Numeraires and Martingale Measures in the Black-Scholes Models

Shunlong Luo and Jiaan Yan
Institute of Applied Mathematics
Academia Sinica, Beijing, 100080, P. R. China

Qiang Zhang
Department of Economics and Finance
City University of Hong Kong
Tat Chee Avenue, Kowloon, Hong Kong

Abstract. In the Black-Scholes models, complete or incomplete, for any positive self-financing wealth process, there is a simple and natural probability measure under which all security prices denominated by this self-financing wealth process are martingales. Thus, if we take this self-financing wealth process as the numeraire and the new probability as the equivalent martingale measure (risk-neutral measure), we obtain a simple arbitrage price system which has an intuitive probabilistic interpretation. Moreover, all this kind of arbitrage price systems with different numeraires are equivalent. In particular, if we choose the growth-optimal wealth as the numeraire (the so-called numeraire portfolio), then the objective probability itself becomes a martingale measure. This provides an intuitive, conceptually clear and an analytically tractable martingale method to price contingent claims when the market is incomplete. Applications to stochastic volatility models are developed.

Keywords: numeraires, martingale measures, arbitrage price systems, growth-optimal wealth process, objective probability

1. Introduction

The arbitrage argument for contingent claims valuation, initiated by Ross (1976), standardized by Harrison and Kreps (1979), Harrison and Pliska (1981), is an elegant and powerful apparatus of modern asset pricing theory. The fundamental theorem of asset pricing states that the absence of arbitrage opportunities is essentially equivalent to the existence of a probability (called an equivalent martingale measure or a risk-neutral measure) such that all discounted security prices are martingales under this new probability. Implicitly incorporated in this statement is a *numeraire* used to discount the security prices. See Dybvig and Ross (1987), Duffie (1994), Delbaen and Schachermayer (1994), and Yan (1998) for extensive study and review of this perspective.

When market is complete, as is the classic Black-Scholes models (1973), the martingale measure is unique. Accordingly, the valuation issue is unambiguous, and any contingent claim can be perfectly hedged. The cost of the hedge defines the fair price of the contingent claim, which can be computed as a mathematical expectation with respect to the equivalent martingale measure.

On the other hand, if the market is incomplete, there may exist many martingale measures, thus many different arbitrage price systems. The valuation and hedging issues are not definite in this situation. Many authors studied these issues from various perspectives. For example, Davis (1997) proposed a pricing rule based on marginal substitution and utility maximization. Föllmer and Sondermann (1986), Föllmer and Schweizer (1991) proposed the notion of minimal martingale measure and related it to an entropy rationale. Hofmann, Platen and Schweizer (1992) applied the theory to incomplete markets with stochastic volatility. Delbaen and Schachermayer (1998) studied the variance-optimal martingale measures. When the incompleteness comes from the stochastic volatility, the models are extensively studied by Hull and White (1987), Wiggins (1987) from the theoretical and emperical aspects, and by Heston (1993) from the analytical aspect. But in producing the option pricing formula, they have to rely on an exogeneously specified risk premium (market price of risk).

The change of numeraire technique, well known in financial community, is a powerful tool for contingent claims valuation, and often sheds insightful light on the relevant problems. Traditionally, the concern is around two extreme cases: (1) take the riskless bond price as the numeraire, the corresponding martingale measure is a Girsanov transform of the objective probability; (2) keep the objective probability as the martingale measure, the numeraire is the reciprocal of state price density process, also known as the state price deflator, or the pricing kernel, or the Arrow-Debreu price (Duffie, 1994), or the numeraire portfolio (Long, 1990). The relation between the numeraire changes and the changes of probabilities has been studied by Geman, El Karoui and Rachet (1995) in an abstract context. Following Long (1990), Bajeux-Besnainou and Portait (1997) have studied extensively the properties of the numeraire portfolio and their applications to financial markets. Yan et al. (1998) have used the numeraire portfolio to study the pricing issue when the stock return is described by a discontinuous process (jump-diffusion like process) or a Lévy process. We emphasize that numeraires and martingale measures should be put in a dual dual position.

In this paper, we show that in Black-Scholes models, complete or incomplete, for any positive *self-financing wealth process* which serves as the numeraire, there is a simple equivalent martingale measure which can be easily identified from an application of Itô's formula. The combination of this numeraire and the martingale measure provides an arbitrage price system. Moreover, all this kind of arbitrage price systems with different numeraires are equivalent. In particular, if we take the growth-optimal wealth process as the numeraire, the objective probability itself becomes a martingale measure. Thus the growth-optimal wealth process is an instance of the numeraire portfolio (Long, 1990). It is interesting to note that in the classic Black-Scholes models, the growth-optimal wealth process is determined by a proportional portfolio, which is widely used in practice (Perold and Sharpe, 1988) and enjoys many appealing properties (Browne, 1997). This paper is more pedagogical than original.

The paper is structured as follows. In Section 2, we review the notion of arbitrage price system. In Section 3, we identify a class of arbitrage price system based on self-financing process numeraires in the Black-Scholes models, and prove the equivalence of these arbitrage price systems. We also present the intuitive probabilistic interpretation of these arbitrage price systems. In Section 4, we apply the theory to some stochastic volatility models. Finally, Section 5 concludes.

2. Arbitrage Price Systems in General

Consider a security market with n trading securities and finite time horizon $[0,\ T]$. The price dynamics is described as

$$S = \{S_t\}_{t\in[0,\ T]} = \{S_t^{(1)}, S_t^{(2)}, \cdots, S_t^{(n)}\}_{t\in[0,\ T]}.$$

The uncertainty of the economy is described by a filtered probability space

$$(\Omega,\ \mathcal{F},\ \mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,\ T]},\ \mathrm{P}).$$

The market can be complete or incomplete. The probability P is referred to as an objective probability.

Following Geman et al. (1995), by a numeraire we mean any positive self-financing wealth process with initial value 1.

Definition 1. *An arbitrage price system is a pair* $\{N, \mathrm{Q}_N\}$ *of a numeraire* $N = \{N_t\}_{t\in[0,\ T]}$ *and a probability measure* Q_N *equivalent to* P *such that the security price dynamics discounted by the numeraire, i.e.,* $\left\{\frac{S_t^{(i)}}{N_t}\right\}_{t\in[0,\ T]}$, *is a* $(\mathrm{Q}_N, \mathbf{F})$*-martingale for* $i = 1, 2, \cdots, n$.

Definition 2. *Under the arbitrage price system* (N, Q_N), *the price dynamics* $\{\xi_t\}_{t\in[0,T]}$ *of any integrable European contingent claim* ξ_T *matured at time* T *is defined as*

$$\xi_t = N_t \mathrm{E}^{\mathrm{Q}_N}\left(\frac{\xi_T}{N_T}\middle|\mathcal{F}_t\right), \qquad t\in[0,\ T], \tag{1}$$

where $\mathrm{E}^{\mathrm{Q}_N}$ *is the expectation with respect to the probability* Q_N.

Consequently, the price dynamics $\{\xi_t\}_{t\in[0,\ T]}$ discounted by the numeraire $\{N_t\}_{t\in[0,\ T]}$, i.e., $\{\frac{\xi_t}{N_t}\}_{t\in[0,\ T]}$, is a $(\mathrm{Q}_N, \mathbf{F})$-martingale. (1) is referred to as the martingale equation.

Definition 3. *Two arbitrage price systems* (N, Q_N) *and* (M, Q_M) *are said to be equivalent, if they give the same price dynamics to any integrable European contingent claim.*

The following result shows that the martingale property is carried over from trading securities to any self-financing wealth process. It is a particular case of Yan (1998, Theorem 2.5).

Lemma 1. *Let (N, Q_N) be an arbitrage price system, $\{X_t\}_{t\in[0,\ T]}$ be any positive self-financing wealth process, then $\{\frac{X_t}{N_t}\}_{t\in[0,\ T]}$ is a $(\mathrm{Q}_N, \mathbf{F})$-local martingale.*

3. Arbitrage Price Systems in Black-Scholes Models

Consider the Black-Scholes model

$$\begin{aligned} dB_t &= B_t r_t dt \\ dS_t &= S_t(\mu_t dt + \sigma_t dW_t), \quad t \in [0,\ T] \end{aligned}$$

where $\{W_t\}_{t\in[0,\ T]}$ is a one dimensional $(\mathrm{P}, \mathbf{F})$-Brownian motion (note here that the filtration $\mathbf{F}$ may be larger than the natural filtration generated by the process $\{W_t\}_{t\in[0,T]}$). $\{r_t\}_{t\in[0,T]}$, $\{\mu_t\}_{t\in[0,T]}$ and $\{\sigma_t\}_{t\in[0,T]}$ are all stochastic processes adapted to $\mathbf{F}$. We assume that σ_t is strictly positive for any $t \in [0, T]$. The market is in general incomplete since the market coefficients may depend on other random factors. The most popular examples are the stochastic volatility models (Hull and White 1987, Wiggins 1987). To avoid technical subtlety such as the conditions for a local martingale to be a martingale, we assume that all the coefficients $\{r_t\}_{t\in[0,T]}$, $\{\mu_t\}_{t\in[0,T]}$, $\{\sigma_t\}_{t\in[0,T]}$ and $\{\sigma_t^{-1}\}_{t\in[0,T]}$ are uniformly bounded and integrable on $[0,\ T] \times \Omega$.

Note the stock price has the explicit expression

$$S_t = S_0 \exp\Big\{ \int_0^t \Big(\mu_s - \frac{1}{2}\sigma_s^2\Big)ds + \int_0^t \sigma_s dW_s \Big\}, \qquad t \in [0, T]. \tag{2}$$

Let $\pi = \{\pi_t\}_{t\in[0,\ T]}$ be a portfolio process. π_t is the proportion of wealth invested in the stock S at time t. The remaining proportion $1 - \pi_t$ is invested in the bond B at time t. The corresponding self-financing wealth process is

$$\begin{aligned} dX_t &= \frac{(1-\pi_t)X_t}{B_t} dB_t + \frac{\pi_t X_t}{S_t} dS_t \\ &= X_t \Big\{ [r_t + \pi_t(\mu_t - r_t)]dt + \pi_t \sigma_t dW_t \Big\}. \end{aligned}$$

Without loss of generality, we assume that $X_0 = 1$. The explicit solution is

$$X_t = \exp\Big\{ \int_0^t \Big(r_s + \pi_s(\mu_s - r_s) - \frac{1}{2}\pi_s^2\sigma_s^2\Big)ds + \int_0^t \pi_s \sigma_s dW_s \Big\}, \quad t \in [0, T]. \tag{3}$$

By Itô's formula,

$$
\begin{aligned}
d\Big(\frac{B_t}{X_t}\Big) &= \frac{1}{X_t}dB_t - \frac{B_t}{X_t^2}dX_t + \frac{B_t}{X_t^3}d[X]_t \\
&= \frac{B_t r_t}{X_t}dt - \frac{B_t X_t}{X_t^2}\{[r_t + \pi_t(\mu_t - r_t)]dt + \pi_t\sigma_t dW_t\} + \frac{B_t}{X_t^3}X_t^2\pi_t^2\sigma_t^2 dt \\
&= \frac{B_t}{X_t}\{-\pi_t(\mu_t - r_t) + \pi_t^2\sigma_t^2\}dt - \frac{B_t}{X_t}\pi_t\sigma_t dW_t \\
&= -\frac{B_t}{X_t}\pi_t\sigma_t\Big\{\Big(\frac{\mu_t - r_t}{\sigma_t} - \pi_t\sigma_t\Big)dt + dW_t\Big\} \\
&= -\frac{B_t}{X_t}\pi_t\sigma_t d\widehat{W_t}.
\end{aligned}
$$

Here

$$\widehat{W_t} = W_t + \int_0^t \widehat{\pi_s}ds, \quad t \in [0,\ T].$$

with

$$\widehat{\pi_s} := \frac{\mu_s - r_s}{\sigma_s} - \pi_s\sigma_s, \qquad s \in [0, T]. \tag{4}$$

$$
\begin{aligned}
d\Big(\frac{S_t}{X_t}\Big) &= \frac{1}{X_t}dS_t - \frac{S_t}{X_t^2}dX_t + \frac{S_t}{X_t^3}d[X]_t - \frac{1}{X_t^2}d[S,X]_t \\
&= \frac{S_t}{X_t}(\mu_t dt + \sigma_t dW_t) - \frac{S_t X_t}{X_t^2}\{(r_t + \pi_t(\mu_t - r_t))dt + \pi_t\sigma_t dW_t\} \\
&\quad + \frac{S_t}{X_t^3}X_t^2\pi_t^2\sigma_t^2 dt - \frac{S_t}{X_t}\pi_t\sigma_t^2 dt \\
&= \frac{S_t}{X_t}(\mu_t - r_t - \pi_t(\mu_t - r_t) + \pi_t^2\sigma_t^2 - \pi_t\sigma_t^2)dt + \frac{S_t}{X_t}(\sigma_t - \pi_t\sigma_t^2)dW_t \\
&= -\frac{S_t}{X_t}\sigma_t(\pi_t - 1)\Big\{\Big(\frac{\mu_t - r_t}{\sigma_t} - \pi_t\sigma_t\Big)dt + dW_t\Big\} \\
&= -\frac{S_t}{X_t}\sigma_t(\pi_t - 1)d\widehat{W_t}.
\end{aligned}
$$

We assume that $\{\pi_t\}_{t\in[0,T]}$ satisfies the Novikov's condition $\mathrm{E}\Big(\exp\frac{1}{2}\int_0^T \pi_s^2 ds\Big) < \infty$, so that we can define a new probability P_π by

$$\frac{d\mathrm{P}_\pi}{d\mathrm{P}} = \exp\Big\{-\frac{1}{2}\int_0^T \widehat{\pi_t}^2 dt - \int_0^T \widehat{\pi_t}dW_t\Big\}. \tag{5}$$

Then by Girsanov's formula, $\{\widehat{W_t}\}_{t\in[0,\ T]}$ is a $(\mathrm{P}_\pi, \mathbf{F})$-Brownian motion. Consequently, both $\{\frac{B_t}{X_t}\}_{t\in[0,\ T]}$ and $\{\frac{S_t}{X_t}\}_{t\in[0,\ T]}$ are $(\mathrm{P}_\pi, \mathbf{F})$-martingales. Thus, we have proved the following result.

Theorem 2. *Let $\{X_t\}_{t\in[0,\ T]}$ be any self-financing wealth process determined by the portfolio π. Then both $\{\frac{B_t}{X_t}\}_{t\in[0,\ T]}$ and $\{\frac{S_t}{X_t}\}_{t\in[0,\ T]}$ are $(\mathrm{P}_\pi, \mathbf{F})$-martingales.*

By Lemma 1 and Theorem 2, we have

Corollary 3. *Let $\{X_t\}_{t\in[0,\ T]}$ be any positive self-financing wealth process determined by the portfolio π. Then for any self-financing wealth process $\{Y_t\}_{t\in[0,\ T]}$, the discounted process $\{\frac{Y_t}{X_t}\}_{t\in[0,\ T]}$ is a $(\mathrm{P}_\pi, \mathbf{F})$-local martingale.*

Remark. If we put $\widehat{\pi}_t = 0,\ \forall\ t \in [0, T]$, then from Eq. (4) we have $\pi_t = \frac{\mu_t - r_t}{\sigma_t^2},\ \forall\ t \in [0, T]$, which is a proportional stategy, the corresponding self-financing wealth process is the growth-optimal wealth process. Now from Eq. (5) we have $\mathrm{P}_\pi = \mathrm{P}$. Thus if we choose the growth-optimal wealth process as the numeraire, then the objective probability itself becomes a martingale measure. This is extensively studied by Long (1990), Bajeux-Besnainou and Portait (1997).

In the sequel, we take $X^\pi = \{X_t\}_{t\in[0,\ T]}$ in Eq. (3) as the numeraire (we put a superscript π to X_t to indicate the dependence of X_t on π), and the corresponding P_π as the martingale measure. Thus we obtain the arbitrage price system $\{X^\pi, \mathrm{P}_\pi\}$. Under this price system, the price dynamics of any integrable European contingent claim ξ_T matured at T is

$$\xi_t = X_t^\pi \mathrm{E}^{\mathrm{P}_\pi}\Big(\frac{\xi_T}{X_T^\pi}\Big|\mathcal{F}_t\Big), \qquad t \in [0,\ T]. \tag{6}$$

provided the integral exists.

Theorem 4. *Let $\pi^{(i)},\ i = 1, 2$ be two portfolio processes, $(X^{\pi^{(i)}}, \mathrm{P}_{\pi^{(i)}}),\ i = 1, 2$, be two arbitrage price systems based on the numeraires $X^{\pi^{(i)}},\ i = 1, 2$, respectively. Then they are equivalent.*

Proof. We have to show that for any integrable European contingent claim ξ_T,

$$X_t^{\pi^{(1)}} \mathrm{E}^{\mathrm{P}_{\pi^{(1)}}}\Big(\frac{\xi_T}{X_T^{\pi^{(1)}}}\Big|\mathcal{F}_t\Big) = X_t^{\pi^{(2)}} \mathrm{E}^{\mathrm{P}_{\pi^{(2)}}}\Big(\frac{\xi_T}{X_T^{\pi^{(2)}}}\Big|\mathcal{F}_t\Big), \quad \forall\ t \in [0,\ T],$$

where by Eqs. (3) and (4), for $i = 1, 2$,

$$\begin{aligned} X_t^{\pi^{(i)}} &= \exp\Big\{\int_0^t \Big(r_s + \pi_s^{(i)}(\mu_s - r_s) - \frac{1}{2}(\pi_s^{(i)}\sigma_s)^2\Big)ds + \int_0^t \pi_s^{(i)}\sigma_s dW_s\Big\}, \\ &= \exp\Big\{\int_0^t \Big(r_s + \frac{(\mu_s - r_s)^2}{2\sigma_s^2} - \frac{1}{2}\widehat{\pi_s^{(i)}}^2\Big)ds + \int_0^t \Big(\frac{\mu_s - r_s}{\sigma_s} - \widehat{\pi_s^{(i)}}\Big)dW_s\Big\}. \end{aligned}$$

Since by Eq. (5),

$$\frac{d\mathrm{P}_{\pi^{(i)}}}{d\mathrm{P}}\Big|_{\mathcal{F}_t} = \exp\Big\{-\frac{1}{2}\int_0^t \widehat{\pi_s^{(i)}}^2 ds - \int_0^t \widehat{\pi_s^{(i)}} dW_s\Big\}, \qquad \widehat{\pi_s^{(i)}} = \frac{\mu_s - r_s}{\sigma_s} - \sigma_s\pi_s^{(i)},\ i = 1, 2.$$

We have

$$\begin{aligned}
\eta_t &\equiv \frac{d\mathrm{P}_{\pi^{(1)}}}{d\mathrm{P}_{\pi^{(2)}}}\Big|_{\mathcal{F}_t} \\
&= \exp\Big\{-\frac{1}{2}\int_0^t \Big(\widehat{\pi_s^{(1)}}^2 - \widehat{\pi_s^{(2)}}^2\Big) ds - \int_0^t (\widehat{\pi_s^{(1)}} - \widehat{\pi_s^{(2)}}) dW_s\Big\} \\
&= \frac{X_t^{\pi^{(1)}}}{X_t^{\pi^{(2)}}}, \qquad t \in [0,\ T].
\end{aligned}$$

Thus by Bayes rule of conditional probability,

$$X_t^{\pi^{(1)}} \mathrm{E}^{\mathrm{P}_{\pi^{(1)}}}\Big(\frac{\xi_T}{X_T^{\pi^{(1)}}}\Big|\mathcal{F}_t\Big) = \frac{X_t^{\pi^{(1)}}}{\eta_t} \mathrm{E}^{\mathrm{P}_{\pi^{(2)}}}\Big(\frac{\xi_T}{X_T^{\pi^{(1)}}}\eta_T\Big|\mathcal{F}_t\Big) = X_t^{\pi^{(2)}} \mathrm{E}^{\mathrm{P}_{\pi^{(2)}}}\Big(\frac{\xi_T}{X_T^{\pi^{(2)}}}\Big|\mathcal{F}_t\Big).$$

In this context, we have the following simple observations:

(1) When $t = 0$, the pricing formula (6) coincides with the formula in Davis (1997) in which the utility is taken to be logarithmic.

(2) Let us consider the classic Black-Scholes model. Then $r_t = r$, $\mu_t = \mu$ and $\sigma_t = \sigma$ are all constants. Let $\pi_t = \frac{\mu - r}{\sigma^2}$, $\forall\ t \in [0,\ T]$, then by Eq. (4), $\widehat{\pi}_t = 0$ and $\mathrm{P}^\pi = \mathrm{P}$. Moreover,

$$\begin{aligned}
S_t &= S_0 \exp\Big\{\Big(\mu - \frac{1}{2}\sigma^2\Big)t + \sigma W_t\Big\} \\
X_t^\pi &= \exp\Big\{\Big(r + \frac{(\mu - r)^2}{2\sigma^2}\Big)t + \frac{\mu - r}{\sigma} W_t\Big\}, \qquad t \in [0,\ T].
\end{aligned}$$

Thus (X^π, P) is an arbitrage price system. Consider a European call option on the stock with maturity T and striking price K, i.e., contingent claim $\xi_T = (S_T - K)^+$. Now by formula (6), the price of this option at time 0 is (note that $-\frac{W_T}{\sqrt{T}}$ follows a standard Gaussian distribution)

$$\begin{aligned}
\xi_0 &= \mathrm{E}^{\mathrm{P}}\Big(\frac{\xi_T}{X_T}\Big) \\
&= \mathrm{E}^{\mathrm{P}}\Big(e^{-(r + \frac{(\mu - r)^2}{2\sigma^2})T - \frac{\mu - r}{\sigma} W_T} (S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T} - K)^+\Big) \\
&= \int_{S_0 e^{(\mu - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}x} \geq K} e^{-(r + \frac{(\mu - r)^2}{2\sigma^2})T + \frac{\mu - r}{\sigma}\sqrt{T}x} (S_0 e^{(\mu - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}x} - K) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= S_0 N(d_+) - K e^{-rT} N(d_-).
\end{aligned}$$

Here

$$N(\cdot) = \int_{-\infty}^{\cdot} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, \quad d_{\pm} = \frac{\log(\frac{S_0}{K}) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

This drivation of the Black-Scholes formula is very simple and intuitive.

4. Specifying to Stochastic Volatility Models

We apply the general results of Section 3 to two popular stochastic volatility models. We also consider an elementary stochastic interest rate model.

First, we consider a Black-Scholes model with a diffusive stochastic volatility. Specifically,

$$\begin{aligned} dB_t &= B_t r dt \\ dS_t &= S_t(\mu dt + \sigma_t dW_t) \\ d\sigma_t &= b(\sigma_t) + \delta(\sigma_t) d\Sigma_t, \qquad t \in [0,\ T] \end{aligned}$$

where $\{\Sigma_t\}_{t\in[0,T]}$ is another standard Brownian motion independent of $\{W_t\}_{t\in[0,\ T]}$, and $r,\ \mu$ are constants, $b(\cdot),\ \delta(\cdot)$ are functions.

Remark. *Wiggins (1987) has studied this kind of incomplete markets from both theoretical and emperical perspectives. A partial differential equation determining the containgent claims valuation is derived by exogeneously specifying a risk premiun (Wiggins attributed this result to Garman 1976, a working paper). A well studied stochastic volatility model (Stein and Stein 1991, Heston 1993) is* $b(\sigma_t) = a(m - \sigma_t)$, $\delta(\sigma_t) = \delta$, *here* a, m *and* δ *are constants. In this case, the stochastic differential equation describing the volatility is an Ornstein-Uhlenbeck equation, its explicit solution is*

$$\sigma_t = \sigma_0 e^{-at} + m(1 - e^{-at}) + \delta e^{-at} \int_0^t e^{as} d\Sigma_s, \qquad t \in [0,\ T].$$

Thus σ_t *follows a normal law with mean and variance given by*

$$\begin{aligned} \mathrm{E}^{\mathrm{P}}\sigma_t &= \sigma_0 e^{-at} + m(1 - e^{-at}) \\ \mathrm{Var}\sigma_t &= \frac{\delta^2}{2a(1 - e^{-2at})} \end{aligned}$$

respectively. In this case, many calculations can be carried out explicitly.

Let $\pi_t = \frac{\mu - r}{\sigma_t^2}$, $t \in [0, T]$, by Eqs. (2) and (3), we have

$$\begin{aligned} S_t &= S_0 \mathrm{exp}\Big\{ \int_0^t \Big(\mu - \frac{1}{2}\sigma_s^2\Big) ds + \int_0^t \sigma_s dW_s \Big\} \\ X_t &= \mathrm{exp}\Big\{ \int_0^t \Big(r + \frac{(\mu - r)^2}{2\sigma_s^2}\Big) ds + \int_0^t \frac{\mu - r}{\sigma_s} dW_s \Big\}. \end{aligned}$$

Take this process $\{X_t\}_{t\in[0,T]}$ as the numeraire. Let $h(S_T)$ be a European contingent claim with maturity T, then by the pricing formula (6), we have

$$\xi_t = X_t \mathrm{E}^{\mathrm{P}}\Big(\frac{h(S_T)}{X_T}\Big| \mathcal{F}_t\Big), \qquad t \in [0,\ T].$$

Consider the system of stochastic differential equations

$$\begin{aligned} dS_t &= S_t(\mu dt + \sigma_t dW_t) \\ dX_t &= X_t \Big\{ \Big(r + \frac{(\mu - r)^2}{\sigma_t^2}\Big) dt + \frac{\mu - r}{\sigma_t} dW_t \Big\} \\ d\sigma_t &= b(\sigma_t) dt + \delta(\sigma_t) d\Sigma_t. \end{aligned}$$

The corresponding diffusion generating operator is

$$\begin{aligned} A &= \frac{1}{2}\Big(s^2\sigma^2 \frac{\partial^2}{\partial s^2} + \frac{(\mu - r)^2}{\sigma^2} x^2 \frac{\partial^2}{\partial x^2} + \delta^2(\sigma)\frac{\partial^2}{\partial \sigma^2} + 2(\mu - r) sx \frac{\partial^2}{\partial s \partial x}\Big) \\ &\quad + \mu s \frac{\partial}{\partial s} + \Big(r + \frac{(\mu - r)^2}{\sigma^2}\Big) x \frac{\partial}{\partial x} + b(\sigma)\frac{\partial}{\partial \sigma}. \end{aligned}$$

Let

$$\mathrm{E}^{\mathrm{P}}\Big(\frac{h(S_T)}{X_T}\Big| \mathcal{F}_t\Big) = v(t; S_t, X_t, \sigma_t), \qquad t \in [0,\ T]. \tag{7}$$

Then by the Feynman-Kac formula, v solves the partial differential equation

$$\begin{aligned} \Big(\frac{\partial}{\partial t} + A\Big) v(t; s, x, \sigma) &= 0 \\ v(T; s, x, \sigma) &= \frac{h(s)}{x}. \end{aligned}$$

From foumula (6), we know that v is of the form

$$v(t; s, x, \sigma) = \frac{u(t; s, \sigma)}{x}.$$

Accordingly $\xi_t = u(t; s, \sigma)$, and from the partial differential equation of v, we know that u satisfies

$$\begin{aligned} u_t + \frac{1}{2} s^2 \sigma^2 u_{ss} + \frac{1}{2} \delta^2(\sigma) u_{\sigma\sigma} + rsu_s + b(\sigma) u_\sigma - ru &= 0 \\ u(T; s, \sigma) &= h(s). \end{aligned}$$

It is surprising to note that the above partial differential equation corresponds to Wiggins' (1987) with the risk premium to be zero. A careful reflection shows that this is reasonable since growth-optimal wealth is optimal under the logarithmic utility, which determines the risk premiun to be zero in a representative agent equilibrium economy.

Remark. *When deriving the above partial differential equation, we have deliberately avoided the explicit use of Girsonav's formula since we want to work on the objective probability space. Alternatively, If we define a new probability* $\widehat{\mathrm{P}}$ *by*

$$\frac{d\widehat{\mathrm{P}}}{d\mathrm{P}} = \exp\Big\{ -\frac{1}{2} \int_0^T \frac{(\mu - r)^2}{\sigma_t^2} dt - \int_0^T \frac{\mu - r}{\sigma_t} dW_t \Big\},$$

then by Girsonav's formula, $\widehat{W}_t = W_t + \int_0^t \frac{\mu - r}{\sigma_\tau} dW_\tau$ *is a Brownian motion under* $\widehat{\mathrm{P}}$*, and* $\xi_t = e^{-r(T-t)} \mathrm{E}^{\widehat{\mathrm{P}}}(h(S_T)|\mathcal{F}_t)$*. To evaluate* $\mathrm{E}^{\widehat{\mathrm{P}}}(h(S_T)|\mathcal{F}_t) = \phi(t; S_t, \sigma_t)$*, applying the Feynman-Kac formula to the system of stochastic differential equations*

$$\begin{aligned} dS_t &= S_t(\mu dt + \sigma_t dW_t) \\ &= S_t(r dt + \sigma_t d\widehat{W}_t) \\ d\sigma_t &= b(\sigma_t) dt + \delta(\sigma_t) d\Sigma_t, \end{aligned}$$

we conclude that ϕ *satisfies*

$$\begin{aligned} \phi_t + \frac{1}{2} s^2 \sigma^2 \phi_{ss} + \frac{1}{2} \delta^2(\sigma) \phi_{\sigma\sigma} + rs\phi_s + b(\sigma) \phi_\sigma &= 0 \\ \phi(T; s, \sigma) &= h(s). \end{aligned}$$

Consequently, $u(t, s, \sigma)$ *and* $e^{-r(T-t)} \phi(t; s, \sigma)$ *satisfy the same partial differential equation.*

When $b(\cdot) = \delta(\cdot) = 0$, and consequently the volatility σ_t, $t \in [0, T]$, is a constant, we recover the classical Black-Scholes model and the Black-Scholes partial differential equation.

Next, we consider a shock stochastic volatility model. The incomplete Black-Scholes model is described as

$$\begin{aligned} dB_t &= B_t r dt \\ dS_t &= S_t(\mu dt + \sigma_t dW_t) \\ \sigma_t &= 1_{[0,\tau]}(t)\sigma_0 + 1_{(\tau,T]}(t)Z. \end{aligned}$$

Here $\tau \in [0,\ T]$ is fixed, Z is a random variable independent of $\{W_t\}_{t\in[0,\ T]}$. A particular example is $Z(\omega) = \sigma_1 1_E(\omega) + \sigma_2 1_{E^c}(\omega)$, i.e., a Bernoulli variable, where E is an event set. We see that the stock price evolves like a usual geometric Brownian motion up to the date τ, then a random event occurs(such as an election, or an anouncement), which influences the volatility parameter whose value may depend on this event. This model with Z being a Bernoulli variable was first studied by Harrison and Pliska (1981), and was further elaborated by Müller (1983), Föllmer and Schweizer (1991). But they did not study the valuation problem. We will use the result of Section 3 to derive a interesiting pricing formula.

Note in this case, we have

$$\begin{aligned} S_T &= S_0 \mathrm{exp}\Big\{ \int_0^T \Big(\mu - \frac{1}{2}\sigma_s^2\Big)ds + \int_0^T \sigma_s dW_s \Big\} \\ &= S_0 \mathrm{exp}\Big\{ \Big(\mu - \frac{1}{2}\sigma_0^2\Big)\tau + \Big(\mu - \frac{1}{2}Z^2\Big)(T-\tau)\Big\} \times e^{\sigma_0 W_\tau + Z(W_T - W_\tau)} \\ &\equiv \alpha(Z) e^{\sigma_0 W_\tau + Z(W_T - W_\tau)}, \end{aligned}$$

and

$$\begin{aligned} X_T &= \mathrm{exp}\Big\{ \int_0^T \Big(r + \frac{(\mu-r)^2}{2\sigma_t^2}\Big)dt + \int_0^T \frac{\mu - r}{\sigma_t} dW_t \Big\} \\ &= \mathrm{exp}\Big\{ \Big(r + \frac{(\mu-r)^2}{2\sigma_0^2}\Big)\tau + \Big(r + \frac{(\mu-r)^2}{2Z^2}\Big)(T-\tau)\Big\} \\ &\quad \times \mathrm{exp}\Big\{ \frac{\mu - r}{\sigma_0} W_\tau + \frac{\mu - r}{Z}(W_T - W_\tau)\Big\} \\ &\equiv \beta(Z) e^{a\sigma_0 W_\tau + b(Z) Z (W_T - W_\tau)}, \end{aligned}$$

where for notational simplicity, we have put

$$\alpha(Z) = S_0 \mathrm{exp}\Big\{ \Big(\mu - \frac{1}{2}\sigma_0^2\Big)\tau + \Big(\mu - \frac{1}{2}Z^2\Big)(T-\tau)\Big\}$$

$$
\begin{aligned}
\beta(Z) &= \exp\Big\{\Big(r+\frac{(\mu-r)^2}{2\sigma_0^2}\Big)\tau+\Big(r+\frac{(\mu-r)^2}{2Z^2}\Big)(T-\tau)\Big\}\\
a &= \frac{\mu-r}{\sigma_0^2}, \quad b(Z)=\frac{\mu-r}{Z^2}.
\end{aligned}
$$

Consider the valuation of a European call option with payoff $(S_T-K)^+$. Note that the random variables Z, $-\sigma_0 W_\tau$ and $-z(W_T-W_\tau)$ are independent (z is a sample point of Z), the later two have probability density functions

$$
\frac{1}{\sqrt{2\pi\tau}\sigma_0}\exp\Big\{-\frac{x^2}{2\tau\sigma_0^2}\Big\}, \qquad \frac{1}{\sqrt{2\pi(T-\tau)}z}\exp\Big\{-\frac{y^2}{2(T-\tau)z^2}\Big\}
$$

respectively. Let the cumulative distribution of Z be $F(\cdot)$. Then by the pricing formula (6), we have

$$
\begin{aligned}
\xi_0 &= \mathrm{E}^{\mathrm{P}}\Big(\frac{(S_T-K)^+}{X_T}\Big)\\
&= \int_{R^3}\frac{1}{\beta(z)}(\alpha(z)e^{-(x+y)}-K)^+e^{ax+by}\frac{1}{2\pi\sqrt{\tau(T-\tau)}\sigma_0 z}e^{-\frac{x^2}{2\tau\sigma_0^2}-\frac{y^2}{2(T-\tau)z^2}}dxdydF(z)\\
&= \int_{R^3}\frac{1}{2\pi\beta(z)\sigma_0 z\sqrt{\tau(T-\tau)}}(\alpha(z)e^{-u}-K)^+\\
&\quad\times\exp\Big\{-\frac{(u-y)^2}{2\tau\sigma_0^2}-\frac{y^2}{2(T-\tau)z^2}+a(u-y)+b(z)y\Big\}dudydF(z)\\
&= \int_{R^3}\frac{1}{2\pi\beta(z)\sigma_0 z\sqrt{\tau(T-\tau)}}(\alpha(z)e^{-u}-K)^+\\
&\quad\times\exp\Big\{-\frac{1}{2}\Big(\frac{1}{\tau\sigma_0^2}+\frac{1}{(T-\tau)z^2}\Big)y^2+\Big(\frac{u}{\tau\sigma_0^2}+b(z)-a\Big)y-\frac{u^2}{2\tau\sigma_0^2}+au\Big\}dydudF(z)\\
&= \int_{R}\frac{1}{\sqrt{2\pi}\beta(z)\sqrt{\tau\sigma_0^2+(T-\tau)z^2}}\int_{u<\ln\frac{\alpha(z)}{K}}(\alpha(z)e^{-u}-K)\\
&\quad\times\exp\Big\{-\frac{u^2}{2\tau\sigma_0^2}+\frac{\tau\sigma_0^2(T-\tau)z^2}{2(\tau\sigma_0^2+(T-\tau)z^2)}\Big(\frac{u}{\tau\sigma_0^2}+b(z)-a\Big)^2+au\Big\}dudF(z)\\
&= \int_{R}\frac{1}{\sqrt{2\pi}\beta(z)\sqrt{\tau\sigma_0^2+(T-\tau)z^2}}\\
&\qquad\times\int_{u<\ln\frac{\alpha(z)}{K}}\Big(\alpha(z)\exp\Big\{-\frac{1}{2}\frac{u^2}{\tau\sigma_0^2+(T-\tau)z^2}+\Big(\frac{T(\mu-r)}{\tau\sigma_0^2+(T-\tau)z^2}-1\Big)u\Big\}\\
&\qquad-K\exp\Big\{-\frac{1}{2}\frac{u^2}{\tau\sigma_0^2+(T-\tau)z^2}+\frac{T(\mu-r)}{\tau\sigma_0^2+(T-\tau)z^2}u\Big\}\Big)\\
&\qquad\times\exp\Big\{\frac{\tau\sigma_0^2(T-\tau)z^2(b(z)-a)^2}{2(\tau\sigma_0^2+(T-\tau)z^2)}\Big\}dudF(z).
\end{aligned}
$$

After some cumerbsome but straightforward calculation, we have the final result

$$\xi_0 = \int_R \Big(S_0 N(d_+) - Ke^{-rT} N(d_-)\Big) dF(z), \tag{8}$$

with

$$d_\pm = \frac{1}{\sqrt{\tau\sigma_0^2 + (T-\tau)z^2}} \Big(\ln\frac{S_0}{K} \pm \frac{1}{2}(\tau\sigma_0^2 + (T-\tau)z^2) + Tr\Big).$$

The pricing formula (8) is intuitively appealing: it is just an average of the classical Black-Scholes prices with deterministic variable volatility. Apparently, due to the randomness of Z, perfect replication of the payoff $(S_T - K)$ at T is impossible. The formula (8) degenerates to the classic Black-Scholes formula in any of the following cases: (1) $Z = \sigma_0$, (2) $\tau = T$.

More generaly, consider the following model

$$\begin{aligned} dB_t &= B_t r dt \\ dS_t &= S_t(\mu dt + \sigma_t dW_t). \end{aligned}$$

Here r and μ are constants, $\{\sigma_t\}_{t\in[0,\ T]}$ is a stochastic process.

We take $\pi_t = \frac{\mu - r}{\sigma_t^2}$, $t \in [0, T]$, then by Eq. (4), $\widehat{\pi}_t = 0$, $\forall\ t \in [0,\ T]$, and $\mathrm{P}_\pi = \mathrm{P}$. Eqs. (2) and (3) become

$$\begin{aligned} S_t &= S_0 \exp\Big\{\int_0^t \Big(\mu - \frac{1}{2}\sigma_s^2\Big) ds + \int_0^t \sigma_s dW_s\Big\} \\ X_t &= \exp\Big\{\int_0^t \Big(r + \frac{(\mu - r)^2}{2\sigma_s^2}\Big) ds + \int_0^t \frac{\mu - r}{\sigma_s^2} dW_s\Big\}. \end{aligned}$$

Take this $\{X_t\}_{t\in[0,T]}$ as the numeraire. Let $h(S_T)$ be a European contingent claim with maturity T, then by finite approximation and pricing formula (6), we have

$$\xi_0 = X_t \mathrm{E}^{\mathrm{P}}\Big(\frac{h(S_T)}{X_T}\Big) = \int_{\mathcal{P}} \Big(N(d_+) - Ke^{-rT} N(d_-)\Big) d\mathcal{D}(\sigma_{\cdot}(\omega)), \tag{9}$$

where

$$d_\pm = \frac{1}{\sqrt{\int_0^T \sigma_t(\omega)^2 dt}} \Big(\ln\frac{S_0}{K} \pm \frac{1}{2}\int_0^T \sigma_t(\omega)^2 dt + Tr\Big),$$

and $\mathcal{P}$ is the sample path space of $\{\sigma_t\}_{t\in[0,T]}$, $\mathcal{D}$ its distribution.

Finally, if we allow the interest rate rather than the volatility to be random, we have an incomplete Black-Scholes model

$$\begin{aligned} dB_t &= B_t r_t dt \\ dS_t &= S_t(\mu dt + \sigma dW_t) \\ r_t &= 1_{[0,\tau]}(t) r_0 + 1_{(\tau,T]}(t) Z. \end{aligned}$$

Here $\tau \in [0,\ T]$ is fixed, Z is a random variable independent of $\{W_t\}_{t\in[0,\ T]}$.

By Eqs. (2) and (3), we have

$$\begin{aligned} S_T &= S_0 \exp\Big\{\int_0^T \Big(\mu - \frac{1}{2}\sigma^2\Big) ds + \int_0^T \sigma dW_s\Big\} \\ &= S_0 \exp\Big\{\Big(\mu - \frac{1}{2}\sigma^2\Big) T + \sigma W_T\Big\}, \\ X_T &= \exp\Big\{\int_0^T \Big(r_t + \frac{(\mu - r_t)^2}{2\sigma^2}\Big) dt + \int_0^T \frac{\mu - r_t}{\sigma} dW_t\Big\} \\ &= \exp\Big\{\Big(r_0 + \frac{(\mu - r_0)^2}{2\sigma^2}\Big)\tau + \Big(Z + \frac{(\mu - Z)^2}{2\sigma^2}\Big)(T-\tau)\Big\} \\ &\quad \times \exp\Big\{\frac{\mu - r_0}{\sigma} W_\tau + \frac{\mu - Z}{\sigma}(W_T - W_\tau)\Big\}. \end{aligned}$$

Proceed similarly as before, we have

$$\begin{aligned} \xi_0 &\equiv \mathrm{E}^{\mathrm{P}}\Big(\frac{(S_T - K)^+}{X_T}\Big) \\ &= \int_R \Big(S_0 N(d_+) - K e^{-(\tau r_0 + (T-\tau) z)} N(d_-)\Big) dF(z), \end{aligned}$$

with

$$d_\pm = \frac{1}{\sqrt{T}\sigma}\Big(\ln\frac{S_0}{K} \pm \frac{1}{2} T\sigma^2 + (\tau r_0 + (T-\tau) z)\Big).$$

5. Conclusion

We have shown that in an incomplete Black-Scholes model, although the equivalent martingale measures are not unique, there is a *simple* martingale measure corresponding to any self-financing numeraire process. This generates a whole class of arbitrage price systems. Moreover, all these arbitrage price systems are equivalent. In particular, the growth-optmal wealth process and the objective probability constitute a natural and intuitively appealing arbitrage price system with clear probabilistic interpretation. This approach provides a simple pricing rule for the incomplete Black-Scholes models as is illustrated by several stochastic volatility models.

References

1. Bajeux-Besnainou I. Portait, R., 1997, "The numeraire portfolio, a new perspective on financial theory,' *European Journal of Finance*, 3, 291-309.

2. Black, F., and M. Scholes, 1973, "The pricing of options and corporate liabilities," *Journal of Political Economy*, 81, 637-659.

3. Browne, S., 1998, "The rate of return from proportional portfolio strategies," *Adv. Appl. Prob.* 30, 216-238.

4. Cox, J., and S. Ross, 1976, "The valuation of options for alternative stochastic processes," *Journal of Financial Economy,* 3, 145-166.

5. Davis, M. H. A., 1997, "Option pricing in incomplete markets," in A. H. Dempster and S. R. Pliska (eds.), *Mathematics of Derivative Securities*, Publication of the Newton Institute.

6. Delbaen F., and W. Schachermayer, 1994, "A general version of the fundamental theorem of asset pricing," *Math. Ann.* 300, 463-520.

7. Delbaen, F., and W, Schachermayer, 1998, "The variance-optimal martingale measure for continuous processes," *Bernoulli*, 2, 81-105.

8. D. Duffie, 1994, "Dynamic Asset Pricing Theory," Princeton Univ. Press, New Jersey.

9. Dybvig, P. H., and S. Ross, 1987, "Arbitrage," in J. Eatwell, M. Milgate and P. Newman(eds.),*The New Palgrave, Finance,* pp. 57-71, Macmillan Press Limited.

10. Föllmer, H., and D. Sondermann, 1985, "Hedging of non-redundant contingent claims," in W. Hildbrand and A. Mas-Colell(eds.), *Contributions to Mathematical Economics*, pp.203-223, Amsterdan, North-Holland, 1986.

11. Föllmer, H., and M. Schweizer, 1991, "Hedging of contingent claims under incomplete information," in M. H. A. Davis and R. J. Elliot(eds.), *Applied Stochastic Analysis,* pp. 389-414, Gordon and Breach, London.

12. Geman, H., El Karoui, N., and J. Rochet, 1995, "Changes of numeraire, changes of probability and option pricing," *Journal of Applied Probability*, 32, 443-458.

13. Harrison, M. J., and D. M. Kreps, 1979, "Martingales and arbitrage in multiperiod securities markets," *Journal of Economic Theory,* 20, 381-408.

14. Harrison, M. J., and S. R. Pliska, 1981, "Martingales and stochastic integrals in the theory of continuous trading," *Stochastic Processes and Their Applications*, 11, 215-260.

15. Heston, L. S., 1993, "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *The Review of Financial Studies,* 6, 327-343.

16. Hofmann, N. Platen, E., and M. Schweizer, 1992, "Option pricing under incomplete and stochastic volatility," *Mathematical Finance*, 2, 153-197.

17. Hull, J. C., and A. White, 1987, "The pricing of options on assets with stochastic volatilities," *Journal of Finance,* 42, 281-300.

18. Long, J., 1990, "The numeraire portfolio," *J. Fin. Econ.*, 26, 29-69.

19. Merton, R., 1971, "Optimum consumption and portfolio rules in a continuous time model," *J. Econ. Th.* 3, 373-413.

20. Müller, S., 1985, *Arbitrage Pricing of Contingent Claims,* Springer-Verlag, Berlin.

21. Perold, A. F., and W. F. Sharpe, 1988, "Dynamic strategies for asset allocation," *Financial Analyst Journal*, Jan/Feb, 16-27.

22. Ross, S., 1976, "An arbitrage theory of capital asset pricing," *Journal of Economic Theory*, 13, 341-360.

23. Wiggins, J. B., 1987, "Option values under stochastic volatility," *Journal of Financial Economics,* 19, 351-372.

24. Yan, J. A., 1998, "A New Look at the Fundamental Theorem of Asset Pricing," J. Korean Math. Soc., 35, 659-673.

25. Yan, J. A., Zhang, S. G., and Q. Zhang, 1998, "A new pricing rule for contingent claims in incomplete markets," working paper.