

01/25/13 Lauren Williams: "An Introduction to Cluster Algebras"  
3:00pm

Overview:

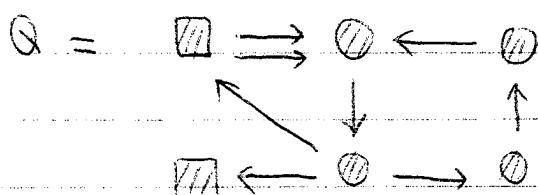
- + Cluster algebras are commutative rings with distinguished generators (called cluster variables) having rich combinatorial structure
- + Structure is encoded by a quiver & relations among generators encoded by quiver mutation
- + Introduced by Fomin + Zelevinsky 2000 in the context of Lie Theory, connections to other fields.

Outline:

- + What is a cluster algebra?
- + Example
- + Main results + open problems
- + Quantum + non-commutative analogues.

Quiver  $Q$ :

- finite directed graph
- multiple edges allowed
- oriented cycles of length 1 and 2 forbidden
- 2 types of vertices : "frozen" and "mutable"
- Ignore edges connecting frozen vertices

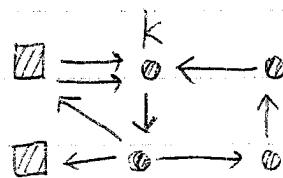
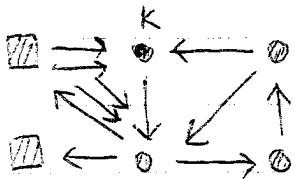


### Quiver mutation:

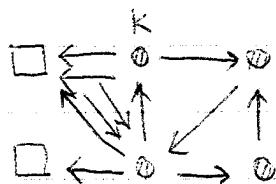
let  $k$  be mutable vertex of  $Q$

Quiver mutation  $\mu_k: Q \rightarrow Q'$  is computed by

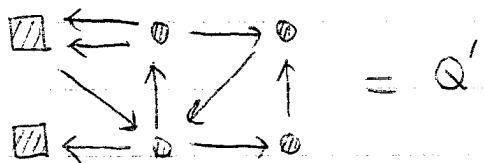
- For each instance of  $j \rightarrow k \rightarrow l$ , introduce edge  $j \rightarrow l$



- Reverse the direction of all edges incident to  $k$



- Remove oriented 2-cycles



Remark:  $\mu_k(\mu_k(Q)) = Q$ ,  $\forall k$

Defn: 2 quivers are mutation-equivalence if one can get between via a sequence of mutations.

### Seeds:

Defn: Let  $F$  be field of rational functions in  $m$  independent variables /  $\mathbb{C}$ . A seed in  $F$  is a pair  $(Q, \underline{x})$  consisting of

- Quiver  $Q$  on  $m$  vertices

- (extended) cluster  $\underline{x}$ , an  $m$ -tuple of alg. indep. (over  $\mathbb{Q}$ ) elements of  $F$  indexed by vertices of  $Q$ .
- where Frozen vertices  $\leftrightarrow$  coefficient variables
- Mutable vertices  $\leftrightarrow$  Cluster variables
- Cluster = {cluster variables}
- Extended cluster = {cluster + coeff. variables}

### Seed mutation:

let  $k$  be a mutable vertex of  $Q$  and let  $x_k$  denote a corresp. cluster variable. Then seed mutation

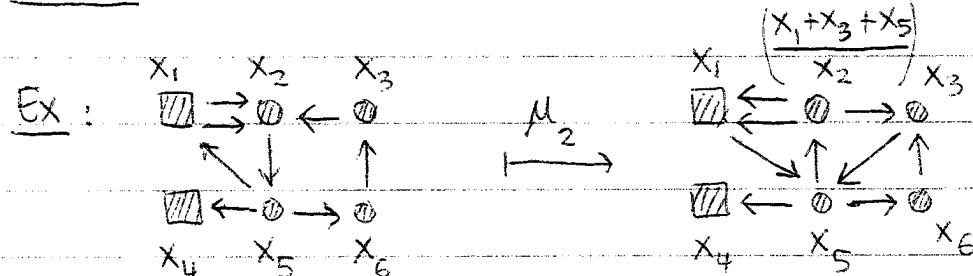
$\mu_k : (Q, \underline{x}) \mapsto (Q', \underline{x}')$  is defined by

$$Q' = \mu_k(Q)$$

$\underline{x}' = \underline{x} \cup \{x'_k\} \setminus \{x_k\}$ , where  $x'_k$  is defined by :

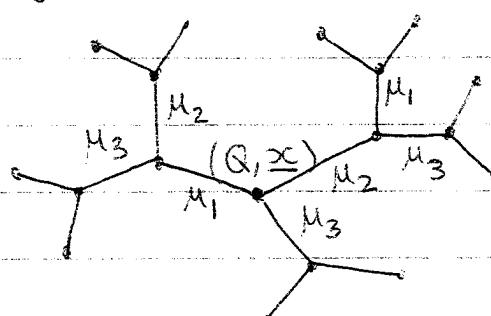
$$x'_k = \prod_{\substack{j \leftarrow k \\ \text{in } Q}} x_j + \prod_{\substack{j \rightarrow k \\ \text{in } Q}} x_j$$

Remark: Seed mutation is involution.



seed mutation

$$\mathbb{C}(x_1, \dots, x_6)$$



We can keep doing seed mutations to form a tree

Defn. (of Cluster Algebra)

let  $(Q, \succeq)$  be a seed in  $F$ , where  $Q$  has  $n$  mutable vertices.

Consider  $n$ -regular tree  $T_n$  with vertices labeled by seeds,

obtained by applying all possible sequences of mutations

let  $X = \text{union of all cluster + coeff. variables at all nodes of } T_n$

The cluster algebra  $A = A(Q)$  is the subring

of  $F$  which is generated by  $X$ .

$n = \text{rank of } A$

$$\text{Ex: Consider } (Q, \succeq) = \begin{array}{c} 1 \\ \bullet \rightarrow \\ x_1 \end{array} \xrightarrow{\quad} \begin{array}{c} 2 \\ \bullet \\ x_2 \end{array}$$

let's compute all cluster variables

$$\begin{array}{c} 1 \\ \bullet \rightarrow \\ x_1 \end{array} \xrightarrow{M_1} \begin{array}{c} 1 \\ \bullet \\ \frac{x_2+1}{x_1} \end{array} \xleftarrow{M_2} \begin{array}{c} 1 \\ \bullet \\ \frac{x_2+1}{x_1} + 1 \end{array} = \frac{x_1+x_2+1}{x_1 x_2}$$

$$\begin{array}{c} 1 \\ \bullet \\ x_2 \end{array} \xleftarrow{M_1} \begin{array}{c} 1 \\ \bullet \\ \frac{1+x_1}{x_2} \end{array} \xleftarrow{M_2} \begin{array}{c} 1 \\ \bullet \\ \frac{1+x_1+x_2}{x_1 x_2} \end{array}$$

up to relabelling the vertices, we get exactly the same seed as before

↪ The cluster algebra  $A(Q)$  is the subring of  $\mathbb{C}(x_1, x_2)$  generated by  $X = \{x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1 x_2}, \frac{1+x_1}{x_2}\}$

Remarks:

- 1) Each cluster variable is a Laurent polynomial in  $x_1, x_2$
- 2) Each Laurent poly. has positive coeffs.
- 3) There are finitely many cluster variables

4) The 2-regular tree closes up to pentagon.

Theorems + Conjectures: (F. + Z.)

let  $\mathcal{A} = \mathcal{A}(Q)$  be arbitrary cluster algebra with initial set  $(Q, \underline{x})$

Laurent Phen.: Each cluster variable is Laurent poly. in  $\underline{x}$ .

Positivity Conj.: All coeffs. of Laurent poly. above are positive

Finite Type Classification Theorem:

Say  $\mathcal{A}$  has finite type if there are finitely many cluster variables

Finite type cluster algebras are classified by Dynkin diagrams

When  $\mathcal{A}$  has finite type,  $\Pi_n$  closes up and becomes 1-skeleton  
of conv. polytope (generalized associahedron)

Generalizations:

+ Quantum cluster algebra (Berenstein + Zelevinsky) cluster  
variables in cluster quasi-commute Laurent phen. still holds (BZ.)  
(Quantum) positivity holds if cluster algebra has a cyclic seed  
(Kimura - Qin)

So far: No noncomm. cluster algebra

Quantum rank 2 cluster algebra

$\bullet \Rightarrow \bullet$  Then  $F = \text{skew-field of fractions of quantum torus}$   
 $y_1 \Rightarrow y_2$  w/ gen.  $y_1, y_2$  such that  $y_1 y_2 = q y_2 y_1$

Quantum cluster variables  $\leftrightarrow \mathbb{Z}$

$$\{y_m : m \in \mathbb{Z}\}$$

$$y_{m-1} y_{m+1} = q^{r/2} y_m^r + 1, \text{ clusters are pairs } \{y_{m-1}, y_{m+1}\}$$