar{B} \cdot \nabla^{\varphi} (\mathrm{curl}^{\varphi} b)_1 \\
\equiv\ & \bar{B}_h \cdot \nabla_h^{\varphi} (\mathrm{curl}^{\varphi} b)_1 + \bar{B}_3 \partial_3^{\varphi} (\mathrm{curl}^{\varphi} b)_1 \\
=\ & \bar{B}_h \cdot \nabla_h^{\varphi} (\mathrm{curl}^{\varphi} b)_1 + \bar{B}_3 \partial_1^{\varphi} (\mathrm{curl}^{\varphi} b)_3 + \bar{B}_3 (\mathrm{curl}^{\varphi} \mathrm{curl}^{\varphi} b)_2 \\
=\ & \bar{B}_h \cdot \nabla_h^{\varphi} (\mathrm{curl}^{\varphi} b)_1 + \bar{B}_3 \partial_1^{\varphi} (\mathrm{curl}^{\varphi} b)_3 + \frac{\bar{B}_3}{k} (-\partial_t^{\varphi} b_2 + \bar{B} \cdot \nabla^{\varphi} u_2 + \cdots) \\
=\ & \bar{B}_h \cdot \nabla_h^{\varphi} (\mathrm{curl}^{\varphi} b)_1 + \bar{B}_3 \partial_1^{\varphi} (\mathrm{curl}^{\varphi} b)_3 - \frac{\bar{B}_3^2}{k} (\mathrm{curl}^{\varphi} u)_1 \\
& + \frac{\bar{B}_3}{k} (-\partial_t^{\varphi} b_2 + \bar{B}_h \cdot \nabla_h^{\varphi} u_2 + \bar{B}_3 \partial_2^{\varphi} u_3 + \cdots) \\
=\ & \bar{B}_h \cdot \nabla_h (\mathrm{curl} b)_1 + \bar{B}_3 \partial_1 (\mathrm{curl}^{\varphi} b)_3 - \frac{\bar{B}_3^2}{k} (\mathrm{curl}^{\varphi} u)_1 \\
& + \frac{\bar{B}_3}{k} (\partial_t b_2 + \bar{B}_h \cdot \nabla_h u_2 + \bar{B}_3 \partial_2 u_3) + \cdots
\end{aligned}
$$

Similar computations hold for $\bar{B}\cdot\nabla^{\varphi}(\mathrm{curl}^{\varphi}b)_2$. Thus we get the following equation for $(\mathrm{curl}^{\varphi}u)_h$, for $i=1,2$:

$$\begin{aligned}
& \partial_t^{\varphi}(\mathrm{curl}^{\varphi}u)_i + u\cdot\nabla^{\varphi}(\mathrm{curl}^{\varphi}u)_i + \frac{\bar{B}_3^2}{k}(\mathrm{curl}^{\varphi}u)_i \\
= \; & \bar{B}_h\cdot\nabla_h(\mathrm{curl} b)_i + \bar{B}_3\partial_i(\mathrm{curl} b)_3 \\
& +(-1)^{i+1}\frac{\bar{B}_3}{k}(-\partial_t b_{3-i} + \bar{B}_h\cdot\nabla_h u_{3-i} + \bar{B}_3\partial_{3-i}u_3) + \cdots
\end{aligned} \tag{3.17}$$

Since $\bar{B}_3\neq 0$, $k>0$, so (3.17) yields the desired transport-damping structure for $(\mathrm{curl}^{\varphi}u)_h$, which provides the key mechanism for global-in-time estimates!!!

III. Ideas of Analysis

- Estimating those terms on the right hand side of (3.17) by $\bar{E}_{2N}$ in (3.7), one can get estimates in E_{2N} as:

$$\begin{cases} \begin{aligned} & \frac{d}{dt}\|(\mathrm{curl}^{\varphi} u)_h\|_{2N-1}^2 + \|(\mathrm{curl}^{\varphi} u)_h\|_{2N-1}^2 + \sum_{j=0}^{2N} \|\partial_t^j u\|_{2N-j}^2 \\ & + \sum_{j=0}^{2N} \|\partial_t^j b\|_{2N-j+1}^2 + \sum_{j=0}^{2N} \|\partial_t^j \hat{b}\|_{2N-j+1}^2 \\ \lesssim \ & \bar{E}_{2N} + \bar{\mathcal{D}}_{2N} + \mathcal{E}_{N+4} E_{2N}, \\ & \sum_{j=0}^{2N} \|\partial_t^j u\|_{2N-j}^2 + \sum_{j=0}^{2N-1} \|\partial_t^j b\|_{2N-j+1}^2 + \|\partial_t^{2N} b\|_0^2 \\ & + \sum_{j=0}^{2N-1} \|\partial_t^j \hat{b}\|_{2N-j+1}^2 + \|\partial_t^{2N} \hat{b}\|_0^2 \\ \lesssim \ & \bar{E}_{2N} + \|(\mathrm{curl}^{\varphi} u)_h\|_{2N-1}^2 + \mathcal{E}_{N+4} E_{2N} \end{aligned} \end{cases} \tag{3.18}$$

- Estimating those terms on the right hand side of (3.17) by (3.15) (the tangential dissipation estimates), one can estimate the terms in D_n as: for $n = N+4, \cdots, 2N$,

$$
\begin{cases}
\frac{d}{dt}||(\mathrm{curl}^{\varphi} u)_h||_{n-2}^2 + \sum_{j=0}^{n-1} ||\partial_t^j u||_{n-j-1}^2 + \sum_{j=0}^{n-2} ||\partial_t^j b||_{n-j}^2 \\
+ \sum_{j=0}^{n} ||\partial_t^j b||_{1,n-j}^2 + \sum_{j=0}^{n} ||\partial_t^j \hat{b}||_{n-j+1}^2 \lesssim \bar{\mathcal{D}}_n + D_{N+4} E_{2N}, \\
||u||_{n-1}^2 + ||u||_{0,n}^2 + \sum_{j=1}^{n} ||\partial_t^j u||_{n-j}^2 + ||b||_n^2 + \sum_{j=1}^{n-1} ||\partial_t^j b||_{n-j+1}^2 \\
+ ||\partial_t^n b||_0^2 + ||\hat{b}||_n^2 + \sum_{j=1}^{n-1} ||\partial_t^j \hat{b}||_{n-j+1}^2 + ||\partial_t^n \hat{b}||_0^2 \\
\lesssim \bar{E}_n + ||(\mathrm{curl}^{\varphi} u)_h||_{n-2}^2 + \mathcal{E}_{N+4} E_{2N}
\end{cases}
\quad (3.19)
$$

- The energy and dissipation estimates for the pressure p and the free surface function η can be obtained by the elliptic estimates as for $n = N + 4, \cdots 2N$,
  - Energy estimates:

$$\begin{aligned}
&\sum_{j=0}^{n-1} ||\partial_t^j p||_{n-j}^2 + \sum_{j=0}^{n-1} |\partial_t^j \eta|_{n-j+\frac{3}{2}}^2 + |\partial_t^n \eta|_1^2 + |\partial_t^{n+1}\eta|_{-\frac{1}{2}}^2 \\
&\leq \bar{E}_n + \sum_{j=1}^{n} ||\partial_t^j u||_{n-j}^2 + \sum_{j=0}^{n-1} ||\partial_t^j b||_{n-j}^2 + \mathcal{E}_{N+4} E_{2N}.
\end{aligned} \tag{3.20}$$

- Dissipation estimates:

$$\begin{aligned}&\sum_{j=0}^{n-2}||\partial_t^j p||_{n-j-1}^2+\sum_{j=0}^{n-2}|\partial_t^j\eta|_{n-j+\frac{1}{2}}^2+|\partial_t^{n-1}\eta|_1^2+|\partial_t^n\eta|_0^2\\&\leq\bar{\mathcal{D}}_n+\sum_{j=1}^{n-1}||\partial_t^j u||_{n-j-1}^2+\sum_{j=0}^{n-2}||\partial_t^j b||_{n-j-1}^2+D_{N+4}E_{2N}.\end{aligned}\tag{3.21}$$

Global Boundedness of High-order Energy:

Collecting all the tangential and normal estimates and using them recessively, one can get that for E_{2N} suitably small, then

$$E_{2N}(t) + \int_0^t D_{2N}(s)ds \lesssim E_{2N}(0) + \int_0^t \sqrt{\mathcal{E}_{N+4}} E_{2N}(s)ds \quad (3.22)$$

and

$$\frac{d}{dt}\mathcal{E}_n + D_n \leq 0, \quad n = N+4, \cdots, 2N-2 \quad (3.23)$$

- The global energy bound will be achieved of $\mathcal{E}_{N+4}(t)$ decays fast in time. However, note that $\mathcal{E}_n \lesssim D_n$ does not hold, as there is no hope to get exponential decays also for either temporal or spatial regularities, D_n cannot control $\mathcal{E}_n$, so it is impossible to derive the algebraic decay as Guo-Tice.

Decay of the Lower-order Energy:
The key observation is that $\mathcal{E}_l \leq D_{l+1}$. This and (3.23) will yield the desired decay of $\mathcal{E}_{N+4}$ by a time weighted argument:

- Rewrite (3.23) as

$$\frac{d}{dt}\mathcal{E}_{N+4+j} + D_{N+4+j} \leq 0, \quad j = 0, \cdots, N-6. \tag{3.24}$$

Multiplying (3.24) by $(1+t)^{N-5-j}$ and using $\mathcal{E}_{N+4+j} \leq D_{N+5+j}$ yield

$$\begin{aligned} & \frac{d}{dt}(1-(t)^{N-5-j}\mathcal{E}_{N+4+j}) + (1+t)^{N-5-j}D_{N+4+j} \\ \leq\ & (N-5-j)(1+t)^{N-6-j}\mathcal{E}_{N+4+j} \\ \leq\ & (N-5-j)(1+t)^{N-6-j}D_{N+5+j} \\ \lesssim\ & (1+t)^{N-5-(j+1)}D_{N+4+(j+1)} \end{aligned} \tag{3.25}$$

Integrating (3.25) in time and making a suitable linear combination of the resulting inequalities, one can get

$$\begin{aligned} &\sum_{j=0}^{N-5}(1+t)^{N+5-j}\mathcal{E}_{N+4+j}(t) \\ &+\sum_{j=0}^{N-6}\int_0^t(1+s)^{N-5-j}D_{N+4+j}(s)ds \\ \lesssim\ & E_{2N}(0)+\int_0^t D_{2N-1}(s)ds \end{aligned} \tag{3.26}$$

The a priori Estimates

Then we arrive at the final energy estimates

Proposition

Let $N \geq 8$. $\exists$ a universal constant $\bar{\delta} > 0$ni if

$$E_{2N}(t) \leq \bar{\sigma}, \quad \forall t \in [0, T] \tag{3.27}$$

Proposition (continued)

Then

$$E_{2N}(t) + \int_0^t D_{2N}(s)ds \leq cE_{2N}(0) \forall \in [0, T] \tag{3.28}$$

and

$$\begin{aligned} & \sum_{j=0}^{n-6} (1+t)^{N-5-j} \mathcal{E}_{N+4+j}(t) \\ & + \sum_{j=0}^{N-6} \int_0^t (1+s)^{N-5-j} D_{N+4+j}(s)ds \\ \lesssim \; & E_{2N}(0), \end{aligned} \tag{3.29}$$

where c is a universal constant independent of T.

IV. Results on Plasma-Plasma Interface

As in Figure 2, consider two immiscible plasmas occupying the two regions $\Omega_\pm(t)$ respectively, with corresponding velocities $u_\pm$, pressure $p_\pm$, and magnetic field $B_\pm$, which are assumed to solve the following plasma-plasma interface problem:

$$\begin{cases} \partial_t u_\pm + u_\pm \cdot \nabla u_\pm + \nabla p_\pm = \mathrm{curl} B_\pm \times B_\pm & \text{in } \Omega_\pm(t) \\ \mathrm{div}\, u_\pm = 0 & \text{in } \Omega_\pm(t) \\ \partial_t B_\pm = \mathrm{curl} E_\pm, E_\pm = u_\pm \times B_\pm - k_\pm \mathrm{curl} B_\pm & \text{in } \Omega_\pm(t) \\ \mathrm{div}\, B_\pm = 0 & \text{in } \Omega_\pm(t) \\ \partial_t \eta = u_\pm \cdot \mathcal{N} & \text{on } \Sigma(t) \\ p_+ = p_- + \sigma H, B_+ = B_-, E_+ \times \mathcal{N} = E_- \times \mathcal{N} & \text{on } \Sigma(t) \\ u_+ \cdot e_3 = 0, B_+ \times e_3 = \bar{B} \times e_3 & \text{on } \Sigma_+ \\ u_- \cdot e_3 = 0, B_- \cdot e_3 = \bar{B} \cdot e_3, E_- \times e_3 = 0 & \text{on } \Sigma_- \end{cases} \quad (4.1)$$

where $k_\pm > 0$ and $\bar{B}$ is a uniform transversal magnetic field ($\bar{B}_3 \neq 0$).

IV. Results on Plasma-Plasma Interface

- Using the same flatten map Φ defined in (2.3) and in flatten coordinates, one has

$$\begin{cases} \partial_t^{\varphi} u + u \cdot \nabla^{\varphi} u + \nabla^{\varphi} p = \mathrm{curl}^{\varphi} b \times (\bar{B} + b) & \text{in } \Omega \\ \mathrm{div}^{\varphi} u = 0 & \text{in } \Omega \\ \partial_t^{\varphi} b = \mathrm{curl}^{\varphi} E, E = u \times B - k\mathrm{curl}^{\varphi} b & \text{in } \Omega \\ \mathrm{div}^{\varphi} b = 0 & \text{in } \Omega \\ \partial_t \eta = u \cdot \mathcal{N} & \text{on } \Sigma \\ [p] = \sigma H, [b] = 0, [E] \times \mathcal{N} = 0 & \text{on } \Sigma \\ u_3 = 0, b \times e_3 = 0 & \text{on } \Sigma_+ \\ u_3 = 0, b_3 = 0, E \times e_3 = 0 & \text{on } \Sigma_- \\ (u, b, \eta)|_{t=0} = (u_0, b_0, \eta_0) \end{cases} \quad (4.2)$$

where $f = f_{\pm}$ on $\Omega_{\pm}$, and $[f] = f_+|_{\Sigma} - f_-|_{\Sigma}$.

The initial data are required to satisfy the $2N$-th order compatibility conditions:

$$\begin{cases} \operatorname{div}^{\varphi_0} u_0 = 0 \text{ in } \Omega, [u_0]\cdot \mathcal{N}_0 = 0 \text{ on } \Sigma, u_{0,3} = 0 \text{ on } \Sigma_{\pm}; \\ \operatorname{div}^{\varphi_0} b_0 = 0 \text{ in } \Omega, [b_0] = 0 \text{ on } \Sigma, b_0 \times e_3 = 0 \text{ on } \Sigma_+, b_{0,3} = 0 \text{ on } \Sigma_-; \\ \left[\partial_t^j b(0)\right] \times \mathcal{N}_0 = 0 \text{ on } \Sigma, \partial_t^j b(0) \times e_3 = 0 \text{ on } \Sigma_+, j = 1, \cdots, 2N-1; \\ \partial_t^j([E]\times \mathcal{N})(0) = 0 \text{ on } \Sigma, \partial_t^j E(0) \times e_3 = 0 \text{ on } \Sigma_-, j = 0, \cdots, 2N-1. \end{cases} \quad (4.3)$$

- Using the notation $||f||_k^2 = ||f_+||^2_{H^k(\Omega_+)} + ||f_-||^2_{H^k(\Omega)}$, $|f|_3^2 = ||f_+||_{H^s(\Sigma)} + ||f_-||_{H^s(\Sigma)}$.

- For $N \geq 4$, define the higher order energy functional, lower-order energy functional, and the corresponding dissipation functional as

$$
\begin{aligned}
E_{2N} := & \sum_{j=0}^{2N} ||\partial_t^j u||_{2N-j}^2 + |\partial_t^{2N} u|_{-\frac{1}{2}}^2 + \sum_{j=0}^{2N-1} ||\partial_t^j b||_{2N-j+1}^2 \\
& + ||\partial_t^{2N} b||_0^2 + \sum_{j=0}^{2N-1} ||\partial_t^j p||_{2N-j}^2 \\
& + \sum_{j=0}^{2N-1} |\partial_t^j \eta|_{2N-j+\frac{3}{2}}^2 + |\partial_t^{2N} \eta|_1^2 + |\partial_t^{2N+1} \eta|_{-\frac{1}{2}}^2
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
\mathcal{E}_n := \quad & ||u||_{n-1}^2 + ||u||_{0,n}^2 + \sum_{j=1}^{n} ||\partial_t^j u||_{n-j}^2 + ||b||_n^2 \\
& + \sum_{j=1}^{n-1} ||\partial_t^j b||_{n-j+1}^2 + ||\partial_t^n b||_0^2 + \sum_{j=0}^{n-1} ||\partial_t^j p||_{n-j}^2 \\
& + \sum_{j=0}^{n-1} |\partial_t^j \eta|_{n-j+\frac{3}{2}}^2 + |\partial_t^n \eta|_1^2 + |\partial_t^{n+1} \eta|_{-\frac{1}{2}}^2
\end{aligned}
\tag{4.5}
$$

where $n = N + 4, \cdots, 2N$, and

$$D_n := \sum_{j=0}^{n-1} ||\partial_t^j u||^2_{n-j-1} + \sum_{j=0}^{n-2} ||\partial_t^j b||^2_{n-j} + \sum_{j=0}^{n} ||\partial_t^j b||^2_{1,n-j} + \sum_{j=0}^{n-2} ||\partial_t^j p||^2_{n-j-1} + \sum_{j=0}^{n-2} |\partial_t^j \eta|^2_{n-j+\frac{1}{2}} + |\partial_t^{n-1}\eta|^2_1 + |\partial_t^n \eta|^2_0 \tag{4.6}$$

- Then the main results are as follows stated as exactly as the main theorem for the plasma-vacuum interface before except the condition (2.10) is replaced by (4.3).

Remark (4.1)

The main strategy of the proof is similar as before except two points: the highest temporal derivative estimates are different, and the local well-posedness is proved by a different regularization procedure!

Thank you for your attention!