Martingale Approach to Option Pricing –A Brief Review with Examples *

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Abstract

The option pricing theory has its origins in the seminal papers by Black and Scholes (1973) and by Merton (1973). Harrison and Kreps (1979) and Harrison and Pliska (1981) have showed that a natural mathematical framework for the analyse of financial markets is martingale theory and stochastic analysis. Since then this framework has played a dominating role in option pricing. In this note, I will give a brief review on the martingale approach to option pricing and illustrate this approach through several examples: pricing foreign currency option, exchange option, lookback options and Asian options.

0 Introduction

The option pricing has its origins in the seminal papers by Black and Scholes (1973) and by Merton (1973), where the Itô's formula has been used for deriving the Black-Scholes equation and the risk neutral valuation principle was introduced. Harrison and Kreps (1979) and Harrison and Pliska (1981) have further showed that a natural mathematical framework for the analyse of financial markets is martingale theory and

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stochastic analysis. Since then this framework has played a dominating role in option pricing. This theory has nowadays become a powerful and effective tool for quantitative analysis in finance and in many economical problems.

An option or contingent claim is a financial contract written on an underlying asset (such as stock, foreign currency, stock-index, or derivetive security). In option pricing theory, a basic assumption is that the market is *arbitrage-free*, meaning that there is no riskless and profitable opportunity in the market. In such a market, it is reasonable to price a contingent claim in such a manner that the market augmented with the resulting derivetive security remains arbitrage-free. Such a pricing principle is called *pricing by arbitrage* or *arbitrage pricing*. It turned out that the martingale theory and stochastic analysis is a proper framework for characterizing arbitrage-free market and for pricing contingent claims.

In this note, I will give a brief review on the martingale approach to option pricing. For clarity and simplicity, we only consider European contingent claims in a *frictionless* market. By latter we mean there are no transaction costs, no bid-ask spread, no restrictions on trade such as margin requirements or short sale restrictions, there are no taxes, borrowing and lending are at the same interest rate, and asset shares are divisible. In addition, we assume that trading in assets takes place continually in time.

The following two theorems will be frequently used in the sequel.

Theorem 0.1 Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, $g(x,y)$ a non-negative Borel function on $\mathbf{R}^m \times \mathbf{R}^n$. If X is a $\mathcal{G}$-measurable $\mathbf{R}^m$-valued r.v. and Y is an $\mathbf{R}^n$-valued r.v. independent of $\mathcal{G}$, then

$$\mathbf{E}[g(X,Y) \mid \mathcal{G}] = \mathbf{E}[g(x,Y)]|_{x=X}. \tag{0.1}$$

Theorem 0.2 (*Bayes rule*) Let $\mathbf{Q}$ be a probability measure equivalent to $\mathbf{P}$ and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. We put $\xi = \frac{d\mathbf{Q}}{d\mathbf{P}}$, $\eta = \mathbf{E}[\xi \,|\, \mathcal{G}]$. Then for a $\mathbf{Q}$-integrable r.v. X we have

$$\mathbf{E}_{\mathbf{Q}}[X \,|\, \mathcal{G}] = \eta^{-1}\mathbf{E}[X\xi \,|\, \mathcal{G}]. \tag{0.2}$$

The rest of this note is organized as follows: In Section 1 we present the Black-Scholes model and the arbitrage pricing of contingent claims. In Sections 2 and 3 we introduce semimartingale model and Itô process model, respectively, and establish a relationship between no-arbitrage (fairness) and existence of martingale measures. The martingale approach to option pricing in semimartingale model is presented in Section 4. Finally, in Section 5 we illustrate some technical points for the martingale approach to option pricing through several examples: pricing foreign currency option, exchange option, lookback options and Asian options.

1 Black-Scholes Economy

1.1 The Black-Scholes Model

We consider the Black-Scholes economy in which there are two financial instruments: a risky asset (stock), and a riskless asset (savings account). Assume that the stock pays no dividends and its price process satisfies the Itô SDE

$$dS_t = S_t(\mu dt + \sigma dB_t), \tag{1.1}$$

where $S_0 > 0, \mu$ and σ are constants, (B_t) is a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$, where $\mathcal{F}_t = \sigma(B_s, s \leq t)$. Such a process (S_t) is called a *geometric Brownian motion*. It is also called a *log-normal* process because

$$S_t = S_0 \exp\left\{(\mu - \frac{\sigma^2}{2})t + \sigma B_t\right\}, \tag{1.2}$$

so that $\log(S_t)$ is normally distributed. We call μ the (*instantaneous*) *expected rate of return* and σ the *volatility* of S. The value process (β_t) of the savings account is assumed to satisfy

$$d\beta_t = r\beta_t dt, \tag{1.3}$$

where r is the constant interest rate. In the sequel, we always assume $\beta_0 = 1$ so that $\beta_t = e^{rt}$.

A *trading strategy* is a pair of $\mathcal{F}_t$-adapted processes $\{a, b\}$, where $a(t)$ denotes the number of units of the asset held at time t, $b(t)\beta_t$ is the total money invested in the bank at time t. So the *wealth* V_t at time t of a portfolio $\{a(t), b(t)\}$ is given by

$$V_t = a(t)S_t + b(t)\beta_t.$$

A trading strategy $\{a, b\}$ is said to be *self-financing* if the change of its wealth is only due to the changes in the assets prices, meaning that

$$dV_t = a(t)dS_t + b(t)d\beta_t. \tag{1.4}$$

A trading strategy is said to be *admissible*, if its wealth process (V_t) is non-negative.

1.2 Martingale Measure and Pricing by Arbitrage

As stated in the introduction, in option pricing theory a basic assumption on the market is the absence of arbitrage opportunity. We will show that the Black-Scholes economy has no-arbitrage. More precisely, we will show that there is no admissible self-financing strategy which has initial wealth zero and a non-negative terminal wealth V_T with $\mathbf{P}(V_T > 0) > 0$.

Lemma 1.1 A trading strategy $\{a, b\}$ is self-financing if and only if its discounted wealth process $(\widetilde{V}_t)$ satisfies

$$d\widetilde{V}_t = a(t)d\widetilde{S}_t. \tag{1.5}$$

Proof Assume $\{a, b\}$ is self-financing. Since $\widetilde{V}_t = e^{-rt}V_t$, by (1.3) we have

$$\begin{aligned} d\widetilde{V}_t &= -V_t r e^{-rt}dt + e^{-rt}dV_t \\ &= -[a(t)S_t + b(t)e^{rt}]re^{-rt}dt + e^{-rt}[a(t)dS_t + b(t)e^{rt}rdt] \\ &= a(t)[S_t d(e^{-rt}) + e^{-rt}dS_t] = a(t)d\widetilde{S}_t. \end{aligned}$$

Similarly, we can prove "if" part.

Now we show that there exists a probability measure $\mathbf{P}^*$ equivalent to $\mathbf{P}$ such that the process $\widetilde{S}_t$ is a $\mathbf{P}^*$-martingale. In fact, we can rewrite (1.1)as

$$d\widetilde{S}_t = \widetilde{S}_t[(\mu - r)dt + \sigma dB_t].$$

Consequently, if we put $\frac{d\mathbf{P}^*}{d\mathbf{P}}|_{\mathcal{F}_T} = \exp\left\{ -\frac{\mu-r}{\sigma}B_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T\right\}$, then by the Girsanov's theorem $B_t^* = B_t + \frac{\mu-r}{\sigma}t$ is a $\mathbf{P}^*$-Brownian motion and

$$d\widetilde{S}_t = \widetilde{S}_t \sigma dB_t^*. \tag{1.6}$$

Thus $(\widetilde{S}_t)$ is a $\mathbf{P}^*$-martingale. We call this probability measure $\mathbf{P}^*$ a *martingale measure* for the market. By contrast, $\mathbf{P}$ is called the *objective* or *physical* probability measure. It is easy to see that the martingale measure for the Black-Scholes economy is unique.

On the other hand, (1.1) can be rewritten as

$$dS_t = S_t[(rdt + \sigma dB_t^*]. \tag{1.7}$$

It means that under this measure $\mathbf{P}^*$ the expected rate of return of the risky asset is equal to the interest rate of the savings account. For this reason the martingale measure $\mathbf{P}^*$ is also called a *risk-neutral probability measure.*

By Lemma 1.1, the discounted wealth process of a self-financing strategy is a local martingale under $\mathbf{P}^*$. Thus, for any admissible self-financing strategy, its discounted wealth process is a $\mathbf{P}^*$-supermartingale, because it is a non-negative local $\mathbf{P}^*$-martingale. Consequently, if the initial wealth of an admissible self-financing strategy is zero then its wealth at any time must be zero. This implies that the market has no arbitrage.

The following theorem is the main result of the arbitrage pricing in the Black-Scholes economy.

Theorem 1.2 Let ξ be a European contingent claim which is integrable under $\mathbf{P}^*$. Then there exists an admissible self-financing strategy $\{a, b\}$ replicating ξ such that its wealth process (V_t) is given by

$$V_t = \mathbf{E}^*[e^{-r(T-t)}\xi|\mathcal{F}_t], \tag{1.8}$$

or equivalently, $(\widetilde{V}_t)$ is a $\mathbf{P}^*$-martingale. If $V_t = F(t, S_t)$ with $F \in C^{1,2}([0,T) \times \mathbf{R}_+$, then

$$a(t) = F_x(t, S_t). \tag{1.9}$$

Proof We define V_t by (1.8). Then $(\widetilde{V}_t)$ is a $\mathbf{P}^*$- martingale. Since $(\mathcal{F}_t)$ is also the natural filtration of (B_t^*), by the martingale representation theorem for Brownian motion there exists an $H \in \mathcal{L}^2$ such that

$$\widetilde{V}_t = V_0 + \int_0^t H_s dB_s^*, \quad t \in [0, T]. \tag{1.10}$$

Put

$$a(t) = H_t/(\sigma \widetilde{S}_t), \quad b(t) = \widetilde{V}_t - a(t)\widetilde{S}_t. \tag{1.11}$$

Then $\{a, b\}$ is a replicating strategy for ξ and (V_t) is its wealth process. By (1.6) and (1.11) we have

$$a(t)d\widetilde{S}_t = a(t)\widetilde{S}_t \sigma dB_t^* = H_t dB_t^* = d\widetilde{V}_t.$$

Thus by Lemma 1.1 the strategy $\{a, b\}$ is self-financing and admissible.

Now assume that $V_t = F(t, S_t)$, where $F \in C^{1,2}([0,T) \times \mathbf{R}^+$. By Itô's formula,

$$\begin{aligned} d\widetilde{V}_t &= d\Big(e^{-rt}F(t,S_t)\Big) = e^{-rt}F_x(t,S_t)dS_t + \text{“}dt\text{” term} \\ &= F_x(t,S_t)d\widetilde{S}_t + \text{“}dt\text{” term.} \end{aligned}$$

Since $(\widetilde{V}_t)$ and $\widetilde{S}_t$ are $\mathbf{P}^*$-martingales, the “dt” term must vanish. Thus by Lemme 1.1 we know (4.2) holds.

Remark It is natural to define V_t as the *fair price* at time t of the contingent claim ξ, because with this price there does not exist any arbitrage opportunity for both the seller and buyer of the contingent claim. Equation (1.8) is called the *risk-neutral valuation* formula.

Corollary 1.3 If $\xi = f(S_T)$, then $V_t = F(t, S_t)$, where

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} f\left(xe^{(r-\sigma^2/2)(T-t)+\sigma y\sqrt{T-t}}\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy. \tag{1.12}$$

Proof We express S_T as

$$S_T = S_t(S_T S_t^{-1}) = S_t \exp\{(r - \sigma^2/2)(T-t) + \sigma(B_T^* - B_t^*)\}.$$

Since S_t is measurable w.r.t. $\mathcal{F}_t$ and $B_T^* - B_t^*$ is independent of $\mathcal{F}_t$, by (1.8) and Theorem 0.1 we have

$$V_t = \mathbf{E}^*\Big[e^{-r(T-t)} f(x \exp\{(r-\sigma^2/2)(T-t) + \sigma(B_T^* - B_t^*)\}\Big]\Big|_{x=S_t},$$

from which we get $V_t = F(t, S_t)$.

Now we consider a European call option $\xi = (S_T - K)^+$, whose price at time t is denoted by $V_t = C(t, S_t)$. From (1.12) we get immediately the following Black-Scholes formula :

$$C(t,x) = xN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{1.13}$$

where $N(z)$ is the cumulative standard normal distribution function and

$$d_1 = \frac{\log(x/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\log(x/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

2 Semimartingale Model

We fix a finite time-horizon $[0,T]$ and consider a security market which consists of $m+1$ assets whose price processes $(S_t^i), i = 0, \cdots, m$ are assumed to be strictly positive semimartingales, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ satisfying the usual conditions. Moreover, we assume that $\mathcal{F}_0$ is the trivial σ-algebra. In option pricing one needs to choose one asset as a common unit, on the basis of which the prices of other assets are expressed. The resulting relative prices are called the *deflated prices* or *denominated prices*. Such an asset is called a *numeraire asset* or *numeraire*, and its price process is called a *numeraire*. If one takes the savings account as a numeraire asset, the relative prices are usually called *discounted prices*.

For notational convenience, we take asset 0 as the numeraire asset. We set $\gamma_t \hat{=} (S_t^0)^{-1}$ and call γ_t the *deflator* at time t. We set $S_t = (S_t^1, \cdots, S_t^m)$ and $\widetilde{S}_t = (\widetilde{S}_t^1, \cdots, \widetilde{S}_t^m)$, where $\widetilde{S}_t^i = \gamma_t S_t^i, 1 \le i \le m$. We call $(\widetilde{S}_t)$ the *deflated* price process of the assets. Note that the deflated price process of asset 0 is the constant 1.

In order to be able to define a trading strategy we need the notion of integration w.r.t. a vector-valued semimartingale. Such integral is defined globally and not componentwise. If $H = (H^0, \cdots, H^m)$ is integrable w.r.t. a semimartingale $(X^0, \cdots, X^m)$ and H^0 is integrable w.r.t. X^0, then we have

$$H.(X^0, \cdots, X^m) = H^0.X^0 + (H^1, \cdots, H^m).(X^1, \cdots, X^m). \tag{2.1}$$

A *trading strategy* is a $\mathbf{R}^{m+1}$-valued $\mathcal{F}_t$-predictable process $\phi = \{\theta^0, \theta\}$, where

$$\theta(t) = (\theta^1(t), \cdots, \theta^m(t)),$$

such that ϕ is integrable w.r.t semimartingale (S^0, S) with $S = (S^1, \cdots, S^m)$. $\theta^i(t)$ represents the numbers of units of asset i held at time t. This notion of trading strategy is not very realistic. However it is convenient for mathematical studies. The wealth $V_t(\phi)$ at time t of a trading strategy $\phi = \{\theta^0, \theta\}$ is

$$V_t(\phi) = \theta^0(t) S_t^0 + \theta(t) \cdot S_t, \tag{2.2}$$

where $\theta(t) \cdot S_t = \sum_{i=1}^m \theta^i(t) S_t^i$. The deflated wealth at time t is $\widetilde{V}_t(\phi) = V_t(\phi)\gamma_t$. A trading strategy $\{\theta^0, \theta\}$ is said to be *self-financing*, if

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi(u) d(S_u^0, S_u). \tag{2.3}$$

We use notation $(H.X)_t$ to stand for the integral of H w.r.t. X over the interval $(0, t]$. In particular, we have $(H.X)_0 = 0$.

Definition 2.1 A security market is said to be *fair* if there exists a probability measure $\mathbf{Q}$ equivalent to the "objective" probability measure $\mathbf{P}$ such that the deflated price processes $(\widetilde{S}_t)$ is a (vector-valued) $\mathbf{Q}$-martingale. We call such a $\mathbf{Q}$ an *equivalent martingale measure* for the market.

We denote by $\mathcal{M}^j$ the set of all equivalent martingale measures for the market, if asset j is taken as the numeraire asset.

The following theorem shows that the definition of fair market does not depend on the choice of numeraire.

Theorem 2.2 The fairness of a market is invariant under the change of numeraire.

Proof Assume that $\mathcal{M}^0 \neq \emptyset$. For a $\mathbf{P}^* \in \mathcal{M}^0$ we define a probability measure $\mathbf{Q}$ by

$$\frac{d\mathbf{Q}}{d\mathbf{P}^*} = \frac{S_0^0}{S_0^j} (S_T^0)^{-1} S_T^j. \tag{2.4}$$

We denote $\mathbf{Q}$ by $h_j(\mathbf{P}^*)$. We are going to show that h_j is a bijection from $\mathcal{M}^0$ onto $\mathcal{M}^j$. Let $\gamma_t' = (S_t^j)^{-1}$ and put

$$\widehat{S}_t^i = \gamma_t' S_t^i, \quad 0 \leq i \leq m.$$

Since $(S_t^0)^{-1} S_t^j = \widetilde{S}_t^j$ is a $\mathbf{P}^*$-martingale, we must have

$$M_t := \mathbf{E}^* \Big[\frac{d\mathbf{Q}}{d\mathbf{P}^*} \Big| \mathcal{F}_t \Big] = \frac{S_0^0}{S_0^j} (S_t^0)^{-1} S_t^j, \quad 0 \leq t \leq T.$$

¿From the fact that

$$M_t \widehat{S}_t^i = M_t (S_t^j)^{-1} S_t^i = \frac{S_0^j}{S_0^0} \widetilde{S}_t^i, \quad 0 \leq i \leq m$$

we know that $\mathbf{Q} \in \mathcal{M}^j$. The theorem is proved.

A strategy is said to be *admissible*, if its wealth process is non-negative. A strategy is said to be *tame*, if its deflated wealth process is bounded from below by some real constant. The weakness of the notion of tame strategy is that it is not invariant under the change of numeraire. We propose below to extend the notion of tame strategy to a notion of "allowable strategy", which does not involve the numeraire.

Definition 2.3 A strategy $\phi = \{\theta^0, \theta\}$ is said to be *allowable*, if there exists a positive constant c such that the wealth $(V_t(\phi))$ at any time t is bounded from below by $-c\sum_{i=0}^m S_t^i$.

Definition 2.4 A market is said to have no arbitrage with allowable strategies if there exists no allowable self-financing strategy with initial wealth zero and a non-negative terminal wealth V_T such that $\mathbf{P}(V_T > 0) > 0$.

A key point of arbitrage pricing of contingent claims is the following characterization of the self-financing strategy (see Yan (1998b) for a proof).

Theorem 2.5 A strategy $\phi = \{\theta^0, \theta\}$ is self-financing if and only if its wealth process (V_t) satisfies

$$d\widetilde{V}_t = \theta(t) d\widetilde{S}_t, \tag{2.5}$$

where $\widetilde{V}_t = V_t\gamma_t$. In particular, the deflated wealth process of an allowable self-financing strategy is a local $\mathbf{Q}$-martingale and a $\mathbf{Q}$-supermartingale for any $\mathbf{Q} \in \mathcal{M}^0$.

As a corollary we obtain:

Theorem 2.6 A fair market has no arbitrage with allowable strategies.

Proof Let $\mathbf{Q} \in \mathcal{M}^0$. Let $\{\theta, \theta^0\}$ be an allowable self-financing strategy with initial wealth zero. By Theorem 2.5 the deflated wealth process of ϕ is a $\mathbf{Q}$-supermartingale. Therefore, we must have $\mathbf{E}_{\mathbf{Q}}[\widetilde{V}_T] \le 0$. So the market has no arbitrage with allowable strategies.

¿From a general version of the fundamental theorem of asset pricing, due to Delbaen and Schachermayer (1994), we can deduce the following characterization of the fairness of a market (see Yan (1998b)).

Theorem 2.7 The market is fair if and only if there is no sequence (ϕ_n) of allowable self-financing strategies with initial wealth 0 such that $V_T(\phi_n) \ge -\frac{1}{n}\sum_{i=0}^m S_T^i$ a.s., for all $n \geq 1$ and such that $V_T(\phi_n)$ a.s. tends to a non-negative random variable ξ satisfying $\mathbf{P}(\xi > 0) > 0$.

In general, for a fair market the martingale measures are not unique. If the martingale measure is unique, then according to the martingale representation theorem in stochastic analysis, every European contingent claim is replicatable by an admissible self-financing strategy. A market having this property is called a *complete market.* In principle, a fair market is complete if and only if the martingale measure is unique (see Stricker and Yan (1998) for a general result in this direction).

3. The Itô Process Model

Now we consider a concrete case of semimartingale model: the Itô process model. This model is widely adopted in financial economy. We fix a finite time-horizon T. Let $B = (B^1, \cdots, B^d)$ be a Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We denote by $(\mathcal{F}_t)$ the natural filtration of (B_t) and by $\mathcal{L}$ the set of all measurable $(\mathcal{F}_t)$-adapted processes. We consider a financial market which consists of $m+1$ assets. The price process (S_t^i) of each asset i is assumed to be a strictly positive Itô process. Since its logarithm is also an Itô process, we can represent (S_t^i) as

$$dS_t^i = S_t^i\Big[\sigma^i(t)dB_t + \mu^i(t)dt\Big], \ S_0^i = p_i, \quad 0 \leq i \leq m. \tag{3.1}$$

We call $\mu = (\mu^0, \cdots, \mu^m)$ the *vector of expected rate of return* and σ the *volatility matrix.*

We specify asset 0 as the numeraire asset and set $\gamma_t := (S_t^0)^{-1}$. By Itô's formula we have

$$d\gamma_t = -\gamma_t\Big[\sigma^0(t)dB_t + (\mu^0(t) - |\sigma^0(t)|^2)dt\Big], \tag{3.2}$$

$$d\widetilde{S}_t^i = \widetilde{S}_t^i\Big[a^i(t)dB_t + b^i(t)dt\Big], \quad 1 \leq i \leq m, \tag{3.3}$$

where

$$a^i(t) = \sigma^i(t) - \sigma^0(t); \quad b^i(t) = \mu^i(t) - \mu^0(t) + |\sigma^0(t)|^2 - \sigma^i(t)\cdot\sigma^0(t).$$

In particular, If asset 0 is a savings account with interest rate process $(r(t))$, then

$$a^i(t) = \sigma^i(t), \quad b^i(t) = \mu^i(t) - r(t).$$

One raises naturally a question: what conditions we should impose on coefficients a and b of the diffusion $(\widetilde{S}_t)$ such that the market is fair? The following theorem gives a partial answer to this question (see Karatzas (1996) or Yan (1998b) for a proof).

Theorem 3.1 If the market is fair, the linear equation

$$a(t)\psi(t) = b(t), \;\; dt \times d\mathbf{P}-\text{a.e., a.s. on } [0,T] \times \Omega \tag{3.4}$$

has a solution $\psi \in (\mathcal{L}^2)^d$, where $\mathcal{L}^2$ stands for the set of all adapted process ϕ with $\int_0^T \phi^2(u)du < \infty$. Conversely, if

$$\mathbf{E}\left[\exp\left\{\frac{1}{2}\int_0^T |a^i(t)|^2 dt\right\}\right] < \infty, \;\; 1 \le i \le m, \tag{3.5}$$

and equation (3.4) has a solution $\psi \in (\mathcal{L}^2)^d$ satisfying

$$\mathbf{E}\left[\exp\left\{\frac{1}{2}\int_0^T |\psi(t)|^2 dt\right\}\right] < \infty, \tag{3.6}$$

then the probability measure $\mathbf{Q}$ with Radon-Nikodym derivative $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(-\psi.B)_T$ is an equivalent martingale measure.

Remark Without condition (3.5) $\mathbf{Q}$ is an equivalent local martingale measure but not necessarily a martingale measure. We refer the reader to Ansel and Stricker (1993) for an investication on this subject.

The following theorem provides a sufficient condition for the existence of a unique equivalent martingale measures.

Theorem 3.2 Assume that $m \ge d$, a satisfies (3.5) and $a^T(t)a(t)$ are non-degenerated for a.e. t, where $a^T(t)$ stands for the transpose of $a(t)$. Put $\psi(t) = (a^T(t)a(t))^{-1}a^T(t)b(t)$. If ψ satisfies (3.4) and (3.6), then there exists a unique equivalent martingale measure $\mathbf{P}^*$ for the market. Moreover, we have

$$\mathbf{E}\Big[\frac{d\mathbf{P}^*}{d\mathbf{P}}\Big|\,\mathcal{F}_t\Big] = \exp\Big\{-\int_0^t \psi(s)dB_s - \frac{1}{2}\int_0^t |\psi(s)|^2 ds\Big\}, \;\; 0 \le t \le T.$$

Proof By Theorem 3.1 there exists an equivalent martingale measure. To prove the uniqueness, let $\mathbf{Q}$ be an equivalent martingale measure. There exists a $\theta \in (\mathcal{L}^2)^d$ such that $\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}(-\theta.B)_T$. By Theorem 3.1, we have $a(t)\theta(t) = b(t)$. Consequently, applying $(a^T(t)a(t))^{-1}a^T(t)$ to the both sides of this equation we get $\theta(t) = \psi(t)$. The uiqueness is thus proved.

Remark If $m = d$, then ψ satisfies (3.4) automatically. In this case if (3.5)-(3.6) are satisfied, then there exists a unique equivalent martingale measure if and only if $a(t, \omega)$ is non-singular, for $(t, \omega) \in [0, T] \times \Omega$, a.e., a.s.. See Karatzas (1996).

4 Pricing European Contingent Claims

In this section we will present the martingale approach to pricing European contingent claims in a fair market with semimartingale model.

Let ξ be a contingent claim. Assume that $\gamma_T \xi$ is $\mathbf{P}^*$-integrable for some $\mathbf{P}^* \in \mathcal{M}^0$. We put

$$V_t = \gamma_t^{-1} \mathbf{E}^*[\gamma_T \xi \,|\, \mathcal{F}_t]. \tag{4.1}$$

If we consider (V_t) as the price process of an asset, then the market augmented with this asset is still fair, because the deflated price process of this asset is a $\mathbf{P}^*$-martingale. So it seems that (V_t) can be considered as a candidate for a "fair" price process of ξ. This definition depends on the choice of martingale measure. We will show that for replicatable contingent claims the fair price is unique.

Definition 4.1 A European contingent claim ξ is said to be *replicatable* (or *attainable*) if there exists a $\mathbf{P}^* \in \mathcal{M}^0$, such that $\gamma_T \xi$ is $\mathbf{P}^*$-integrable and there exists an admissible self-financing strategy ϕ such that its terminal wealth is equal to ξ and its deflated wealth process is a $\mathbf{P}^*$-martingale (i.e. $\mathbf{E}^*[\gamma_T \xi] = \gamma_0 V_0(\phi)$). In this case, we say that ξ is $\mathbf{P}^*$-replicatable and call ϕ a $\mathbf{P}^*$-*replicating strategy* for ξ.

The following theorem shows that the "fair" price process of a replicatable contingent claim is uniquely determined.

Theorem 4.2 Let ξ be a replicatable contingent claim, and $\mathbf{P}^*, \mathbf{P}' \in \mathcal{M}^0$ be such that ξ is $\mathbf{P}^*$- and $\mathbf{P}'$-replicable. Let (V_t) (resp. (U_t)) be the wealth process of a $\mathbf{P}^*$- (resp. $\mathbf{P}'$-)replicating strategy for ξ. Then (V_t) and (U_t) are the same. Moreover, V_t is given by (4.1).

Proof Put $\widetilde{V}_t = \gamma_t V_t$, $\widetilde{U}_t = \gamma_t U_t$. Then $(\widetilde{V}_t)$ is a $\mathbf{P}^*$-martingale and a $\mathbf{P}'$-supermartingale and $(\widetilde{U}_t)$ is a $\mathbf{P}'$-martingale and a $\mathbf{P}^*$-supermartingale. Note that

$U_T = V_T = \xi$ and we have

$$\mathbf{E}^*[\widetilde{V}_T|\,\mathcal{F}_t] = \widetilde{V}_t \geq \mathbf{E}'[\widetilde{V}_T|\,\mathcal{F}_t] = \mathbf{E}'[\widetilde{U}_T|\,\mathcal{F}_t] = \widetilde{U}_t.$$

Thus we have $V_t \geq U_t$, a.s.. Similarly, we have $U_t \geq V_t$, a.s.. Hence $V = U$.

According to Theorem 4.2, for a $\mathbf{P}^*$-replicatable contingent claim ξ it is natural to define its *"fair" price* at time t by (4.1). We call this method of pricing the *arbitrage pricing* (or *pricing by arbitrage*). The following theorem shows that the replicatabilty of a contingent claim and the arbitrage pricing of replicatable contingent claims are independent of the choice of numeraire.

Theorem 4.3 Let $\mathbf{P}^* \in \mathcal{M}^0$ and ξ be a $\mathbf{P}^*$-replicatable contingent claim and ϕ be a $\mathbf{P}^*$-replicating strategy for ξ. Then for any $0 \leq j \leq m$ ξ is an $h_j(\mathbf{P}^*)$-replicatable contingent claim, and its "fair" price process remains the same.

Proof We keep the notations in the proof of Theorem 2.2. We have by (4.1)

$$\mathbf{E}_{\mathbf{Q}}[\gamma'_T\xi] = \mathbf{E}^*[M_T\gamma'_T\xi] = \frac{S_0^0}{S_0^j}\mathbf{E}^*[\gamma_T\xi] = \gamma'_0 V_0.$$

This implies that a $\mathbf{P}^*$-hedging strategy for ξ is also a $\mathbf{Q}$-hedging strategy for ξ. So ξ is a $\mathbf{Q}$-replicatable contingent claim. Moreover, by the Bayes rule (Theorem 0.2) we have

$$\begin{aligned}(\gamma'_t)^{-1}\mathbf{E}_{\mathbf{Q}}[\gamma'_T\xi|\,\mathcal{F}_t] &= (\gamma'_t)^{-1}M_t^{-1}\mathbf{E}^*[M_T\gamma'_T\xi|\,\mathcal{F}_t] \\ &= \gamma_t^{-1}\mathbf{E}^*[\gamma_T\xi|\,\mathcal{F}_t].\end{aligned}$$

This proves that the "fair" price process of ξ is invariant under the change of numeraire.

5 Examples

In this section we illustrate some technical points for the martingale approach to option pricing through several examples. These technical points are: find a suitable multiplicative decomposition of the contingent claim so that one can use Theorem 0.1 to compute the conditional expectation (see examples 5.2 and 5.4); choose a proper numeraire for the problem (see examples 5.1 and 5.3); derive a partial diffirential equation from Itô's formula (see example 5.5). For reader's convenience we give the details of all derivations.

5.1 Foreign currency option. Consider a contract which give its owner the right to buy M units of a foreign currency at a prespecified exchange rate K and at the date T. We assume that both domestic and foreign risk-free interest rates are non-negative constants, say r^d and r^f, and the exchange rate Q satisfies the following equation:

$$dQ_t = Q_t[\mu dt + \sigma dB_t], \tag{5.1}$$

where μ and σ are constants. From a domestic point of view, there are two basic assets: one is the domestic savings account (regarded as a risk-free asset), whose price process is

$$d\beta_t^d = r^d \beta_t^d dt;$$

another is the foreign savings account in domestic term (regarded as a risky asset), whose price process is $S_t := \beta_t^f Q_t$, where

$$d\beta_t^f = r^f \beta_t^f dt.$$

By Itô's formula, we get

$$dS_t = S_t[(r^f + \mu)dt + \sigma dB_t].$$

Now the contract can be regarded as a European call option in this domestic market, whose payoff at expiry date T is $\xi = M(Q_T - K)^+$.

We take β_t^d as a numeraire, and let $\widetilde{S}_t = (\beta_t^d)^{-1} S_t$, then

$$d\widetilde{S}_t = \widetilde{S}_t[(r^f - r^d + \mu)dt + \sigma dB_t].$$

Consequently, if we put $\frac{d\mathbf{P}^*}{d\mathbf{P}}|_{\mathcal{F}_T} = \mathcal{E}(-\theta.B)_T$ with $\theta(t) = \theta = (r^f - r^d + \mu)/\sigma$, then by Girsanov's theorem $B_t^* = B_t + \theta t$ is a $\mathbf{P}^*$-Brownian motion and $(\widetilde{S}_t)$ is a $\mathbf{P}^*$-martingale. We call $\mathbf{P}^*$ the *domestic martingale measure*. By (1.8) the price at time t of the option ξ is given by

$$V_t = \mathbf{E}^*[e^{-r^d(T-t)} M(Q_T - K)^+ | \mathcal{F}_t]. \tag{5.2}$$

We can rewrite (5.1) and (5.2) as

$$dQ_t = Q_t[(r^d - r^f)dt + \sigma dB_t^*],$$

$$V_t = Me^{-r^f(T-t)}\mathbf{E}^*[e^{-(r^d-r^f)(T-t)}(Q_T-K)^+|\mathcal{F}_t].$$

Thus by the Black-Scholes formula we obtain immediately the valuation formula for option ξ:

$$V_t = Me^{-r^f(T-t)}C(t,Q_t),$$

where $C(t,x)$ is given by (2.9) with r being replaced by $r^d - r^f$. This formula is due to Geman-Kohlhagen (1983). Remark that just like the Black-Scholes formula, the parameter μ does not inter the valuation formula and even μ is time-dependet and random.

5.2 Lookback options We consider a Black-Scholes economy. By a lookback option we mean an option whose payoff depends on the maximum or minimum realized asset price over the option's life. There are two types of options: the *lookback strike* and *lookback rate*, in both call and put varieties. We only consider the lookback strike put option, whose payoff is defined by

$$\eta = \max{}_{0\le s\le T}S_s - S_T.$$

We denote its price at time t by P_t. By (1.8), we have

$$P_t = e^{-r(T-t)}\mathbf{E}^*[\eta \mid \mathcal{F}_t].$$

We are going to derive an explicit expression for P_t. We let $\lambda = r - \frac{1}{2}\sigma^2$ and set

$$M_t = \max{}_{0\le s\le t}S_s, \quad L_t = \max{}_{t\le s\le T}S_s.$$

Then M_t is $\mathcal{F}_t$-measurable. Since

$$S_t^{-1}L_t = \exp\{\max{}_{t\le s\le T}(\sigma(B_s^* - B_t^*) + \lambda(s-t)\},$$

$S_t^{-1}L_t$ is independent of $\mathcal{F}_t$. Using these notations and by Theorem 0.1 we have

$$\begin{aligned} P_t &= e^{-r(T-t)}\mathbf{E}^*[M_T - S_T|\mathcal{F}_t] = e^{-r(T-t)}S_t\mathbf{E}^*[\max(S_t^{-1}M_t, S_t^{-1}L_t)|\mathcal{F}_t] - S_t \\ &= e^{-r(T-t)}S_t\mathbf{E}^*[\max(y, S_t^{-1}L_t)]|_{y=S_t^{-1}M_t} - S_t\ . \end{aligned}$$

We denote $\mathbf{P}\Big(\max_{s\le t}(\sigma B_s + \lambda s) \le x\Big)$ by $F_t(x)$, which is given by the following expression due to Shepp (1966):

$$F_t(x) = N\Big(\frac{x-\lambda t}{\sigma\sqrt{t}}\Big) - e^{2\lambda x/\sigma^2}N\Big(\frac{-x-\lambda t}{\sigma\sqrt{t}}\Big),$$

where $x \geq 0$ and $N(z)$ is the cumulative standard normal distribution. Then we have

$$
\begin{aligned}
\mathbf{E}^*[\max(y, S_t^{-1}L_t)] &= \mathbf{E}^*[\exp\{\max(\log y, \max_{t\leq s\leq T}[\sigma(B_s^* - B_t^*) + \lambda(s-t)])\}] \\
&= \mathbf{E}[\exp\{\max(\log y, \max_{0\leq s\leq T-t}(\sigma B_s + \lambda s))\}] \\
&= yF_{T-t}(\log y) + \int_{\log y}^{\infty} e^x F'_{T-t}(x)dx.
\end{aligned}
$$

Finally we get the following formula, due to Goldman-Sosin-Gatto (1979):

$$
\begin{aligned}
P_t = \ & S_t(-1 + N(d_3)(1+\sigma^2/2r)) \\
&+ M_t e^{-r(T-t)}\Big(N(d_1) - \tfrac{\sigma^2}{2r}(S_t^{-1}M_t)^{(2r/\sigma^2)-1}N(d_2)\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\log(M_t/S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= \frac{-\log(M_t/S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\
d_3 &= \frac{-\log(M_t/S_t) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}
$$

5.3 Exchange option Consider a market consisting of $d+1$ assets with $d \geq 2$: a savings account with interest rate r and d stocks whose price processes $(S_t^i), i = 1, 2, \cdots, d$ satisfies the following Itô's equation:

$$
dS_t^i = S_t^i[\mu^i dt + \sum_{j=1}^{d} \sigma_{ij} dB_t^j], \ i = 1, 2, \cdots, d,
$$

where μ^i, σ_{ij} are constanats and $(B_t^1, B_t^2, \cdots, B_t^d)$ is a d-dimensional Brownian motion. We consider an option which gives its owner the right but no obligation to exchange stock 2 for stock 1 at time T. The payoff of this option is $\xi = (S_T^1 - S_T^2)^+$. The exchange option was first studied by Margrabe (1978) in the Black-Scholes economy. The change of numeraire approach to this problem is due to Davis (1994). We follow closely Karatzas (1996).

Assume that the matrix (σ_{ij}) is positive definite. Put

$$
Z_0(t) = \exp\Big\{-\sum_{i=1}^{d}\theta^i B_t^i - \frac{1}{2}\sum_{i=1}^{d}(\theta^i)^2 t\Big\}, \ 0 \leq t \leq T
$$

and

$$\frac{d\mathbf{P}^*}{d\mathbf{P}} = Z_0(T),$$

where $(\theta^1, \theta^2)^\tau = (\sigma_{ij})(\mu^1 - r, \mu^2 - r)^\tau$. Then (Z_t) is a $\mathbf{P}$-martingale and $\mathbf{P}^*$ is the unique martingale measure for the market. Put

$$B_t^{i*} = B_t^i + \theta^i t, \quad S_t^{i*} = e^{-rt} S_t^i, \ i = 1, 2, \cdots, d,$$

then $(B_t^{1*}, B_t^{2*}, \cdots, B_t^{d*})$ is a d-dimensional Brownian motion under $\mathbf{P}^*$ and

$$dS_t^{i*} = S_t^{i*}[\sum_{j=1}^{d} \sigma_{ij} dB_t^{j*}].$$

By (4.1) the value process of the exchange option is given by

$$V_t = \gamma_t^{-1} \mathbf{E}^*[\gamma_T (S_T^1 - S_T^2)^+ | \mathcal{F}_t],$$

where $\gamma_t = e^{-rt}$. Now we take asset 2 as a numeraire, then the associated martingale measure $\mathbf{Q}$ is given by

$$\frac{d\mathbf{Q}}{d\mathbf{P}^*} = \frac{\gamma_T}{S_0^2} S_T^2 = \frac{1}{S_0^2} S_T^{2*}.$$

Put

$$\widetilde{B}_t^i = B_t^{i*} - \sigma_{2i} t, \quad S_t = \frac{S_t^1}{S_t^2} = \frac{S_t^{1*}}{S_t^{2*}}.$$

Then $(\widetilde{B}_t^1, \widetilde{B}_t^2, \cdots, \widetilde{B}_t^d)$ is a d-dimensional Brownian motion under $\mathbf{Q}$, and

$$dS_t = S_t d[\sum_{j=1}^{2} (\sigma_{1j} - \sigma_{2j}) d\widetilde{B}_t^j] = S_t \sigma dB_t,$$

where $\sigma = \sqrt{\sum_{j=1}^d (\sigma_{1j} - \sigma_{2j})^2}$ and (B_t) is a one-dimentional Brownian motion under $\mathbf{Q}$. Thus we have

$$\begin{aligned} V_t &= S_t^2 \mathbf{E}_{\mathbf{Q}}[(\frac{S_T^1}{S_T^2} - 1)^+ | \mathcal{F}_t] = S_t^2 \mathbf{E}_{\mathbf{Q}}[(S_T - 1)^+ | \mathcal{F}_t] \\ &= S_t^2 [S_t N(d_1) - N(d_2)] = S_t^1 N(d_1) - S_t^2 N(d_2), \end{aligned}$$

where

$$d_1 = \frac{\log(S_t) + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{T-t}}, \ d_2 = \frac{\log(S_t) - \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{T-t}}.$$

Remark that in the above valuation formula the paprameters μ^i and r do not appear.

5.4 Asian options (geometric average) We consider a Black-Scholes economy. By an Asian option we mean an option whose payoff depends on a suitably defined average of the asset price. There are two kinds of average: geometric and arithmetic. For each kind of average there are two types of options: the *average strike* and *average rate* options.

Here we only consider the average rate call options with continuous sampling. We consider the geometric average case first. The payoff of an geometric average call option is given by

$$\xi_1 = \left(\exp\left\{\frac{1}{T}\int_0^T \log(S_u)du\right\} - K\right)^+.$$

We denote by $C_t^{(1)}$ its price at time t. In this case a closed form expression for option price is known (see Rubinstein (1992)). The martingale approch presented below was suggested by Lamberton-Lapeyre (1996). We denote by $\mathbf{P}^*$ the equivalent martingale measure for $\widetilde{S}$. By (1.8) we have

$$C_t^{(1)} = e^{-r(T-t)}\mathbf{E}^*[\xi_1|\mathcal{F}_t].$$

Put $I_t = \int_0^t \log(S_u)du$. Then

$$\begin{aligned}\xi_1 &= \left(\exp\left\{\frac{1}{T}I_t + \frac{1}{T}\int_t^T \log(S_u S_t^{-1})du + \frac{T-t}{T}\log S_t\right\} - K\right)^+, \\ &= (X_t Y_t - K)^+,\end{aligned}$$

where

$$X_t = e^{I_t/T}S_t^{(T-t)/T}, \quad Y_t = \exp\left\{\frac{1}{T}\int_t^T \log S_u S_t^{-1} du\right\}.$$

Since $S_t = S_0\exp\left\{(r-\frac{\sigma^2}{2})t + \sigma B_t^*\right\}$, we have

$$Y_t = \exp\left\{\frac{1}{T}\int_t^T \left[(r-\frac{\sigma^2}{2})(u-t) + \sigma(B_u^* - B_t^*)\right]du\right\} = e^{r^*(T-t)+Z_t},$$

where

$$r^* = (r-\frac{\sigma^2}{2})\frac{T-t}{2T}, \quad Z_t = \frac{1}{T}\int_t^T \sigma(B_u^* - B_t^*)du.$$

Since Z_t is independent of $\mathcal{F}_t$ and X_t is $\mathcal{F}_t$-measurable, by Theorem 1.1 we have

$$C_t^{(1)} = e^{-r(T-t)}\mathbf{E}^*[(X_t e^{r*(T-t)+Z_t} - K)^+|\mathcal{F}_t] = e^{-r(T-t)}F(t, X_t),$$

where

$$F(t,x) = \mathbf{E}^*[(xe^{r^*(T-t)+Z_t} - K)^+].$$

Note that Z_t is a Gaussian random variable with mean zero and variance $\sigma^{*2}(T-t)$, with

$$\sigma^{*2} = \frac{\sigma^2(T-t)^2}{3T}.$$

We have

$$\begin{aligned} F(t,x) &= e^{xr^*(T-t)} \int_{-\infty}^{\infty} \left(e^{\sigma^*\sqrt{T-t}y} - Ke^{-r^*(T-t)}\right)^+ \tfrac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &= xe^{(r^*+\frac{\sigma^{*2}}{2})(T-t)} N(d_1^*) - KN(d_2^*), \end{aligned}$$

where

$$d_1^* = \frac{\log(x/K) + (r^* + \sigma^{*2})(T-t)}{\sigma^*\sqrt{T-t}}, \quad d_2^* = \frac{\log(x/K) + r^*(T-t)}{\sigma^*\sqrt{T-t}}.$$

5.5 Asian options (arithmetic average) Now we turn to the pricing of an arithmetic average rate call option. The payoff of an arithmetic average call option is given by

$$\xi_2 = \left(\frac{1}{T}\int_0^T S_u du - K\right)^+.$$

We denote by $C_t^{(2)}$ its price at time t. By (1.8) we have

$$C_t^{(2)} = \mathbf{E}^*\Big[e^{-r(T-t)}\Big(\frac{1}{T}\int_0^T S_t dt - K\Big)^+\Big|\,\mathcal{F}_t\Big].$$

Following Rogers and Shi (1995) we put

$$M_t = \mathbf{E}^*\Big[\Big(\int_0^T S_u du - TK\Big)^+\Big|\,\mathcal{F}_t\Big].$$

Since $\int_t^T S_t^{-1} S_u du$ is independent of $\mathcal{F}_t$, we have

$$M_t = S_t\mathbf{E}^*\Big[\Big(\int_t^T S_t^{-1}S_u du - S_t^{-1}(TK - \int_0^t S_u du)\Big)^+\Big|\mathcal{F}_t\Big] = S_t f(t, Y_t),$$

where
$$f(t,x) = \mathbf{E}^*\Big[\Big(\int_t^T S_t^{-1} S_u du - x\Big)^+\Big]$$
and $Y_t = S_t^{-1}(TK - \int_0^t S_u du)$. Since
$$dS_t^{-1} = -S_t^{-2} dS_t + S_t^{-3} d\langle S, S\rangle_t = S_t^{-1}[(\sigma^2 - r)dt - \sigma dB_t^*],$$
we have
$$dY_t = \Big(TK - \int_0^t S_u du\Big) dS_t^{-1} - dt = Y_t[(\sigma^2 - r)dt - \sigma dB_t^*] - dt.$$
Consequently,
$$d\langle Y, Y\rangle_t = Y_t^2 \sigma^2 dt$$
and
$$d\langle S, f(\cdot, Y.)\rangle_t = f_x(t, Y_t) d\langle S, Y\rangle_t = -f_x(t, Y_t) S_t Y_t \sigma^2 dt.$$
In the following we use the notation $A \sim B$ to mean the fact that $A - B$ is a local martingale under $\mathbf{P}^*$. Therefore, by Itô's formula we obtain
$$\begin{aligned} dM_t &= S_t df(t, Y_t) + f(t, Y_t) dS_t + d\langle S, f(\cdot, Y.)\rangle \\ &= S_t\Big[f_t(t, Y_t)dt + f_x(t, Y_t)dY_t + \frac{1}{2} f_{xx}(t, Y_t) d\langle Y, Y\rangle_t\Big] \\ &\quad + f(t, Y_t) dS_t + d\langle S, f(\cdot, Y.)\rangle_t \\ &\sim S_t\Big(f_t - (1 + rY_t) f_x + \frac{1}{2}\sigma^2 Y_t^2 f_{xx} + rf\Big)(t, Y_t) dt. \end{aligned}$$
Since (M_t) is a martingale under $\mathbf{P}^*$, the above term on the right hand side must vanish. This leads to the following PDE:
$$f_t - (1 + rx) f_x + \frac{\sigma^2 x^2}{2} f_{xx} + rf = 0.$$
The first boundary condition is
$$f(T, x) = x^-.$$
The second one is
$$f(t, 0) = r^{-1}(e^{r(T-t)} - 1),$$
because
$$\mathbf{E}^*[S_t^{-1} S_u] = \mathbf{E}^*\Big[\exp\Big\{\sigma(B_u^* - B_t^*) - (\frac{\sigma^2}{2} - r)(u - t)\Big\}\Big] = e^{r(u-t)}.$$

In addition, since

$$f_x(t,x) = -\mathbf{P}^*\Big(\int_t^T S_t^{-1} S_u du \geq x\Big),$$

we have the third boundary condition:

$$\lim_{x\to\infty} f_x(t,x) = 0.$$

Unfortunately, deriving an analytic expression for the option price seems impossible. A lower bound for the option price was provided in Rogers and Shi (1995).

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