On Comparison Theorems for Diffusion Processes

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Abstract: It is well known that when two diffusion processes in R^1 have same diffusion terms, one can get the comparison theorems for their sample path. In this talk, we will deal with the comparison problems for two diffusion processes in R^d with different diffusion terms. In general, we cannot obtain the comparison results for their sample path in this case. We will discuss the problems of comparison between their distributions and their convex order. Several comparison theorems are given for two uniformly elliptic diffusion processes and for two uniformly elliptic diffusion processes with jumps. We also discuss the comparison problems for one kind of infinite dimensional diffusion, named supperdiffusion.

Key Words: Comparison Theorems, Uniformly Elliptic Diffusion Processes, Uniformly Elliptic Diffusion Processes with Jumps, Superprocesses.

I. Introduction

The comparison theorems for diffusion processes is a interesting subject. Many authors have devoted their interest to this area. For example, Anderson[1], Ikeda and Watanabe[13], Skorokhod[17], etc. When the two diffusion processes have the same diffusion coefficients, one can obtain the following well known comparison theorem.

Theorem 1.1. Suppose that we are given the following:
a real continuous function $\sigma(t, x)$ defined on $[0, \infty) \times R^1$ such that

$$|\sigma(t, x) - \sigma(t, y)| \leq \rho(|x - y|), \quad x, y \in R^1 \quad t > 0,$$

where ρ is a strictly increasing function defined on $[0,\infty)$ such that $\rho(0)=0$ and

$$\int_{0+}\rho(x)\,dx=\infty,$$

(ii) two real continuous functions $b_1(t,x)$ and $b_2(t,x)$ defined on $[0,\infty)\times R^1$ such that

$$b_1(t,x)\le b_2(t,x)\quad t\ge 0, x\in R^1.$$

Suppose that $X_i(t)$, $i=1,2$ are the (pathwise unique) solutions to the stochastic differential equations

$$dX_i(t)=\sigma(t,X_i(t))dB(t)+b_i(t,X_i(t));\quad X_i(0)=x_i,$$

where $B(t)$ is a one-dimensional Brownian motion with $B(0)=0$. If $X_1(0)\le X_2(0)$, then

$$X_1(t)\le X_2(t)\quad \text{for all t, P a.s..}$$

When the two diffusion processes have the different diffusion term, O'Brien[15] first proved the same type comparison theorem as following.

Theorem 1.2[15] Let $B(t)$ be a one-dimensional Brownian motion with $B(0)=0$ and $\sigma_i:R^1\to(0,\infty)$, $i=1,2$ be continuously differentiable functions. Let z^i, $i=1,2$, be constants. Suppose that $X_i(t)$, $i=1,2$, be the global solutions of stochastic differential equations

$$dX_i(t)=\sigma_i(X_i(t))dB(t)+\frac{1}{2}\sigma_i(X_i(t))\sigma_i'(X_i(t))dt,$$

with initial data $X_i(0)=z^i$. If

$$\int_{z^1}^{y}\frac{dx}{\sigma_1(x)}\ge\int_{z^2}^{y}\frac{dx}{\sigma_2(x)},\quad \text{for all } y\in R^1.$$

Then $X_1(t)\le X_2(t)$ a.s. for all $t\ge 0$.

Obviously, the drift term have deeply depedent on the diffusion term in O'Brien's results. As for the case of $d>1$, Borkar[3] first obtain in 1987 a new type of comparison theorem for two diffusion processes in R^d with different diffusion coefficients and without drift.

Theorem 1.3[3] Let $X(t)$ be a d-dimensiaonal diffusion process described by the stochastic differential equation

$$X(t) = x + \int_0^t \sigma(X(s))\, dB(t), \quad x \in R^d,\ t \geq 0$$

where $B(t)$ is a d-dimentional standard Brownian motion and $\sigma : R^d \to R^{d\times d}$ is bounded continuous and staisfies

$$\lambda^2 ||y||^2 \leq ||\sigma(x)^T y||^2 \leq \Lambda^2 ||y||^2, \quad x, y \in R^d,$$

for some $0 < \lambda < \Lambda$. Let

$$G = \{f : R^d \to R | f \text{ is convex function and statisfies (1.1) below}\},$$

$$\lim_{||x||\to\infty} e^{-\alpha||x||^2} |f(x)| = 0 \ \text{ for all } \alpha > 0. \tag{1.1}$$

Then for any $T > 0$ and $f_1,\ f_2 \in G$

$$\begin{array}{rl} & E[\int_0^T f_1(x + \lambda B(t))dt + f_2(x + \lambda B(T))] \\ \leq & E[\int_0^T f_1(X(t)dt + f_2(X(T)) \\ \leq & E[\int_0^T f_1(x + \Lambda B(t))dt + f_2(x + \Lambda B(T))] \end{array}$$

In recently, Chen and Wang[4] discussed the problem order-preservation and positive correlations for multidimensional diffusion processes. They obtained the comparison results for two diffusion semigroups for all monotone functions.

Let

$$A_l = \frac{1}{2}\sum_{i,j}^d a_{i,j}^{(l)} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i^{(l)} \frac{\partial}{\partial x_i}, \quad l = 1, 2.$$

and let $\{P_t^{(l)}\}$ be the semigroup generated by $A^{(l)}$. We call $P_t^{(1)} \geq P_t^{(2)}$, if for all bounded continuous monotone functions f, and all $x \geq y$ and $t \geq 0$,

$$P_t^{(1)} f(x) \geq P_t^{(2)} f(y)$$

Theorem 1.4[4] $P_t^{(1)} \geq P_t^{(2)}$ if and only if the following two conditions hold:

(i) for all i and j, $a_{i,j}^{(1)} = a_{i,j}^{(2)}$ and $a_{i,j}^{(1)}(x)$ depends only on x_i and x_j.

(ii) for all i, $b_i^{(1)}(x) \geq b_i^{(2)}(y)$ whenever $x \geq y$ with $x_i = y_i$.

In this talk, We will extend the Borkar's result to uniformal elliptic diffusion with jumps. we will give another type of comparison theorem for the d-dimensioal difffusion $X(t)$ without drift term. Two comparison theorems are also given for one kind of infinite dimensional diffusion, named supperdiffusion.

The main tool we used is following integral by part formula:

Lamma 1[18] Let T_t, S_t be strongly continuous semigroups on the Banach space E, with infinitesimal generators A and B, respectively. Suppose that $x \in \mathcal{D}(A)$. If $T_s x \in \mathcal{D}(B)$ for $0 < s < t$ and $\int_0^t ||BT_s x||\, ds < \infty$, then we have

$$S_t - T_t = \int_0^t S_{t-s}(B - A)T_s x\, ds$$

Throught this paper, we will suppose that $\sigma : R^d \to R^{d\times d}$ be bounded continuous matrices value functions. Let $a(x) = \sigma(x)\sigma^T(x)$, $a(x) = (a_{i,j}(x))_{d\times d}$ staisfies the following

Condition (A):

$$\lambda^2 ||y||^2 \leq y^T a(x) y \leq \Lambda^2 ||y||^2, \quad x, y \in R^d, \tag{1.2}$$

for some $0 < \lambda < \Lambda$, and $a_{i,j}(x)$, $\quad 1 \leq i \leq d, 1 \leq j \leq d$ are bounded countiunous functons on R^d. Let $B(t)$ be a d-dimentional standard Brownian motion. We call $X(t)$ is uniformly elliptic diffusion in R^d, if $X(t)$ is the solution of the following stochastic differential equation:

$$dX(t) = \sigma(X(t))dB(t) + b(X(t))dt.$$

II. A Comparison Theorem for Uniformly Elliptic Diffusion without Drift

Let $X(t)$ be d-dimensional diffusion processes without drift, i.e., $X(t)$ staisfies the following stochastic differential equation:

$$dX(t) = \sigma(X(s))\, dB(t), \quad \text{with } X(0) = x.$$

We have the following new type of comparison theorem:

Theorem 2.1 Let $X(t)$ be a d-dimensional uniformal elliptic diffusion i.e., σ satisfies (1.2). If $d \geq 2$, then $\forall t > 0, \quad \forall C > 0, \quad \forall x \in R^d$

$$P_x(|X(t)| \geq C) \leq 2P_x(|B(t)| \geq C/\Lambda),$$

$$P_x(\sup_{0 \leq t \leq T} |X(t)| \geq C) \leq 2P_x(\sup_{0 \leq t \leq T} |B(t)| \geq C/\Lambda), \quad \forall T > 0.$$

Sketch of Proof Set $R(t) = |B(t)|$, and $\bar{R}(t) = |X(t)|$. By appplying Itô formula, we can express $R(t)$ and $\bar{R}(t)$ as following

$$R(t) = |x| + \int_0^t d\beta(u) + \frac{1}{2}\int_0^t (d-1)R(u)^{-1}\, du,$$

where $\beta(u)$ is a one-dimensional standard Brownian motion.

$$\bar{R}(t) = |x| + \int_0^t H(u)\, dW(u) + \int_0^t h(X(u))\, du,$$

where

$$H(u)^2 = \bar{R}(u)^{-2} X(u)^T \sigma(X(u))\sigma^T(X(u))X(u),$$

$$h(u) = [Trace\ \sigma(X(u))\sigma^T(X(u)) - \bar{R}(u)^{-2} X(u)^T \sigma(X(u))\sigma^T(X(u))X(u)]/(2\bar{R}(u)).$$

For $\forall \varepsilon > 0$, set

$$b(x) = \begin{cases} (d-1)/(2y), & y \geq \varepsilon \\ (d-1)y/(2\varepsilon^2), & y \in (0, \varepsilon) \\ 0, & y \leq 0. \end{cases}$$

We can suppose that $\Lambda = 1$. Now, consider the following stochastic differential equation:

$$Z(t)^\varepsilon = |x| + \int_0^t H(u)\, dW(u) + \int_0^t b(Z(u)^\varepsilon)\, du.$$

Let $\bar{\tau}_\varepsilon = inf\{t > 0,\ \bar{R}(t) \leq \varepsilon\}, \quad \tau_\varepsilon = \{t > 0,\ Z^\varepsilon(t) \leq \varepsilon\}$, $Y(t) = \bar{R}(t \wedge \tau_\varepsilon \wedge \bar{\tau}_\varepsilon) - Z^\varepsilon(t \wedge \tau_\varepsilon \wedge \bar{\tau}_\varepsilon)$. By applying Tanaka formula we can proof

$$P_x(Z^\varepsilon(s) \geq \bar{R}(s), \quad s \leq \tau_\varepsilon \wedge \bar{\tau}_\varepsilon) = 1$$

Now by using the idea of Hajek[12], we consider the following stochastic differential equation:

$$Y_i^\varepsilon(t) = |x| + \int_0^t b(Y_i^\varepsilon(u))du + \int_0^t H(u)dW(u) + (-1)^i \int_0^t (1-H^2(u))^{\frac{1}{2}} dB(u), \;\; i = 1,2$$

where $B(u)$ is a one-dimensioanal Brownian motion and independent of $W(u)$. Let $\bar{Y}^\varepsilon(t) = \frac{1}{2}[Y_1^\varepsilon(t) + Y_2^\varepsilon(t)]$, $\;T_i^\varepsilon = \inf\{t > 0, \;\; Y_i^\varepsilon(t) \le \varepsilon\}$, $i = 1,2$. We can proof:

$$P_x(Z^\varepsilon(s) \le \bar{Y}^\varepsilon(s), \quad s \le T_1^\varepsilon \wedge T_2^\varepsilon \wedge \tau_\varepsilon) = 1,$$

and for fixed $t > 0$, $\forall \delta > 0$, $\exists \varepsilon > 0$ such that

$$P_x(T_1^\varepsilon \wedge T_2^\varepsilon \wedge \tau_\varepsilon \le t) \le \delta, \quad x \in R^d,\, x \ne 0.$$

So, for $\forall C > 0$, we have

$$\begin{aligned}
& P_x(\bar{R}(t) \ge C) \\
\le\ & P_x(\bar{R}(t) \ge C, \tau_\varepsilon \wedge \bar{\tau}_\varepsilon > t) + \delta \\
\le\ & P_x(Z^\varepsilon(t) \ge C, \tau_\varepsilon \wedge \bar{\tau}_\varepsilon > t) + \delta \\
\le\ & P_x(Z^\varepsilon(t) \ge C, T_1^\varepsilon \wedge T_2^\varepsilon \wedge \tau_\varepsilon > t) + 2\delta \\
\le\ & P_x(max(Y_1^\varepsilon(t), Y_2^\varepsilon(t)) \ge C, T_1^\varepsilon \wedge T_2^\varepsilon \wedge \tau_\varepsilon > t) + 2\delta \\
\le\ & P_x(Y_1^\varepsilon(t) \ge C, T_1^\varepsilon \wedge T_2^\varepsilon \wedge \tau_\varepsilon > t) + P_x(Y_2^\varepsilon(t) \ge C, T_1^\varepsilon \wedge T_2^\varepsilon \wedge \tau_\varepsilon > t) + 2\delta \\
\le\ & 2P_x(R(t) \ge C, T_1^\varepsilon \wedge T_2^\varepsilon \wedge \tau_\varepsilon > t) + 2\delta.
\end{aligned}$$

Let $\delta \to 0$, we get

$$P_x(|X(t)| \ge C) \le 2P_x(|B(t)| \ge C), \quad \forall C > 0, \quad |x| > 0$$

Similarly, we can proof that

$$P_x(\sup_{0 \le t \le T} |X(t)| \ge C) \le 2P_x(\sup_{0 \le t \le T} |B(t)| \ge C/\Lambda), \quad \forall T > 0.$$

As one application of this comparison theorem, we can get the following upbound estimation of packing measure for the image of $X(t)$.

Theorem 2.2 Let $X(t)$ be a d-dimensional uniformly elliptic diffusion $X(t)$ satisfies the conditions of theorem 2.1. If $d \ge 3$, then we have the following upbound estimation of packing measure for the image of $X(t)$.

$$\psi - p(ImX[0,1]) \le C < \infty, \qquad P \;\; a.s.,$$

where $\psi(t) = \frac{t^2}{\log\log(1/t)}$, $ImX[0,1] = \{X(s):\ s \in [0,1]\}$.

III. A Comparison Theorem for Uniformly Elliptic Diffusion with jumps

Now we will consider the diffusion process $X(t)$ with jumps in R^d, whose infinitesimal generator is defined by

$$Lf(x) = \frac{1}{2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \int_{R^d}[f(x+y) - f(x) - \sum_{j=1}^{d} y_j \frac{\partial f}{\partial x_j} 1_{\{|y|<1\}}]\nu(dy)$$

where $(a_{i,j}(x))_{d\times d} = a(x)$ is same as in introduction and ν is a Lévy measure which is a measure on R^d with $\nu(\{0\}) = 0$ and $\int_R^d (1 \wedge |y|^2)\nu(dy) < \infty$.

Let $X_1(t)$ and $X_2(t)$ be two Lévy processes in R^d, whose infinitesimal generators are defined by

$$L_1 f(x) = \frac{\lambda^2}{2}\sum_{i=1}^{d}\frac{\partial^2 f}{\partial x_i^2}(x) + \int_{R^d}[f(x+y) - f(x) - \sum_{j=1}^{d} y_j \frac{\partial f}{\partial x_j} 1_{\{|y|<1\}}]\nu(dy),$$

$$L_2 f(x) = \frac{\Lambda^2}{2}\sum_{i=1}^{d}\frac{\partial^2 f}{\partial x_i^2}(x) + \int_{R^d}[f(x+y) - f(x) - \sum_{j=1}^{d} y_j \frac{\partial f}{\partial x_j} 1_{\{|y|<1\}}]\nu(dy),$$

respectively. For $X(t)$, $X_1(t)$ and $X_2(t)$, we have the following comparison theorem.

Theorem 3.1: Let f be a convex function from R^d to R with at most polynomial growth at infinity. Then

$$E_x f(X_1(t)) \le E_x f(X(t)) \le E_x f(X_2(t)) \quad \forall t > 0, \quad \forall x \in R^d$$

Sketch of the proof

Let $T(t)$, $T_1(t)$ and $T_2(t)$ be the transition semigroups of $X(t)$, $X_1(t)$ and $X_2(t)$, respectively. It is well known that the Lévy processes are homogeneous in space, i.e. $T_i(t),\ \ i = 1,2$ commute with an arbitrary shift operator. So, if f is convex function, then $T_i(t)f$ is also a convex function.

By using the following modification: Let $f: R^d \to R^1$ be convex, set $\rho_n(x) = n^d \rho(nx), \quad x \in R^d, \quad n \geq 1$ with

$$\rho(x) = \begin{cases} c\exp\left\{-\frac{1}{1-||x||^2}\right\}, & ||x|| < 1 \\ 0, & \text{otherwise} \end{cases}$$

where c is a positive constant such that $\int_{R^d} \rho(x)\,dx = 1$. Define

$$f_n(x) = \int_{R^d} f(y)\rho_n(x-y)\,dy \qquad x \in R^d, \quad n \geq 1.$$

Then it is easy to prove that

(i) $f_n \in C_0^\infty(R^d), \quad n \geq 1$ and $\frac{\partial^2 f_n}{\partial x_i \partial x_j} \in C_0^\infty(R^d), \quad 1 \leq i,\, j \leq d, \quad n \geq 1.$

(ii) f_n is convex on R^d, $n \geq 1$.

(iii) f_n converges to f uniformly on any compact subset of R^d as $n \to \infty$.

So, we only to prove our result for a convex function f with $f \in C_0^\infty(R^d)$. In this case, one can check that all conditions of Lemma 1 hold. By lemma 1 and the following lemma 2[3], we have

$$T(t)f(x) - T_i(t)f(x) = \int_0^t T(s)(L-L_i)T_i(s)f(x)ds \quad i = 1,\, 2.$$

Note that:

$$\begin{aligned} &(L-L_1)T_1(s)f(x) \\ &\quad = \sum_{i,j} a_{i,j}(x)\frac{\partial^2 T_1(s)f(x)}{\partial x_i \partial x_j} - \lambda^2 \triangle T_1(s)f(x) \\ &\quad \geq \lambda^2 \triangle T_1(s)f(x) - \lambda^2 \triangle T_1(s)f(x) \\ &\quad = 0. \end{aligned}$$

$$\begin{aligned} &(L-L_2)T_2(s)f(x) \\ &\quad = \sum_{i,j} a_{i,j}(x)\frac{\partial^2 T_2(s)f(x)}{\partial x_i \partial x_j} - \Lambda^2 \triangle T_2(s)f(x) \\ &\quad \leq \Lambda^2 \triangle T_2(s)f(x) - \Lambda^2 \triangle T_2(s)f(x) \\ &\quad = 0. \end{aligned}$$

Therefore:

$$T(t)f(x) \geq T_1(t)f(x), \qquad T(t)f(x) \leq T_2(t)f(x).$$

i.e:

$$E_x f(X_1(t)) \leq E_x f(X_t) \leq E_x f(X_2(t)).$$

Lemma 2[2] Let $B = (b_{ij})_{d\times d}$ $C = (c_{ij})_{d\times d}$ be non-negative matrices. Moreover, B satisfies the following condition :$\exists 0 < \lambda < \Lambda < +\infty$ for $\forall \xi \in R^d$,

$$\lambda^2 \xi^T \xi \leq \xi^T A \xi \leq \Lambda^2 \xi^T \xi$$

Then

$$\lambda^2 \sum_{i=1}^{d} b_{ii} \leq \sum_{i,j} a_{ij} b_{ij} \leq \Lambda^2 \sum_{i=1}^{d} b_{ii}.$$

IV. Two Comparison Theorems for Dawson-Watanabe superprocess

(A) Dawson-Watanabe superprocess

Let $(E, \mathcal{E})$ be a polish space. Let $\mathcal{M}(E)$ denote the set of all finite measure on E, and $bC(E)$ denote (resp. $bpC(E)$) the family of bounded continuous functions(resp. bounded positive continuous functions) on E. Set:

$$\Psi(x,\lambda) = -b(x)\lambda - c(x)\lambda^2 + \int_0^\infty (1 - e^{-\lambda u} - \lambda u) n(x, du)$$

where $c \geq 0$, and b are bounded measurable functions and $n(x, du)$ is a kernel from $E \times R_+$ to R_+ such that $\int_0^\infty u \wedge u^2 n(x, du)$ is a bounded function on E. Here we give two special cases of $\Psi(x, \lambda)$:

(1) choose $b(x) = 0$, $n(x, du) = 0$, then $\Psi(x, \lambda) = -c(x)\lambda^2$;

(2) choose $b(x) = \gamma(x)(1 + \beta/\Gamma(1-\beta))$, $n(x, du) = \frac{\beta(1+\beta)\gamma(x)}{\Gamma(1-\beta)} \frac{1}{u^{\beta+2}} du$, $0 < \beta < 1$, $c(x) \equiv 0$, then $\Psi(x, \lambda) = -\gamma(x)\lambda^{1+\beta}$.

Let ξ_t be Borel right process on E, T_t denote its transition semigroup. By Fitzsimmons's [9] results, there exists a unique $\mathcal{M}(E)$-valued Markov process X_t, whose Laplace functional is determined by the following formula:

$$E_\mu \exp\{- < X_t, f >\} = \exp\{- < \mu, u(t) f >\}, \quad f \in bpC(E), \quad \mu \in \mathcal{M}(E),$$

where $u(t)f$ is the solution of the integral equation:

$$u(t)f = T_t f + \int_0^t T_{t-s} \Psi(u(s)f) ds.$$

Moreover, X_t has the following branching property:

$$(X_t : t \geq 0 | X_0 = \mu + \nu) \stackrel{d}{=} (X_t' + X_t'' : t \geq 0 | X_0' = \mu, X_0'' = \nu),$$

" $\stackrel{d}{=}$ " means the random variables of the two sides of equation have the same distribution. Following Dynkin [8], we call X_t a Dawson-Watanabe superprocess with parameter $(\xi,\ \Psi)$.

Let:

$$D(\mathcal{L}) = \{\phi(<\mu, f_1>, \cdots, <\mu, f_n>), \quad \phi \in C_0^\infty(R^n), \quad f_1, \cdots, f_n \in \mathcal{D}(L)\},$$

where $\mathcal{D}(L)$ is the domain of operator L, $<\mu, f_i>= \int_E f_i d\mu$, $i = 1, \cdots, n$.

For $\forall F \in D(\mathcal{L})$, set:

$$\begin{array}{rcl} \mathcal{L}F(\mu) & = & \int_{R^d} \mu(dx) c(x) F''(\mu, x) + \int_E \mu(dx)[LF'(\mu, x) - b(x) F'(\mu, x)] \\ & + & \int_E \mu(dx) \int_0^\infty n(x, du)[F(\mu + u\delta_x) - F(\mu) - uF'(\mu, x)], \end{array}$$

where $F'(\mu, x) = \lim_{\epsilon \to 0} \frac{F(\mu + \epsilon\delta_x) - F(\mu)}{\epsilon}$, δ_x denote the Dirac measure of point x. $F''(\mu, x)$ can be defined similarly.

By Fitgsinmmon's results(see[9]), we know that $\mathcal{L}$ is the generalized infinitesmall operator of X_t, i.e.

$$F(X_t) - F(X_0) - \int_0^t \mathcal{L}F(X_s) ds, \quad t \geq 0$$

is a locally bounded $\mathcal{P}_\mu$-martingale.

(B) Two Comparison Theorems

Let $\xi = \{\xi_t,\ t \geq 0\}$ be a Borel right process on E and

$$\Psi_1(x, \lambda) \equiv -b_1(x)\lambda - c_1(x)\lambda^2 + \int_0^\infty (1 - e^{-\lambda u} - \lambda u) n_1(x, du),$$

$$\Psi_2(x, \lambda) \equiv -b_2(x)\lambda - c_2(x)\lambda^2 + \int_0^\infty (1 - e^{-\lambda u} - \lambda u) n_2(x, du).$$

Theorem 1 establish the relationship between two Dawson-Watanabe superprocess with same ξ but different Ψ.

Theorem 4.1: Suppose $X_t^{(i)}$ $i = 1, 2$ be D-W superprocess with parameters $(\xi,\ \Psi_i)$. T_t and L are the transition semigroup and infinitesimal generator of

ξ_t, respectively. If $\mathcal{D}(L)$(the domain of operator L) is dense in $bpC(E)$ and $\Psi_1(x,\lambda) \leq \Psi_2(x,\lambda)$, then

$$E_\mu \exp\{-<f, X_t^{(1)}>\} \geq E_\mu \exp\{-<f, X_t^{(2)}>\}, \quad f \in bpC(E), \mu \in \mathcal{M}(E).$$

Proof:

Let $\mathcal{P}_t^{(i)}$ be the transition semigroup of $X_t^{(i)}$ and $\mathcal{L}_i$ be the infinitesimal generator of $\mathcal{P}_t^{(i)}$. Note that $\mathcal{P}_t^{(i)}$ is the strongly continuous semigroups on Banach space $B(\mathcal{M}(E))$. Here,

$$B(\mathcal{M}(E)) = \{F(\mu) : \sup_{\mu\in\mathcal{M}(E)} |F(\mu)| < +\infty\}.$$

For $\forall f \in \mathcal{D}(L)_+$, set $F(\mu) = \exp\{-<\mu, f>\}$. It is easy to check that $F(\mu) \in D(\mathcal{L}_i), \quad i = 1, 2$. Since

$$\mathcal{P}_t^{(i)} F(\mu) = E_\mu \exp\{-<X_t^{(i)}, f>\} = \exp\{-<\mu, u(t)^{(i)} f>\},$$

where

$$u^{(i)}(t)f = T_t f + \int_0^t T_{t-s}\Psi(u^{(i)}(s)f)\,ds, \qquad i = 1,2.$$

Hence

$$\mathcal{P}_t^{(1)} F(\mu) \in D(\mathcal{L}_2) \quad (f \in \mathcal{D}(L)_+ \Longrightarrow u^{(1)}(t)f \in \mathcal{D}(L)).$$

On the other hand,

$$\begin{aligned} F'(\mu, x) &= \lim_{\epsilon\downarrow 0} \tfrac{F(\mu+\epsilon\delta_x)-F(\mu)}{\epsilon} \\ &= \lim_{\epsilon\downarrow 0} \tfrac{\exp\{-<f,\mu+\delta_x\epsilon>\}-\exp\{-<f,\mu>\}}{\epsilon} \\ &= -e^{-<f,\mu>} f(x) \end{aligned}$$

and $F''(\mu,\, x) = f^2(x)e^{-<f,\mu>}$. Therefore:

$$\mathcal{L}_i F(\mu) = -e^{-<f,\mu>} \int_E \{\Psi_i(f) + Af\} d\mu, \quad i = 1,2. \tag{4.1}$$

By [9], we know that

$$F(X_t^{(i)}) - F(X_0^{(i)}) - \int_0^t \mathcal{L}_i F(X_s^{(i)}) ds, \quad t \geq 0, \quad i = 1,2 \tag{4.2}$$

is a locally bounded $\mathcal{P}_\mu$-martingale. Take expectation in two side of (4) and applying the martingale property, we have

$$\mathcal{P}_t^{(i)} F(\mu) - F(\mu) - \int_0^t \mathcal{P}_s^{(i)} \mathcal{L}_i F(\mu) ds = 0, \quad i = 1, 2.$$

Therefore,

$$\begin{array}{rcl} ||\frac{1}{t}(\mathcal{P}_t^{(i)} F - F) - \mathcal{L}_i F|| & = & ||\frac{1}{t}\int_0^t [\mathcal{P}_s^{(i)} \mathcal{L}_i F - \mathcal{L}_i F]\, ds|| \\ & \leq & \frac{1}{t}\int_0^t ||\mathcal{P}_s^{(i)} \mathcal{L}_i F - \mathcal{L}_i F||\, ds. \end{array}$$

By (4.1), we have

$$\begin{array}{rl} & \mathcal{P}_s^{(i)} \mathcal{L}_i F - \mathcal{L}_i F \\ = & \mathcal{L}_i \mathcal{P}_s^{(i)} F - \mathcal{L}_i F \\ = & -e^{-<u^{(i)}(t)f,\mu>} \int_E \{\Psi_i(u^{(i)}(t)f) + A(u^{(i)}(t)f)\} d\mu \\ + & e^{-<f,\mu>} \int_E \{\Psi_i(f) + Af\} d\mu, \end{array} \tag{4.3}$$

where $u^{(i)}(t)f$ is the solution of the following integral equation:

$$u^{(i)}(t)f = T_t f + \int_0^t T_{t-s} \Psi_i(u^{(i)}(s)f) ds \quad i = 1, 2.$$

From the above equalities and $\mathcal{P}_t^{(i)}$ is the strongly continuous semigroup on Banach space $B(\mathcal{M}(E))$, we can conclude that

$$\frac{1}{t}\int_0^t ||\mathcal{P}_s^{(i)} \mathcal{L}_i F - \mathcal{L}_i F||\, ds \to 0 \text{ as } t \to 0, \quad i = 1, 2.$$

Also,

$$\begin{array}{rl} & \int_0^t ||\mathcal{L}_2 \mathcal{P}_s^{(1)} F|| ds \\ = & \int_0^t || - e^{-<u^{(1)}(s)f,\mu>} \int_E \{\Psi_2(u^{(1)}(s)f) + A(u^{(1)}(t)f)\} d\mu || ds \\ < & \infty. \end{array}$$

By lemma 1:

$$\begin{array}{rl} & \mathcal{P}_t^{(2)} F(\mu) - \mathcal{P}_t^{(1)} F(\mu) \\ = & \int_0^t \mathcal{P}_{t-s}^{(2)} (\mathcal{L}_2 - \mathcal{L}_1) \mathcal{P}_s^{(1)} F(\mu) ds \\ = & \int_0^t \mathcal{P}_{t-s}^{(2)} e^{-<\mu,u^{(1)}(s)f>} \int_E [\Psi_1(u^{(1)}(s)f) - \Psi_2(u^{(1)}(s)f)] d\mu\, ds \\ \leq & 0, \end{array}$$

i.e:

$$\exp\{- < \mu, X_t^{(2)} >\} \leq \exp\{- < \mu, X_t^{(1)} >\}.$$

Note that $\mathcal{D}(L)_+$ is dense in $bpC(E)$, we can obtain our result.

When the two Dawson-Watanabe superprocess have the same Ψ and different ξ, we have the following comparison theorem.

Theorem 4.2: Suppose $X_t^{(i)}$ $i = 1, 2$ be D-W superprocess with parameters $(\xi^{(i)}, \Psi)$. i.e.

$$E_\mu \exp\{- < f, X_t^{(i)} >\} = \exp\{- < u^{(i)}(t)f, \mu >\}, \quad f \in bpC(E), \ \mu \in \mathcal{M}(E),$$

where

$$u^{(i)}(t)f = T_t^{(i)} f + \int_0^t T_{t-s}\Psi(u^{(i)}(s)f)ds, \quad i = 1, 2.$$

$T_t^{(i)}$ and L_i are the transition semigroup and infinitesimal generator of $\xi_t^{(i)}$, respectively. If $f \in C_0^\infty(E)$ and $L_1 u^{(1)}(t)f \geq L_2 u^{(1)}(t)f$ (or $L_1 u^{(2)}(t)f \geq L_2 u^{(2)}(t)f$), then

$$E_\mu \exp\{- < f, X_t^{(1)} >\} \leq E_\mu \exp\{- < f, X_t^{(2)} >\}, \quad f \in bpC(E), \ \mu \in \mathcal{M}(E).$$

Proof:

The theorem 4.2 can be proved by the same way as in the proof of theorem 4.1. First, we need to substitute (4.1) by

$$\mathcal{L}_i F(\mu) = -e^{-<\mu, f>} \int_{R^d} \{\Psi(f) + L_i f\} d\mu, \quad i = 1, 2. \tag{4.4}$$

By (4.4), we have

$$\begin{aligned} & \mathcal{P}_s^{(i)} \mathcal{L}_i F - \mathcal{L}_i F \\ = \ & \mathcal{L}_i \mathcal{P}_s^{(i)} F - \mathcal{L}_i F \\ = \ & -e^{-<u^{(i)}(t)f, \mu>} \textstyle\int_E \{\Psi(u^{(i)}(t)f) + L_i(u^{(i)}(t)f)\} d\mu \\ + \ & e^{-<f,\mu>} \textstyle\int_E \{\Psi(f) + L_i f\} d\mu, \end{aligned}$$

where $u^{(i)}(t)f$ is the solution of the following integral equation:

$$u^{(i)}(t)f = T_i(t)f + \int_0^t T_i(t-s)\Psi(u^{(i)}(s)f)ds \quad i = 1, 2.$$

From the above equalities and $\mathcal{P}_t^{(i)}$ is the strongly continuous semigroup on Banach space $B(\mathcal{M}(E))$, we can conclude that

$$\frac{1}{t}\int_0^t ||\mathcal{P}_s^{(i)}\mathcal{L}_i F - \mathcal{L}_i F||\, ds \to 0 \text{ as } t \to 0, \quad i = 1, 2.$$

Also,

$$\begin{array}{rl} & \int_0^t ||\mathcal{L}_2\mathcal{P}_s^{(1)}F||ds \\ = & \int_0^t || - e^{-<u^{(1)}(s)f,\mu>} \int_E\{\Psi(u^{(1)}(s)f) + L_2(u^{(1)}(t)f)\}d\mu||ds \\ < & \infty. \end{array}$$

By lemma 1:

$$\begin{array}{rl} & \mathcal{P}_t^{(2)}F(\mu) - \mathcal{P}_t^{(1)}F(\mu) \\ = & \int_0^t \mathcal{P}_{t-s}^{(2)}[\mathcal{L}_2 - \mathcal{L}_1]\mathcal{P}_s^{(1)}F(\mu)ds \\ = & \int_0^t \mathcal{P}_{t-s}^{(2)}e^{-<\mu,u^{(1)}(s)f>} \int_E[L_1(u^{(1)}(s)f) - L_2(u^{(1)}(s)f)]d\mu\, ds \\ \geq & 0, \end{array}$$

i.e:

$$\exp\{- < \mu, X_t^{(2)} >\} \geq \exp\{- < \mu, X_t^{(1)} >\}.$$

Note that $\mathcal{D}(L)_+$ is dense in $bpC(E)$, we can obtain our result.

(C) Applications

(i) Let X_t be D-W superprocess with parameter $(\xi, \Psi(x,\lambda))$, where ξ is a Markov process and $\Psi(x,\lambda) = -\gamma(x)\lambda^2, \quad \gamma(x) > 0$. Let $\mathcal{P}(\mu, t, d\nu)$ be a transition probability of X_t. In [8], Dynkin proved that: if γ is a constant, then

$$\mathcal{P}(\mu, t, \{0\}) = \exp\{- < 1, \mu > \frac{\gamma}{2}t^{-1}\}.$$

Applying Theorem 1, we can obtain the following result:if $0 < c_1 \leq \gamma(x) \leq c_2$,then

$$\exp\{- < 1, \mu > \frac{c_2}{2}t^{-1}\} \leq \mathcal{P}(\mu, t, \{0\}) \leq \exp\{- < 1, \mu > \frac{c_1}{2}t^{-1}\}.$$

(ii) Let X_t be D-W superprocess with parameter $(W_t, \Psi(x,\lambda))$, where W_t is d-dimensional Brownian motion and $\Psi(x,\lambda) = -\gamma(x)\lambda^{1+\beta}$, $0 < \beta \leq$

1, $\gamma(x) > 0$. Set:

$$\mathcal{M}_I = \{\nu : \nu \text{ is Borel measure on } R^d, \text{ satisfies } 0 < \lim_{k\to\infty} \frac{\nu(A_k)}{\lambda_d(A_k)} < \infty\},$$

where A_k denotes the cube centered at the origin in R^d of side k and λ_d is d-dimensional Lebesgue measure. In [5] Dawson proved that: when $d \leq 2$, $\lim_{t\to\infty} \mathcal{P}(X_t(K) > \varepsilon) = 0, \quad \forall \varepsilon > 0, \forall \nu \in \mathcal{M}_I(R^d)$, for any compact set $K \subset R^d$. We can extend this result to more general branching characteristic function Ψ by applying theorem 1.

Theorem 4.3: Let X_t be a D-W superprocess with parameter (W, Ψ) where $\Psi(x, \lambda) = -\gamma(x)\lambda^{1+\beta} - c(x)\lambda^2$ and W_t is a Brownian motion on R^d, where $c(x)$ is a arbitrary nonnegative bounded measurable function. Then, for any compact set $K \subset R^d, \forall \varepsilon > 0, \forall \nu \in \mathcal{M}_I(R^d)$,

$$\lim_{t\to\infty} \mathcal{P}_\nu(X_t(K) > \varepsilon) = 0.$$

Proof: Let K be a compact set in R^d. Let $\bar{X}_t$ be a D-W superprocess with parameter (ξ, ψ), where xi is a Brownian motion on R^d and $\psi(x, \lambda) = -\gamma(x)\lambda^{1+\beta}$. Since $\forall x, \forall \lambda, \Psi(x, \lambda) \leq \psi(x, \lambda)$, by theorem 1 for $\forall \mu \in \mathcal{M}(E)$, we have

$$E_\mu \exp\{- < f, X_t >\} \geq E_\mu \exp\{- < f, \bar{X}_t >\}, \quad f \in bpC(E)$$

Therefore,

$$\exp\{- < u(t)f, \mu >\} \geq \exp\{- < \bar{u}(t)f, \mu >\}, \quad f \in bpC(E)$$

where

$$\bar{u}(t)f = T_t f + \int_0^t T_t \psi(\bar{u}(s)f) ds,$$

and

$$u(t) = T_t f + \int_0^t T_t \Psi(u(s)f) ds.$$

Take $\mu = \delta_x$, we have

$$u(t)f \leq \bar{u}(t)f$$

and for $\forall \nu \in \mathcal{M}_I(R^d)$

$$\int_{R^d} u(t) f \nu(dx) \leq \int_{R^d} \bar{u}(t) f \nu(dx).$$

Since the above inequality is hold for all continuous function with compact support, moreover, by [7] for $\forall \nu \in \mathcal{M}_I(R^d)$, and all f that is continuous with compact support

$$\lim_{t\to\infty} \int_{R^d} u^{(i)}(t) f \nu(dx) = 0, \quad i = 1, 3.$$

Therefore,

$$\lim_{t\to\infty} \int_{R^d} u(t) f \nu(dx) = 0, \quad \forall \nu \in \mathcal{M}_I(R^d).$$

We conclude our result.

(iii) We can also use our comparison theorems to deduce some comparison results for solutions of nonlinear evolution equation. Note that

$$E_\mu \exp\{- < X_t, f >\} = \exp\{- < \mu, u(t)f >\}, \quad f \in bpC(E), \quad \mu \in \mathcal{M}(E),$$

where $u(t)f$ is the solution of the integral equation:

$$u(t)f = T_t f + \int_0^t T_{t-s} \Psi(u(s)f) ds. \tag{4.5}$$

Taking $\mu = \delta_x$, we have $\exp\{- < \mu, u(t)f >\} = \exp(-u(t)f(x))$. Therefore one can get the following comparison results for $u(t)f$.

Theorem 4.1': Let $u^{(1)}(t)f$ and $u^{(2)}(t)f$ be the unique solutions of the following two nonlinear evolutions (4.6) and (4.7) respectively,

$$u^{(1)}(t)f = T_t f + \int_0^t T_{t-s} \Psi_1(u^{(1)}(s)f) ds, \tag{4.6}$$

$$u^{(2)}(t)f = T_t f + \int_0^t T_{t-s} \Psi_2(u^{(2)}(s)f) ds, \tag{4.7}$$

Let L be the infinitesimal generator of semigroup T_t. If $\mathcal{D}(L)$(the domain of operator L) is dense in $bpC(E)$ and $\Psi_1(x,\lambda) \leq \Psi_2(x,\lambda)$, $\forall x \in R^d$, $\lambda \in R^1$, then for $0 \leq f(x) \leq g(x)$ we have

$$u^{(1)}(t)f \leq u^{(2)}(t)f, \quad \forall t \geq 0.$$

Theorem 4.2': Let $u^{(1)}(t)f$ and $u^{(2)}(t)f$ be the unique solutions of the following two nonlinear evolutions (4.8) and (4.9) respectively,

$$u^{(1)}(t)f = T_1(t)f + \int_0^t T_1(t-s)\Psi(u^{(1)}(s)f)ds, \tag{4.8}$$

$$u^{(2)}(t)f = T_2(t)f + \int_0^t T_2(t-s)\Psi(u^{(2)}(s)f)ds, \tag{4.9}$$

Let L_i be the infinitesimal generator of semigroup $T_i(t)$ respectively. If $\mathcal{D}(L_1)\cap\mathcal{D}(L_2)$ is dense in $bpC(E)$ and $L_1u^{(1)}(t)f \geq L_2u^{(1)}(t)f$ (or $L_1u^{(2)}(t)f \geq L_2u^{(2)}(t)f$), then

$$u^{(1)}(t)f \geq u^{(2)}(t)f, \quad \forall t \geq 0.$$

Let $\mathcal{D}(L)$ be the domain of operator L, where L is the infintesimall generator of semigroup T_t. It is well known that if $f \in \mathcal{D}(L)$, then $u(t)f$ is also the solution of the following abstract Cauchy problem:

$$\begin{cases} \frac{du(t)f}{dt} = Lu(t)f + \Psi(u(t)f) \\ u(0)f = f \end{cases}$$

From now on, we will only consider the case $E = R^d$.

Let $\Psi_i(x,\lambda) = -\gamma_i(x)\lambda^{1+\beta}$, $0 < \beta \leq 1$, $\gamma_i(x) > 0$, $\quad i = 1,2$. Suppose that $u_1(t)f$ and $u_2(t)f$ be the solutions of the following equation (4.10) and equation (4.11) respectively,

$$\begin{cases} \frac{du_1(t)f}{dt} = Lu_1(t)f - \gamma_1(x)(u_1(t)f)^{1+\beta} \\ u_1(0)f = f, \qquad f \in \mathcal{D}(L) \end{cases} \tag{4.10}$$

$$\begin{cases} \frac{du_2(t)f}{dt} = Lu_2(t)f - \gamma_2(x)(u_2(t)f)^{1+\beta} \\ u_2(0)f = f, \qquad f \in \mathcal{D}(L) \end{cases} \tag{4.11}$$

If $\gamma_1(x) \leq \gamma_2(x)$, then $u_1(t)f(x) \geq u_2(t)f(x) \;\; \forall t > 0, \; \forall x \in R^d$.

We have another interesting result.

Theorem 4.4 Let $u_1(t)$ and $u_2(t)$ be the positive solutions of the following initial value problem (4.12) and (4.13) respectively,

$$\begin{cases} \frac{du_1(t)}{dt} = Lu_1(t) - \gamma(x)(u_1(t))^{1+\beta_1} \\ u_1(0) = f, \qquad f \in \mathcal{D}(L) \end{cases} \tag{4.12}$$

$$\begin{cases} \frac{du_2(t)}{dt} = Lu_2(t) - \gamma(x)(u_2(t))^{1+\beta_2} \\ u_2(0)f = f, \quad f \in \mathcal{D}(L) \end{cases}, \tag{4.13}$$

If $0 \le f \le 1$ and $\beta_1 \le \beta_2$, then $u_1(t) \le u_2(t)$.

proof: Note that $0 \le u_i(t) \le T_t f \le 1$, therefore

$$\gamma(x)(u_1(t))^{1+\beta_2} \le \gamma(x)(u_1(t))^{1+\beta_1}.$$

From the proof of theorem 1, we have

$$\begin{aligned} & \mathcal{P}_t^{(2)}F(\mu) - \mathcal{P}_t^{(1)}F(\mu) \\ = & \int_0^t \mathcal{P}_{t-s}^{(2)} e^{-<\mu,u^{(1)}(s)f>} \int_{R^d} \{[-\gamma(x)(u_1(t))^{1+\beta_1}] - [-\gamma(x)(u_1(t))^{1+\beta_2}]\} d\mu\, ds \\ \le & \ 0. \end{aligned}$$

Hence,

$$\exp\{- < u_1(t), \mu >\} \ge \exp\{- < u_2(t), \mu >\}.$$

Taking $\mu = \delta_x$, we obtain $u_1(t) \le u_2(t)$.

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