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Global well-posedness of regular solutions to the three-dimensional isentropic compressible Navier-Stokes equations with degenerate viscosities and vacuum

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ABSTRACT

In this paper, the Cauchy problem for the three-dimensional (3-D) isentropic compressible Navier-Stokes equations with degenerate viscosities is considered. By introducing some new variables and making use of the "quasi-symmetric hyperbolic"–"degenerate elliptic" coupled structure to control the behavior of the fluid velocity, we prove the global-in-time well-posedness of regular solutions with vacuum for a class of smooth initial data that are of small density but possibly large velocities. Here the initial mass density is required to decay to zero in the far field, and the spectrum of the Jacobi matrix of the initial velocity are all positive. The result here applies to a class of degenerate density-dependent viscosity coefficients, is independent of the BD-entropy, and seems to be the first on the global existence of smooth solutions which have large velocities and contain vacuum state for such degenerate system in three space dimensions.
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1. Introduction

The time evolution of the mass density $\rho \geq 0$ and the velocity $u = \left(u^{(1)}, u^{(2)}, u^{(3)}\right)^\top$ of a general viscous isentropic compressible fluid occupying a spatial domain $\Omega \subset \mathbb{R}^3$ is governed by the following isentropic compressible Navier-Stokes equations (**ICNS**):

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P = \mathrm{div}\mathbb{T}. \end{cases} \tag{1.1}$$

Here, $x = (x_1, x_2, x_3) \in \Omega$, $t \geq 0$ are the space and time variables, respectively. In considering the polytropic gases, the constitutive relation, which is also called the equations of state, is given by

$$P = A\rho^\gamma, \quad \gamma > 1, \tag{1.2}$$

where $A > 0$ is an entropy constant and γ is the adiabatic exponent. $\mathbb{T}$ denotes the viscous stress tensor with the form

$$\mathbb{T} = \mu(\rho)\Big(\nabla u + (\nabla u)^\top\Big) + \lambda(\rho)\mathrm{div}u\, \mathbb{I}_3, \tag{1.3}$$

where $\mathbb{I}_3$ is the 3×3 identity matrix,

$$\mu(\rho) = \alpha\rho^\delta, \quad \lambda(\rho) = \beta\rho^\delta, \tag{1.4}$$

for some constant $\delta \geq 0$, $\mu(\rho)$ is the shear viscosity coefficient, $\lambda(\rho) + \frac{2}{3}\mu(\rho)$ is the bulk viscosity coefficient, α and β are both constants satisfying

$$\alpha > 0, \quad \text{and} \quad 2\alpha + 3\beta \geq 0. \tag{1.5}$$

Let $\Omega = \mathbb{R}^3$. We look for smooth solutions, $(\rho(t,x), u(t,x))$ to the Cauchy problem for (1.1)-(1.5) with the initial data and far field behavior:

$$(\rho, u)|_{t=0} = (\rho_0(x) \geq 0,\ u_0(x)) \quad \text{for} \quad x \in \mathbb{R}^3, \tag{1.6}$$

$$\rho(t,x) \to 0, \quad \text{as} \quad |x| \to \infty \qquad \text{for} \quad t \geq 0. \tag{1.7}$$

In the theory of gas dynamics, the compressible Navier-Stokes equations can be derived from the Boltzmann equations through the Chapman-Enskog expansion, cf. Chapman-Cowling [3] and Li-Qin [19]. Under some proper physical assumptions, the viscosity coefficients and the heat conductivity coefficient κ are not constants but functions of the absolute temperature θ such as:

$$\mu(\theta) = a_1\theta^{\frac{1}{2}}F(\theta), \quad \lambda(\theta) = a_2\theta^{\frac{1}{2}}F(\theta), \quad \kappa(\theta) = a_3\theta^{\frac{1}{2}}F(\theta) \tag{1.8}$$

for some constants a_i $(i = 1, 2, 3)$. Actually in [3] for the cut-off inverse power force model, if the intermolecular potential varies as r^{-a}, where r is intermolecular distance, then in (1.8): $F(\theta) = \theta^b$ with $b = \frac{2}{a} \in [0, +\infty)$. In particular, for Maxwellian molecules, $a = 4$ and $b = \frac{1}{2}$; while for elastic spheres, $a = \infty$ and $b = 0$. As a typical model whose F is not a power function of θ, the Sutherland's model is well known where

$$F(\theta) = \frac{\theta}{\theta + s_0}, \quad (s_0 > 0 : \text{ Sutherland's constant}). \tag{1.9}$$

According to Liu-Xin-Yang [24], if we restrict the gas flow to be isentropic, such dependence is inherited through the laws of Boyle and Gay-Lussac:

$$P = R\rho\theta = A\rho^\gamma, \quad \text{for constant} \quad R > 0,$$

i.e., $\theta = AR^{-1}\rho^{\gamma-1}$, and one finds that the viscosity coefficients are functions of the density. In this paper, we will focus on the cut-off inverse power force model whose viscosities have the forms shown in (1.4). The corresponding conclusion for the isentropic flow of the Sutherland's model will be shown in our forthcoming paper Xin-Zhu [32].

Throughout this paper, we adopt the following simplified notations, most of them are for the standard homogeneous and inhomogeneous Sobolev spaces:

$$\|f\|_s = \|f\|_{H^s(\mathbb{R}^3)}, \quad |f|_p = \|f\|_{L^p(\mathbb{R}^3)}, \quad \|f\|_{m,p} = \|f\|_{W^{m,p}(\mathbb{R}^3)},$$

$$|f|_{C^k} = \|f\|_{C^k(\mathbb{R}^3)}, \quad \|f\|_{L^pL^q_t} = \|f\|_{L^p([0,t];L^q(\mathbb{R}^3))}, \quad \|f\|_{X_1\cap X_2} = \|f\|_{X_1} + \|f\|_{X_2},$$

$$D^{k,r} = \{f \in L^1_{loc}(\mathbb{R}^3) : |f|_{D^{k,r}} = |\nabla^k f|_r < \infty\}, \quad D^k = D^{k,2}, \quad \int_{\mathbb{R}^3} f\mathrm{d}x = \int f,$$

$$\Xi = \{f \in L^1_{loc}(\mathbb{R}^3) : |\nabla f|_\infty + \|\nabla^2 f\|_2 < \infty\}, \quad \|f\|_\Xi = |\nabla f|_\infty + \|\nabla^2 f\|_2,$$

$$\|f(t,x)\|_{T,\Xi} = \|\nabla f(t,x)\|_{L^\infty([0,T]\times\mathbb{R}^3)} + \|\nabla^2 f(t,x)\|_{L^\infty([0,T];H^2(\mathbb{R}^3))}.$$

A detailed study of homogeneous Sobolev spaces can be found in [9].

There is a lot of literature on the well-posedness of solutions to the problem (1.1)–(1.6) in multi-dimensional space. For 3-D constant viscous flow ($\delta = 0$ in (1.4)) with $\inf_x \rho_0(x) > 0$, it is well-known that the local existence of classical solutions has been obtained by a standard Banach fixed point argument by Nash [28], which has been extended to be a global one by Matsumura-Nishida [26] for initial data close to a nonvacuum equilibrium in some Sobolev space H^s $(s > \frac{5}{2})$. Hoff [12] studied the global weak solutions with strictly positive initial density and temperature for discontinuous initial data. However, these approaches do not work when $\inf_x \rho_0(x) = 0$, which occurs when some physical requirements are imposed, such as finite total initial mass and energy in the whole space. The main breakthrough for the well-posedness of solutions with generic data and vacuum is due to Lions [23], where he established the global existence of weak solutions with finite energy to the isentropic compressible flow provided that $\gamma > \frac{9}{5}$ (see also Feireisl-Novotný-Petzeltová [8] for the case $\gamma > \frac{3}{2}$). However, the uniqueness problem of these weak solutions is widely open due to their fairly low regularities. Recently, Xin-Yan [31] proved that any classical solutions of viscous non-isentropic compressible fluids without heat conduction will blow up in finite time, if the initial data has an isolated mass group.

For density-dependent viscosities ($\delta > 0$ in (1.4)) which degenerate at vacuum, system (1.1) has received extensive attentions in recent years. In this case, the strong degeneracy of the momentum equations in (1.1) near vacuum creates serious difficulties for well-posedness of both strong and weak solutions. A mathematical entropy function was proposed by Bresch-Desjardins [2] for $\lambda(\rho)$ and $\mu(\rho)$ satisfying the relation

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)), \tag{1.10}$$

which offers an estimate $\mu'(\rho)\nabla\sqrt{\rho} \in L^\infty([0,T]; L^2(\mathbb{R}^d))$ provided that $\mu'(\rho_0)\nabla\sqrt{\rho_0} \in L^2(\mathbb{R}^d)$ for any $d \geq 1$. This observation plays an important role in the development of the global existence of weak solutions with vacuum for system (1.1) and some related models, see Bresch-Desjardins [1], Li-Xin [18], Mellet-Vasseur [27] and some other interesting results, c.f. [15,33]. However, the regularities and uniquness of such weak solutions remain open especially in multi-dimensional cases.

In this paper, we study the global well-posedness of regular solutions to the system (1.1) with density-dependent viscosities given in (1.4) and initial data such that $\inf_x \rho_0(x) = 0$. Then the analysis of the degeneracies in momentum equations $(1.1)_2$ requires some special attentions. The major concerns are:

(1) The degeneracy of time evolution in momentum equations $(1.1)_2$. Note that the leading coefficient of u_t in momentum equations vanishes at vacuum, and this leads to infinitely many ways to define velocity (if it exits) when vacuum appears. Mathematically, this degeneracy leads to a difficulty that it is hard to find a reasonable way to extend the definition of velocity into vacuum region. For constant viscosities, a remedy was suggested by Cho-Choe-Kim (for example [4]), where they imposed initially a *compatibility condition*

$$-\text{div}\mathbb{T}_0 + \nabla P(\rho_0) = \sqrt{\rho_0} g, \quad \text{for some} \quad g \in L^2(\mathbb{R}^3),$$

which, is roughly equivalent to the L^2-integrability of $\sqrt{\rho}u_t(t=0)$, and plays a key role in deducing that $(\sqrt{\rho}u_t, \nabla u_t) \in L^\infty([0,T_*]; L^2(\mathbb{R}^3))$ for a short time $T_* > 0$. Then they established successfully the local well-posedness of smooth solutions with non-negative density in $\mathbb{R}^3$, which, recently, has been shown to be a global one with small energy but large oscillations by Huang-Li-Xin [13] in $\mathbb{R}^3$ and Li-Xin [17] in $\mathbb{R}^2$.

(2) The strong degeneracy of the elliptic operator $\text{div}\mathbb{T}$ caused by vacuum for $\delta > 0$. In [4,13] for $\delta = 0$, the uniform ellipticity of the Lamé operator L defined by

$$Lu = -\alpha \triangle u - (\alpha + \beta)\nabla \mathtt{div} u$$

plays an essential role in the high order regularity estimates on u. One can use standard elliptic theory to estimate $|u|_{D^{k+2}}$ by the D^k-norm of all other terms in momentum equations. However, for $\delta > 0$, viscosity coefficients vanish as density function connects to vacuum continuously. This degeneracy makes it difficult to adapt the approach in the constant viscosity case in [4,13] to the current case.

(3) The strong nonlinearity for the variable coefficients of the viscous term due to $\delta > 0$. It should be pointed out here that unlike the case of constant viscosities, despite the weak regularizing effect on solutions, the elliptic part div$\mathbb{T}$ will also cause some troubles in the high order regularity estimates. For example, to establish some uniform a priori estimates independent of the lower bound of density in H^3 space, the key is to handle the extra nonlinear terms such as

$$\mathrm{div}\big(\nabla^k \rho^\delta \mathbb{S}(u)\big) \quad \text{for} \quad \mathbb{S}(u) = \alpha(\nabla u + (\nabla u)^\top) + \beta \mathtt{div} u \mathbb{I}_3,$$

where $k = 0, 1, 2, 3$. Therefore, much attention need to be paid in order to control these strong nonlinearities, especially for establishing the global well-posedenss of classical solutions.

Recently, there have some interesting works to overcome these difficulties mentioned above. In Li-Pan-Zhu [21] for the case $\delta = 1$, the degeneracies of the time evolution and the viscosity can be transferred to the possible singularity of the special source term $\left(\frac{\nabla\rho}{\rho}\right) \cdot \mathbb{S}(u)$. Based on this observations, via establishing a uniform a priori estimates in $L^6 \cap D^1 \cap D^2$ space for the quantity $\frac{\nabla\rho}{\rho}$, the existence of the unique local classical solution in 2-D space (see also Zhu [34] for 3-D case) to (1.1) has been obtained under the assumptions

$$\rho_0(x) \to 0, \quad \text{as} \quad |x| \to \infty,$$

which also applies to the 2-D shallow water equations.

However, this result only allows vacuum at the far field, and the corresponding problem with vacuum appearing in some open sets, or even at a single point is still unsolved. Later, via introducing a proper class of solutions and establishing the uniform weighted a priori estimates for the higher order term $\rho^{\frac{\delta-1}{2}}\nabla^4 u$, the same authors [22,35] gave the existence of 3-D local classical solutions to system (1.1) for the cases

$$1 < \delta \leq \min\{3, (\gamma + 1)/2\}.$$

Some other interesting results can also be seen in Ding-Zhu [5] and Li-Pan-Zhu [20].

1.1. Symmetric formulation

In order to deal with the issues mentioned above, we first need to analyze the structure of the momentum equations $(1.1)_2$ carefully, which can be decomposed into hyperbolic, elliptic and source parts as follows:

$$\underbrace{\rho(u_t + u\cdot\nabla u) + \nabla P}_{\text{Hyperbolic}} = \underbrace{-\rho^\delta Lu}_{\text{Elliptic}} + \underbrace{\nabla\rho^\delta\cdot\mathbb{S}(u)}_{\text{Source}}. \tag{1.11}$$

For smooth solutions (ρ, u) away from vacuum, these equations could be written into

$$\underbrace{u_t + u\cdot\nabla u + \frac{A\gamma}{\gamma-1}\nabla\rho^{\gamma-1} - \frac{\delta}{\delta-1}\nabla\rho^{\delta-1}\cdot\mathbb{S}(u)}_{\text{Principal order}} = \underbrace{-\rho^{\delta-1}Lu}_{\text{Higher order}}. \tag{1.12}$$

Note that if ρ is smooth enough, one could pass to the limit as $\rho\to 0$ on both sides of (1.12) and formally have

$$u_t + u\cdot\nabla u = 0 \quad \text{as} \quad \rho(t,x) = 0. \tag{1.13}$$

So (1.12)-(1.13) indicate that the velocity u could be governed by a nonlinear degenerate parabolic system when vacuum appears in some open sets or at the far field.

Recently, in Geng-Li-Zhu [10], via introducing two new quantities:

$$\varphi = \rho^{\frac{\delta-1}{2}}, \quad \text{and} \quad \phi = \sqrt{\frac{4A\gamma}{(\gamma-1)^2}}\rho^{\frac{\gamma-1}{2}},$$

system (1.1) can be rewritten into a system that consists of a transport equation for φ, and a "quasi-symmetric hyperbolic"–"degenerate elliptic" coupling system for $U = (\phi, u)^\top$:

$$\begin{cases} \underbrace{\varphi_t + u\cdot\nabla\varphi + \frac{\delta-1}{2}\varphi\text{div}u = 0,}_{\text{Transport equations}} \\ \underbrace{U_t + \sum_{j=1}^{3} A_j(U)\partial_j U}_{\text{Symmetric hyperbolic}} = \underbrace{-\varphi^2\mathbb{L}(u)}_{\text{Degenerate elliptic}} + \underbrace{\mathbb{H}(\varphi)\cdot\mathbb{Q}(u)}_{\text{First order source}}, \end{cases} \tag{1.14}$$

where $\partial_j U = \partial_{x_j} U$,

$$
A_j(U) = \begin{pmatrix} u^{(j)} & \frac{\gamma-1}{2}\phi e_j \\ \frac{\gamma-1}{2}\phi e_j^\top & u^{(j)}\mathbb{I}_3 \end{pmatrix}, \quad j=1,2,3, \quad \mathbb{L}(u) = \begin{pmatrix} 0 \\ Lu \end{pmatrix},
$$
$$
\mathbb{H}(\varphi) = \begin{pmatrix} 0 \\ \nabla\varphi^2 \end{pmatrix}, \quad \mathbb{Q}(u) = \begin{pmatrix} 0 & 0 \\ 0 & Q(u) \end{pmatrix}, \quad Q(u) = \frac{\delta}{\delta-1}\mathbb{S}(u), \tag{1.15}
$$

and $e_j = (\delta_{1j}, \delta_{2j}, \delta_{3j})$ $(j=1,2,3)$ is the Kronecker symbol satisfying $\delta_{ij}=1$, when $i=j$ and $\delta_{ij}=0$, otherwise. Based on this reformulation, a local well-posedness for arbitrary $\delta>1$ in some non-homogeneous Sobolev spaces has been obtained for the compressible degenerate viscous flow in [10], which could be shown as follows:

Theorem 1.1. *[10] Let $\gamma>1$ and $\delta>1$. If initial data (ρ_0,u_0) satisfy*

$$
\rho_0 \geq 0, \quad \left(\rho_0^{\frac{\gamma-1}{2}}, \rho_0^{\frac{\delta-1}{2}}, u_0\right) \in H^3, \tag{1.16}
$$

then there exists a time $T_*>0$ independent of the viscosity coefficients and a unique regular solution (ρ,u) in $[0,T_*]\times\mathbb{R}^3$ to the Cauchy problem (1.1)-(1.7), where the regular solution (ρ,u) satisfies this problem in the sense of distribution and:

$$
\begin{aligned}
&(A) \quad \rho\geq 0,\ \rho^{\frac{\delta-1}{2}} \in C([0,T_*];H^3),\ \rho^{\frac{\gamma-1}{2}} \in C([0,T_*];H^3);\\
&(B) \quad u \in C([0,T_*];H^{s'}) \cap L^\infty([0,T_*];H^3), \quad \rho^{\frac{\delta-1}{2}}\nabla^4 u \in L^2([0,T_*];L^2);\\
&(C) \quad u_t + u\cdot\nabla u = 0 \quad as \quad \rho(t,x)=0,
\end{aligned}
$$

where $s' \in [2,3)$ is an arbitrary constant.

1.2. Singularity formation

We consider first whether the local regular solution in Theorem 1.1 can be extended globally in time. In contrast to the classical theory for the constant viscosity case, we show the following somewhat surprising phenomenon that such an extension is impossible if the velocity field decays to zero as $t\to\infty$ and the initial total momentum is non-zero. More precisely, let

$$
\mathbb{P}(t) = \int \rho u \quad \text{(total momentum)}.
$$

Theorem 1.2. *Let $1<\min\{\gamma,\delta\}\leq 2$. Assume $|\mathbb{P}(0)|>0$. Then there is no global regular solution (ρ,u) obtained in Theorem 1.1 satisfying the following decay*

$$
\limsup_{t\to\infty} |u(t,\cdot)|_\infty = 0. \tag{1.17}
$$

Second, in the presence of vacuum region, there is another possibility of finite singularity defined as follows.

Definition 1.1. The non-empty open set $V \subset \mathbb{R}^3$ is called a hyperbolic singularity set of (ρ_0, u_0), if V satisfies

$$\begin{cases} \rho_0(x) = 0, \ \forall\ x \in V; \\ \\ Sp(\nabla u_0) \cap \mathbb{R}^- \neq \ \emptyset, \ \forall\ x \in V, \end{cases} \tag{1.18}$$

where $Sp(\nabla u_0(x))$ denotes the spectrum of the matrix $\nabla u_0(x)$.

By exploring the hyperbolic structure in (1.1) in vacuum region, one can confirm exactly that hyperbolic singularity set does generate singularities from local regular solutions in finite time.

Theorem 1.3. *Let $\gamma > 1$ and $\delta > 1$. If the initial data $(\rho_0, u_0)(x)$ have a non-empty hyperbolic singularity set V, then the regular solution (ρ, u) obtained in Theorem 1.1 with maximal existence time T_m blows up in finite time, i.e., $T_m < \infty$.*

1.3. Global-in-time well-posedness of smooth solutions

We now come to the major task to construct global smooth solutions for the system (1.1). Due to Theorems 1.2-1.3, one needs to identify a class of initial data and a proper energy space to avoid the two singularity mechanisms shown above.

Let $\widehat{u} = (\widehat{u}^{(1)}, \widehat{u}^{(2)}, \widehat{u}^{(3)})^\top$ be the solution in $[0, T] \times \mathbb{R}^3$ of the following Cauchy problem:

$$\widehat{u}_t + \widehat{u} \cdot \nabla \widehat{u} = 0, \quad \widehat{u}(t = 0, x) = u_0(x) \tag{1.19}$$

for $x \in \mathbb{R}^3$. We give the definition of regular solutions considered in this paper:

Definition 1.2. Let $T > 0$ be a finite constant. A solution (ρ, u) to the Cauchy problem (1.1)-(1.7) is called a regular solution in $[0, T] \times \mathbb{R}^3$ if (ρ, u) satisfies this problem in the sense of distribution and:

(A) $\rho \geq 0, \ (\rho^{\frac{\delta-1}{2}}, \rho^{\frac{\gamma-1}{2}}) \in C([0, T]; H^{s'}_{loc}) \cap L^\infty([0, T]; H^3)$;

(B) $u - \widehat{u} \in C([0, T]; H^{s'}_{loc}) \cap L^\infty([0, T]; H^3), \quad \rho^{\frac{\delta-1}{2}} \nabla^4 u \in L^2([0, T]; L^2)$;

(C) $u_t + u \cdot \nabla u = 0 \quad \text{as} \quad \rho(t, x) = 0$,

where $s' \in [2, 3)$ is an arbitrary constant. If ρ is compactly supported, the space $H^{s'}_{loc}$ can be replaced by $H^{s'}$.

The main result on the global well-posedness of regular solutions to the three-dimensional isentropic compressible Navier-Stokes equations with degenerate viscosities and vacuum could be stated as follows.

Theorem 1.4. *Let parameters* $(\gamma, \delta, \alpha, \beta)$ *satisfy*

$$\gamma > 1, \quad \delta > 1, \quad \alpha > 0, \quad 2\alpha + 3\beta \geq 0, \tag{1.20}$$

and any one of the following conditions (P_1)-(P_3):

(P_1) $2\alpha + 3\beta = 0$; (P_2) $\delta \geq 2\gamma - 1$; (P_3) $\delta = \gamma$.

If the initial data (ρ_0, u_0) *satisfy*

(A_1) $u_0 \in \Xi$ *and there exists a constant* $\kappa > 0$ *such that,*

$$Dist\big(Sp(\nabla u_0(x)), \mathbb{R}_-\big) \geq \kappa \quad \text{for all} \quad x \in \mathbb{R}^3,$$

(A_2) $\rho_0 \geq 0$ *and* $\left\|\rho_0^{\frac{\gamma-1}{2}}\right\|_3 + \left\|\rho_0^{\frac{\delta-1}{2}}\right\|_3 \leq D_0(\gamma, \delta, \alpha, \beta, A, \kappa, \|u_0\|_\Xi)$,

where $D_0 > 0$ *is some constant depending on* $(\alpha, \beta, \delta, A, \gamma, \kappa, \|u_0\|_\Xi)$, *then the Cauchy problem (1.1)-(1.7) has a unique smooth solution* (ρ, u) *in* $[0, \infty) \times \mathbb{R}^3$ *satisfying that, for any* $0 < T < \infty$, (ρ, u) *is a regular one in* $[0, T] \times \mathbb{R}^3$ *as defined in Definition 1.2. Particularly, when condition* (P_2) *holds, the smallness assumption on* $\rho_0^{\frac{\delta-1}{2}}$ *could be removed.*

Moreover, if $1 < \min(\gamma, \delta) \leq 3$, (ρ, u) *is a classical solution to the Cauchy problem (1.1)-(1.7) in* $[0, T] \times \mathbb{R}^3$.

Remark 1.1. The conditions (A_1)-(A_2) identify a class of admissible initial data that provides unique solvability to the Cauchy problem (1.1)-(1.7). Such initial data contain the following examples:

$$\rho_0(x) = \frac{\epsilon_1}{(1+|x|)^{2\sigma_1}}, \quad \epsilon_1 g^{2\sigma_2}(x), \quad \epsilon_1 \exp\{-x^2\}, \quad \frac{\epsilon_1 |x|}{(1+|x|)^{2\sigma_3}},$$

where $\epsilon_1 > 0$ is a sufficiently small constant, $0 \leq g(x) \in C_c^3(\mathbb{R}^3)$, and

$$\sigma_1 > \frac{3}{2}\max\left\{\frac{1}{\delta-1}, \frac{1}{\gamma-1}\right\}, \ \sigma_2 > 3\max\left\{\frac{1}{\delta-1}, \frac{1}{\gamma-1}\right\},$$
$$\sigma_3 > \frac{3}{2}\max\left\{\frac{1}{\delta-1}, \frac{1}{\gamma-1}\right\} + \frac{1}{2};$$
$$u_0 = Ax + \mathbf{b} + \epsilon_2 f(x),$$

where A is a 3×3 constant matrix whose eigenvalues are all greater than 2κ, $\epsilon_2 > 0$ is a sufficiently small constant, $f \in \Xi$ and $\mathbf{b} \in \mathbb{R}^3$ is a constant vector.

Remark 1.2. It is worth pointing out that, for any $\gamma > 1$, even $2\alpha + 3\beta \neq 0$, that is to say (P_1) fails, we can still deal with the corresponding well-posedness problem for a relatively wide class of (α, β, δ) under the following condition:

(P$_0$) $0 < M_1 = \frac{2\alpha+3\beta}{2\alpha+\beta} < \frac{3}{2} - \frac{1}{\delta}$ and
$$M_2 = -3\delta + 1 + \frac{1}{2}\left(\frac{(\delta-1)^2}{4(2\alpha+\beta)} + \frac{4\delta^2(2\alpha+\beta)}{(\delta-1)^2}M_1^2 + 2M_1\delta\right) < -1,$$

which can be seen in §5.

Next we indicate that the set of parameters (α, β, δ) satisfying (P_0) must be non-empty. First, for fixed $\delta > 1$, let

$$\alpha = a_1\eta, \quad \beta = a_2\eta,$$

where $\eta > 0$, $a_1 > 0$ and $2a_1 + 3a_2 \geq 0$ are all constants. Thus $M_1 = \frac{2a_1+3a_2}{2a_1+a_2}$. Then one can adjust the values of a_1 and a_2 to ensure that $M_1 < \frac{3}{2} - \frac{1}{\delta}$ holds.

Second, let $x = (\delta - 1)^2/(4(2a_1 + a_2)\eta)$, and consider the following function:

$$F(x) = x + \frac{1}{x}\delta^2 M_1^2 + 2M_1\delta - 6\delta + 4,$$

whose minimum value is

$$F(\delta M_1) = 4M_1\delta - 6\delta + 4 < 0$$

due to $M_1 < \frac{3}{2} - \frac{1}{\delta}$. Thus, one needs only to choose η to ensure that x belongs to a small neighbor of δM_1 and $F < 0$, which is equivalent to $M_2 < -1$.

Remark 1.3. Without generality, we can assume that $u_0(0) = 0$, which can be achieved by the following Galilean transformation:

$$t' = t, \quad x' = x + u_0(0)t, \quad \rho'(t', x') = \rho(t, x), \quad \widehat{u}'(t', x') = \widehat{u}(t, x) - u_0(0).$$

The above well-posedness theory is still available for some other models such as the ones shown in the following two theorems.

Theorem 1.5. *Let the viscous stress tensor* $\mathbb{T}$ *in (1.1) be given by*

$$\mathbb{T} = \rho^\delta(2\alpha\nabla u + \beta \mathrm{div} u\mathbb{I}_3).$$

Then the same well-posedness theory as in Theorem 1.4 holds in this case.

Theorem 1.6. *Assume* $\alpha > 0$, $\delta > 1$ *and* $\gamma > 1$. *If the viscous term* $\mathrm{div}\mathbb{T}$ *in* (1.1) *is replaced by*

$$\alpha\rho^\delta \triangle u,$$

then under the initial conditions shown in Theorem 1.4, the Cauchy problem (1.1)-(1.2) with (1.6)-(1.7) has a unique smooth solution (ρ, u) *in* $[0, \infty) \times \mathbb{R}^3$ *satisfying properties* $(A) - (C)$ *in Definition 1.2 for any* $0 < T < \infty$.

Moreover, if $1 < \min(\gamma, \delta) \leq 3$, *the solution* (ρ, u) *solves the Cauchy problem (1.1)-(1.2) with (1.6)-(1.7) in* $[0, \infty) \times \mathbb{R}^3$ *classically.*

The rest of the paper is organized as follows: In §2, we show some decay estimates for the classical solutions of the multi-dimensional Burgers equations and the global well-posedenss of an ordinary differential equation, which will be used later. In §3, we prove the finite time singularity formation in Theorems 1.2-1.3. §4-§8 are devoted to establishing the global well-posedness of regular solutions stated in Theorem 1.4. We start with the reformulation of the original problem (1.1)-(1.7) as (4.1) in terms of the new variables, and establish the local-in-time well-posedness of smooth solutions to (4.1) in the case that the initial density is compactly supported in §4. Note that one cannot apply the result in [10] directly here, the initial velocity here does not have a uniform upper bound in the whole space and stays in a homogeneous Sobolev space. It is also worth pointing out that recently, Li-Wang-Xin [16] prove that classical solutions with finite energy to the Cauchy problem of the compressible Navier-Stokes systems with constant viscosities do not exist in general inhomogeneous Sobolev space for any short time, which indicates in particular that the homogeneous Sobolev space is crucial as studying the well-posedness (even locally in time) for the Cauchy problem of the compressible Navier-Stokes systems in the presence of vacuum. This local smooth solution to (4.1) is shown to be global in time in §5-§6 by deriving a uniform (in time) a priori estimates independent of the size of the initial density's support through energy methods based on suitable choice of time weights. Here, we have employed some arguments due to Grassin [11] and Serre [29] to deal with the nonlinear convection term $u \cdot \nabla u$. Next, the assumption that the initial density has compact support is removed in §7. Finally, in §8, the proof of Theorem 1.4 is completed by making use of the results obtained in §7, and the proof of Theorem 1.6 is given in §9. Furthermore, we give an appendix to list some lemmas that are used in our proof, and outline some proofs of properties shown in §2 and some inequalities used frequently in this paper.

2. Preliminary

This section will be devoted to show some decay estimates for the classical solutions of the multi-dimensional Burgers equations and the global well-posedenss of an ordinary differential equation, which will be used frequently in our proof.

First, let $\widehat{u}$ be the solution to the problem (1.19) in d-dimensional space. Then, along the particle path $X(t; x_0)$ defined as

$$\frac{d}{dt}X(t;x_0)=\widehat{u}(t,X(t;x_0)),\quad x(0;x_0)=x_0, \tag{2.1}$$

$\widehat{u}$ is a constant in t: $\widehat{u}(t,X(t;x_0))=u_0(x_0)$ and

$$\nabla\widehat{u}(t,X(t;x_0))=\big(\mathbb{I}_d+t\nabla u_0(x_0)\big)^{-1}\nabla u_0(x_0).$$

Based on this observation, one can have the following decay estimates of $\widehat{u}$, which play important roles in establishing the global existence of the smooth solution to the problem considered in this paper, and their proof could be found in [11] or the appendix.

Proposition 2.1. *[11] Let $m>1+\frac{d}{2}$. Assume that*

$$\nabla u_0\in L^\infty(\mathbb{R}^d),\quad \nabla^2 u_0\in H^{m-1}(\mathbb{R}^d),$$

and there exists a constant $\kappa>0$ such that for all $x\in\mathbb{R}^d$,

$$Dist\big(Sp(\nabla u_0(x)),\mathbb{R}_-\big)\geq\kappa,$$

then there exists a unique global classical solution $\widehat{u}$ to the problem (1.19), which satisfies

(1) *$\nabla\widehat{u}(t,x)=\frac{1}{1+t}\mathbb{I}_d+\frac{1}{(1+t)^2}K(t,x)$ for all $x\in\mathbb{R}^d$ and $t\geq 0$, where the matrix $K(t,x)=K_{ij}:\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}^{d\times d}$ satisfies*

$$\|K\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)}\leq C_0\big(1+\kappa^{-d}\|\nabla u_0\|^{d-1}_{L^\infty(\mathbb{R}^d)}\big);$$

(2) *$\|\nabla^l\widehat{u}(\cdot,t)\|_{L^2(\mathbb{R}^d)}\leq C_{0,l}(1+t)^{\frac{d}{2}-(l+1)},\quad for\quad 2\leq l\leq m+1;$*
(3) *$\|\nabla^2\widehat{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}\leq C_0(1+t)^{-3}\|\nabla^2u_0\|_{L^\infty(\mathbb{R}^d)}$.*

Here, C_0 is a constant depending only on m, d, κ and u_0, and $C_{0,l}$ are all constants depending on C_0 and l. Moreover, if $u_0(0)=0$, then it holds that

$$|\widehat{u}(t,x)|\leq\|\nabla\widehat{u}(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}|x|\quad for\quad any\quad (t,x)\in[0,\infty)\times\mathbb{R}^d. \tag{2.2}$$

Secondly, we give a global well-posedness to the Cauchy problem of an ordinary differential equation:

Proposition 2.2. *For the constants b, C_i and D_i $(i=1,2)$ satisfying*

$$a>1,\quad D_1-(a-1)b<-1,\quad D_2<-1,\quad C_i\geq 0,\quad for\quad i=1,2, \tag{2.3}$$

there exists a constant Λ such that there exists a global smooth solution to the following Cauchy problem

$$\begin{cases} \dfrac{dZ}{dt}(t) + \dfrac{b}{1+t} Z(t) = C_1(1+t)^{D_1} Z^a(t) + C_2(1+t)^{D_2} Z, \\ Z(x,0) = Z_0 < \Lambda. \end{cases} \tag{2.4}$$

Moreover, one has

$$Z(t) = \frac{(1+t)^{-b} \exp\left(\frac{C_2}{D_2+1}\left((1+t)^{D_2+1} - 1\right)\right)}{\left(Z_0^{-(a-1)} - (a-1)C_1 \int_0^t (1+s)^M \exp\left(\frac{(a-1)C_2}{D_2+1}\left((1+s)^{D_2+1} - 1\right)\right) ds\right)^{\frac{1}{a-1}}}, \tag{2.5}$$

where $M = D_1 - (a-1)b$.

Its proof can be found in the appendix.

For simplicity, we use the following notations. For matrices $A_1, A_2, A_3, B = (b_{ij}) = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$, a vector $N = (n_1, n_2, n_3)^\top$, set $A = (A_1, A_2, A_3)$,

$$\begin{cases} \text{div} A = \displaystyle\sum_{j=1}^{3} \partial_j A_j, \quad W \cdot BW = \sum_{i,j=1}^{3} b_{ij} w_i w_j, \\ |B|_2^2 = B : B = \displaystyle\sum_{i,j=1}^{3} b_{ij}^2, \quad N \cdot B = n_1 \mathbf{b}_1 + n_2 \mathbf{b}_2 + n_3 \mathbf{b}_3. \end{cases} \tag{2.6}$$

3. Singularity formation

In order to prove Theorem 1.2, we define:

$$m(t) = \int \rho \quad \text{(total mass)}, \quad E_k(t) = \frac{1}{2} \int \rho |u|^2 \quad \text{(total kinetic energy)}.$$

Then the regular solution $(\rho, u)(t,x)$ in $[0,T] \times \mathbb{R}^3$ obtained in Theorem 1.1 has finite mass $m(t)$, momentum $\mathbb{P}(t)$ and kinetic energy $E_k(t)$. Indeed, it follows from $1 < \gamma \leq 2$, $\phi = \sqrt{\frac{4A\gamma}{(\gamma-1)^2}} \rho^{\frac{\gamma-1}{2}} \in C([0,T]; H^3(\mathbb{R}^3))$ and the Sobolev imbedding theorem that

$$m(t) = \int \rho \leq C \int \phi^{\frac{2}{\gamma-1}} \leq C|\phi|_\infty^{\frac{2}{\gamma-1}-2} |\phi|_2^2 \leq C\|\phi\|_3^{\frac{2}{\gamma-1}-2} |\phi|_2^2 < \infty$$

for some positive constant C depending only on A and γ, which, together with the regularity shown in Theorem 1.1, implies that

$$\begin{aligned} \mathbb{P}(t) &= \int \rho u \leq |\rho|_2 |u|_2 < \infty, \\ E_k(t) &= \int \frac{1}{2} \rho |u|^2 \leq \frac{1}{2} |\rho|_\infty |u|_2^2 < \infty. \end{aligned} \tag{3.1}$$

The case for $1 < \delta \leq 2$ can be verified similarly.

Next it is shown that the total mass and momentum are conserved.

Lemma 3.1. *Let* $1 < \min\{\gamma, \delta\} \leq 2$, *and* (ρ, u) *be the regular solution obtained in Theorem 1.1 with* $|\mathbb{P}(0)| > 0$, *then*

$$\mathbb{P}(t) = \mathbb{P}(0), \quad m(t) = m(0), \quad \textit{for} \quad t \in [0, T].$$

Proof. The momentum equations imply that

$$\mathbb{P}_t = -\int \operatorname{div}(\rho u \otimes u) - \int \nabla P + \int \operatorname{div}\mathbb{T} = 0, \tag{3.2}$$

where one has used the fact that

$$\rho u^{(i)} u^{(j)}, \quad \rho^\gamma \quad \text{and} \quad \rho^\delta \nabla u \in W^{1,1}(\mathbb{R}^3), \quad \text{for} \quad i,\ j = 1,\ 2,\ 3,$$

due to the regularities of the solutions.

Similarly, one can also get the conservation of the total mass. □

Now we are ready to prove Theorem 1.2. It follows from Lemma 3.1 that

$$|\mathbb{P}(0)| \leq \int \rho(t,x)|u|(t,x) \leq \sqrt{2} m^{\frac{1}{2}}(t) E_k^{\frac{1}{2}}(t) = \sqrt{2} m^{\frac{1}{2}}(0) E_k^{\frac{1}{2}}(t), \tag{3.3}$$

which yields that there exists a unique positive lower bound for $E_k(t)$,

$$E_k(t) \geq \frac{|\mathbb{P}(0)|^2}{2m(0)} = C_0 > 0 \quad \text{for} \quad t \in [0, T]. \tag{3.4}$$

Thus one gets that

$$C_0 \leq E_k(t) \leq \frac{1}{2} m(0) |u(t)|_\infty^2 \quad \text{for} \quad t \in [0, T].$$

Obviously, that there exists a positive constant C_u such that

$$|u(t)|_\infty \geq C_u \quad \text{for} \quad t \in [0, T].$$

Then Theorem 1.2 follows.

Finally, we prove Theorem 1.3. It follows from the definition of regular solutions given in Theorem 1.2 that in the vacuum domain, the velocity satisfies $u_t + u \cdot \nabla u = 0$, which, along with the formula

$$\nabla u(t, X(t; x_0)) = \left(\mathbb{I}_d + t \nabla u_0(x_0)\right)^{-1} \nabla u_0(x_0)$$

and (1.18), yields the desired conclusion.

4. Local-in-time well-posedness with compactly supported density

The rest of this paper is devoted to proving Theorem 1.4. In this section, we first reformulate the original Cauchy problem (1.1)-(1.7) as (4.1) below in terms of some variables, and then establish the local well-posedness of the smooth solutions to (4.1) in the case that the initial density has compact support.

Let $\widehat{u}$ be the unique classical solution to (1.19) obtained in Proposition 2.1. In terms of the new variables

$$(\varphi, W=(\phi, w=u-\widehat{u}))=\left(\rho^{\frac{\delta-1}{2}}, \sqrt{\frac{4A\gamma}{(\gamma-1)^2}}\rho^{\frac{\gamma-1}{2}}, u-\widehat{u}\right)$$

with $w=(w^{(1)}, w^{(2)}, w^{(3)})^\top$, the Cauchy problem (1.1)-(1.7) can be reformulated into

$$\begin{cases}\varphi_t+(w+\widehat{u})\cdot\nabla\varphi+\dfrac{\delta-1}{2}\varphi\mathrm{div}(w+\widehat{u})=0,\\ W_t+\displaystyle\sum_{j=1}^3 A_j^*(W,\widehat{u})\partial_j W+\varphi^2\mathbb{L}(w)=\mathbb{H}(\varphi)\cdot\mathbb{Q}(w+\widehat{u})+G(W,\varphi,\widehat{u}),\\ (\varphi, W)|_{t=0}=(\varphi_0, W_0)=(\varphi_0,\phi_0,0),\quad x\in\mathbb{R}^3,\\ (\varphi, W)=(\varphi,\phi,w)\to(0,0,0)\qquad \text{as}\qquad |x|\to\infty\quad \text{for}\quad t\geq 0,\end{cases}\tag{4.1}$$

where

$$A_j^*(W,\widehat{u})=\begin{pmatrix} w^{(j)}+\widehat{u}^{(j)} & \frac{\gamma-1}{2}\phi e_j\\ \frac{\gamma-1}{2}\phi e_j^\top & (w^{(j)}+\widehat{u}^{(j)})\mathbb{I}_3\end{pmatrix},\quad j=1,2,3,$$

$$G(W,\varphi,\widehat{u})=-B(\nabla\widehat{u},W)-D(\varphi^2,\nabla^2\widehat{u}),\tag{4.2}$$

$$B(\nabla\widehat{u},W)=\begin{pmatrix}\frac{\gamma-1}{2}\phi\mathrm{div}\widehat{u}\\ (w\cdot\nabla)\widehat{u}\end{pmatrix},\quad D(\varphi^2,\nabla^2\widehat{u})=\begin{pmatrix}0\\ \varphi^2L\widehat{u}\end{pmatrix},$$

and $\mathbb{L}$, $\mathbb{H}$ and $\mathbb{Q}$ are given in (1.15).

The main result in this section can be stated as follows:

Theorem 4.1. *Let (1.20) hold. If initial data (φ_0,ϕ_0,u_0) satisfy*

(H_1) $\varphi_0\geq 0$, $\phi_0\geq 0$ *and* $(\varphi_0,\phi_0)\in H^3$*;*
(H_2) $u_0\in\Xi$ *and there exists a constant* $\kappa>0$ *such that for all* $x\in\mathbb{R}^3$*:*

$$Dist\big(Sp(\nabla u_0(x)),\mathbb{R}_-\big)\geq\kappa;$$

(H$_3$) *φ_0 and ϕ_0 are both compactly supported:* $\text{supp}_x\varphi_0 = \text{supp}_x\phi_0 \subset B_R$*;*

where B_R is the ball centered at the origin with radius $R > 0$, then there exist a time $T_* = T_*(\alpha, \beta, A, \gamma, \delta, \varphi_0, W_0) > 0$ independent of R and a unique classical solution (φ, ϕ, w) in $[0, T_*] \times \mathbb{R}^3$ to (4.1) satisfying

$$(\varphi, \phi) \in C([0, T_*]; H^3),\ w \in C([0, T_*]; H^{s'}) \cap L^\infty([0, T_*]; H^3),\ \varphi\nabla^4 w \in L^2([0, T_*]; L^2),$$

for any constant $s' \in [2, 3)$. Moreover, for $t \in [0, T_*]$,

$$\varphi(t, z(t; \xi_0)) = \phi(t, z(t; \xi_0)) = 0, \quad \textit{and} \quad w(t, z(t; \xi_0)) = 0 \quad \textit{for} \quad \xi_0 \in \mathbb{R}^3/\text{supp}\varphi_0, \tag{4.3}$$

where the curve $z(t; \xi_0)$ is given via

$$\frac{d}{dt} z(t; \xi_0) = (w + \widehat{u})(t, z(t; \xi_0)), \quad z(0; \xi_0) = \xi_0. \tag{4.4}$$

The next three subsections will be devoted to prove Theorem 4.1.

4.1. Uniform a priori estimates for the linear problem

In order to show the local well-posedenss for (4.1), we will consider the following linearized approximate problem:

$$\begin{cases} \varphi_t + (v + \widehat{u}^N) \cdot \nabla\varphi + \dfrac{\delta - 1}{2} h\text{div}(v + \widehat{u}) = 0, \\ W_t + \displaystyle\sum_{j=1}^3 A_j^*(V, \widehat{u}^N)\partial_j W + (\varphi^2 + \eta^2)\mathbb{L}(w) = \mathbb{H}(\varphi) \cdot \mathbb{Q}(v + \widehat{u}) + G(W, \varphi, \widehat{u}), \\ (\varphi, W)|_{t=0} = (\varphi_0, W_0) = (\varphi_0, \phi_0, 0), \quad x \in \mathbb{R}^3, \\ (\varphi, W) = (\varphi, \phi, w) \to (0, 0, 0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0, \end{cases} \tag{4.5}$$

where

$$A_j^*(V, \widehat{u}^N) = \begin{pmatrix} v^{(j)} + \left(\widehat{u}^N\right)^{(j)} & \frac{\gamma-1}{2}\psi e_j \\ \frac{\gamma-1}{2}\psi e_j^\top & \left(v^{(j)} + \left(\widehat{u}^N\right)^{(j)}\right)\mathbb{I}_3 \end{pmatrix}, \quad j = 1, 2, 3, \tag{4.6}$$

with the vector $\widehat{u}^N = \widehat{u}F(|x|/N)$. $F(x) \in C_c^\infty(\mathbb{R}^3)$ is a truncation function satisfying

$$0 \leq F(x) \leq 1, \quad \text{and} \quad F(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2, \end{cases} \tag{4.7}$$

and $N \geq 1$ is a sufficiently large constant. $\eta > 0$ is a constant, and $V = (\psi, v)^\top$. (h, ψ) are both known functions and $v = (v^{(1)}, v^{(2)}, v^{(3)})^\top \in \mathbb{R}^3$ is a known vector satisfying:

$$\begin{aligned}&(h, \psi, v)(0, x) = (\varphi_0, \phi_0, 0), \quad h \in C([0, T]; H^3), \quad \psi \in C([0, T]; H^3),\\&v \in C([0, T]; H^{s'}) \cap L^\infty([0, T]; H^3), \quad h\nabla^4 v \in L^2([0, T]; L^2),\end{aligned} \tag{4.8}$$

for any constant $s' \in [2, 3)$. We also assume that for $t \in [0, T_*]$,

$$\psi(t, X(t; \xi_0)) = h(t, X(t; \xi_0)) = 0 \quad \text{and} \quad v(t, X(t; \xi_0)) = 0 \quad \text{for} \quad \xi_0 \in \mathbb{R}^3 \setminus \text{supp}\varphi_0. \tag{4.9}$$

Now the following global well-posedness in $[0, T] \times \mathbb{R}^3$ of a classical solution $(\varphi^{N\eta}, W^{N\eta}) = (\varphi^{N\eta}, \phi^{N\eta}, w^{N\eta})$ to (4.5) can be obtained by the standard theory [4,7] at least when $0 < \eta < \infty$ and $1 \leq N < \infty$.

Lemma 4.1. *Let $\eta > 0$ and (H_1)-(H_3) in Theorem 4.1 hold. Then there exists a unique classical solution $(\varphi^{N\eta}, W^{N\eta})$ in $[0, T] \times \mathbb{R}^3$ to (4.5) satisfying*

$$(\varphi^{N\eta}, \phi^{N\eta}) \in C([0, T]; H^3), \quad w^{N\eta} \in C([0, T]; H^3) \cap L^2([0, T]; H^4). \tag{4.10}$$

Next we give some a priori estimates for solutions $(\varphi^{N\eta}, \phi^{N\eta}, w^{N\eta})$ in H^3 in the following Lemmas 4.2-4.3, which are independent of (R, N, η). For simplicity, we denote $(\varphi^{N\eta}, \phi^{N\eta}, w^{N\eta})$ as (φ, ϕ, w), and $W^{N\eta}$ as W in the rest of Subsection 4.1. For this purpose, we fix a positive constant c_0 large enough such that

$$2 + \|\varphi_0\|_3 + \|\phi_0\|_3 + \|u_0\|_\Xi \leq c_0, \tag{4.11}$$

and

$$\sup_{0 \leq t \leq T^*} \left(\|h(t)\|_3^2 + \|\psi(t)\|_3^2 + \|v(t)\|_2^2\right) + \operatorname{ess}\sup_{0 \leq t \leq T^*} |v(t)|_{D^3}^2 + \int_0^{T^*} |h\nabla^4 v|_2^2 \mathrm{d}t \leq c_1^2, \tag{4.12}$$

for some constant

$$c_1 \geq c_0 > 1$$

and time $T^* \in (0, T)$, which will be determined later (see (4.23)) and depend only on c_0 and the fixed constants $(\alpha, \beta, \gamma, A, \delta, T)$.

In the rest of this section, $C \geq 1$ will denote a generic positive constant depending only on fixed constants $(\alpha, \beta, \gamma, A, \delta, T)$, but independent of (R, N, η), which may be different from line to line. It follows from (10.14)-(10.15) (the estimate on $|\nabla \widehat{u}|_\infty$), $(\mathbf{IH}(\mathbf{j})_1)$ and (10.19)(the estimate on $\|\nabla^2 \widehat{u}\|_2$) in the proof of Proposition 2.1 that

$$\|\widehat{u}\|_{T_0,\Xi} \leq Cc_0^4 \quad \text{for} \quad T_0 = \min\{T^*, c_1^{-1}\}. \tag{4.13}$$

Based on this fact, one can establish the following estimates for φ.

Lemma 4.2. *Let (φ, W) be the unique classical solution to (4.5) in $[0,T] \times \mathbb{R}^3$. Then*

$$1 + \|\varphi(t)\|_3^2 \leq Cc_0^2 \quad \textit{for} \quad 0 \leq t \leq T_1 = \min(T_0, c_1^{-5}).$$

Proof. Applying ∇^k $(0 \leq k \leq 3)$ to $(4.5)_1$, multiplying both sides by $\nabla^k\varphi$, and integrating over $\mathbb{R}^3$, one gets

$$\frac{1}{2}\frac{d}{dt}|\nabla^k\varphi|_2^2 \leq C|\text{div}(v+\widehat{u}^N)|_\infty|\nabla^k\varphi|_2^2 + C\Lambda_1^k|\nabla^k\varphi|_2 + C\Lambda_2^k|\nabla^k\varphi|_2, \tag{4.14}$$

where

$$\Lambda_1^k = |\nabla^k((v+\widehat{u}^N)\cdot\nabla\varphi) - (v+\widehat{u}^N)\cdot\nabla^{k+1}\varphi|_2, \quad \Lambda_2^k = |\nabla^k(h\text{div}(v+\widehat{u}))|_2.$$

It follows from Lemmas 10.1, 10.3, 10.6 and Hölder's inequality that

$$\begin{aligned} |\Lambda_1^k|_2 \leq & C\big(\|\widehat{u}\|_{T,\Xi} + \|v\|_3\big)\|\varphi(t)\|_3, \\ |\Lambda_2^k|_2 \leq & C\big(\|\widehat{u}\|_{T,\Xi} + \|v\|_3\big)\|h(t)\|_3 + C|h\nabla^4 v|_2, \end{aligned} \tag{4.15}$$

where one has used Proposition 2.1 and the fact that for $0 \leq t \leq T^*$,

$$|\widehat{u}(t,x)| \leq 2N|\nabla\widehat{u}|_\infty \quad \text{and} \quad |\nabla F(|x|/N)| \leq CN^{-1} \quad \text{for} \quad N \leq |x| \leq 2N. \tag{4.16}$$

Then, it follows from (4.13)-(4.15), Gronwall's inequality and (4.12) that

$$\|\varphi(t)\|_3 \leq \Big(\|\varphi_0\|_3 + c_1^5 t + c_1 t^{\frac{1}{2}}\Big)\exp(Cc_1^4 t) \leq Cc_0 \quad \text{for}$$
$$0 \leq t \leq T_1 = \min\{T_0, c_1^{-5}\}. \quad \square$$

Lemma 4.3. *Let (φ, W) be the unique classical solution to (4.5) in $[0,T] \times \mathbb{R}^3$. Then*

$$\|W(t)\|_3^2 + \int_0^t |\varphi\nabla^4 u|_2^2 \text{d}s \leq Cc_0^2 \quad \textit{for} \quad 0 \leq t \leq T_2 = \min\{T_1, c_1^{-10}\}.$$

Proof. Applying ∇^k to $(4.5)_2$, multiplying both sides by $\nabla^k W$, and integrating over $\mathbb{R}^3$ by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla^k W|^2+\alpha|\sqrt{\varphi^2+\eta^2}\nabla^{k+1}w|_2^2+(\alpha+\beta)|\sqrt{\varphi^2+\eta^2}\mathrm{div}\nabla^k w|_2^2\\
=&\int(\nabla^k W)^\top \mathrm{div}A_j^*(V,\widehat{u}^N)\nabla^k W-\int\nabla\varphi^2\cdot\nabla^k\mathbb{S}(w)\cdot\nabla^k w\\
&-\sum_{j=1}^3\int\Big(\nabla^k(A_j^*(V,\widehat{u}^N)\partial_j W)-A_j^*(V,\widehat{u}^N)\partial_j\nabla^k W\Big)\cdot\nabla^k W\\
&+\int\Big(-\nabla^k((\varphi^2+\eta^2)Lw)+(\varphi^2+\eta^2)L\nabla^k w+\nabla\varphi^2\cdot Q(\nabla^k v)\Big)\cdot\nabla^k w\\
&+\int\Big(\nabla^k(\nabla\varphi^2\cdot Q(v+\widehat{u}))-\nabla\varphi^2\cdot Q(\nabla^k(v+\widehat{u}))\Big)\cdot\nabla^k w\\
&+\int\Big(\nabla\varphi^2\cdot Q(\nabla^k\widehat{u})\Big)\cdot\nabla^k w+\int\nabla^k G(W,\varphi,\widehat{u})\cdot\nabla^k W\equiv\sum_{i=1}^8 I_i.
\end{aligned}\tag{4.17}$$

Now consider the terms on the right-hand side of (4.17). It follows from Lemmas 10.1 and 4.2, Proposition 2.1, Hölder's and Young's inequalities, (4.13) and (4.16) that

$$\begin{aligned}
I_1=&\int(\nabla^k W)^\top\mathrm{div}A_j^*(V,\widehat{u}^N)\nabla^k W\\
\le& C(|\nabla V|_\infty+|\nabla\widehat{u}|_\infty)|\nabla^k W|_2^2\le Cc_1^4|\nabla^k W|_2^2,\\
I_2=&-\int\Big(\nabla\varphi^2\cdot\mathbb{S}(\nabla^k w)\Big)\cdot\nabla^k w\\
\le& C|\nabla\varphi|_\infty|\varphi\nabla^{k+1}w|_2|\nabla^k w|_2\le\frac{\alpha}{20}|\varphi\nabla^{k+1}w|_2^2+Cc_0^2|\nabla^k w|_2^2,\\
I_3=&-\sum_{j=1}^3\int\Big(\nabla^k(A_j^*(V,\widehat{u}^N)\partial_j W)-A_j^*(V,\widehat{u}^N)\partial_j\nabla^k W\Big)\nabla^k W\\
\le& C(|\nabla V|_\infty+|\nabla\widehat{u}|_\infty)|\nabla W|_2^2\le Cc_1^4|\nabla W|_2^2\quad\text{when}\quad k=1,\\
I_3\le& C\big((|\nabla V|_\infty+|\nabla\widehat{u}|_\infty)\|\nabla W\|_1+(|\nabla^2 V|_3+|\nabla^2\widehat{u}|_3)|\nabla W|_6\big)|\nabla^2 W|_2\\
\le& Cc_1^4\|\nabla W\|_1^2\quad\text{when } k=2,\\
I_3\le& C\big((|\nabla V|_\infty+\|\nabla\widehat{u}\|_{W^{1,\infty}})\|\nabla W\|_2+(|\nabla^3 V|_2+|\nabla^3\widehat{u}|_2)|\nabla W|_\infty\big)|\nabla^3 W|_2\\
&+C(|\nabla^2 V|_3+|\nabla^2\widehat{u}|_3)|\nabla^2 W|_6|\nabla^3 W|_2\le Cc_1^4\|\nabla W\|_2^2\quad\text{when } k=3,\\
I_4=&-\int\Big(\nabla^k((\varphi^2+\eta^2)Lw)-(\varphi^2+\eta^2)L\nabla^k w\Big)\cdot\nabla^k w\\
\le& C|\varphi\nabla\varphi|_\infty|\nabla^2 w|_2|\nabla w|_2\le Cc_0^2\|\nabla w\|_1^2\quad\text{when}\quad k=1,\\
I_4\le& C\big(|\varphi\nabla\varphi|_\infty|\nabla^3 w|_2+\big(|\nabla\varphi\cdot\nabla\varphi|_3+|\varphi\nabla^2\varphi|_3\big)|\nabla^2 w|_6\big)|\nabla^2 w|_2\\
\le& Cc_0^2\|\nabla^2 w\|_1^2\quad\text{when}\quad k=2,
\end{aligned}\tag{4.18}$$

and

$$\begin{aligned}
I_4 \leq & C\big(|\nabla^3\varphi|_2|\varphi\nabla^3 w|_6|\nabla^2 w|_3+|\nabla\varphi|_\infty|\nabla^2\varphi|_3|\nabla^2 w|_6|\nabla^3 w|_2\big)\\
&+C\big(|\nabla^2\varphi|_3|\varphi\nabla^3 w|_6+|\nabla\varphi|^2_\infty|\nabla^3 w|_2+|\nabla\varphi|_\infty|\varphi\nabla^4 w|_2\big)|\nabla^3 w|_2\\
\leq & \frac{\alpha}{20}|\varphi\nabla^4 w|_2^2+Cc_0^2\|\nabla^2 w\|_1^2+Cc_0^2 \quad \text{when } k=3,\\
I_5 = & \int \Big(\nabla\varphi^2\cdot Q(\nabla^k v)\Big)\cdot\nabla^k w\\
\leq & C|\varphi|_\infty|\nabla\varphi|_\infty|\nabla^{k+1}v|_2|\nabla^k w|_2\leq Cc_0^2c_1|\nabla^k w|_2, \quad \text{when } k\leq 2,\\
I_5 = & \int \Big(\nabla\varphi^2\cdot Q(\nabla^k v)\Big)\cdot\nabla^k w \qquad (4.19)\\
\leq & C\big(|\nabla\varphi|^2_\infty|\nabla^3 w|_2+|\nabla^2\varphi|_3|\varphi\nabla^3 w|_6+|\nabla\varphi|_\infty|\varphi\nabla^4 w|_2\big)|\nabla^3 v|_2\\
\leq & Cc_0^2c_1|\nabla^3 w|_2+Cc_0^2c_1^2+\frac{\alpha}{20}|\varphi\nabla^4 w|_2^2 \quad \text{when } k=3,\\
I_6 = & \int \Big(\nabla^k(\nabla\varphi^2\cdot Q(v+\widehat{u}))-\nabla\varphi^2\cdot Q(\nabla^k(v+\widehat{u}))\Big)\cdot\nabla^k w\\
\leq & C\big(\|\widehat{u}\|_{T,\Xi}+\|v\|_3\big)\|\varphi\|_3^2\|w\|_2\leq Cc_1^6\|w\|_2 \quad \text{when} \quad k\leq 2,\\
I_6 \leq & C\big(\|\widehat{u}\|_{T,\Xi}+\|v\|_3\big)\|\varphi\|_3\big(\|\varphi\|_3|\nabla^3 w|_2+|\varphi\nabla^3 w|_6\big)+I_6^*\\
\leq & \frac{\alpha}{20}|\varphi\nabla^4 w|_2^2+C\big(c_1^6\|w\|_3+c_1^{10}\big)+I_6^* \quad \text{when} \quad k=3,\\
I_7 = & \int \Big(\nabla\varphi^2\cdot Q(\nabla^k\widehat{u})\Big)\cdot\nabla^k w\leq C|\nabla\varphi|_\infty|\varphi|_\infty|\nabla^{k+1}\widehat{u}|_2|\nabla^k w|_2\leq Cc_1^6|\nabla^k w|_2,\\
I_8 = & -\frac{\gamma-1}{2}\int\nabla^k(\phi\mathrm{div}\widehat{u})\nabla^k\phi-\int\nabla^k(w\cdot\nabla\widehat{u})\nabla^k w-\int\nabla^k(\varphi^2L\widehat{u})\nabla^k w\\
\leq & C\|\widehat{u}\|_{T,\Xi}\big(\|W\|_3^2+\|W\|_3\|\varphi\|_3^2\big)+I_8^*\delta_{3,k},
\end{aligned}$$

where integrations by parts have been used for the terms I_5-I_6 and I_8 when $k=3$. And the terms I_6^* and $I_8^*\delta_{3,k}$ can be estimated similarly by integration by parts as

$$\begin{aligned}
I_6^* = & \int \varphi\nabla^4\varphi\cdot Q(v+\widehat{u}))\cdot\nabla^3 w\leq\frac{\alpha}{20}|\varphi\nabla^4 w|_2^2+Cc_1^6\|w\|_3+Cc_1^{10},\\
I_8^*\delta_{3,k} = & \int \varphi^2L\nabla^3\widehat{u}\cdot\nabla^3 w\leq\frac{\alpha}{20}|\varphi\nabla^4 w|_2^2+Cc_0^6\|w\|_3+Cc_0^{10}.
\end{aligned}$$

Due to (4.17)-(4.19), one gets that

$$\frac{1}{2}\frac{d}{dt}\int|\nabla^k W|^2+\frac{1}{2}\alpha|\sqrt{\varphi^2+\eta^2}\nabla^4 w|_2^2\leq Cc_1^6\|W\|_3^2+Cc_1^{10}, \qquad (4.20)$$

which, along with Gronwall's inequality, implies that for $0\leq t\leq T_2=\min\{T_1,c_1^{-10}\}$,

$$\|W(t)\|_3^2+\int_0^t(\varphi^2+\eta^2)|\nabla^4 w|_2^2\mathrm{d}s\leq\big(\|W_0\|_3^2+Cc_1^{10}t\big)\exp(Cc_1^6t)\leq Cc_0^2. \quad \square \qquad (4.21)$$

Combining the estimates obtained in Lemmas 4.2-4.3 shows that

$$1+\|\varphi(t)\|_3^2+\|W(t)\|_3^2+\sum_{k=0}^{3}\int_0^t(\varphi^2+\eta^2)|\nabla^{k+1}w|_2^2\mathrm{d}s \leq Cc_0^2, \tag{4.22}$$

for $0\leq t\leq \min\{T^*,c_1^{-10}\}$. Therefore, defining the constant c_1 and time T^* by

$$c_1=C^{\frac{1}{2}}c_0, \quad T^*=\min\{T,c_1^{-10}\}, \tag{4.23}$$

we then deduce that for $0\leq t\leq T^*$,

$$\|\varphi(t)\|_3^2+\|\phi(t)\|_3^2+\|w(t)\|_3^2+\sum_{k=0}^{3}\int_0^t(\varphi^2+\eta^2)|\nabla^{k+1}w|_2^2\mathrm{d}s\leq c_1^2. \tag{4.24}$$

In other words, given fixed c_0 and T, there exist positive constant c_1 and time T^*, depending solely on c_0, T and the generic constant C such that if (4.12) holds for h and V, then (4.24) holds for the solution to (4.5) in $[0,T^*]\times\mathbb{R}^3$. Here, it should be noted that the definitions of (c_1,T^*) are all independent of the parameters (R,N,η).

4.2. Passing to the limits as $\eta\to 0$ and $N\to\infty$

Due to the a priori estimate (4.24), one can solve the following problem:

$$\begin{cases}\varphi_t+(v+\widehat{u})\cdot\nabla\varphi+\dfrac{\delta-1}{2}h\mathrm{div}(v+\widehat{u})=0,\\ W_t+\displaystyle\sum_{j=1}^{3}A_j^*(V,\widehat{u})\partial_j W+\varphi^2\mathbb{L}(w)=\mathbb{H}(\varphi)\cdot\mathbb{Q}(v+\widehat{u})+G(W,\varphi,\widehat{u}),\\ (\varphi,W)|_{t=0}=(\varphi_0,W_0)=(\varphi_0,\phi_0,0),\quad x\in\mathbb{R}^3,\\ (\varphi,W)=(\varphi,\phi,w)\to(0,0,0)\qquad \text{as}\qquad |x|\to\infty\quad\text{for}\quad t\geq 0.\end{cases} \tag{4.25}$$

Lemma 4.4. *Assume that the initial data (φ_0,ϕ_0,u_0) satisfy conditions (H_1)-(H_3) shown in Theorem 4.1. Then there exist a time $T^*=T^*(\alpha,\beta,A,\gamma,\delta,\varphi_0,W_0)>0$ independent of R and a unique classical solution (φ,W) in $[0,T^*]\times\mathbb{R}^3$ to (4.25) such that*

$$(\varphi,\phi)\in C([0,T^*];H^3),\quad w\in C([0,T^*];H^{s'})\cap L^\infty([0,T^*];H^3),$$
$$\varphi\nabla^4 w\in L^2([0,T^*];L^2),$$

for any constant $s'\in[2,3)$. Moreover, (φ,W) satisfies (4.24), and for $t\in[0,T^*]$,

$$\varphi(t,y(t;\xi_0))=\phi(t,y(t;\xi_0))=0\quad \textit{and}\quad w(t,y(t;\xi_0))=0\quad \textit{for}\quad \xi_0\in\mathbb{R}^3\setminus\mathrm{supp}\varphi_0. \tag{4.26}$$

Proof. Step 1: Passing to the limit as $\eta \to 0$. Let $N \geq 1$ be a fixed constant. Due to Lemma 4.1, for every $\eta > 0$, there exists a unique classical solution $(\varphi^{N\eta}, W^{N\eta})$ to the linear Cauchy problem (4.5) satisfying (4.24) in $[0, T^*] \times \mathbb{R}^3$, where these estimates and the life span $T^* > 0$ are all independent of (N, R, η).

Due to the estimate (4.24) and the equations in (4.5), it holds that

$$\|\varphi_t^{N\eta}\|_2 + \|\phi_t^{N\eta}\|_2 + \|w_t^{N\eta}\|_1 + \int_0^t \|w_t^{N\eta}\|_2^2 \mathrm{d}s \leq C_0(N), \quad \text{for} \quad 0 \leq t \leq T^*, \tag{4.27}$$

where the constant $C_0(N)$ depends only on the generic constant C, (φ_0, ϕ_0, u_0) and N, and is independent of (R, η).

By virtue of the uniform estimate (4.24) and (4.27) independent of (R, η), there exists a subsequence of solutions (still denoted by) $(\varphi^{N\eta}, W^{N\eta})$, which converges to a limit $(\varphi^N, W^N) = (\varphi^N, \phi^N, w^N)$ in weak or weak* sense as $\eta \to 0$:

$$\begin{aligned}
(\varphi^{N\eta}, W^{N\eta}) \rightharpoonup (\varphi^N, W^N) \quad & \text{weakly* in} \quad L^\infty([0, T^*]; H^3(\mathbb{R}^3)),\\
(\varphi_t^{N\eta}, \phi_t^{N\eta}) \rightharpoonup (\varphi_t^N, \phi_t^N) \quad & \text{weakly* in} \quad L^\infty([0, T^*]; H^2(\mathbb{R}^3)),\\
w_t^{N\eta} \rightharpoonup w_t^N \quad & \text{weakly* in} \quad L^\infty([0, T^*]; H^1(\mathbb{R}^3)),\\
w_t^{N\eta} \rightharpoonup w_t^N \quad & \text{weakly in} \quad L^2([0, T^*]; H^2(\mathbb{R}^3)).
\end{aligned} \tag{4.28}$$

The lower semi-continuity of weak convergence implies that (φ^N, W^N) also satisfies the corresponding estimates (4.24) and (4.27) except that of $\varphi^N \nabla^4 w^N$.

Again by the uniform estimate (4.24) and (4.27) independent of η, and the compactness in Lemma 10.8 (see [30]), it holds that for any $R_0 > 0$, there exists a subsequence of solutions (still denoted by) $(\varphi^{N\eta}, W^{N\eta})$, which converges to the same limit (φ^N, W^N) as above in the following strong sense:

$$(\varphi^{N\eta}, W^{N\eta}) \to (\varphi^N, W^N) \quad \text{in} \ \ C([0, T^*]; H^2(B_{R_0})), \quad \text{as} \quad \eta \to 0. \tag{4.29}$$

Let $\beta_i = 1, 2, 3$ for $i = 1, 2, 3, 4$. Because $\|\varphi^{N\eta} \nabla^4 w^{N\eta}\|_{L^2 L^2_{T^*}}$ is uniformly bounded with respect to N and η, there exists a vector $g = (g_1, g_2, g_3) \in L^2([0, T^*]; L^2(\mathbb{R}^3))$ such that,

$$\int_{\mathbb{R}^3} \int_0^{T^*} \varphi^{N\eta} \partial_{\beta_1\beta_2\beta_3\beta_4} w^{N\eta} f \mathrm{d}x\mathrm{d}t \to \int_{\mathbb{R}^3} \int_0^{T^*} g f \mathrm{d}x\mathrm{d}t \quad \text{as} \quad \eta \to 0 \tag{4.30}$$

for any $f \in L^2([0, T^*]; L^2(\mathbb{R}^3))$.

For any $f^* \in C_c^\infty([0, T^*] \times \mathbb{R}^3)$, it follows from integration by parts, (4.28) and (4.29) that

$$
\begin{aligned}
&\lim_{\eta\to 0}\int_{\mathbb{R}^3}\int_0^{T^*}\varphi^{N\eta}\partial_{\beta_1\beta_2\beta_3\beta_4}w^{N\eta}f^*\mathrm{d}x\mathrm{d}t\\
=&\lim_{\eta\to 0}\Big(-\int_{\mathbb{R}^3}\int_0^{T^*}\Big(\partial_{\beta_1}\varphi^{N\eta}\partial_{\beta_2\beta_3\beta_4}w^{N\eta}f^*+\varphi^{N\eta}\partial_{\beta_2\beta_3\beta_4}w^{N\eta}\partial_{\beta_1}f^*\Big)\mathrm{d}x\mathrm{d}t\Big)\\
=&-\int_{\mathbb{R}^3}\int_0^{T^*}\Big(\partial_{\beta_1}\varphi^{N}\partial_{\beta_2\beta_3\beta_4}w^{N}f^*+\varphi^{N}\partial_{\beta_2\beta_3\beta_4}w^{N}\partial_{\beta_1}f^*\Big)\mathrm{d}x\mathrm{d}t\\
=&-\int_{\mathbb{R}^3}\int_0^{T^*}\partial_{\beta_2\beta_3\beta_4}w^{N}\partial_{\beta_1}(\varphi^N f^*)\mathrm{d}x\mathrm{d}t=\int_{\mathbb{R}^3}\int_0^{T^*}\varphi^{N}\partial_{\beta_1\beta_2\beta_3\beta_4}w^{N}f^*\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{4.31}
$$

Because $C_c^\infty([0,T^*]\times\mathbb{R}^3)$ is dense in $L^2([0,T^*];L^2(\mathbb{R}^3))$, one can get

$$
\varphi^{N\eta}\nabla^4 w^{N\eta}\rightharpoonup\varphi^N\nabla^4 w^N\quad\text{weakly in}\ \ L^2([0,T^*]\times\mathbb{R}^3),\quad\text{as}\quad\eta\to 0. \tag{4.32}
$$

Thus, (φ^N,W^N) satisfies (4.24).

Now, it is easy to show that (φ^N,W^N) is a weak solution in the sense of distributions to the following problem:

$$
\begin{cases}
\varphi^N_t+(v+\widehat{u}^N)\cdot\nabla\varphi^N+\dfrac{\delta-1}{2}h\mathrm{div}(v+\widehat{u})=0,\\
W^N_t+\displaystyle\sum_{j=1}^3 A_j^*(V,\widehat{u}^N)\partial_j W^N+(\varphi^N)^2\mathbb{L}(w^N)=\mathbb{H}(\varphi^N)\cdot\mathbb{Q}(v+\widehat{u})+G(W^N,\varphi^N,\widehat{u}),\\
(\varphi^N,W^N)|_{t=0}=(\varphi_0,W_0)=(\varphi_0,\phi_0,0),\quad x\in\mathbb{R}^3,\\
(\varphi^N,W^N)=(\varphi^N,\phi^N,w^N)\to(0,0,0)\qquad\text{as}\qquad|x|\to\infty\quad\text{for}\quad t\geq 0,
\end{cases}
\tag{4.33}
$$

and has the following regularities

$$
\begin{aligned}
&(\varphi^N,\phi^N,w^N)\in L^\infty([0,T^*];H^3),\quad(\varphi^N_t,\phi^N_t)\in L^\infty([0,T^*];H^2),\\
&\varphi^N\nabla^4 w^N\in L^2([0,T^*];L^2),\qquad w^N_t\in L^\infty([0,T^*];H^1)\cap L^2([0,T^*];H^2).
\end{aligned}
\tag{4.34}
$$

Step 2: Passing to the limit as $N\to\infty$. One can prove the existence, uniqueness and time continuity of the classical solutions to (4.25) as follows.

Step 2.1: Existence. According to Step 1, for every $N\geq 1$, there exists a solution (φ^N,W^N) to (4.33) with the a priori estimate (4.24) independent of (R,N).

Due to (4.24) and (4.33), for any $R_0 > 0$, it holds that

$$\|\varphi^N_t\|_{H^2(B_{R_0})} + \|\phi^N_t\|_{H^2(B_{R_0})} + \|w^N_t\|_{H^1(B_{R_0})} + \int_0^t \|\nabla^2 w^N_t\|^2_{L^2(B_{R_0})}\mathrm{d}s \leq C_0(R_0), \tag{4.35}$$

where the constant $C_0(R_0)$ is independent of (R, N).

It follows from (4.24) and (4.35) that there exists a subsequence of solutions (still denoted by) (φ^N, W^N), which converges to a limit (φ, W) in weak or weak* sense as $\eta \to 0$:

$$\begin{aligned} (\varphi^N, W^N) \rightharpoonup (\varphi, W) \quad & \text{weakly* in } \ L^\infty([0, T^*]; H^3(\mathbb{R}^3)),\\ (\varphi^N_t, \phi^N_t) \rightharpoonup (\varphi_t, \phi_t) \quad & \text{weakly* in } \ L^\infty([0, T^*]; H^2(B_{R_0})),\\ w^N_t \rightharpoonup w_t \quad & \text{weakly* in } \ L^\infty([0, T^*]; H^1(B_{R_0})),\\ w^N_t \rightharpoonup w_t \quad & \text{weakly in } \ L^2([0, T^*]; H^2(B_{R_0})), \end{aligned} \tag{4.36}$$

for any $R_0 > 0$. The lower semi-continuity of weak convergence implies that (φ, W) also satisfies the estimate (4.24) except that for $\varphi\nabla^4 w$.

Again by virtue of the uniform estimates (4.24) and (4.35) independent of N and the compactness in Lemma 10.8 (see [30]), for any $R_0 > 0$, there exists a subsequence of solutions (still denoted by) (φ^N, W^N), which converges to the same limit $(\varphi, W) = (\varphi, \phi, w)$ in the following strong sense:

$$(\varphi^N, W^N) \to (\varphi, W) \quad \text{in } \ C([0, T^*]; H^2(B_{R_0})), \quad \text{as} \quad N \to \infty. \tag{4.37}$$

Then, (4.36)-(4.37) and a similar proof as for (4.32) show that

$$\varphi^N \nabla^4 w^N \rightharpoonup \varphi\nabla^4 w \quad \text{weakly in } \ L^2([0, T^*] \times \mathbb{R}^3), \quad \text{as} \quad N \to \infty. \tag{4.38}$$

Thus, (φ, W) satisfies (4.24).

It is easy to show that (φ, W) is a weak solution in the sense of distributions to (4.25) with the properties that

$$\begin{aligned} &(\varphi, \phi, w) \in L^\infty([0, T^*]; H^3), \quad (\varphi_t, \phi_t) \in L^\infty([0, T^*]; H^2_{loc}),\\ &\varphi\nabla^4 w \in L^2([0, T^*]; L^2), \quad w_t \in L^\infty([0, T^*]; H^1_{loc}) \cap L^2([0, T^*]; H^2_{loc}). \end{aligned} \tag{4.39}$$

Step 2.2: Time-continuity. (4.25) and (4.9) imply that

$$\frac{d}{dt}\varphi(t, X(t; \xi_0)) = 0, \quad \frac{d}{dt}\phi(t, X(t; \xi_0)) = 0, \quad \frac{d}{dt}w(t, X(t; \xi_0)) = -w \cdot \nabla\widehat{u},$$

for $t \in [0, T^*]$ and $\xi_0 \in \mathbb{R}^3 \setminus \mathrm{supp}\varphi_0$, which yields that

$$\varphi(t, X(t;\xi_0)) = \phi(t, X(t;\xi_0)) = 0, \quad w(t, X(t;\xi_0)) = 0,$$

for $t \in [0, T^*]$ and $\xi_0 \in \mathbb{R}^3 \setminus \text{supp}\varphi_0$. Thus for any positive time $t \in [0, T^*]$, the solution (φ, ϕ, w) is compactly supported with respect to the space variable x.

Let $\xi_0 \in \partial\text{supp}\varphi_0$ and $|\xi_0| \le R$. It follows from Proposition 2.1 that

$$|X(t;\xi_0)| \le R + \|\nabla \widehat{u}\|_{L^\infty([0,T^*]\times\mathbb{R}^3)} \int_0^t |X(\tau;\xi_0)| \mathrm{d}\tau. \tag{4.40}$$

Then the Gronwall's inequality implies that for $0 \le t \le T^*$

$$|X(t;\xi_0)| \le R \exp\left(\|\nabla \widehat{u}\|_{L^\infty([0,T^*]\times\mathbb{R}^3)} T^*\right) \le C_0(R, T^*), \tag{4.41}$$

where $C_0(R, T^*)$ depends on the generic constant C, T^*, (φ_0, ϕ_0, u_0) and R. Then for any $0 < R < \infty$, it follows from (4.25) that

$$(\varphi_t, \phi_t) \in L^\infty([0, T^*]; H^2), \qquad w_t \in L^\infty([0, T^*]; H^1) \cap L^2([0, T^*]; H^2). \tag{4.42}$$

(4.39) and (4.42) imply that for any $s' \in [0, 3)$,

$$(\varphi, \phi) \in C([0, T^*]; H^{s'} \cap \text{weak-}H^3), \ w \in C([0, T^*]; H^{s'} \cap \text{weak-}H^3). \tag{4.43}$$

Using the same arguments as in the proof of Lemma 4.2-4.3, one has

$$\limsup_{t\to 0} \|\varphi(t)\|_3 \le \|\varphi_0\|_3, \quad \limsup_{t\to 0} \|\phi(t)\|_3 \le \|\phi_0\|_3, \tag{4.44}$$

which, together with Lemma 10.9 and (4.43), implies that (φ, ϕ) is right continuous at $t = 0$ in H^3 space. The time reversibility of equations in (4.25) for (ϕ, φ) yields that

$$(\varphi, \phi) \in C([0, T^*]; H^3). \tag{4.45}$$

Step 2.3: Uniqueness. Let $(\varphi_i, W_i) = (\varphi_i, \phi_i, w_i)$, $i = 1, 2$, be two solutions obtained in Step 2.1, and

$$\overline{\varphi} = \varphi_1 - \varphi_2, \ \overline{\phi} = \phi_1 - \phi_2,$$
$$\overline{w} = w_1 - w_2, \ \overline{W} = W_1 - W_2.$$

Then $\varphi_1 = \varphi_2$ follows from $\overline{\varphi}_t + v \cdot \nabla\overline{\varphi} = 0$, and $W_1 = W_2$ follows from

$$\overline{W}_t + \sum_{j=1}^{3} A_j^*(V, \widehat{u})\partial_j \overline{W} + \varphi_1^2 \mathbb{L}(\overline{w}) = G(\overline{W}, \varphi_1, \widehat{u}). \quad \square \tag{4.46}$$

4.3. Proof of Theorem 4.1

The proof is based on the classical iteration scheme and the existence results for the linearized problem obtained in §4.2. As in §4.2, we define constants c_0 and c_1, and assume that

$$1+\|\varphi_0\|_3++\|\phi_0\|_3+\|u_0\|_\Xi \leq c_0.$$

Let $(\varphi^0, W^0=(\phi^0,w^0))$ with the regularities

$$\begin{aligned}&(\varphi^0,\phi^0)\in C([0,T^*];H^3),\ \ w^0\in C([0,T^*];H^{s'})\cap L^\infty([0,T^*];H^3),\\&\varphi^0\nabla^4 w^0\in L^2([0,T^*];L^2)\end{aligned}$$

for any $s'\in[2,3)$ be the solution to the following problem:

$$\begin{cases}X_t+\widehat{u}\cdot\nabla X=0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^3,\\ Y_t+\widehat{u}\cdot\nabla Y=0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^3,\\ \Phi_t-X^2\triangle\Phi=0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^3,\\ (X,Y,\Phi)|_{t=0}=(\varphi_0,\phi_0,0) \quad \text{in} \quad \mathbb{R}^3,\\ (X,Y,\Phi)\to(0,0,0) \quad \text{as} \quad |x|\to\infty, \quad t>0.\end{cases} \tag{4.47}$$

Choose a time $T^{**}\in(0,T^*]$ small enough such that for $0\leq t\leq T^{**}$,

$$||\varphi^0(t)||_3^2+||\phi^0(t)||_3^2+||w^0(t)||_3^2+\int_0^t|\varphi^0\nabla^4 w^0|_2^2\mathrm{d}s\leq c_1^2. \tag{4.48}$$

Now the existence, uniqueness and time continuity can be proved as follows.

Step 1: Existence. Let $(h,\psi,v)=(\varphi^0,\phi^0,w^0)$ and (φ^1,W^1) be a classical solution to (4.25). Then we construct approximate solutions $(\varphi^{k+1},W^{k+1})=(\varphi^{k+1},\phi^{k+1},w^{k+1})$ inductively as follows: Assume that (φ^k,W^k) has been defined for $k\geq 1$, and let (φ^{k+1},W^{k+1}) be the unique solution to (4.25) with (h,V) replaced by (φ^k,W^k) as:

$$\begin{cases}\varphi_t^{k+1}+(w^k+\widehat{u})\cdot\nabla\varphi^{k+1}+\dfrac{\delta-1}{2}\varphi^k\mathrm{div}(w^k+\widehat{u})=0,\\ W_t^{k+1}+\displaystyle\sum_{j=1}^3 A_j^*(W^k,\widehat{u})\partial_j W^{k+1}+(\varphi^{k+1})^2\mathbb{L}(w^{k+1})\\ =\mathbb{H}(\varphi^{k+1})\cdot\mathbb{Q}(w^k+\widehat{u})+G(W^{k+1},\varphi^{k+1},\widehat{u}),\\ (\varphi^{k+1},W^{k+1})|_{t=0}=(\varphi_0,W_0), \quad x\in\mathbb{R}^3,\\ (\varphi^{k+1},W^{k+1})\to(0,0) \quad \text{as} \quad |x|\to\infty, \quad t>0.\end{cases} \tag{4.49}$$

Here the sequence (φ^k, W^k) satisfies the uniform a priori estimate (4.24) for $0 \le t \le T^{**}$, and is also compactly supported as long as it exists.

Now we prove the convergence of the whole sequence (φ^k, W^k) to a limit (φ, W) in some strong sense. Set

$$\begin{aligned} \overline{\varphi}^{k+1} &= \varphi^{k+1} - \varphi^k, \ \overline{W}^{k+1} = (\overline{\phi}^{k+1}, \overline{w}^{k+1})^\top, \\ \overline{\phi}^{k+1} &= \phi^{k+1} - \phi^k, \ \overline{w}^{k+1} = w^{k+1} - w^k, \end{aligned}$$

then it follows from (4.49) that

$$\begin{cases} \overline{\varphi}_t^{k+1} + (w^k + \widehat{u}) \cdot \nabla \overline{\varphi}^{k+1} + \overline{w}^k \cdot \nabla \varphi^k \\ = -\dfrac{\delta - 1}{2} (\overline{\varphi}^k \mathrm{div}(w^{k-1} + \widehat{u}) + \varphi^k \mathrm{div} \overline{w}^k), \\ \overline{W}_t^{k+1} + \displaystyle\sum_{j=1}^3 A_j^*(W^k, \widehat{u}) \partial_j \overline{W}^{k+1} + (\varphi^{k+1})^2 \mathbb{L}(\overline{w}^{k+1}) \\ = -\displaystyle\sum_{j=1}^3 A_j(\overline{W}^k) \partial_j W^k - \overline{\varphi}^{k+1} (\varphi^{k+1} + \varphi^k) \mathbb{L}(w^k) \\ \quad + \big(\mathbb{H}(\varphi^{k+1}) - \mathbb{H}(\varphi^k)\big) \cdot \mathbb{Q}(w^k + \widehat{u}) + \mathbb{H}(\varphi^k) \cdot \mathbb{Q}(\overline{w}^k) \\ \quad - B(\nabla \widehat{u}, \overline{W}^{k+1}) - D(\overline{\varphi}^{k+1} (\varphi^{k+1} + \varphi^k), \nabla^2 \widehat{u}). \end{cases} \tag{4.50}$$

First, multiplying $(4.50)_1$ by $2\overline{\varphi}^{k+1}$ and integrating over $\mathbb{R}^3$ yield that

$$\begin{aligned} \frac{d}{dt} |\overline{\varphi}^{k+1}|_2^2 \le & C\big(|\nabla w^k|_\infty + |\nabla \widehat{u}|_\infty\big) |\overline{\varphi}^{k+1}|_2^2 + C |\overline{\varphi}^{k+1}|_2 |\overline{w}^k|_2 |\nabla \varphi^k|_\infty \\ & + C |\overline{\varphi}^{k+1}|_2 \big(\big(|\nabla w^{k-1}|_\infty + |\nabla \widehat{u}|_\infty\big) |\overline{\varphi}^k|_2 + |\varphi^k \mathrm{div} \overline{w}^k|_2\big) \\ \le & C \nu^{-1} |\overline{\varphi}^{k+1}(t)|_2^2 + \nu \big(|\overline{w}^k|_2^2 + |\overline{\varphi}^k|_2^2 + |\varphi^k \mathrm{div} \overline{w}^k|_2^2\big), \end{aligned} \tag{4.51}$$

where $0 < \nu \le \frac{1}{10}$ is a constant to be determined.

Second, multiplying $(4.50)_2$ by $2\overline{W}^{k+1}$ and integrating over $\mathbb{R}^3$, one has

$$\begin{aligned} & \frac{d}{dt} \int |\overline{W}^{k+1}|^2 + 2\alpha |\varphi^{k+1} \nabla \overline{w}^{k+1}|_2^2 + 2(\alpha + \beta) |\varphi^{k+1} \mathrm{div} \overline{w}^{k+1}|_2^2 \\ = & \int (\overline{W}^{k+1})^\top \mathrm{div} A^*(W^k, \widehat{u}) \overline{W}^{k+1} - 2 \int \sum_{j=1}^3 (\overline{W}^{k+1})^\top A_j(\overline{W}^k) \partial_j W^k \\ & - 2 \frac{\delta - 1}{\delta} \int \nabla (\varphi^{k+1})^2 \cdot Q(\overline{w}^{k+1}) \cdot \overline{w}^{k+1} - 2 \int \overline{\varphi}^{k+1} (\varphi^{k+1} + \varphi^k) L w^k \cdot \overline{w}^{k+1} \\ & + 2 \int \nabla (\overline{\varphi}^{k+1} (\varphi^{k+1} + \varphi^k)) \cdot Q(w^k + \widehat{u}) \cdot \overline{w}^{k+1} + 2 \int \nabla (\varphi^k)^2 \cdot Q(\overline{w}^k) \cdot \overline{w}^{k+1} \end{aligned} \tag{4.52}$$

$$-2\int B(\nabla\widehat{u},\overline{W}^{k+1})\cdot\overline{W}^{k+1}-2\int\overline{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k)L(\widehat{u})\cdot\overline{w}^{k+1}:=\sum_{i=1}^{8}J_i.$$

The terms J_1-J_8 above can be estimated as follows

$$\begin{aligned}
J_1=&\int(\overline{W}^{k+1})^\top\mathrm{div}A^*(W^k,\widehat{u})\overline{W}^{k+1}\leq C|\nabla(w^k+\widehat{u})|_\infty|\overline{W}^{k+1}|_2^2\leq C|\overline{W}^{k+1}|_2^2,\\
J_2=&-2\int\sum_{j=1}^{3}A_j(\overline{W}^k)\partial_jW^k\cdot\overline{W}^{k+1}\\
\leq&C|\nabla W^k|_\infty|\overline{W}^k|_2|\overline{W}^{k+1}|_2\leq C\nu^{-1}|\overline{W}^{k+1}|_2^2+\nu|\overline{W}^k|_2^2,\\
J_3=&-2\frac{\delta-1}{\delta}\int\nabla(\varphi^{k+1})^2\cdot Q(\overline{w}^{k+1})\cdot\overline{w}^{k+1}\\
\leq&C|\nabla\varphi^{k+1}|_\infty|\varphi^{k+1}\nabla\overline{w}^{k+1}|_2|\overline{w}^{k+1}|_2\leq C|\overline{W}^{k+1}|_2^2+\frac{\alpha}{20}|\varphi^{k+1}\nabla\overline{w}^{k+1}|_2^2,\\
J_4=&-2\int\overline{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k)Lw^k\cdot\overline{w}^{k+1}\\
\leq&C|\overline{\varphi}^{k+1}|_2|\varphi^{k+1}\overline{w}^{k+1}|_6|Lw^k|_3+C|\overline{\varphi}^{k+1}|_2|\varphi^kLw^k|_\infty|\overline{w}^{k+1}|_2\\
\leq&C|\overline{\varphi}^{k+1}|_2^2+\frac{\alpha}{20}|\varphi^{k+1}\nabla\overline{w}^{k+1}|_2^2+C(1+|\varphi^k\nabla^4w^k|_2^2)|\overline{W}^{k+1}|_2^2,\\
J_5=&2\int\nabla(\overline{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k))\cdot Q(w^k+\widehat{u})\cdot\overline{w}^{k+1}\\
\leq&C(|\nabla^2w^k|_6+|\nabla^2\widehat{u}|_6)|\varphi^{k+1}\overline{w}^{k+1}|_3|\overline{\varphi}^{k+1}|_2\\
&+C(|\varphi^k\nabla^2w^k|_\infty+|\varphi^k\nabla^2\widehat{u}|_\infty)|\overline{w}^{k+1}|_2|\overline{\varphi}^{k+1}|_2\\
&+C(|\nabla w^k|_\infty+|\nabla\widehat{u}|_\infty)|\overline{\varphi}^{k+1}|_2|\varphi^{k+1}\nabla\overline{w}^{k+1}|_2+J_5^*,\\
J_6=&4\int\varphi^k\nabla\varphi^k\cdot Q(\overline{w}^k)\cdot\overline{w}^{k+1}\\
\leq&C|\nabla\varphi^k|_\infty|\varphi^k\nabla\overline{w}^k|_2|\overline{w}^{k+1}|_2\leq\nu|\varphi^k\nabla\overline{w}^k|_2^2+C\nu^{-1}|\overline{w}^{k+1}|_2^2,\\
J_7=&-2\int B(\nabla\widehat{u},\overline{W}^{k+1})\cdot\overline{W}^{k+1}\leq C|\nabla\widehat{u}|_\infty|\overline{W}^{k+1}|_2^2,\\
J_8=&-2\int\overline{\varphi}^{k+1}(\varphi^{k+1}+\varphi^k)L\widehat{u}\cdot\overline{w}^{k+1}\leq C|\overline{\varphi}^{k+1}|_2|\varphi^{k+1}+\varphi^k|_\infty|L\widehat{u}|_\infty|\overline{w}^{k+1}|_2,
\end{aligned}\tag{4.53, 4.54}$$

where one has used the inequality (10.2) for the term $|\varphi^kLw^k|_\infty$, and

$$\begin{aligned}
J_5^*=&-2\int\sum_{i,j}\overline{\varphi}^{k+1}(\varphi^k-\varphi^{k+1}+\varphi^{k+1})Q_{ij}(w^k+\widehat{u})\partial_i(\overline{w}^{k+1})^{(j)}\\
\leq&C(|\nabla w^k|_\infty+|\nabla\widehat{u}|_\infty)\Big(|\nabla\overline{w}^{k+1}|_\infty|\overline{\varphi}^{k+1}|_2^2+|\varphi^{k+1}\nabla\overline{w}^{k+1}|_2|\overline{\varphi}^{k+1}|_2\Big).
\end{aligned}\tag{4.55}$$

Then it follows from (4.52)-(4.55) and Young's inequality that

$$\begin{aligned}&\frac{d}{dt}\int |\overline{W}^{k+1}|^2+\alpha|\varphi^{k+1}\nabla \overline{w}^{k+1}|_2^2\\ \le& C(\nu^{-1}+|\varphi^k\nabla^4 w^k|_2^2)|\overline{W}^{k+1}|_2^2+C|\overline{\varphi}^{k+1}|_2^2+\nu(|\varphi^k\nabla \overline{w}^k|_2^2+|\overline{\varphi}^k|_2^2+|\overline{W}^k|_2^2).\end{aligned} \tag{4.56}$$

Finally, define

$$\Gamma^{k+1}(t)=\sup_{s\in[0,t]}|\overline{W}^{k+1}(s)|_2^2+\sup_{s\in[0,t]}|\overline{\varphi}^{k+1}(s)|_2^2.$$

It follows from (4.51) and (4.56) that

$$\begin{aligned}&\frac{d}{dt}\int \Big(|\overline{W}^{k+1}|^2+|\overline{\varphi}^{k+1}(t)|_2^2\Big)+\alpha|\varphi^{k+1}\nabla \overline{w}^{k+1}|_2^2\\ \le& E^k_\nu(|\overline{W}^{k+1}|_2^2+|\overline{\varphi}^{k+1}|_2^2)+\nu(|\varphi^k\nabla \overline{w}^k|_2^2+|\overline{\varphi}^k|_2^2+|\overline{W}^k|_2^2),\end{aligned}$$

for some E^k_ν such that

$$\int_0^t E^k_\nu(s)\mathrm{d}s\le C+C_\nu t\quad \text{for}\quad 0\le t\le T^{**}.$$

Then Gronwall's inequality implies that

$$\begin{aligned}&\Gamma^{k+1}+\int_0^t \alpha|\varphi^{k+1}\nabla \overline{w}^{k+1}|_2^2\mathrm{d}s\\ \le&\Big(\nu\int_0^t |\varphi^k\nabla \overline{w}^k|_2^2\mathrm{d}s+t\nu\sup_{s\in[0,t]}\big(|\overline{W}^k|_2^2+|\overline{\varphi}^k|_2^2\big)\Big)\exp{(C+C_\nu t)}.\end{aligned}$$

Choose $\nu_0>0$ and $T_*\in(0,\min(1,T^{**}))$ small enough such that

$$\nu_0\exp C=\frac{1}{8}\min\big\{1,\alpha\big\},\quad \exp(C_{\nu_0}T_*)\le 2,$$

which implies that

$$\sum_{k=1}^{\infty}\Big(\Gamma^{k+1}(T_*)+\int_0^{T_*}\alpha|\varphi^{k+1}\nabla \overline{w}^{k+1}|_2^2\mathrm{d}t\Big)\le C<+\infty.$$

Thus, by the above estimate for $\Gamma^{k+1}(T_*)$ and (4.24), the whole sequence (φ^k,W^k) converges to a limit $(\varphi,W)=(\varphi,\phi,w)$ in the following strong sense:

$$(\varphi^k, W^k) \to (\varphi, W) \text{ in } L^\infty([0,T_*]; H^2(\mathbb{R}^3)). \tag{4.57}$$

Due to (4.24) and the lower-continuity of norm for weak convergence, (φ, W) still satisfies (4.24). Now (4.57) implies that (φ, W) satisfies problem (4.1) in the sense of distributions. So the existence of a classical solution is proved.

Step 2: Uniqueness. Similarly to the proof of Lemma 4.4, for any $t \in [0,T_*]$, the solution (φ, ϕ, w) is still compactly supported with respect to the space variable x, and the size of their supports is uniformly bounded by the constant $C_0(R, T_*)$.

Let (φ_1, W_1) and (φ_2, W_2) be two solutions to (4.1) satisfying the uniform a priori estimate (4.24). Set

$$\overline{\varphi} = \varphi_1 - \varphi_2, \quad \overline{W} = (\overline{\phi}, \overline{w}) = (\phi_1 - \phi_2, w_1 - w_2).$$

Then according to (4.50), $(\overline{\varphi}, \overline{W})$ solves the following system

$$\begin{cases} \overline{\varphi}_t + (w_1 + \widehat{u}) \cdot \nabla \overline{\varphi} + \overline{w} \cdot \nabla \varphi_2 + \dfrac{\delta - 1}{2}(\overline{\varphi}\text{div}(w_1 + \widehat{u}) + \varphi_2 \text{div}\overline{w}) = 0, \\ \overline{W}_t + \sum\limits_{j=1}^{3} A_j^*(W_1, \widehat{u})\partial_j \overline{W} + \varphi_1^2 \mathbb{L}(\overline{w}) = -\sum\limits_{j=1}^{3} A_j(\overline{W})\partial_j W_2 \\ \quad -\overline{\varphi}(\varphi_1 + \varphi_2)\mathbb{L}(w_2) + \big(\mathbb{H}(\varphi_1) - \mathbb{H}(\varphi_2)\big) \cdot \mathbb{Q}(W_2) \\ \quad +\mathbb{H}(\varphi_1) \cdot \mathbb{Q}(\overline{W}) - B(\nabla \widehat{u}, \overline{w}) - D(\overline{\varphi}(\varphi_1 + \varphi_2), \nabla^2 \widehat{u}). \end{cases} \tag{4.58}$$

Using the same arguments as in the derivation of (4.51)-(4.56), and letting

$$\Lambda(t) = |\overline{W}(t)|_2^2 + |\overline{\varphi}(t)|_2^2,$$

one can get that

$$\frac{d}{dt}\Lambda(t) + C|\varphi_1 \nabla \overline{w}(t)|_2^2 \le I(t)\Lambda(t), \tag{4.59}$$

for some $I(t)$ such that

$$\int_0^t I(s)ds \le C \quad \text{for} \quad 0 \le t \le T_*.$$

So the Gronwall's inequality yields $\overline{\varphi} = \overline{\phi} = \overline{w} = 0$. Then the uniqueness is obtained.

Step 3. The time-continuity can be obtained via the same arguments as in Lemma 4.4.

5. Global-in-time well-posedness with compactly supported initial density under (P_0) or (P_1)

In this section, we will consider the global-in-time well-posedness of classical solutions to the Cauchy problem (4.1) with compactly supported (φ_0, ϕ_0). To this end, we first rewrite (4.1) as

$$\begin{cases} \varphi_t + w\cdot\nabla\varphi + \dfrac{\delta-1}{2}\varphi \mathrm{div} w = -\widehat{u}\cdot\nabla\varphi - \dfrac{\delta-1}{2}\varphi\mathrm{div}\widehat{u}, \\ W_t + \displaystyle\sum_{j=1}^{3} A_j(W)\partial_j W + \varphi^2\mathbb{L}(w) = \mathbb{H}(\varphi)\cdot\mathbb{Q}(w+\widehat{u}) + G^*(W,\varphi,\widehat{u}), \\ (\varphi,\phi,w)(t=0,x) = (\varphi_0, W_0) = (\varphi_0,\phi_0,0), \quad x\in\mathbb{R}^3, \\ (\varphi,\phi,w)\to(0,0,0) \quad \text{as} \quad |x|\to\infty \quad \text{for} \quad t\geq 0, \end{cases} \tag{5.1}$$

where

$$G^*(W,\varphi,\widehat{u}) = -B(\nabla\widehat{u}, W) - \sum_{j=1}^{3}\widehat{u}^{(j)}\partial_j W - D(\varphi^2,\nabla^2\widehat{u}). \tag{5.2}$$

Then the main result in this section can be stated as:

Theorem 5.1. *Let (1.20) and any one of conditions* (P_0)-(P_3) *hold. If the initial data* (φ_0,ϕ_0,u_0) *satisfies* (H_1)-(H_3) *in Theorem 4.1 and*

$$\|\phi_0\|_3 + \|\varphi_0\|_3 \leq D_0(\alpha,\beta,\delta,A,\gamma,\kappa,\|u_0\|_\Xi),$$

where $D_0>0$ *is some constant depending on* $(\alpha,\beta,\delta,A,\gamma,\kappa,\|u_0\|_\Xi)$, *then for any* $T>0$, *there exists a unique global classical solution* (φ,ϕ,u) *in* $[0,T]\times\mathbb{R}^3$ *to (5.1) satisfying*

$$(\varphi,\phi)\in C([0,T];H^3),\ w\in C([0,T];H^{s'})\cap L^\infty([0,T];H^3),\ \varphi\nabla^4 w\in L^2([0,T];L^2), \tag{5.3}$$

for any constant $s'\in[2,3)$. *Moreover, when* (P_2) *holds, the smallness assumption on* φ_0 *could be removed.*

We will prove Theorem 5.1 in the following five Subsections 5.1-5.5 via establishing a uniform-in-time weighted estimates under the condition (P_0) or (P_1) (see Theorem 1.4 and Remark 1.2). The proof for the cases $(P2)$-$(P3)$ will be given in Section 6. In the rest of this section, let T be some positive time, and (φ,ϕ,u) in $[0,T]\times\mathbb{R}^3$ be the unique local-in-time solution to the problem (4.1) obtained in Theorem 4.1. Introduce a time weighed norm $Z(t)$ for classical solutions as

$$
\begin{aligned}
Y_k(t) =&|\nabla^k W(t)|_2, \qquad Y^2(t) = \sum_{k=0}^{3}(1+t)^{2\gamma_k} Y_k^2(t),\\
U_k(t) =&|\nabla^k \varphi(t)|_2, \qquad U^2(t) = \sum_{k=0}^{3}(1+t)^{2\delta_k} U_k^2(t),\\
Z^2(t) =&Y^2(t)+U^2(t), \quad \gamma_k = k-n, \quad \delta_k = k-m,
\end{aligned}
\tag{5.4}
$$

with (n, m) to be determined in Subsections 5.3-5.4. Denote also

$$
Z(0) = Z_0, \quad Y(0) = Y_0, \quad U(0) = U_0.
$$

Hereinafter, $C \geq 1$ will denote a generic constant depending only on fixed constants $(\alpha, \beta, \gamma, A, \delta, \kappa)$, but independent of (φ_0, W_0), which may be different from line to line, and $C_0 > 0$ denotes a generic constant depending on (C, φ_0, W_0). Specially, $C(l)$ (or $C_0(l)$) denotes a generic positive constant depending on (C, l) (or (C, φ_0, W_0, l)).

It then follows from Lemma 10.2 that:

Lemma 5.1.

$$
\begin{aligned}
|W(t)|_\infty \leq& C(1+t)^{\frac{2n-3}{2}} Y(t), \quad \textit{and} \quad |\nabla W(t)|_\infty \leq C(1+t)^{\frac{2n-5}{2}} Y(t),\\
|\varphi(t)|_\infty \leq& C(1+t)^{\frac{2m-3}{2}} U(t), \quad \textit{and} \quad |\nabla \varphi(t)|_\infty \leq C(1+t)^{\frac{2m-5}{2}} U(t).
\end{aligned}
$$

5.1. Energy estimates on W

First, applying ∇^k to $(5.1)_2$, multiplying by $\nabla^k W$ and integrating over $\mathbb{R}^3$, one can get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla^k W|_2^2 + \int \Big(\alpha\varphi^2|\nabla^{k+1}w|^2 + (\alpha+\beta)\varphi^2|\mathrm{div}\nabla^k w|^2\Big)\\
=&R_k(W) + S_k(W, \widehat{u}) + L_k(W, \varphi, \widehat{u}) + Q_k(W, \varphi, \widehat{u}),
\end{aligned}
\tag{5.5}
$$

where

$$
\begin{aligned}
R_k(W) =& -\int \nabla^k W \cdot \Big(\nabla^k\Big(\sum_{j=1}^{3} A^j(W)\partial_j W\Big) - \sum_{j=1}^{3} A^j(W)\partial_j \nabla^k W\Big)\\
&+\frac{1}{2}\int \sum_{j=1}^{3} \nabla^k W \cdot \partial_j A^j(W)\nabla^k W,\\
S_k(W, \widehat{u}) =& -\int \nabla^k W \cdot \nabla^k B(\nabla \widehat{u}, W) + \frac{1}{2}\int \sum_{j=1}^{3} \partial_j \widehat{u}^{(j)} \nabla^k W \cdot \nabla^k W\\
&-\int \nabla^k W \cdot \Big(\nabla^k\Big(\sum_{j=1}^{3} \widehat{u}^{(j)}\partial_j W\Big) - \sum_{j=1}^{3} \widehat{u}^{(j)}\partial_j \nabla^k W\Big),
\end{aligned}
\tag{5.6}
$$

$$L_k(W,\varphi,\widehat{u}) = -\int \Big(\nabla\varphi^2 \cdot \mathbb{S}(\nabla^k w) - \big(\nabla^k(\varphi^2 Lw) - \varphi^2 L\nabla^k w\big)\Big) \cdot \nabla^k w \\ + \int \nabla^k(\varphi^2 L\widehat{u}) \cdot \nabla^k w \equiv: L_k^1 + L_k^2 + L_k^3,$$

and

$$Q_k(W,\varphi,\widehat{u}) = \int \Big(\nabla\varphi^2 \cdot Q(\nabla^k w) + \big(\nabla^k(\nabla\varphi^2 \cdot Q(w)) - \nabla\varphi^2 \cdot Q(\nabla^k w)\big)\Big) \cdot \nabla^k w \\ + \int \nabla^k\big(\nabla\varphi^2 \cdot Q(\widehat{u})\big) \cdot \nabla^k w \equiv: Q_k^1 + Q_k^2 + Q_k^3. \tag{5.7}$$

The right hand side of (5.5) will be estimated in the next lemmas.

Lemma 5.2 *(**Estimates on** R_k **and** S_k).*

$$\Big|R_k(W)(t,\cdot)\Big| \le C|\nabla W|_\infty Y_k^2, \tag{5.8}$$

$$\frac{k+r}{1+t}Y_k^2 + S_k(W,\widehat{u})(t,\cdot) \le C_0 Y_k Z(1+t)^{-\gamma_k-2}, \tag{5.9}$$

for $k=0,1,2,3$*, where the constant* r *is given by:*

$$r = -\frac{1}{2} \quad \textit{if} \quad \gamma \ge \frac{5}{3}, \quad \textit{or} \quad \frac{3}{2}\gamma - 3 \quad \textit{if} \quad 1<\gamma<\frac{5}{3}. \tag{5.10}$$

Proof. Step 1: Estimates on R_k. Noticing that R_k is a sum of terms as

$$\nabla^k W \cdot \nabla^l W \cdot \nabla^{k+1-l} W \quad \text{for} \quad 1 \le l \le k,$$

then (5.8) is obvious when $k=0$, or 1.

For $k \ne 0$, 1, one can apply Lemma 10.5 to ∇W to get that

$$|\nabla^j W|_{p_j} \le C|\nabla W|_\infty^{1-2/p_j}|\nabla^k W|_2^{2/p_j}, \quad \text{for} \quad p_j = 2\frac{k-1}{j-1}. \tag{5.11}$$

If $l \ne k$ and $l \ne 1$, since $1/p_l + 1/p_{k-l+1} = \frac{1}{2}$, then Hölder's inequality implies

$$\int |\nabla^k W \nabla^l W \nabla^{k+1-l} W| \le |\nabla^k W|_2 |\nabla^l W|_{p_l} |\nabla^{k+1-l} W|_{p_{k-l+1}} \le C|\nabla W|_\infty |\nabla^k W|_2^2.$$

The other cases could be handled similarly. Thus (5.8) is proved.

Step 2: Estimates on S_k. The integrand of $S_k(W,\widehat{u})$ in $(5.6)_2$ can be rewritten as

$$\begin{aligned} s_k(W,\widehat{u}) =& -\nabla^k W\cdot B(\nabla\widehat{u},\nabla^k W)+\frac{1}{2}\sum_{j=1}^{3}\partial_j\widehat{u}^{(j)}\nabla^k W\cdot\nabla^k W\\ &-\nabla^k W\cdot\Big(\nabla^k(B(\nabla\widehat{u},W))-B(\nabla\widehat{u},\nabla^k W)\Big)\\ &-\nabla^k W\cdot\Big(\nabla^k\Big(\sum_{j=1}^{3}\widehat{u}^{(j)}\partial_j W\Big)-\sum_{j=1}^{3}\widehat{u}^{(j)}\partial_j\nabla^k W\Big)\equiv: s_k^1+s_k^2, \end{aligned} \tag{5.12}$$

where s_k^1 is a sum of terms with a derivative of order one of $\widehat{u}$, and s_k^2 is a sum of terms with a derivative of order at least two for $\widehat{u}$.

Step 2.1: Estimates on $S_k^1=\int s_k^1$. Let $\nabla^k=\partial_{\beta_1\beta_2\dots\beta_i\dots\beta_k}$ with $\beta_i=1,2,3$. Decompose S_k^1 as:

$$\begin{aligned} S_k^1=&\int\Big(-\nabla^k W\cdot B(\nabla\widehat{u},\nabla^k W)+\frac{1}{2}\sum_{j=1}^{3}\partial_j\widehat{u}^{(j)}\nabla^k W\cdot\nabla^k W\Big)\\ &-\int\partial_{\beta_1\dots\beta_k}W\cdot\sum_{i=1}^{k}\sum_{j=1}^{3}\partial_{\beta_i}\widehat{u}^{(j)}\partial_j\partial_{\beta_1\dots\beta_{i-1}\beta_{i+1}\dots\beta_k}W=I_1+I_2+I_3, \end{aligned} \tag{5.13}$$

where, from Proposition 2.1, I_1-I_3 are given by

$$\begin{aligned} I_1=&-\frac{3(\gamma-1)}{2(1+t)}\int\nabla^k\phi\cdot\nabla^k\phi-\frac{1}{1+t}\int\nabla^k w\cdot\nabla^k w+G_1,\\ I_2=&\frac{3}{2}\frac{1}{1+t}Y_k^2+G_2,\quad I_3=-\frac{k}{1+t}Y_k^2+G_3,\quad |G_j|\le\frac{C_0}{(1+t)^2}Y_k^2,\quad j=1,2,3. \end{aligned} \tag{5.14}$$

Therefore,

$$\begin{aligned} S_k^1(W,\widehat{u})(t,\cdot)\le&\frac{C_0}{(1+t)^2}Y_k^2-\frac{A_k}{1+t}\int\nabla^k w\cdot\nabla^k w-\frac{B_k}{1+t}\int\nabla^k\phi\cdot\nabla^k\phi\\ \le& C_0Y_kZ(1+t)^{-\gamma_k-2}-\frac{k+r}{1+t}Y_k^2, \end{aligned} \tag{5.15}$$

where

$$A_k=k-\frac{1}{2},\qquad B_k=\frac{3}{2}\gamma-3+k,\quad r=\min\big(A_k,B_k\big)-k. \tag{5.16}$$

Then (5.9) for S_k^1 follows from (5.15)-(5.16).

Step 2.2: Estimates on $S_k^2=\int s_k^2$. s_k^2 is a sum of the terms as

$$\begin{aligned} &E_1(W)=\nabla^k W\cdot\nabla^l\widehat{u}\cdot\nabla^{k+1-l}W\quad\text{for}\quad 2\le k\le 3\quad\text{and}\quad 2\le l\le k;\\ &E_2(W)=\nabla^k W\cdot\nabla^{l+1}\widehat{u}\cdot\nabla^{k-l}W\quad\text{for}\quad 1\le k\le 3\quad\text{and}\quad 1\le l\le k. \end{aligned}$$

Hence,

$$\begin{aligned} S_1^2(W) \le& C|W|_\infty |\nabla^2 \widehat{u}|_2 |\nabla W|_2 \le C_0 Y_1 Z(1+t)^{-2-\gamma_1},\\ S_2^2(W) \le& C\big(|\nabla^2 \widehat{u}|_\infty |\nabla W|_2 + |\nabla^3 \widehat{u}|_2 |W|_\infty\big)|\nabla^2 W|_2 \le C_0 Y_2 Z(1+t)^{-2-\gamma_2},\\ S_3^2(W) \le& C\big(|\nabla^2 \widehat{u}|_\infty |\nabla^2 W|_2 + |\nabla^3 \widehat{u}|_2 |\nabla W|_\infty + |\nabla^4 \widehat{u}|_2 |W|_\infty\big)|\nabla^3 W|_2\\ \le& C_0 Y_3 Z(1+t)^{-2-\gamma_3}, \end{aligned} \tag{5.17}$$

which imply that

$$S_k^2(W) \le C_0 Y_k Z(1+t)^{-2-\gamma_k}, \quad \text{for} \quad k=1,2,3.$$

This, together with (5.15), yields (5.9)-(5.10). □

Lemma 5.3 *(**Estimates on** L_k). For any suitably small constant $\eta > 0$, there are two constants $C(\eta)$ and $C_0(\eta)$ such that*

$$\begin{aligned} L_k(W) \le& \eta |\varphi \nabla^{k+1} w|_2^2 \delta_{3k} + C(\eta)(1+t)^{2m-5-\gamma_k} Z^3 Y_k\\ &+ C_0(1+t)^{2m-n-4.5-\gamma_k} Z^2 Y_k + C_0(\eta)(1+t)^{2m-10} Z^2. \end{aligned} \tag{5.18}$$

Proof. Step 1: Estimates on L_k^1. It is easy to check that,

$$\begin{aligned} L_k^1 \le& C|\varphi|_\infty |\nabla \varphi|_\infty |\nabla^{k+1} w|_2 |\nabla^k w|_2 \le C(1+t)^{2m-5-\gamma_k} Z^3 Y_k, \ \text{for } k \le 2,\\ L_3^1 \le& C|\varphi \nabla^4 w|_2 |\nabla \varphi|_\infty |\nabla^3 w|_2 \le \eta |\varphi \nabla^4 w|_2^2 + C(\eta)(1+t)^{2m-5-\gamma_3} Z^3 Y_3. \end{aligned} \tag{5.19}$$

Step 2: Estimates on L_k^3. If $k=0$ or 1, one has

$$\begin{aligned} L_0^3 \le& C|\varphi|_\infty |\nabla^2 \widehat{u}|_\infty |\varphi|_2 |w|_2 \le C_0(1+t)^{2m-n-4.5-\gamma_0} Z^2 Y_0,\\ L_1^3 \le& C\big(|\nabla \varphi|_2 |\nabla^2 \widehat{u}|_\infty + |\varphi|_\infty |\nabla^3 \widehat{u}|_2\big)|\varphi|_\infty |\nabla w|_2\\ \le& C_0(1+t)^{2m-n-4.5-\gamma_1} Z^2 Y_1. \end{aligned} \tag{5.20}$$

For the case $k=2$, decompose $L_2^3 \triangleq L_2^3(0,2) + L_2^3(1,1) + L_2^3(2,0)$. One can get

$$\begin{aligned} L_2^3(0,2) \triangleq& \int \varphi^2 \nabla^2 L\widehat{u} \cdot \nabla^2 w \le C|\varphi|_\infty^2 |\nabla^2 L\widehat{u}|_2 |\nabla^2 w|_2\\ \le& C_0(1+t)^{2m-n-4.5-\gamma_2} Z^2 Y_2,\\ L_2^3(1,1) \triangleq& \int \nabla \varphi^2 \cdot \nabla L\widehat{u} \cdot \nabla^2 w \le C|\varphi|_\infty |\nabla \varphi|_\infty |\nabla L\widehat{u}|_2 |\nabla^2 w|_2\\ \le& C_0(1+t)^{2m-n-4.5-\gamma_2} Z^2 Y_2,\\ L_2^3(2,0) \triangleq& \int \nabla^2 \varphi^2 \cdot L\widehat{u} \cdot \nabla^2 w \le C(|\nabla \varphi|_\infty^2 |L\widehat{u}|_2 + |\varphi|_\infty |\nabla^2 \varphi|_2 |L\widehat{u}|_\infty)|\nabla^2 w|_2\\ \le& C_0(1+t)^{2m-n-4.5-\gamma_2} Z^2 Y_2. \end{aligned} \tag{5.21}$$

For the case $k=3$, decompose $L_3^3 \triangleq L_3^3(0,3)+L_3^3(1,2)+L_3^3(2,1)+L_3^3(3,0)$. Then, by integration by parts, one can obtain

$$
\begin{aligned}
L_3^3(0,3) \triangleq & \int \varphi^2 \cdot \nabla^3 L\widehat{u} \cdot \nabla^3 w \leq C|\varphi|_\infty |\nabla^2 L\widehat{u}|_2 \big(|\varphi \nabla^4 w|_2 + |\nabla\varphi|_\infty |\nabla^3 w|_2\big)\\
\leq & \eta |\varphi \nabla^4 W|_2^2 + C_0(1+t)^{2m-n-4.5-\gamma_3} Z^2 Y_3 + C_0(\eta)(1+t)^{2m-10} Z^2,\\
L_3^3(1,2) \triangleq & \int \nabla\varphi^2 \cdot \nabla^2 L\widehat{u} \cdot \nabla^3 w \leq C|\varphi|_\infty |\nabla\varphi|_\infty |\nabla^2 L\widehat{u}|_2 |\nabla^3 w|_2\\
\leq & C_0(1+t)^{2m-n-4.5-\gamma_3} Z^2 Y_3,\\
L_3^3(2,1) \triangleq & \int \nabla^2\varphi^2 \cdot \nabla L\widehat{u} \cdot \nabla^3 w \leq C\big(|\nabla\varphi|_\infty^2 |\nabla^3 \widehat{u}|_2 + |\varphi|_\infty |\nabla^2\varphi|_6 |\nabla^3 \widehat{u}|_3\big)|\nabla^3 w|_2\\
\leq & C_0(1+t)^{2m-n-4.5-\gamma_3} Z^2 Y_3,\\
L_3^3(3,0) \triangleq & \int \nabla^3\varphi^2 \cdot L\widehat{u} \cdot \nabla^3 w \leq C\big(|\varphi|_\infty |\nabla^3\varphi|_2 + |\nabla\varphi|_\infty |\nabla^2\varphi|_2\big)|L\widehat{u}|_\infty |\nabla^3 w|_2\\
\leq & C_0(1+t)^{2m-n-4.5-\gamma_3} Z^2 Y_3,
\end{aligned}
\tag{5.22}
$$

with $\eta>0$ being any sufficiently small constant. It follows from (5.20)-(5.22) that

$$
L_k^3 \leq \eta |\varphi \nabla^{k+1} w|_2^2 \delta_{3k} + C_0(1+t)^{2m-n-4.5-\gamma_k} Z^2 Y_k + C_0(\eta)(1+t)^{2m-10} Z^2. \tag{5.23}
$$

Step 3: Estimates on L_k^2. If $k=1$, one gets

$$
L_1^2 = \int \nabla\varphi^2 \cdot Lw \cdot \nabla w \leq C|\varphi|_\infty |\nabla\varphi|_\infty |\nabla^2 w|_2 |\nabla w|_2 \leq C(1+t)^{2m-5-\gamma_1} Z^3 Y_1. \tag{5.24}
$$

Next for $k=2$, decompose $L_2^2 \triangleq L_2^2(1,1)+L_2^2(2,0)$. In a similar way for L_2^1, one has

$$
\begin{aligned}
L_2^2(1,1) \triangleq & \int \nabla\varphi^2 \cdot \nabla Lw \cdot \nabla^2 w \leq C(1+t)^{2m-5-\gamma_2} Z^3 Y_2,\\
L_2^2(2,0) \triangleq & \int \nabla^2\varphi^2 \cdot Lw \cdot \nabla^2 w \leq C(|\nabla\varphi|_\infty^2 |\nabla^2 w|_2 + |\varphi|_6 |\nabla^2 w|_6 |\nabla^2\varphi|_6)|\nabla^2 w|_2\\
\leq & C(1+t)^{2m-5-\gamma_2} Z^3 Y_2.
\end{aligned}
\tag{5.25}
$$

At last for $k=3$, decompose $L_3^2 \triangleq L_3^2(1,2)+L_3^2(2,1)+L_3^2(3,0)$. In a similar way for L_3^1, one can get

$$
\begin{aligned}
L_3^2(1,2) \triangleq & \int \nabla\varphi^2 \cdot \nabla^2 Lw \cdot \nabla^3 w \leq \eta|\varphi\nabla^4 w|_2^2 + C(\eta)(1+t)^{2m-5-\gamma_3} Z^3 Y_3, \\
L_3^2(2,1) \triangleq & \int \nabla^2\varphi^2 \cdot \nabla Lw \cdot \nabla^3 w \leq C(|\nabla\varphi|_\infty^2 |\nabla^3 w|_2 + |\nabla^2\varphi|_3 |\varphi\nabla Lw|_6)|\nabla^3 w|_2 \\
\leq & \eta|\varphi\nabla^4 w|_2^2 + C(\eta)(1+t)^{2m-5-\gamma_3} Z^3 Y_3, \\
L_3^2(3,0) \triangleq & \int \nabla^3\varphi^2 \cdot Lw \cdot \nabla^3 w \\
\leq & C|\nabla\varphi|_\infty |\nabla^2\varphi|_3 |\nabla^3 w|_2 |\nabla^2 w|_6 + C|\varphi\nabla^3 w|_6 |\nabla^2 w|_3 |\nabla^3\varphi|_2 \\
\leq & \eta|\varphi\nabla^4 w|_2^2 + C(\eta)(1+t)^{2m-5-\gamma_3} Z^3 Y_3.
\end{aligned}
\tag{5.26}
$$

Then combining the estimates (5.24)-(5.26) yields

$$
L_k^2 \leq \eta|\varphi\nabla^{k+1} w|_2^2 \delta_{3k} + C(\eta)(1+t)^{2m-5-\gamma_k} Z^3 Y_k. \tag{5.27}
$$

Thus (5.18) follows from above three steps. □

Lemma 5.4 *(**Estimates on** Q_k). For any suitably small constant $\eta > 0$, there are two constants $C(\eta)$ and $C_0(\eta)$ such that*

$$
\begin{aligned}
Q_k(W,\widehat{u}) \leq & \eta|\varphi\nabla^{k+1} w|_2^2 \delta_{3k} + C(\eta)(1+t)^{2m-5-\gamma_k} Z^3 Y_k \\
& + C_0(1+t)^{2m-n-3.5-\gamma_k} Z^2 Y_k + C_0(\eta)(1+t)^{2m-10} Z^2 \\
& + \frac{(4\alpha+6\beta)\delta}{(\delta-1)(1+t)} |\nabla^3\varphi|_2 |\varphi\nabla^3 \mathit{div} w|_2.
\end{aligned}
\tag{5.28}
$$

Proof. Step 1: Estimates on Q_k^1. In a similar way for L_k^1, it is easy to get

$$
Q_k^1 \leq \eta|\varphi\nabla^{k+1} w|_2^2 \delta_{3,k} + C(1+t)^{2m-5-\gamma_k} Z^3 Y_k.
$$

Step 2: Estimates on Q_3^k. If $k=0$, one has

$$
Q_0^3 \leq C|\varphi|_\infty |\nabla\widehat{u}|_\infty |\nabla\varphi|_2 |w|_2 \leq C_0(1+t)^{2m-n-3.5-\gamma_0} Z^2 Y_0. \tag{5.29}
$$

For $k=1$, in a similar way for L_1^3, one obtains

$$
\begin{aligned}
Q_1^3 \leq & C\Big(|\varphi|_\infty |\nabla^2\varphi|_2 |\nabla\widehat{u}|_\infty + |\nabla\varphi|_6 |\nabla\varphi|_3 |\nabla\widehat{u}|_\infty + |\varphi|_\infty |\nabla\varphi|_2 |\nabla^2\widehat{u}|_\infty\Big)|\nabla w|_2 \\
\leq & C_0(1+t)^{2m-n-3.5-\gamma_1} Z^2 Y_1.
\end{aligned}
\tag{5.30}
$$

Next for $k=2$, decompose $Q_2^3 \triangleq Q_2^3(0,2) + Q_2^3(1,1) + Q_2^3(2,0)$. Then as for $L_2^3(1,1)$ and $L_2^3(2,0)$, one has

$$
\begin{aligned}
Q_2^3(0,2) \triangleq & C\int \nabla\varphi^2\cdot\nabla^3\widehat{u}\cdot\nabla^2 w \le C_0(1+t)^{2m-n-4.5-\gamma_2}Z^2Y_2,\\
Q_2^3(1,1) \triangleq & C\int \nabla^2\varphi^2\cdot\nabla^2\widehat{u}\cdot\nabla^2 w \le C_0(1+t)^{2m-n-4.5-\gamma_2}Z^2Y_2,\\
Q_2^3(2,0) \triangleq & C\int \nabla^3\varphi^2\cdot\nabla\widehat{u}\cdot\nabla^2 w \le C\big(|\nabla^3\varphi|_2|\varphi|_\infty+|\nabla^2\varphi|_6|\nabla\varphi|_3\big)|\nabla\widehat{u}|_\infty|\nabla^2 w|_2\\
\le & C_0(1+t)^{2m-n-3.5-\gamma_2}Z^2Y_2.
\end{aligned}
\tag{5.31}
$$

For $k=3$, let $Q_3^3 \triangleq Q_3^3(0,3)+Q_3^3(1,2)+Q_3^3(2,1)+Q_4^3(3,0)$. Then as for $L_3^3(1,2)$, $L_3^3(2,1)$ and $L_3^3(3,0)$, one can get

$$
\begin{aligned}
\sum_{i=0}^{2} Q_3^3(i,3-i) \triangleq & C\int \nabla\varphi^2\cdot\nabla^4\widehat{u}\cdot\nabla^3 w + C\int \nabla^2\varphi\cdot\nabla^3\widehat{u}\cdot\nabla^3 w\\
& + C\int \nabla^3\varphi^2\cdot\nabla^2\widehat{u}\cdot\nabla^3 w \le C_0(1+t)^{2m-n-4.5-\gamma_3}Z^2Y_3.
\end{aligned}
\tag{5.32}
$$

Finally, it remains to handle the term $Q_3^3(3,0)$ defined below, for which some additional information is needed. Using integration by parts and Proposition 2.1, one can get

$$
\begin{aligned}
Q_3^3(3,0) \triangleq & \frac{\delta}{\delta-1}\sum_{i,j=1}^{3}\int \Big(\alpha\nabla^3\partial_j\varphi^2(\partial_i\widehat{u}^{(j)}+\partial_j\widehat{u}^{(i)})+\beta\nabla^3\partial_i\varphi^2\mathrm{div}\widehat{u}\Big)\cdot\nabla^3 w^{(i)}\\
= & Q_3^3(A)-\frac{\delta}{\delta-1}\sum_{i,j=1}^{3}\int \alpha\nabla^3\varphi^2(\partial_i\widehat{u}^{(j)}+\partial_j\widehat{u}^{(i)})\cdot\nabla^3\partial_j w^{(i)}\\
& -\frac{\delta}{\delta-1}\int\Big(\beta\nabla^3\varphi^2\mathrm{div}\widehat{u}\Big)\cdot\nabla^3\mathrm{div}w\\
= & Q_3^3(A)+Q_3^3(B)+Q_3^3(D)-\frac{\delta}{(\delta-1)(1+t)}\int(4\alpha+6\beta)\varphi\nabla^3\varphi\cdot\nabla^3\mathrm{div}w\\
\le & Q_3^3(A)+Q_3^3(B)+Q_3^3(D)+\frac{(4\alpha+6\beta)\delta}{(\delta-1)(1+t)}|\nabla^3\varphi|_2|\varphi\nabla^3\mathrm{div}w|_2,
\end{aligned}
$$

where

$$
\begin{aligned}
Q_3^3(A) = & C\int \nabla^3\varphi^2\cdot\nabla^2\widehat{u}\cdot\nabla^3 w = C\int\Big(\varphi\nabla^3\varphi+\nabla\varphi\cdot\nabla^2\varphi\Big)\cdot\nabla^2\widehat{u}\cdot\nabla^3 w\\
\le & C\big(|\varphi|_\infty|\nabla^3\varphi|_2+|\nabla\varphi|_\infty|\nabla^2\varphi|_2\big)|\nabla^2\widehat{u}|_\infty|\nabla^3 w|_2\\
\le & C_0(1+t)^{2m-n-4.5-\gamma_3}Z^2Y_3,
\end{aligned}
\tag{5.33}
$$

and

$$\begin{aligned}
Q_3^3(B) =& C\int \nabla\varphi\cdot\nabla^2\varphi\cdot\nabla\widehat{u}\cdot\nabla^4 w\\
=& C\int \Big(\big(\nabla\varphi\cdot\nabla^3\varphi+\nabla^2\varphi\cdot\nabla^2\varphi\big)\cdot\nabla\widehat{u}+\nabla\varphi\cdot\nabla^2\varphi\cdot\nabla^2\widehat{u}\Big)\cdot\nabla^3 w\\
\leq& C\big(|\nabla\varphi|_\infty|\nabla^3\varphi|_2|\nabla\widehat{u}|_\infty+|\nabla^2\varphi|_6\big(|\nabla^2\varphi|_6|\nabla\widehat{u}|_6+|\nabla\varphi|_3|\nabla^2\widehat{u}|_\infty\big)\big)|\nabla^3 w|_2\\
\leq& C_0(1+t)^{2m-n-3.5-\gamma_3}Z^2Y_3,\\
Q_3^3(D) =& -\frac{\delta}{(\delta-1)(1+t)^2}\sum_{i,j=1}^{3}\int 2\alpha\varphi\nabla^3\varphi(K_{ij}+K_{ji})\cdot\nabla^3\partial_j w^{(i)}\\
&-\frac{\delta}{(\delta-1)(1+t)^2}\sum_{i=1}^{3}\int 2\beta\varphi\nabla^3\varphi K_{ii}\cdot\nabla^3\mathrm{div}w\\
\leq& \eta|\varphi\nabla^4 w|_2^2+C_0(\eta)(1+t)^{2m-10}Z^2,
\end{aligned}\tag{5.34}$$

where the matrix $K=\{K_{ij}\}$ is defined in Proposition 2.1.

Then combining the estimates (5.29)-(5.34) yields

$$\begin{aligned}
Q_k^3 \leq& \eta|\varphi\nabla^{k+1}w|_2^2\delta_{3k}+C_0(1+t)^{2m-n-3.5-\gamma_k}Z^2Y_k\\
&+C_0(\eta)(1+t)^{2m-10}Z^2+\frac{(4\alpha+6\beta)\delta}{(\delta-1)(1+t)}|\nabla^3\varphi|_2|\varphi\nabla^3\mathrm{div}w|_2.
\end{aligned}\tag{5.35}$$

Step 3: Estimates on Q_k^2. For $k=1$, direct estimates give

$$\begin{aligned}
Q_1^2 =& C\int \big(\varphi\nabla^2\varphi+\nabla\varphi\cdot\nabla\varphi\big)\cdot\nabla w\cdot\nabla w\\
\leq& C(|\varphi|_3|\nabla^2\varphi|_6|\nabla w|_\infty+|\nabla\varphi|_\infty^2|\nabla w|_2\big)|\nabla w|_2\leq C(1+t)^{2m-5-\gamma_1}Z^3Y_1.
\end{aligned}\tag{5.36}$$

For $k=2$, in a similar way for $L_2^2(2,0)$, one gets easily

$$\begin{aligned}
Q_2^2 =& C\int \big(\nabla^3\varphi^2\cdot\nabla w+\nabla^2\varphi^2\cdot\nabla^2 w\big)\cdot\nabla^2 w\\
\leq& C\big(|\varphi|_\infty|\nabla^3\varphi|_2|\nabla w|_\infty+|\nabla w|_\infty|\nabla\varphi|_6|\nabla^2\varphi|_3\big)|\nabla^2 w|_2\\
&+C(1+t)^{2m-5-\gamma_2}Z^3Y_2\leq C(1+t)^{2m-5-\gamma_2}Z^3Y_2.
\end{aligned}\tag{5.37}$$

For $k=3$, as for $L_3^2(3,0)$ and $L_3^2(2,1)$, it follows from integration by parts that

$$\begin{aligned}
Q_3^2 =& C\int \big(\nabla^4\varphi^2\cdot\nabla w+\nabla^3\varphi^2\cdot\nabla^2 w+\nabla^2\varphi^2\cdot\nabla^3 w\big)\cdot\nabla^3 w\\
\leq& \eta|\varphi\nabla^4 w|_2^2+C(\eta)(1+t)^{2m-5-\gamma_3}Z^3Y_3+Q_3^2(A),
\end{aligned}\tag{5.38}$$

with

$$
\begin{aligned}
Q_3^2(A) \triangleq & C\int \nabla^3\varphi^2\cdot\nabla w\cdot\nabla^4 w = C\int \Big(\varphi\nabla^3\varphi+\nabla\varphi\cdot\nabla^2\varphi\Big)\cdot\nabla w\cdot\nabla^4 w\\
\leq & C|\nabla^3\varphi|_2|\nabla w|_\infty|\varphi\nabla^4 w|_2+Q_3^2(B)\\
\leq & \eta|\varphi\nabla^4 w|_2^2+C(\eta)(1+t)^{2m-5-\gamma_3}Z^3Y_3+Q_3^2(B),
\end{aligned}
\tag{5.39}
$$

where the term $Q_3^2(B)$ can be estimated by using integration by parts again,

$$
\begin{aligned}
Q_3^2(B) \triangleq & C\int \nabla\varphi\cdot\nabla^2\varphi\cdot\nabla w\cdot\nabla^4 w\\
= & C\int \Big(\nabla^2\varphi\cdot\nabla^2\varphi\cdot\nabla w+\nabla\varphi\cdot\nabla^3\varphi\cdot\nabla w+\nabla\varphi\cdot\nabla^2\varphi\cdot\nabla^2 w\Big)\cdot\nabla^3 w\\
\leq & C\big(|\nabla^2\varphi|_6^2|\nabla w|_6+|\nabla^3\varphi|_2|\nabla\varphi|_\infty|\nabla w|_\infty+|\nabla\varphi|_6|\nabla^2\varphi|_6|\nabla^2 w|_6\big)|\nabla^3 w|_2\\
\leq & C(1+t)^{2m-5-\gamma_3}Z^3Y_3.
\end{aligned}
\tag{5.40}
$$

Then collecting the estimates (5.36)-(5.40) shows that

$$
Q_k^2 \leq \eta|\varphi\nabla^{k+1}w|_2^2\delta_{3k}+C(\eta)(1+t)^{2m-5-\gamma_k}Z^3Y_k. \tag{5.41}
$$

Then (5.28) follows directly from above estimates. □

It follows from (5.5) and Lemmas 5.2-5.4 that

Lemma 5.5.

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}Y_k^2+\frac{k+r}{1+t}Y_k^2+\alpha\int|\varphi\nabla^{k+1}w|^2+(\alpha+\beta)\int|\varphi\nabla^k divw|^2\\
\leq & C(\eta)(1+t)^{2m-5-\gamma_k}Z^3Y_k+C_0\Big((1+t)^{2m-n-3.5-\gamma_k}+(1+t)^{n-2.5-\gamma_k}\Big)Z^2Y_k\\
&+C_0(\eta)\Big((1+t)^{2m-10}Z^2+(1+t)^{-\gamma_k-2}ZY_k\Big)\\
&+\eta|\varphi\nabla^{k+1}w|_2^2\delta_{3k}+\frac{(4\alpha+6\beta)\delta}{(\delta-1)(1+t)}|\nabla^3\varphi|_2|\varphi\nabla^k divw|_2\delta_{3k}.
\end{aligned}
\tag{5.42}
$$

5.2. Estimates on φ

φ is estimated in the following lemma.

Lemma 5.6.

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla^k\varphi|_2^2+\frac{\frac{3}{2}\delta-3+k}{1+t}|\nabla^k\varphi|_2^2\\
\leq & C(1+t)^{n-2.5-\delta_k}Z^2U_k+C_0(1+t)^{-2-\delta_k}ZU_k+\frac{\delta-1}{2}|\varphi\nabla^3 divw|_2|\nabla^3\varphi|_2\delta_{3k}.
\end{aligned}
\tag{5.43}
$$

Proof. Step 1. Applying ∇^k to $(5.1)_1$, multiplying by $\nabla^k\varphi$ and integrating over $\mathbb{R}^3$, one gets

$$\frac{1}{2}\frac{d}{dt}|\nabla^k\varphi|_2^2=\int\Big(w\cdot\nabla^{k+1}\varphi+\frac{\delta-1}{2}\nabla^k\varphi\mathrm{div}w\Big)\cdot\nabla^k\varphi+S_k^*(\varphi,\widehat{u})-\Lambda_1^k-\Lambda_2^k,\quad(5.44)$$

where $S_k^*(\varphi,\widehat{u})$ is given in Step 2 below, and

$$\Lambda_1^k=\Big(\nabla^k(w\cdot\nabla\varphi)-w\cdot\nabla^{k+1}\varphi\Big)\cdot\nabla^k\varphi,\quad\Lambda_2^k=\frac{\delta-1}{2}\Big(\nabla^k(\varphi\mathrm{div}w)-\nabla^k\varphi\mathrm{div}w\Big)\cdot\nabla^k\varphi.$$

Integration by parts yields immediately that

$$\int\Big(w\cdot\nabla^{k+1}\varphi+\frac{\delta-1}{2}\nabla^k\varphi\mathrm{div}w\Big)\cdot\nabla^k\varphi\leq C|\nabla w|_\infty|\nabla^k\varphi(t)|_2^2.\quad(5.45)$$

Step 2: Estimates on $S_k^*=\int s_k^*$ with the integrand defined as

$$\begin{aligned}s_k^*(\varphi,\widehat{u})=&-\frac{\delta-1}{2}\nabla^k\varphi\cdot\nabla^k\varphi\mathrm{div}\widehat{u}+\frac{1}{2}\mathrm{div}\widehat{u}\nabla^k\varphi\cdot\nabla^k\varphi\\&-\frac{\delta-1}{2}\nabla^k\varphi\cdot\Big(\nabla^k(\varphi\mathrm{div}\widehat{u})-\mathrm{div}\widehat{u}\nabla^k\varphi\Big)\\&-\nabla^k\varphi\cdot\Big(\nabla^k\big(\widehat{u}\cdot\nabla\varphi\big)-\widehat{u}\cdot\nabla^{k+1}\varphi\Big)\equiv: s_k^{*1}+s_k^{*2},\end{aligned}\quad(5.46)$$

where s_k^{*1} is a sum of terms with a derivative of order one of $\widehat{u}$, and s_k^{*2} is a sum of terms with a derivative of order at least two of $\widehat{u}$.

Step 2.1: Estimates on $S_k^{*1}=\int s_k^{*1}$. Let $\nabla^k=\partial_{\beta_1\ldots\beta_k}$ with $0\leq\beta_i=1,2,3$.

$$\begin{aligned}S_k^{*1}=&\int\Big(-\frac{\delta-1}{2}\nabla^k\varphi\cdot\nabla^k\varphi\mathrm{div}\widehat{u}+\frac{1}{2}\mathrm{div}\widehat{u}\nabla^k\varphi\cdot\nabla^k\varphi\Big)\\&-\int\partial_{\beta_1\ldots\beta_k}\varphi\cdot\sum_{i=1}^k\sum_{j=1}^k\partial_{\beta_i}\widehat{u}^{(j)}\partial_j\partial_{\beta_1\ldots\beta_{i-1}\beta_{i+1}\ldots\beta_k}\varphi=J_1+J_2+J_3,\end{aligned}\quad(5.47)$$

where, by Proposition 2.1, J_1-J_3 are estimated by

$$\begin{aligned}&J_1=-\frac{3(\delta-1)}{2(1+t)}U_k^2+N_1,\quad J_2=\frac{3}{2}\frac{1}{1+t}U_k^2+N_2,\\&J_3=-\frac{k}{1+t}U_k^2+N_3,\quad|N_j|\leq\frac{C_0}{(1+t)^2}U_k^2,\quad j=1,2,3.\end{aligned}\quad(5.48)$$

Therefore, it holds that

$$S_k^{*1}(W,\widehat{u})(t,\cdot)\mathrm{d}x\leq\frac{C_0}{(1+t)^2}U_k^2-\frac{\frac{3}{2}\delta-3+k}{1+t}U_k^2.\quad(5.49)$$

Step 2.2: Estimates on $S_k^{*2} = \int s_k^{*2}$. s_k^{*2} is a sum of the terms defined as

$$
\begin{aligned}
E_1(\varphi) &= \nabla^k \varphi \cdot \nabla^l \widehat{u} \cdot \nabla^{k+1-l}\varphi \quad \text{for} \quad 2 \le k \le 3 \quad \text{and} \quad 2 \le l \le k;\\
E_2(\varphi) &= \nabla^k \varphi \cdot \nabla^{l+1} \widehat{u} \cdot \nabla^{k-l}\varphi \quad \text{for} \quad 1 \le k \le 3 \quad \text{and} \quad 1 \le l \le k.
\end{aligned}
$$

Then it holds that

$$
\begin{aligned}
S_1^{*2}(\varphi) \le & C|\varphi|_6|\nabla^2\widehat{u}|_2|\nabla\varphi|_3 \le C_0(1+t)^{-2-\delta_1}ZU_1,\\
S_2^{*2}(\varphi) \le & C(|\nabla^2\widehat{u}|_\infty|\nabla\varphi|_2 + |\nabla^3\widehat{u}|_2|\varphi|_\infty)|\nabla^2\varphi|_2 \le C_0(1+t)^{-2-\delta_2}ZU_2,\\
S_3^{*2}(\varphi) \le & C\big(|\nabla^2\widehat{u}|_\infty|\nabla^2\varphi|_2 + |\nabla^3\widehat{u}|_6|\nabla\varphi|_3 + |\nabla^4\widehat{u}|_2|\varphi|_\infty\big)|\nabla^3\varphi|_2\\
\le & C_0(1+t)^{-2-\delta_3}ZU_3,
\end{aligned}
\tag{5.50}
$$

which implies immediately that

$$
S_k^{*2}(\varphi) \le C_0(1+t)^{-2-\delta_k}ZU_k, \quad \text{for} \quad k = 1,2,3.
$$

Step 3: Estimates on $\Lambda_1^k + \Lambda_2^k$. It follows from Lemma 5.2 and Hölder's inequality that

$$
\begin{aligned}
\Lambda_1^1 + \Lambda_2^1 \le & C|\nabla w|_\infty|\nabla\varphi(t)|_2^2 \le C(1+t)^{n-2.5-\delta_1}Z^2U_1,\\
\Lambda_1^2 + \Lambda_2^2 \le & C(|\nabla\varphi|_\infty|\nabla^2 w(t)|_2 + |\nabla w|_\infty|\nabla^2\varphi(t)|_2)|\nabla^2\varphi(t)|_2\\
\le & C(1+t)^{n-2.5-\delta_2}Z^2U_2,\\
\Lambda_1^3 \le & C\big(|\nabla\varphi|_\infty|\nabla^3 w(t)|_2 + |\nabla^2\varphi|_6|\nabla^2 w(t)|_3 + |\nabla w|_\infty|\nabla^3\varphi(t)|_2\big)|\nabla^3\varphi(t)|_2\\
\le & C(1+t)^{n-2.5-\delta_3}Z^2U_3,\\
\Lambda_2^3 \le & C\Big(|\nabla\varphi|_\infty|\nabla^3 w(t)|_2 + |\nabla^2\varphi|_6|\nabla^2 w(t)|_3 + \frac{\delta-1}{2}|\varphi\nabla^3\mathrm{div}w|_2\Big)|\nabla^3\varphi|_2\\
\le & C(1+t)^{n-2.5-\delta_3}Z^2U_3 + \frac{\delta-1}{2}|\varphi\nabla^3\mathrm{div}w|_2|\nabla^3\varphi|_2.
\end{aligned}
\tag{5.51}
$$

Then (5.44)-(5.51) yield the desired (5.43). □

Finally, set

$$
\begin{aligned}
H(A^*,A,B) = & \alpha(A^*)^2 + (\alpha+\beta)A^2 + \frac{\frac{3}{2}\delta - 3 + m}{1+t}B^2\\
& - \Big(\frac{\delta-1}{2}(1+t)^{n-m} + \frac{2\delta}{\delta-1}(2\alpha+3\beta)(1+t)^{m-n-1}\Big)AB,\\
A^* = & (1+t)^{\gamma_3}|\varphi\nabla^4 w|_2, \quad A = (1+t)^{\gamma_3}|\varphi\nabla^3\mathrm{div}w|_2, \quad B = (1+t)^{\delta_3}|\nabla^3\varphi|_2,
\end{aligned}
\tag{5.52}
$$

then the following lemma holds:

Lemma 5.7.

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}Z^2+\frac{b_m}{1+t}(Z^2-B^2)+H(A^*,A,B)\\
&+\alpha\sum_{k=0}^{2}(1+t)^{2\gamma_k}|\varphi\nabla^{k+1}w(t)|_2^2+(\alpha+\beta)\sum_{k=0}^{2}(1+t)^{2\gamma_k}|\varphi\nabla^k divw(t)|_2^2\\
\le&\eta(1+t)^{2\gamma_3}|\varphi\nabla^4 w|_2^2+C(\eta)(1+t)^{2m-5}Z^4+C_0(1+t)^{2m-n-3.5}Z^3\\
&+C_0(1+t)^{n-2.5}Z^3+C_0(\eta)\Big((1+t)^{2m-2n-4}+(1+t)^{-2}\Big)Z^2,
\end{aligned}
\tag{5.53}
$$

where the constant b_m is given by

$$
b_m=\begin{cases}\min\left\{n-0.5,\frac{3}{2}\delta-3+m\right\} & \text{if } \gamma\ge\frac{5}{3},\\ \min\left\{\frac{3\gamma}{2}-3+n,\frac{3}{2}\delta-3+m\right\} & \text{if } 1<\gamma<\frac{5}{3},\end{cases}
\tag{5.54}
$$

and $\eta>0$ is any suitably small constant.

Proof. (5.53) can be obtained by multiplying (5.42) and (5.43) by $(1+t)^{2\gamma_k}$ and $(1+t)^{2\delta_k}$ respectively and summing the resulting inequalities together. □

5.3. Proof of Theorem 5.1 under the condition (P_0)

Set

$$
\begin{aligned}
M_1=&\frac{2\alpha+3\beta}{2\alpha+\beta},\quad M_2=-3\delta+1+\frac{1}{2}M_3,\\
M_3=&\frac{(\delta-1)^2}{4(2\alpha+\beta)}+\frac{4\delta^2(2\alpha+\beta)}{(\delta-1)^2}M_1^2+2M_1\delta,\\
M_4=&\frac{1}{2}\min\left\{\frac{3\gamma-3}{2},\frac{-M_2-1}{2},1\right\}+M_2,\\
\Phi(A,B)=&(2\alpha+\beta)A^2+\frac{\frac{3}{2}\delta-3+m}{1+t}B^2\\
&-\Big(\frac{\delta-1}{2}(1+t)^{n-m}+\frac{2\delta}{\delta-1}(2\alpha+3\beta)(1+t)^{m-n-1}\Big)AB.
\end{aligned}
\tag{5.55}
$$

Lemma 5.8. *Let the condition (P_0) hold. Then for $m=n+0.5=3$, one has*

$$
\frac{1}{2}\frac{d}{dt}Z^2+\frac{(1-\nu_*)b_*}{1+t}Z^2\le C(1+t)^{1+\epsilon^*}Z^4+C_0(1+t)^{-1-\epsilon^*}Z^2,
\tag{5.56}
$$

where constants ν_*, b_* and ϵ^* are given by

$$
\begin{aligned}
\epsilon^* =&\frac{1}{2}\min\Big\{\frac{3\gamma-3}{2},\frac{-M_2-1}{2},1\Big\}>0,\\
\nu_* =&\min\Big\{\frac{3\gamma-3}{4(3\gamma-1)},\frac{-M_4-1}{6\delta-M_3},\frac{1}{10}\Big\}>0,\\
b_* =&\begin{cases}\min\Big\{2,\frac{3}{2}\delta-\frac{1}{4}M_3\Big\}>1 & \text{if } \ \gamma\geq\frac{5}{3},\\ \min\Big\{\frac{3\gamma}{2}-0.5,\frac{3}{2}\delta-\frac{1}{4}M_3\Big\}>1 & \text{if } \ 1<\gamma<\frac{5}{3}.\end{cases}
\end{aligned}
\tag{5.57}
$$

Moreover, there exists a constant $\Lambda(C_0)$ *such that* $Z(t)$ *is globally well-defined in* $[0,\infty)$ *if* $Z_0\leq\Lambda(C_0)$.

Proof. Step 1. First, note that

$$
\Phi(A,B)=aA^2+bB^2-cAB=a\big(A-\frac{c}{2a}B\big)^2+dB^2, \tag{5.58}
$$

with

$$
\begin{aligned}
a=&(2\alpha+\beta),\quad b=\frac{\frac{3}{2}\delta-3+m}{1+t},\\
c=&\Big(\frac{\delta-1}{2}(1+t)^{n-m}+\frac{2\delta}{\delta-1}(2\alpha+3\beta)(1+t)^{m-n-1}\Big),\\
d=&(1+t)^{-1}\Big(\frac{3}{2}\delta-3+m\Big)-\frac{1}{4}\Big(\frac{(\delta-1)^2}{4(2\alpha+\beta)}(1+t)^{2n-2m}\\
&+\frac{4\delta^2(2\alpha+\beta)}{(\delta-1)^2}M_1^2(1+t)^{2m-2n-2}+2M_1\delta(1+t)^{-1}\Big).
\end{aligned}
$$

Then according to Lemma 10.10 and (5.53), for any $\nu\in(0,1)$, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}Z^2+\frac{b_m}{1+t}(Z^2-B^2)+\nu\alpha(A^*)^2+(1-\nu)dB^2+\nu\Big(b-\frac{c^2}{4(\alpha+\beta)}\Big)B^2\\
&+\alpha\sum_{k=0}^{2}(1+t)^{2\gamma_k}|\varphi\nabla^{k+1}w(t)|_2^2+(\alpha+\beta)\sum_{k=0}^{2}(1+t)^{2\gamma_k}|\varphi\nabla^k\mathrm{div}w(t)|_2^2\\
\leq&\eta(A^*)^2+C(\eta)((1+t)^{2m-5}+(1+t)^{2m-5-2\gamma_3})Z^4\\
&+C_0\big((1+t)^{2m-n-3.5}+(1+t)^{n-2.5}\big)Z^3\\
&+C_0(\eta)\Big((1+t)^{2m-2n-4}+(1+t)^{-2}\Big)Z^2,
\end{aligned}
\tag{5.59}
$$

where one has used the following decomposition:

$$
H(A^*,A,B)=\nu\alpha(A^*)^2+(1-\nu)H(A^*,A,B)+\nu((\alpha+\beta)A^2+bB^2-cAB).
$$

By Proposition 2.2, in order to obtain the uniform estimates on Z, one needs

$$2m-2n-2=2n-2m=-1, \tag{5.60}$$

which implies that

$$d=(1+t)^{-1}d^*(m)=(1+t)^{-1}\Big(\frac{3}{2}\delta-3+m-\frac{1}{4}M_3\Big). \tag{5.61}$$

Choose $\epsilon^*=\frac{1}{2}\min\{\frac{3\gamma-3}{2},\frac{-M_2-1}{2},\frac{1}{10}\}>0$. Then it follows from (5.59) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}Z^2+\frac{b_m}{1+t}(Z^2-B^2)+\nu\alpha(A^*)^2+(1-\nu)\frac{d^*(m)}{1+t}B^2\\&+\frac{\nu}{1+t}\Big(d^*-\frac{1}{4}M_3\frac{\alpha}{\alpha+\beta}\Big)B^2\\ \le&\eta(A^*)^2+C(\eta)((1+t)^{2m-5+\epsilon^*}+(1+t)^{2m-5-2\gamma_3})Z^4+C_0(\eta)(1+t)^{-1-\epsilon^*}Z^2.\end{aligned} \tag{5.62}$$

Step 2. On one hand, for fixed $\delta>1$, a necessary condition to guarantee that the set

$$\Pi=\{(\alpha,\beta)|\alpha>0,\ 2\alpha+3\beta>0,\ \text{and}\ M_2<-1\}$$

is not empty is that $M_1<\frac{3}{2}-\frac{1}{\delta}$.

On the other hand, for fixed $\alpha,\beta,\gamma,\delta$, there always exists a sufficiently large number m such that $d^*(m)>0$. Indeed, one needs that

$$d^*(m)=\frac{3}{2}\delta-3+m-\frac{1}{4}M_3=-\frac{1}{2}M_2+m-\frac{5}{2}>0.$$

Here for the choice $m=n+0.5=3$, one has therefore $d^*=\frac{3}{2}\delta-\frac{1}{4}M_3$.

Step 3. Due to

$$d^*=\frac{3}{2}\delta-\frac{1}{4}M_3>1 \quad \text{and} \quad M_4=\epsilon^*+M_2<-1,$$

so for $\nu_*=\min\Big\{\frac{3\gamma-3}{4(3\gamma-1)},\frac{-M_4-1}{6\delta-M_3},\frac{1}{20}\Big\}$ and $\nu=\min\Big\{\frac{1}{200},\frac{4d^*v_*(\alpha+\beta)}{M_3\alpha}\Big\}$, it hold that

$$\begin{aligned}1+\epsilon^*-2(1-\nu_*)d^*=&M_4+2\nu_*d^*<-1,\\(1-\nu)d^*+\nu\Big(d^*-\frac{1}{4}M_3\frac{\alpha}{\alpha+\beta}\Big)\ge&(1-\nu_*)d^*.\end{aligned}$$

Based on this observation, we choose $\eta=\frac{\nu\alpha}{100}$, which, together with (5.53), $m=n+0.5=3$ and (5.58), implies that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}Z^2+\frac{(1-\nu_*)b_*}{1+t}Z^2+\frac{\nu\alpha}{2}(1+t)^{2\gamma_3}|\varphi\nabla^4 w(t)|_2^2\\&+\alpha\sum_{k=0}^{2}(1+t)^{2\gamma_k}|\varphi\nabla^{k+1}w(t)|_2^2+(\alpha+\beta)\sum_{k=0}^{2}(1+t)^{2\gamma_k}|\varphi\nabla^k \mathrm{div} w(t)|_2^2\\ \le& C(1+t)^{1+\epsilon^*}Z^4+C_0(1+t)^{-1-\epsilon^*}Z^2.\end{aligned} \tag{5.63}$$

Therefore,

$$\frac{d}{dt}Z+\frac{(1-\nu_*)b_*}{1+t}Z\le C(1+t)^{1+\epsilon^*}Z^3+C_0(1+t)^{-1-\epsilon^*}Z. \tag{5.64}$$

According to Proposition 2.2,

$$\epsilon^*>0,\quad 1+\epsilon^*-2(1-\nu_*)b_*<-1,$$

and the classical comparison principle of ODE, then $Z(t)$ satisfies

$$Z(t)\le\frac{(1+t)^{-(1-\nu_*)b_*}\exp\left(\frac{C_0}{-\epsilon^*}\left((1+t)^{-\epsilon^*}-1\right)\right)}{\left(Z_0^{-2}-2C\int_0^t(1+s)^M\exp\left(\frac{2C_0}{-\epsilon^*}\left((1+t)^{-\epsilon^*}-1\right)\right)\mathrm{d}s\right)^{\frac{1}{2}}}, \tag{5.65}$$

where $M=1+\epsilon^*-2(1-\nu_*)b_*<-1$.

Moreover, $Z(t)$ is globally well-defined for $t\ge 0$ if and only if

$$0<Z_0<\frac{1}{\left(2C\int_0^t(1+s)^M\exp\left(\frac{2C_0}{-\epsilon^*}\left((1+t)^{-\epsilon^*}-1\right)\right)\mathrm{d}s\right)^{\frac{1}{2}}}. \tag{5.66}$$

Moreover,

$$Z(t)\le C_0(1+t)^{-(1-\nu_*)b_*}\qquad \text{for all}\ \ t\ge 0, \tag{5.67}$$

which implies that

$$Y_k(t)\le C_0(1+t)^{-\gamma_k-(1-\nu_*)b_*},\ \text{and}\ U_k(t)\le C_0(1+t)^{-\delta_k-(1-\nu_*)b_*}\qquad \text{for}\ \forall\ t\ge 0.\quad\square \tag{5.68}$$

Finally, it follows from (5.63) and (5.67) that for $k=0,1,2,3$:

$$\sum_{k=1}^{3}\int_0^t(1+t)^{2\gamma_k}\int|\varphi\nabla^{k+1}w|^2\mathrm{d}s\le C_0. \tag{5.69}$$

5.4. Proof of Theorem 5.1 under the condition (P_1)

For this case, we first choose $2n-2m=-(1+\epsilon)<-1$ for sufficiently small constant $0<\epsilon<\frac{1}{2}\min\left\{1,3\gamma-3\right\}$. Then it follows from Lemma 5.7 that

Lemma 5.9.

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}Z^2+\frac{b_m}{1+t}Z^2+\frac{\alpha}{2}\sum_{k=0}^{3}(1+t)^{2\gamma_k}|\varphi\nabla^{k+1}w(t)|_2^2\\ \leq &C(1+t)^{2n-4+\epsilon}Z^4+C_0(1+t)^{-1-\epsilon}Z^2,\end{aligned} \tag{5.70}$$

where the constant b_m is given by (5.54).

The proof is routine and thus omitted. The rest of the proof for Theorem 5.1 under the condition (P_1) is similar to the one in Subsection 5.3 for the condition (P_0).

6. Global-in-time well-posedness with compactly supported density under (P_2) or (P_3)

Denote $\iota=(\delta-1)/(\gamma-1)$. It is always assumed that $\iota\geq 2$ or $=1$. Based on the analysis of the previous section, it remains to estimate the following two terms:

$$\begin{aligned}L_k(W,\widehat{u})=&-\int\Big(\nabla\phi^{2\iota}\cdot\mathbb{S}(\nabla^k w)-\big(\nabla^k(\phi^{2\iota}Lw)-\phi^{2\iota}L\nabla^k w\big)\Big)\cdot\nabla^k w\\ &+\int\nabla^k(\phi^{2\iota}L\widehat{u})\cdot\nabla^k w\equiv: L_k^1+L_k^2+L_k^3,\\ Q_k(W,\widehat{u})=&\int\Big(\nabla\phi^{2\iota}\cdot Q(\nabla^k w)+\big(\nabla^k(\nabla\phi^{2\iota}\cdot Q(w))-\nabla\phi^{2\iota}\cdot Q(\nabla^k w)\big)\Big)\cdot\nabla^k w\\ &+\int\nabla^k\big(\nabla\phi^{2\iota}\cdot Q(\widehat{u})\big)\cdot\nabla^k w\equiv: Q_k^1+Q_k^2+Q_k^3.\end{aligned}$$

6.1. Estimates on L_k and Q_k

Lemma 6.1. *For any suitably small constant $\eta>0$, there are two constants $C(\eta)$ and $C_0(\eta)$ such that*

$$\begin{aligned}L_k(W)\leq&\eta|\phi^\iota\nabla^{k+1}w|_2^2\delta_{3,k}+C(\eta)(1+t)^{2\iota n-3\iota-2-\gamma_k}Y^{2\iota+1}Y_k\\ &+C_0(1+t)^{2\iota n-3\iota-n-1.5-\gamma_k}Y^{2\iota}Y_k+C_0(\eta)(1+t)^{2\iota n-3\iota-7}Y^{2\iota},\\ Q_k(W,\bar{u})\leq&\eta|\phi^\iota\nabla^{k+1}w|_2^2\delta_{3,k}+C(\eta)(1+t)^{2\iota n-3\iota-2-\gamma_k}Y^{2\iota+1}Y_k\\ &+C_0(1+t)^{2\iota n-3\iota-n-0.5-\gamma_k}Y^{2\iota}Y_k+C_0(\eta)(1+t)^{2\iota n-3\iota-5}Y^{2\iota}.\end{aligned} \tag{6.1}$$

Proof. Now we give only the corresponding estimates on Q_k. The estimates on L_k can be obtained similarly.

Step 1: Estimates on Q_1^k. Direct estimates yield that

$$\begin{aligned} Q_k^1 \le& C|\phi|_\infty^{2\iota-1}|\nabla\phi|_\infty|\nabla^{k+1}w|_2|\nabla^k w|_2 \le C(1+t)^{2\iota n-3\iota-2-\gamma_k}Y^{2\iota+1}Y_k, \quad \text{for} \quad k\le 2,\\ Q_3^1 \le& C|\phi^\iota\nabla^4 w|_2|\phi|_\infty^{\iota-1}|\nabla\phi|_\infty Y_3 \le \eta|\phi^\iota\nabla^4 w|_2^2 + C(\eta)(1+t)^{2\iota n-3\iota-2-\gamma_3}Y^{2\iota+1}Y_3. \end{aligned} \tag{6.2}$$

Step 2: Estimates on Q_3^k. If $k=0$ or 1, one has

$$\begin{aligned} Q_0^3 \le& C_0|\phi|_\infty^{2\iota-2}|\nabla\phi|_\infty|\nabla\widehat{u}|_\infty|\phi|_2|w|_2 \le C(1+t)^{2\iota n-3\iota-n-0.5-\gamma_0}Y^{2\iota}Y_0,\\ Q_1^3 \le& C|\phi|_\infty^{2\iota-1}|\nabla\phi|_2|\nabla^2\widehat{u}|_\infty|\nabla w|_2 \le C_0(1+t)^{2\iota n-3\iota-n-0.5-\gamma_1}Y^{2\iota}Y_1. \end{aligned} \tag{6.3}$$

For $k=2$, decompose $Q_2^3 \triangleq Q_2^3(L)+Q_2^3(2,0)$, which can be estimated as

$$\begin{aligned} Q_2^3(L) \triangleq& C\int (\nabla\phi^{2\iota}\cdot\nabla^3\widehat{u}+\nabla^2\phi^{2\iota}\cdot\nabla^2\widehat{u})\cdot\nabla^2 w\\ \le& C\big(|\phi|_\infty|\nabla\phi|_\infty|\nabla^3\widehat{u}|_2+|\nabla\phi|_\infty^2|\nabla^2\widehat{u}|_2+|\phi|_\infty|\nabla^2\phi|_2|\nabla^2\widehat{u}|_\infty\big)|\phi|_\infty^{2\iota-2}|\nabla^2 w|_2\\ \le& C_0(1+t)^{2\iota n-3\iota-n-1.5-\gamma_2}Y^{2\iota}Y_2, \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} Q_2^3(2,0) \triangleq& C\int \nabla^3\phi^{2\iota}\cdot\nabla\widehat{u}\cdot\nabla^2 w \le C\big(|\nabla\phi|_\infty|\nabla\phi|_6^2|\nabla\widehat{u}|_6\\ &+|\phi|_\infty|\nabla^2\phi|_6|\nabla\phi|_6|\nabla\widehat{u}|_6+|\phi|_\infty^2|\nabla^3\phi|_2|\nabla\widehat{u}|_\infty\big)|\phi|_\infty^{2\iota-3}|\nabla^2 w|_2\\ \le& C_0(1+t)^{2\iota n-3\iota-n-0.5-\gamma_2}Y^{2\iota}Y_2. \end{aligned} \tag{6.5}$$

For $k=3$, set $Q_3^3 \triangleq Q_3^3(L)+Q_3^3(3,0)$, where

$$\begin{aligned} Q_3^3(L) =& C\int \big(\nabla\phi^{2\iota}\cdot\nabla^4\widehat{u}+\nabla^2\phi^{2\iota}\cdot\nabla^3\widehat{u}+\nabla^3\phi^{2\iota}\cdot\nabla^2\widehat{u}\big)\cdot\nabla^3 w\\ \le& C\Big(|\phi|_\infty^2|\nabla\phi|_\infty|\nabla^4\widehat{u}|_2+|\phi|_\infty\nabla\phi|_\infty^2|\nabla^3\widehat{u}|_2+|\phi|_\infty^2|\nabla^2\phi|_6|\nabla^3\widehat{u}|_3\\ &+\big(|\nabla\phi|_\infty^2|\nabla\phi|_2+|\phi|_\infty|\nabla^2\phi|_2|\nabla\phi|_\infty+|\phi|_\infty^2|\nabla^3\phi|_2\big)|\nabla^2\widehat{u}|_\infty\Big)|\phi|_\infty^{2\iota-3}|\nabla^3 w|_2\\ \le& C_0(1+t)^{2\iota n-3\iota-n-1.5-\gamma_3}Y^{2\iota}Y_3,\\ Q_3^3(3,0) =& C\int \nabla^4\phi^{2\iota}\cdot\nabla\widehat{u}\cdot\nabla^3 w = C\int \nabla^3\phi^{2\iota}\cdot\big(\nabla^2\widehat{u}\cdot\nabla^3 w+\nabla\widehat{u}\cdot\nabla^4 w\big)\\ \le& C_0(1+t)^{2\iota n-3\iota-n-1.5-\gamma_3}Y^{2\iota}Y_3+Q_3^3(A), \end{aligned} \tag{6.6}$$

with

$$\begin{aligned}Q_3^3(A) \triangleq & C\int \nabla^3\phi^{2\iota}\cdot\nabla\widehat{u}\cdot\nabla^4 w\\ \leq & C\big(|\nabla^3\phi|_2|\phi|_\infty^{\iota-1}+|\nabla\phi|_3|\nabla^2\phi|_6|\phi|_\infty^{\iota-2}\big)|\nabla\widehat{u}|_\infty|\phi^\iota\nabla^4 w|_2+Q_3^3(B)\\ \leq & \eta|\phi^\iota\nabla^4 w|_2^2+C_0(\eta)(1+t)^{2\iota n-3\iota-5}Y^{2\iota}+Q_3^3(B),\end{aligned}\tag{6.7}$$

where the term $Q_3^3(B)$ can be estimated by using integration by parts again,

$$\begin{aligned}Q_3^3(B) \triangleq & C\int \phi^{2\iota-3}\nabla\phi\cdot\nabla\phi\cdot\nabla\phi\cdot\nabla\widehat{u}\cdot\nabla^4 w\\ \leq & C\Big(|\nabla\phi|_\infty^3|\phi|_\infty^{2\iota-4}|\nabla\phi|_2+|\nabla^2\phi|_6|\nabla\phi|_6^2|\phi|_\infty^{2\iota-3}\Big)|\nabla\widehat{u}|_\infty|\nabla^3 w|_2\\ & +C(1+t)^{2\iota n-3\iota-n-1.5-\gamma_3}Y^{2\iota}Y_3\leq C_0(1+t)^{2\iota n-3\iota-n-0.5-\gamma_3}Y^{2\iota}Y_3.\end{aligned}\tag{6.8}$$

Then (6.3)-(6.8) give that

$$Q_k^3\leq\eta|\phi^\iota\nabla^{k+1}w|_2^2+C_0(1+t)^{2\iota n-3\iota-n-0.5-\gamma_k}Y^{2\iota}Y_k+C_0(\eta)(1+t)^{2\iota n-3\iota-5}Y^{2\iota}.\tag{6.9}$$

Step 3: Estimates on Q_k^2. For $k=1$, one has

$$\begin{aligned}Q_1^2 = & C\int\big(\phi^{2\iota-1}\nabla^2\phi+\phi^{2\iota-2}\nabla\phi\cdot\nabla\phi\big)\cdot\nabla w\cdot\nabla w\\ \leq & C|\phi|_\infty^{2\iota-2}\big(|\phi|_\infty|\nabla^2\phi|_6|\nabla w|_3+|\nabla\phi|_\infty^2|\nabla w|_2\big)|\nabla w|_2\\ \leq & C(1+t)^{2\iota n-3\iota-2-\gamma_1}Y^{2\iota+1}Y_1.\end{aligned}\tag{6.10}$$

For $k=2$, one has

$$\begin{aligned}Q_2^2 = & C\int\big(\nabla^3\phi^{2\iota}\cdot\nabla w+\nabla^2\phi^{2\iota}\cdot\nabla^2 w\big)\cdot\nabla^2 w\\ \leq & C\big(|\phi|_\infty|\nabla\phi|_\infty|\nabla w|_\infty|\nabla^2\phi|_2+|\phi|_\infty^2|\nabla w|_\infty|\nabla^3\phi|_2\\ & +|\nabla\phi|_\infty^2|\nabla\phi|_3|\nabla w|_6+|\phi|_\infty|\nabla\phi|_\infty^2|\nabla^2 w|_2\\ & +|\nabla^2 w|_6|\nabla^2\phi|_3|\phi|_\infty^2\big)|\phi|_\infty^{2\iota-3}|\nabla^2 w|_2\leq C(1+t)^{2\iota n-3\iota-2-\gamma_2}Y^{2\iota+1}Y_2.\end{aligned}\tag{6.11}$$

For $k=3$, using integration by parts, one can get

$$\begin{aligned}Q_3^2 = & C\int\big(\nabla^4\phi^{2\iota}\cdot\nabla w+\nabla^3\phi^{2\iota}\cdot\nabla^2 w+\nabla^2\phi^{2\iota}\cdot\nabla^3 w\big)\cdot\nabla^3 w\\ \leq & C|\phi|_\infty^{2\iota-3}|\nabla\phi|_6\big(|\nabla\phi|_\infty|\nabla\phi|_6+|\phi|_\infty|\nabla^2\phi|_6\big)|\nabla^2 w|_6|\nabla^3 w|_2\\ & +C|\phi^\iota\nabla^3 w|_6|\nabla^2 w|_3|\phi|_\infty^{\iota-1}|\nabla^3\phi|_2\\ & +C|\phi|_\infty^{\iota-1}(|\phi|_\infty^{\iota-1}|\nabla\phi|_\infty^2|\nabla^3 w|_2+|\nabla^2\phi|_3|\phi^\iota\nabla^3 w|_6)|\nabla^3 w|_2\\ \leq & \eta|\phi^\iota\nabla^4 w|_2^2+C(\eta)(1+t)^{2\iota n-3\iota-2-\gamma_3}Y^{2\iota+1}Y_3+Q_3^2(A),\end{aligned}\tag{6.12}$$

with

$$
\begin{aligned}
Q_3^2(A) =& C\int \nabla^3\phi^{2\iota}\cdot\nabla w\cdot\nabla^4 w\\
\leq& C\big(|\nabla^3\phi|_2|\phi|_\infty+|\nabla\phi|_3|\nabla^2\phi|_6\big)|\phi|_\infty^{\iota-2}|\nabla w|_\infty|\phi^\iota\nabla^4 w|_2+Q_3^2(B)\\
\leq& C(\eta)(1+t)^{2\iota n-3\iota-2-\gamma_3}Y^{2\iota+1}Y_3+\eta|\phi^\iota\nabla^4 w|_2^2+Q_3^2(B),
\end{aligned}
\tag{6.13}
$$

where the term $Q_3^2(B)$ can be estimated by using integration by parts again,

$$
\begin{aligned}
Q_3^2(B) =& C\int \phi^{2\iota-3}\nabla\phi\cdot\nabla\phi\cdot\nabla\phi\cdot\nabla w\cdot\nabla^4 w\\
\leq& C\big(|\nabla\phi|_\infty^2|\nabla\phi|_2+|\nabla^2\phi|_6|\nabla\phi|_3|\phi|_\infty\big)|\phi|_\infty^{2\iota-4}|\nabla\phi|_\infty|\nabla w|_\infty|\nabla^3 w|_2\\
&+C|\phi|_\infty^{2\iota-3}|\nabla\phi|_\infty^3|\nabla^2 w|_6|\nabla^3 w|_2\\
\leq& C(1+t)^{2\iota n-3\iota-2-\gamma_3}Y^{2\iota+1}Y_3.
\end{aligned}
\tag{6.14}
$$

Then the estimates (6.10)-(6.13) lead to

$$
Q_k^2 \leq \eta|\phi^\iota\nabla^{k+1}w|_2^2+C(\eta)(1+t)^{2\iota n-3\iota-2-\gamma_k}Y^{2\iota+1}Y_k. \quad \square \tag{6.15}
$$

6.2. Derivation of a global-in-time well-posedness for an ordinary differential inequality

Then, based on the above energy estimates, one can derive the following lemma.

Lemma 6.2.

$$
\frac{dY(t)}{dt}+\frac{a}{1+t}Y(t)\leq C_0(1+t)^{-1-\eta}Y(t)+C(1+t)^{2\iota n-3\iota-1+2\theta\eta-\eta}Y^{2\iota+1}, \tag{6.16}
$$

holds for any sufficiently small constant $0<\eta<\frac{3\iota(\gamma-1)}{2\iota-1}$, *where* $a=r+n$. *Moreover, there exists a constant* $\Lambda(C_0)$ *such that* $Y(t)$ *is globally well-defined in* $[0,\infty)$ *if* $Y_0\leq\Lambda(C_0)$.

Proof. First, it follows from Lemmas 5.2, 6.1 and (5.5) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}Y_k^2+\frac{k+r}{1+t}Y_k^2+\frac{1}{2}\alpha\int|\phi^\iota\nabla^{k+1}w|^2\\
\leq& CYY_k(1+t)^{-\gamma_k-2}+C(1+t)^{n-2.5}YY_k^2+C(1+t)^{2\iota n-3\iota-2-\gamma_k}Y^{2\iota+1}Y_k\\
&+C_0(1+t)^{2\iota n-3\iota-n-0.5-\gamma_k}Y^{2\iota}Y_k+C_0(1+t)^{2\iota n-3\iota-5}Y^{2\iota}.
\end{aligned}
\tag{6.17}
$$

Second, multiplying (6.17) by $(1+t)^{2\gamma_k}$ and simplifying by Y give

$$
\begin{aligned}
\frac{\mathrm{d}Y(t)}{\mathrm{d}t}+\frac{a}{1+t}Y(t)\leq& C_0(1+t)^{-2}Y+C(1+t)^{n-2.5}Y^2+C(1+t)^{2\iota n-3\iota-2}Y^{2\iota+1}\\
&+C_0(1+t)^{2\iota n-3\iota-n-0.5}Y^{2\iota}+C_0(\eta)(1+t)^{2\iota n-3\iota-2n+1}Y^{2\iota-1}\\
\leq& C_0(1+t)^{-1-\eta}Y(t)+C(1+t)^{2\iota n-3\iota-1+2\theta\eta-\eta}Y^{2\iota+1}
\end{aligned}
\tag{6.18}
$$

for any sufficiently small constant $0 < \eta < 1$. A sufficient condition for the global existence of solutions to (6.18) is

$$K_{a,\iota,\eta} = 2\iota n - 3\iota - 2 + 1 + 2\iota\eta - \eta - 2\iota a < -1.$$

Indeed, it follows from the definition of a and $\iota \geq 2$ or $\iota = 1$ that

$$K_{a,\iota,\eta} = \begin{cases} -2\iota - 1 - \eta + 2\iota\eta < -1 & \text{if } \ \gamma \geq \frac{5}{3}; \\ 3\iota(1-\gamma) + \eta(2\iota - 1) < -1 & \text{if } \ 1 < \gamma < \frac{5}{3}, \end{cases} \tag{6.19}$$

for the sufficiently small constant $0 < \eta < \frac{3\iota(\gamma-1)}{2\iota-1}$.

Then, it follows from (6.18)-(6.19), Proposition 2.2 and the classical comparison principle of ODE that

$$Y(t) \leq \frac{(1+t)^{-a} \exp\left(\frac{C_0}{\eta}(1-(1+t)^{-\eta})\right)}{\left(Y_0^{-2\iota} - 2\iota C \int_0^t (1+s)^{K_{a,\iota,\eta}} \exp\left(\frac{2\iota C_0}{\eta}\left(1-(1+s)^{-\eta}\right)\right) \mathrm{d}s\right)^{\frac{1}{2\iota}}}. \tag{6.20}$$

Thus, similarly to (5.66), there exists a constant $\Lambda(C_0)$ such that $Y(t)$ is globally well-defined in $[0, +\infty)$ if $Y_0 \leq \Lambda(C_0)$. Moreover, it holds that

$$Y(t) \leq C_0(1+t)^{-r-n}, \quad \text{and} \quad Y_k(t) \leq K(1+t)^{-k-r} \qquad \text{for all } \ t \geq 0. \quad \square \tag{6.21}$$

Now we are ready to prove Theorem 5.1. **Step 1:** Estimates on $\varphi\nabla^4 w$. Note that (6.17) and (6.21) for $k = 3$ imply that

$$\frac{1}{2}\frac{d}{dt}Y_3^2 + \frac{3+r}{1+t}Y_3^2 + \frac{1}{2}\alpha \int |\phi^\iota \nabla^4 w|^2 \leq C(1+t)^{-8.5-3r}. \tag{6.22}$$

Multiplying (6.22) by $(1+t)^{2e}$ on both sides gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}((1+t)^{2e}Y_3^2) + \frac{3+r-e}{1+t}((1+t)^{2e}Y_3^2) + \frac{1}{2}\alpha(1+t)^{2e}\int |\phi^\iota \nabla^4 w|^2 \\ \leq& C(1+t)^{-8.5-3r+2e}. \end{aligned} \tag{6.23}$$

Under the assumptions that

$$3 + r - e \geq 0, \quad -8.5 - 3r + 2e < -1,$$

i. e., $e \leq 3 + r$, one can obtain

$$(1+t)^{2e}Y_3^2 + \frac{1}{2}\alpha \int_0^t (1+t)^{2e} \int |\phi^\iota \nabla^4 w|^2 \mathrm{d}s \leq C_0. \tag{6.24}$$

Step 2: Estimates on $\varphi = \phi^\iota$. Here we always assume that $\iota \geq 2$ in the rest of this proof. Due to (6.21), one can get

$$|\nabla^k \varphi|_2 \leq C(1+t)^{-1.5\iota - r\iota + 1.5 - k}. \tag{6.25}$$

Applying ∇^3 to $(5.1)_1$, multiplying by $\nabla^3\varphi$ and integrating over $\mathbb{R}^3$, then similar to Lemma 5.6, one can get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla^3\varphi|_2^2 + \frac{3\delta}{2(1+t)}|\nabla^3\varphi|_2^2\\
\leq & C_0(1+t)^{-r-2.5}|\nabla^3\varphi|_2^2 + C(1+t)^{-0.75\iota-0.5r\iota-r-3.25}|\nabla^3\varphi(t)|_2^{3/2}\\
&+C(1+t)^{-1.5\iota-r\iota-3.5}|\nabla^3\varphi|_2 + C|\varphi\nabla^4 w|_2|\nabla^3\varphi(t)|_2.
\end{aligned} \tag{6.26}$$

Then, multiplying (6.26) by $(1+t)^{2p}$ and simplifying by $G^2 = (1+t)^{2p}|\nabla^3\varphi|_2^2$ lead to

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}G^2}{\mathrm{d}t} + \frac{\frac{3}{2}\delta - p}{1+t}G^2\\
\leq & C_0\big((1+t)^{-r-2.5} + (1+t)^{2p-2e}\big)G^2 + C(1+t)^{-0.75\iota-0.5r\iota-r-3.25+0.5p}G^{3/2}\\
&+C(1+t)^{-1.5\iota-r\iota-3.5+p}G + C(1+t)^{2e}|\varphi\nabla^4 w|_2^2.
\end{aligned} \tag{6.27}$$

Choose p satisfying the following conditions:

$$\begin{aligned}
\frac{3}{2}\delta - p > 0, \quad & -0.75\iota - 0.5r\iota - r - 3.25 + 0.5p < -1,\\
2p - 2e < -1, \quad & -1.5\iota - r\iota - 3.5 + p < -1.
\end{aligned}$$

Then (6.27) yields

$$(1+t)^{2p}|\nabla^3\varphi|_2^2 \leq C_0.$$

7. Global-in-time well-posedness without compactly supported assumption

In this section, we will extend the global-in-time well-posedness in Theorem 5.1 to the more general case whose initial mass density ρ_0 is still small, but is not necessary to be compactly supported, which can be stated as:

Theorem 7.1. *Let (1.20) and any one of the conditions* (P_0)*-*(P_3) *hold. If initial data* (φ_0, ϕ_0, u_0) *satisfies* (A_1)*-*(A_2)*, then for any positive time* $T > 0$*, there exists a unique global classical solution* (φ, ϕ, u) *in* $[0,T]\times\mathbb{R}^3$ *to the Cauchy problem (4.1) satisfying*

$$(\varphi, \phi, w) \in C([0,T]; H^{s'}_{loc}) \cap L^\infty([0,T]; H^3), \ \varphi\nabla^4 w \in L^2([0,T]; L^2), \tag{7.1}$$

for any constant $s' \in [2,3)$. Moreover, when (P_2) holds, the smallness assumption on φ_0 could be removed.

In the following subsections, we will prove Theorem 7.1 under the condition (P_0). The proof for other cases is similar, and so is omitted.

7.1. Existence

According to Theorem 5.1, for the initial data

$$(\varphi_0^R, \phi_0^R, w_0) = (\varphi_0 F(|x|/R), \phi_0 F(|x|/R), 0),$$

there exists the unique global regular solution (φ^R, ϕ^R, w^R) satisfying:

$$\begin{aligned} |\nabla^k \phi^R(t)|_2 + |\nabla^k w^R(t)|_2 \leq & C_0(1+t)^{-(1-\nu_*)b_*+2.5-k};\\ \|(1+t)^{k-2.5}\varphi^R\nabla^{k+1}w^R\|_{L^2L_t^2} \leq C_0,\ |\nabla^k \varphi^R(t)|_2 \leq & C_0(1+t)^{-(1-\nu_*)b_*+3-k}, \end{aligned} \tag{7.2}$$

for any positive time $t > 0$, with C_0 independent of R.

Due to (7.2) and the following relations:

$$\begin{cases} \varphi_t^R = -(w^R+\widehat{u})\cdot\nabla\varphi^R - \dfrac{\delta-1}{2}\varphi^R \mathrm{div}(w^R+\widehat{u}),\\ \phi_t^R = -(w^R+\widehat{u})\cdot\nabla\phi^R - \dfrac{\gamma-1}{2}\phi^R \mathrm{div}(w^R+\widehat{u}),\\ w_t^R = -w^R\cdot\nabla w^R - \dfrac{\gamma-1}{2}\phi^R\nabla\phi^R - (\varphi^R)^2 L w^R\\ \quad +\nabla(\varphi^R)^2\cdot Q(w^R+\widehat{u}) - \widehat{u}\cdot\nabla w^R - w^R\cdot\nabla\widehat{u} - (\varphi^R)^2 L\widehat{u}, \end{cases} \tag{7.3}$$

it holds that for any finite constant $R_0 > 0$ and finite time $T > 0$,

$$\|\varphi_t^R\|_{H^2(B_{R_0})} + \|\phi_t^R\|_{H^2(B_{R_0})} + \|w_t^R\|_{H^1(B_{R_0})} + \int_0^t \|\nabla^2 w_t^R\|_{L^2(B_{R_0})}^2 \mathrm{d}s \leq C_0(R_0,T), \tag{7.4}$$

for $0 \leq t \leq T$, where the constant $C_0(R_0,T) > 0$ depends only on C_0, R_0 and T.

Since (7.2) and (7.4) are independent of R, there exists a subsequence of solutions (still denoted by) (φ^R, ϕ^R, w^R) converging to a limit (φ, ϕ, w) in the sense:

$$(\varphi^R, \phi^R, w^R) \to (\varphi, \phi, w) \quad \text{strongly in } \ C([0,T]; H^2(B_{R_0})). \tag{7.5}$$

For $k = 0, 1, 2, 3$, denote

$$a_k = -(1-\nu_*)b_* + 2.5 - k, \quad b_k = -(1-\nu_*)b_* + 3 - k, \quad c_k = k - 2.5.$$

Again due to (7.2), there exists a subsequence of solutions (still denoted by) (φ^R, ϕ^R, w^R) converging to the same limit (φ, ϕ, w) as above in the following weak* sense (for $k = 0, 1, 2, 3$):

$$
\begin{aligned}
(1+t)^{b_k}\varphi^R \rightharpoonup (1+t)^{b_k}\varphi \quad & \text{weakly* in } \ L^\infty([0,T]; H^3(\mathbb{R}^3)),\\
(1+t)^{a_k}\left(\phi^R, w^R\right) \rightharpoonup (1+t)^{a_k}\left(\phi, w\right) \quad & \text{weakly* in } \ L^\infty([0,T]; H^3(\mathbb{R}^3)).
\end{aligned} \tag{7.6}
$$

Combining the strong convergence in (7.5) and the weak convergence in (7.6) shows that (φ, ϕ, w) also satisfies the corresponding estimates (7.2) and (for $k = 0, 1, 2, 3$):

$$
(1+t)^{c_k}\varphi^R \nabla^{k+1} w^R \rightharpoonup (1+t)^{c_k}\varphi \nabla^{k+1} w \quad \text{weakly in } \ L^2([0,T] \times \mathbb{R}^3). \tag{7.7}
$$

It is then obvious that (φ, W) is a weak solution to problem (4.1) in the sense of distribution.

7.2. Uniqueness

Let (φ_1, W_1) and (φ_2, W_2) be two solutions to (4.1) satisfying the uniform a priori estimates (7.2). Set $F_N = F(|x|/N)$, and

$$
\begin{aligned}
&\overline{\varphi} = \varphi_1 - \varphi_2, \quad \overline{W} = (\overline{\phi}, \overline{w}) = (\phi_1 - \phi_2, w_1 - w_2),\\
&\overline{\varphi}^N = \overline{\varphi} F_N, \quad \overline{W}^N = \overline{W} F_N = (\overline{\phi}^N, \overline{w}^N),
\end{aligned}
$$

then $(\overline{\varphi}^N, \overline{W}^N)$ solves the following problem

$$
\begin{cases}
\quad \overline{\varphi}^N_t + (w_1 + \widehat{u}) \cdot \nabla \overline{\varphi}^N + \overline{w}^N \cdot \nabla \varphi_2 + \dfrac{\delta - 1}{2}(\overline{\varphi}^N \mathrm{div}(w_2 + \widehat{u}) + \varphi_1 \mathrm{div} \overline{w}^N)\\
= \overline{\varphi}(w_1 + \widehat{u}) \cdot \nabla F_N + \dfrac{\delta - 1}{2}\varphi_1 \overline{w} \cdot \nabla F_N,\\
\quad \overline{W}^N_t + \sum\limits_{j=1}^{3} A^*_j(W_1, \widehat{u})\partial_j \overline{W}^N + \varphi_1^2 \mathbb{L}(\overline{w}^N)\\
= -\sum\limits_{j=1}^{3} A_j(\overline{W}^N)\partial_j W_2 - \overline{\varphi}^N(\varphi_1 + \varphi_2)\mathbb{L}(w_2) + F_N\left(\mathbb{H}(\varphi_1) - \mathbb{H}(\varphi_2)\right) \cdot \mathbb{Q}(W_2)\\
\quad + \mathbb{H}(\varphi_1) \cdot \mathbb{Q}(\overline{W}^N) - B(\nabla \widehat{u}, \overline{w}^N) - D(\overline{\varphi}^N(\varphi_1 + \varphi_2), \nabla^2 \widehat{u})\\
\quad + \sum\limits_{j=1}^{3} A^*_j(W_1, \widehat{u})\overline{W}\partial_j F_N + \varphi_1^2(\mathbb{L}(\overline{w}^N) - F_N \mathbb{L}(\overline{w})) - \mathbb{H}(\varphi_1) \cdot \mathbb{Q}(\overline{W}) \cdot \nabla F_N,\\
\quad (\overline{\varphi}^N, \overline{W}^N)|_{t=0} = (0, 0), \quad x \in \mathbb{R}^3,\\
\quad (\overline{\varphi}^N, \overline{W}^N) \to (0, 0) \quad \text{as} \quad |x| \to \infty, \quad t > 0.
\end{cases} \tag{7.8}
$$

It is not hard to obtain that

$$
\begin{aligned}
&\frac{d}{dt}|\overline{\varphi}^N|_2^2 \leq C(|\nabla w_1|_\infty + |\nabla w_2|_\infty + |\nabla \widehat{u}|_\infty)|\overline{\varphi}^N|_2^2 + C|\nabla \varphi_2|_\infty |\overline{w}^N|_2 |\overline{\varphi}^N|_2\\
&\qquad + C|\varphi_1 \mathrm{div} \overline{w}^N|_2 |\overline{\varphi}^N|_2 + I_N^1,\\
&\frac{d}{dt}|\overline{W}^N|_2^2 + \frac{1}{2}\alpha \int \varphi_1^2 |\nabla \overline{w}^N|^2 \leq C\Big(1 + \|W_1\|_3^2 + \|W_2\|_3^2 + \|\nabla \widehat{u}\|_3^2\Big)|\overline{W}^N|_2^2\\
&\quad + C\big(|\varphi_2 \nabla^2 w_2|_\infty |\overline{w}^N|_2 + |\nabla^2 w_2|_3\big(|\varphi_1 \nabla \overline{w}^N|_2 + |\nabla \varphi_1|_\infty |\overline{w}^N|_2\big)\big)|\overline{\varphi}^N|_2 + I_N^2,
\end{aligned}
\tag{7.9}
$$

where the error terms I_N^1-I_N^2 are given and estimated by

$$
\begin{aligned}
I_N^1 =& \int\limits_{N\leq |x| \leq 2N} \frac{1}{N}\Big((|w_1| + |\widehat{u}|)|\overline{\varphi}|^2 + |\overline{\varphi}||\overline{w}||\varphi_1|\Big)\mathrm{d}x\\
\leq & C\Big((|w_1|_\infty + |\nabla \widehat{u}|_\infty)\|\overline{\varphi}\|_{L^2(\mathbb{R}^3\setminus B_N)} + |\varphi_1|_\infty \|\overline{w}\|_{L^2(\mathbb{R}^3\setminus B_N)}\Big)\|\overline{\varphi}\|_{L^2(\mathbb{R}^3\setminus B_N)},\\
I_N^2 =& \int\limits_{N\leq |x| \leq 2N} \frac{1}{N}\Big((|W_1| + |\widehat{u}|)|\overline{W}|^2 + |\varphi_1|^2|\overline{w}|^2 + |\varphi_1|^2|\overline{w}||\nabla \overline{w}|\Big)\mathrm{d}x\\
& + \int\limits_{N\leq |x| \leq 2N} \frac{1}{N}\Big(|\varphi_1 + \varphi_2||\overline{\varphi}||\nabla w_2| + |\nabla \varphi_1||\varphi_1 \nabla \overline{w}|\Big)|\overline{w}|\mathrm{d}x\\
\leq & C(|W_1|_\infty + |\nabla \widehat{u}|_\infty)\|\overline{W}\|_{L^2(\mathbb{R}^3\setminus B_N)}^2 + C|\varphi_1|_\infty^2 \|\overline{w}\|_{L^2(\mathbb{R}^3\setminus B_N)}^2\\
& + C(|\varphi_1|_\infty + |\nabla \varphi_1|_\infty)\|\overline{w}\|_{L^2(\mathbb{R}^3\setminus B_N)}|\varphi_1 \nabla \overline{w}|_2\\
& + C|\varphi_1 + \varphi_2|_\infty |w_2|_\infty \|\overline{w}\|_{L^2(\mathbb{R}^3\setminus B_N)}\|\overline{\varphi}\|_{L^2(\mathbb{R}^3\setminus B_N)},
\end{aligned}
$$

for $0 \leq t \leq T$.

Using the same arguments as in the derivation of (4.51)-(4.56), and letting

$$
\Lambda^N(t) = |\overline{W}^N(t)|_2^2 + \frac{\alpha}{2C}|\overline{\varphi}^N(t)|_2^2,
$$

one can have

$$
\begin{cases}
\dfrac{d}{dt}\Lambda^N(t) + \dfrac{\alpha}{2}|\varphi_1 \nabla \overline{w}^N(t)|_2^2 \leq J(t)\Lambda^N(t) + I_N^1(t) + I_N^2(t),\\
\displaystyle\int_0^t (J(s) + I_N(s))ds \leq C_0 \qquad\qquad \text{for} \quad 0 \leq t \leq T,
\end{cases}
\tag{7.10}
$$

where the constant $C_0 > 0$ is independent of N.

It follows from

$$
\int_0^t \int (I_N^1 + I_N^2)\mathrm{d}t \to 0, \quad \text{as} \quad N \to \infty,
$$

and Gronwall's inequality that $\overline{\varphi} = \overline{\phi} = \overline{w} = 0$. Then the uniqueness is obtained.

7.3. Time continuity

This follows from the uniform estimates (7.2) and equations in (5.1).

8. Proof of Theorem 1.4

Based on Theorem 7.1, now we are ready to give the global well-posedness of the regular solution to the original problem (1.1)-(1.7), i.e., the proof of Theorem 1.4. Moreover, we will show that this regular solution satisfies (1.1) classically in positive time $(0, T_*]$ when $1 < \min(\gamma, \delta) \leq 3$.

Proof. First, Theorem 7.1 shows that there exists a unique global classical solution (φ, ϕ, w) satisfying (7.2) and

$$(\rho^{\frac{\delta-1}{2}}, \rho^{\frac{\gamma-1}{2}}) = (\varphi, \phi) \in C^1((0,T)\times\mathbb{R}^3), \quad \text{and} \quad (u, \nabla u) \in C((0,T)\times\mathbb{R}^3). \tag{8.1}$$

Second, in terms of (φ, ϕ, u), one has

$$\begin{cases} \phi_t + u\cdot\nabla\phi + \dfrac{\gamma-1}{2}\phi \mathrm{div} u = 0, \\ u_t + u\cdot\nabla u + \dfrac{\gamma-1}{2}\phi\nabla\phi + \varphi^2 Lu = \nabla\varphi^2 \cdot Q(u). \end{cases} \tag{8.2}$$

Case 1: $1 < \min\{\gamma, \delta\} \leq 3$. We assume that $1 < \gamma \leq 3$, and the other cases can be dealt with similarly. Since $\rho = \phi^{\frac{2}{\gamma-1}}$ and $\frac{2}{\gamma-1} \geq 1$, so

$$\rho \in C^1((0,T)\times\mathbb{R}^3).$$

Multiplying $(8.2)_1$ by $\frac{\partial\rho}{\partial\phi}(t,x) = \frac{2}{\gamma-1}\phi^{\frac{3-\gamma}{\gamma-1}}(t,x) \in C((0,T)\times\mathbb{R}^3)$ on both sides yields the continuity equation, $(1.1)_1$.

While multiplying $(8.2)_2$ by $\phi^{\frac{2}{\gamma-1}} = \rho(t,x) \in C^1((0,T)\times\mathbb{R}^3)$ on both sides gives the momentum equations, $(1.1)_2$.

Thus, (ρ, u) solves the Cauchy problem (1.1)-(1.7) in the classical sense.

Case 2: $\min\{\gamma, \delta\} > 3$. For definiteness, we assume that $\gamma > 3$, and the other cases could be dealt with similarly. Since $\phi = \rho^{\frac{\gamma-1}{2}}$ and $\frac{2}{\gamma-1} > 0$, so $\rho \in C((0,T)\times\mathbb{R}^3)$.

It follows from $(8.2)_2$ and $\frac{\gamma-1}{2} > 1$ that

$$\rho_t + \mathrm{div}(\rho u) = 0, \quad \text{when} \quad \rho(t,x) > 0. \tag{8.3}$$

Now the whole space could be divided into two domains: the vacuum domain V_0 and its complement $V_p(t)$. Then it holds that for any smooth f,

$$\int_0^t \int_{V_p(s)} (\rho f_t + \rho u \cdot \nabla f)\mathrm{d}x\mathrm{d}s = \int_{V_p(0)} \rho_0 f(0,x)\mathrm{d}x, \tag{8.4}$$

which means that

$$\int_0^t \int_{\mathbb{R}^3} (\rho f_t + \rho u \cdot \nabla f)\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^3} \rho_0 f(0,x)\mathrm{d}x. \tag{8.5}$$

Multiplying $(8.2)_2$ by $\phi^{\frac{2}{\gamma-1}} = \rho(t,x)$ gives the momentum equations, $(1.1)_2$. Thus, (ρ, u) satisfies the Cauchy problem (1.1)-(1.7) in the sense of distributions. Finally, it is easy to show that

$$\rho(t,x) \geq 0, \ \forall (t,x) \in [0,T] \times \mathbb{R}^3.$$

That is to say, (ρ, u) satisfies the Cauchy problem (1.1)-(1.7) in the sense of distributions and has the regularities shown in Definition 1.2, which means that the Cauchy problem(1.1)-(1.7) has a unique regular solution (ρ, u). □

9. Proof of Theorem 1.6

In this section, we prove Theorem 1.6 by modifying the proof of Theorem 1.4. For the Cauchy problem (1.1)-(1.2) with (1.6)-(1.7), if $\mathrm{div}\mathbb{T} = \alpha\rho^\delta \triangle u$, it can be rewritten into

$$\begin{cases} \varphi_t + w \cdot \nabla \varphi + \dfrac{\delta-1}{2}\varphi \mathrm{div} w = -\widehat{u} \cdot \nabla \varphi - \dfrac{\delta-1}{2}\varphi \mathrm{div}\widehat{u}, \\ W_t + \displaystyle\sum_{j=1}^3 A_j(W)\partial_j W - \varphi^2 \triangle w = G_*(W, \varphi, \widehat{u}), \\ (\varphi, \phi, w)(t=0,x) = (\varphi_0, \phi_0, 0), \quad x \in \mathbb{R}^3, \\ (\varphi, \phi, w) \to (0,0,0) \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad t \geq 0, \end{cases} \tag{9.1}$$

where

$$G_*(W, \varphi, \widehat{u}) = -B(\nabla \widehat{u}, W) - \sum_{j=1}^3 \widehat{u}^{(j)} \partial_j W - \begin{pmatrix} 0 \\ \varphi^2 \triangle \widehat{u} \end{pmatrix}. \tag{9.2}$$

Let $\gamma_k = k - n$ and $\delta_k = k - m$, where $n = 2.5$, m will be determined in the end of this section, and $Z(t)$ is defined in (5.4). Then one has

Lemma 9.1.

$$\frac{dZ}{dt}(t) + \frac{b_m}{1+t} Z(t) \leq \begin{cases} C(1+t)^{2m-5} Z^3(t) + C_0(1+t)^{2m-9} Z & \text{if } m \in [3.5, 4), \\ C(1+t)^{2m-5} Z^3(t) + C_0(1+t)^{5-2m} Z & \text{if } m \in (3, 3.5), \end{cases} \tag{9.3}$$

where the constant b_m is given by (5.54).

The proof is similar to that of Lemma 5.7 with $Q_k = 0$. So the details are omitted here. Now we give the proof for Theorem 1.6 in the following subsections.

9.1. The proof for Theorem 1.6 with $m \in [3.5, 4)$

When $m \in [3.5, 4)$, we first consider the following lemma.

Lemma 9.2. *Let $\delta > 1$, $\gamma > 4/3$ and $m \in \big[3.5, \min\{4, \frac{3\gamma+3}{2}\}\big)$. For the following problem*

$$\begin{cases} \dfrac{dZ(t)}{dt} + \dfrac{b_m}{1+t} Z(t) = C_1(1+t)^{2m-5} Z^3(t) + C_2(1+t)^{2m-9} Z, \\ Z(x,0) = Z_0, \end{cases} \tag{9.4}$$

there exists a constant Λ such that $Z(t)$ is globally well-defined in $[0,\infty)$ if $Z_0 \le \Lambda$.

Proof. We want to prove this lemma by Proposition 2.2. First we need

$$M = D_1 - (a-1)b = 2m - 5 - 2b_m < -1, \quad \text{and} \quad D_2 = 2m - 9 < -1. \tag{9.5}$$

It follows from the definition of b_m in (5.54) and $\delta > 1$ that for $\gamma \ge \frac{5}{3}$, it only requires $m < 4$; for $\frac{4}{3} < \gamma < \frac{5}{3}$, it requires $m < \frac{3\gamma+3}{2}$. Then (9.5) obviously holds according to the assumptions on δ, γ and m.

Second, it follows from the formula (2.5) that by choosing Z_0 small enough, one can obtain the global existence of the solution to (9.4). Moreover,

$$Z(t) \le C_0(1+t)^{-b_m} \qquad \text{for all} \ \ t \ge 0. \quad \square \tag{9.6}$$

Next we prove Theorem 1.6 for the cases $\gamma > 4/3$. First, according to Lemma 9.1, when $m \in [3.5, 4)$, the following inequality holds

$$\frac{\mathrm{d}Z}{\mathrm{d}t}(t) + \frac{b_m}{1+t} Z(t) \le C(1+t)^{2m-5} Z^3(t) + C_0(1+t)^{2m-9} Z. \tag{9.7}$$

Then it follows from Lemma 9.2 that for $m \in \big[3.5, \min\{4, \frac{3\gamma+3}{2}\}\big)$, there exists a constant Λ such that $Z(t)$ is globally well-defined in $[0,\infty)$ if $Z_0 \le \Lambda$, and (9.6) holds.

According to Lemma 5.5 and its proof with $Q_k = 0$, it is easy to see that

$$\alpha \sum_{k=1}^{3} \int_0^t (1+t)^{2\gamma_k} \int |\varphi \nabla^{k+1} w|^2 \mathrm{d}s \le C_0, \quad \text{for} \quad t \ge 0. \tag{9.8}$$

The rest of the proof is similar to that for Theorem 1.4, and so is omitted.

9.2. The proof for Theorem 1.6 with $m \in (3, 3.5)$

For the case $m \in (3, 3.5)$, similarly we need to consider the following lemma.

Lemma 9.3. *Let $\delta > 1$, $\gamma > 1$ and $m \in \left(3, \min\{3.5, \frac{3\gamma+3}{2}\}\right)$. For the following problem*

$$\begin{cases} \dfrac{dZ}{dt}(t) + \dfrac{b_m}{1+t} Z(t) = C(1+t)^{2m-5} Z^3(t) + C_0(1+t)^{5-2m} Z, \\ Z(x, 0) = Z_0, \end{cases} \tag{9.9}$$

there exists a constant Λ such that $Z(t)$ is globally well-defined in $[0, \infty)$ if $Z_0 \le \Lambda$.

Proof. We want to prove this lemma by Proposition 2.2. First we need

$$M = D_1 - (a-1)b = 2m - 5 - 2b_m < -1, \quad \text{and} \quad D_2 = 5 - 2m < -1. \tag{9.10}$$

It follows from the definition of b_m in (5.54) and $\delta > 1$ that for $\gamma \ge \frac{5}{3}$, it only requires $3 < m < 4$; for $1 < \gamma < \frac{5}{3}$, it requires $3 < m < \frac{3\gamma+3}{2}$. Then (9.10) obviously holds according to the assumptions on δ, γ and m.

Second, it follows from the formula (2.5) that by choosing Z_0 small enough, one can obtain the global existence of the solution to (9.9). Moreover, (9.6) still holds. □

Remark 9.1. Both in Lemmas 9.2 and 9.3, the choice of m depends on the value of γ.

Next we prove Theorem 1.6 for the cases $\gamma > 1$. First, according to Lemma 9.1, when $m \in (3, 3.5)$, it holds that

$$\frac{\mathrm{d}Z}{\mathrm{d}t}(t) + \frac{b_m}{1+t} Z(t) \le C(1+t)^{2m-5} Z^3(t) + C_0(1+t)^{5-2m} Z. \tag{9.11}$$

Then it follows from Lemma 9.3 that for $m \in \left(3, \min\{3.5, \frac{3\gamma+3}{2}\}\right)$, there exists a constant Λ such that $Z(t)$ is globally well-defined in $[0, \infty)$ if $Z_0 \le \Lambda$, and (9.6) holds.

According to Lemma 5.5 and its proof with $Q_k = 0$, it is easy to see that (9.8) holds.

The rest of the proof is similar to that for Theorem 1.4, and so is omitted.

10. Appendix

In the first part of this appendix, we list some lemmas which were used frequently in the previous sections. The rest of the appendix will be devoted to show the proofs of Propositions 2.1-2.2 and some special Sobolev inequalities.

10.1. Some lemmas

The first one is the well-known Gagliardo-Nirenberg inequality.

Lemma 10.1. *[14] For $p \in [2,6]$, $q \in (1,\infty)$, and $r \in (3,\infty)$, there exists some generic constant $C > 0$ that may depend on q and r such that for*

$$f \in H^1(\mathbb{R}^3), \quad \textit{and} \quad g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3),$$

it holds that

$$|f|_p^p \le C|f|_2^{(6-p)/2}|\nabla f|_2^{(3p-6)/2}, \quad |g|_\infty \le C|g|_q^{q(r-3)/(3r+q(r-3))}|\nabla g|_r^{3r/(3r+q(r-3))}. \tag{10.1}$$

Some useful inequalities that can be obtained from the above lemma are given by

$$\begin{aligned}
|u|_6 \le & C|u|_{D^1}, \quad |u|_\infty \le C\|\nabla u\|_1, \quad |u|_\infty \le C\|u\|_{W^{1,r}}, \quad \text{for} \quad r > 3,\\
|\varphi\nabla w|_6 \le & C|\varphi\nabla w|_{D^1} \le C(|\varphi\nabla^2 w|_2 + |\nabla\varphi|_\infty \nabla w|_2),\\
|\varphi\nabla^2 w|_\infty \le & C|\varphi\nabla^2 w|_6^{\frac12}|\nabla(\varphi\nabla^2 w)|_6^{\frac12} \le C|\varphi\nabla^2 w|_{D^1}^{\frac12}|\nabla(\varphi\nabla^2 w)|_{D^1}^{\frac12}\\
\le & C\|\nabla(\varphi\nabla^2 w)\|_1 \le C(\|\nabla\varphi\|_2\|\nabla^2 w\|_1 + |\varphi\nabla^4 w|_2).
\end{aligned} \tag{10.2}$$

The Sobolev imbedding theorem yields

Lemma 10.2. *[14] Let $p > 3/2$ and $f \in H^p(\mathbb{R}^3)$. Then*

$$|f|_\infty \le C|f|_2^{1-\frac{3}{2p}}|\nabla^p f|_2^{\frac{3}{2p}}, \tag{10.3}$$

where C is a positive constant that may depend on p.

The next one can be found in Majda [25].

Lemma 10.3. *[25] Let r, a and b be constants such that*

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}, \quad \textit{and} \quad 1 \le a,\ b,\ r \le \infty.$$

$\forall s \ge 1$, if $f, g \in W^{s,a} \cap W^{s,b}(\mathbb{R}^3)$, then it holds that

$$|\nabla^s(fg) - f\nabla^s g|_r \le C_s\big(|\nabla f|_a|\nabla^{s-1}g|_b + |\nabla^s f|_b|g|_a\big), \tag{10.4}$$

$$|\nabla^s(fg) - f\nabla^s g|_r \le C_s\big(|\nabla f|_a|\nabla^{s-1}g|_b + |\nabla^s f|_a|g|_b\big), \tag{10.5}$$

where $C_s > 0$ is a constant depending only on s, and $\nabla^s f$ $(s > 1)$ is the set of all $\nabla_x^\zeta f$ with $|\zeta| = s$. Here $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3$ is a multi-index.

Next, the interpolation estimate, product estimate, composite function estimate and so on are given in the following four lemmas.

Lemma 10.4. *[25] Let $u \in H^s$, then for any $s' \in [0, s]$, there exists a constant C_s depending only on s such that*

$$\|u\|_{s'} \leq C_s \|u\|_0^{1-\frac{s'}{s}} \|u\|_s^{\frac{s'}{s}}.$$

Lemma 10.5. *[14] Let $r \geq 0$, $i \in [0, r]$, and $f \in L^\infty \cap H^r$. Then $\nabla^i f \in L^{2r/i}$, and there some generic constants $C_{i,r} > 0$ such that*

$$|\nabla^i f|_{2r/i} \leq C_{i,r} |f|_\infty^{1-i/r} |\nabla^r f|_2^{i/r}. \tag{10.6}$$

Lemma 10.6. *[25] Let functions $u,\ v \in H^s$ and $s > \frac{3}{2}$, then $u \cdot v \in H^s$, and there exists a constant C_s depending only on s such that*

$$\|uv\|_s \leq C_s \|u\|_s \|v\|_s.$$

Lemma 10.7. *[25]*

(1) *For functions $f,\ g \in H^s \cap L^\infty$ and $|\nu| \leq s$, there exists a constant C_s depending only on s such that*

$$\|\nabla^\nu (fg)\|_s \leq C_s (|f|_\infty |\nabla^s g|_2 + |g|_\infty |\nabla^s f|_2). \tag{10.7}$$

(2) *Assume that $g(u)$ is a smooth vector-valued function on Ω, $u(x)$ is a continuous function with $u \in H^s \cap L^\infty$, $u(x) \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega$. Then for $s \geq 1$, there exists a constant C_s depending only on s such that*

$$|\nabla^s g(u)|_2 \leq C_s \Big\| \frac{\partial g}{\partial u} \Big\|_{s-1, \overline{\Omega}_1} |u|_\infty^{s-1} |\nabla^s u|_2. \tag{10.8}$$

As a consequence of the Aubin-Lions Lemma, one has (c.f. [30]),

Lemma 10.8. *[30] Let X_0, X and X_1 be three Banach spaces satisfying $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1.*

(1) *Let G be bounded in $L^p(0, T; X_0)$ with $1 \leq p < \infty$, and $\frac{\partial G}{\partial t}$ be bounded in $L^1(0, T; X_1)$. Then G is relatively compact in $L^p(0, T; X)$.*
(2) *Let F be bounded in $L^\infty(0, T; X_0)$ and $\frac{\partial F}{\partial t}$ be bounded in $L^q(0, T; X_1)$ with $q > 1$. Then F is relatively compact in $C(0, T; X)$.*

The following lemma is useful to improve weak convergence to strong convergence.

Lemma 10.9. *[25] If the function sequence $\{w_n\}_{n=1}^{\infty}$ converges weakly to w in a Hilbert space X, then it converges strongly to w in X if and only if*

$$\|w\|_X \geq \lim sup_{n\to\infty}\|w_n\|_X.$$

The last lemma will be used in the proof shown in Section 5.

Lemma 10.10. *Let (φ, ϕ, w) be the regular the solution to the Cauchy problem (4.1), and Z be the time weighted energy defined in (5.4). One has*

$$|\varphi\nabla^3 divw|_2^2 \leq |\varphi\nabla^4 w|_2^2 + J^*, \tag{10.9}$$

where the term J^* can be controlled by

$$J^* \leq \eta|\varphi\nabla^4 w|_2^2 + C(\eta)(1+t)^{2m-5-2\gamma_3}Z^4,$$

for any constant $\eta > 0$ small enough, and the constant $C(\eta) > 0$.

Proof. According to the definition of div, one directly has

$$|\varphi\nabla^3 \mathrm{div} w|_2^2 = \sum_{i=1}^{3}|\varphi\nabla^3\partial_i w^{(i)}|_2^2 + 2\sum_{i,j=1,\ i<j}^{3}\int \varphi^2\nabla^3\partial_i w^{(i)}\cdot\nabla^3\partial_j w^{(j)}. \tag{10.10}$$

Via the integration by parts, one can obtain that

$$\begin{aligned} J_{ij} = &\int \varphi^2\nabla^3\partial_i w^{(i)}\cdot\nabla^3\partial_j w^{(j)} = \int \varphi^2\nabla^3\partial_i w^{(j)}\cdot\nabla^3\partial_j w^{(i)} \\ &+ \int \big(\partial_i\varphi^2\nabla^3\partial_j w^{(i)}\nabla^3 w^{(j)} - \partial_j\varphi^2\nabla^3\partial_i w^{(i)}\nabla^3 w^{(j)}\big). \end{aligned} \tag{10.11}$$

It is easy to see that

$$\begin{aligned} J_{ij}^* = &\int \big(\partial_i\varphi^2\nabla^3\partial_j w^{(i)}\nabla^3 w^{(j)} - \partial_j\varphi^2\nabla^3\partial_i w^{(i)}\nabla^3 w^{(j)}\big) \\ \leq &C|\nabla\varphi|_\infty|\varphi\nabla^4 w|_2|\nabla^3 w|_2 \\ \leq &\eta|\varphi\nabla^4 w|_2^2 + C(\eta)(1+t)^{2m-5-2\gamma_3}Z^4, \end{aligned} \tag{10.12}$$

for any constant $\eta > 0$ small enough, and the constant $C(\eta) > 0$, which, along with (10.10)-(10.11), quickly implies (10.9). □

10.2. Proof of Proposition 2.1

Here in this subsection, set

$$\|f\|_{\Xi_m} = |\nabla f|_\infty + \|\nabla^2 f\|_{m-2} \ \text{ for any } m > 1 + \frac{d}{2}.$$

Proof. We divide the proof into 4 steps: **(I)**-**(IV)**,

- in **(I)**, we prove the global existence of the solution $\widehat{u}(t,x)$ in the space C^1 to the problem (1.19), which satisfies the property (1) in Proposition 2.1;
- in **(II)**, by giving the estimates (2) in Proposition 2.1, we show that this C^1 solution $\widehat{u}(t,x)$ indeed exists in the space Ξ_m globally in time;
- in **(III)** and **(IV)**, we establish the estimates (3) and (2.2) in Proposition 2.1.

(I) The local existence of $\widehat{u}(t,x)$ in the space C^1 can be established from the characteristic method and the implicit function theorem (see §3 of Evans [6]). For extending $\widehat{u}$ to be a global C^1 solution, since $\widehat{u}(t, X(t;x_0))$ remains invariant along the particle path $X(t;x_0)$, one only needs to give the uniform estimates on $|\nabla_x \widehat{u}|_\infty$ with respect to the time.

For the first order derivative, let $G(t,x) = \nabla \widehat{u}(t,x)$ and $G_0(x_0) = \nabla u_0(x_0)$. Then

$$\begin{aligned} G(t, X(t;x_0)) =& \big(\mathbb{I}_d + tG_0(x_0)\big)^{-1} G_0(x_0) \\ =& G_0(x_0)\big(\mathbb{I}_d + tG_0(x_0)\big)^{-1} \\ =& \frac{1}{t}\Big(\mathbb{I}_d - \big(\mathbb{I}_d + tG_0(x_0)\big)^{-1}\Big), \end{aligned} \tag{10.13}$$

where one has used the fact $\nabla_{x_0} X = \mathbb{I}_d + tG_0(x_0)$. Note that

$$\big(\mathbb{I}_d + tG_0(x_0)\big)^{-1} = \big(\det(\mathbb{I}_d + tG_0)\big)^{-1} (\operatorname{adj}(\mathbb{I}_d + tG_0))^\top, \tag{10.14}$$

where $\operatorname{adj}(\mathbb{I}_d + tG_0)$ stands for the adjugate of $(\mathbb{I}_d + tG_0)$. Then it holds that

$$|\nabla_x \widehat{u}|_\infty \le \frac{(1 + t|\nabla u_0|_\infty)^{d-1}}{(1+t\kappa)^d} |\nabla_{x_0} u_0|_\infty, \tag{10.15}$$

which implies that $\widehat{u}$ is a global C^1 solution to the problem (1.19). Rewrite

$$G(t, X(t;x_0)) = \frac{1}{1+t}\mathbb{I}_d + \frac{1}{(1+t)^2} K(t, x_0),$$

where $K(t,x_0) = (1+t)^2(\mathbb{I}_d + tG_0)^{-1}G_0 - (1+t)\mathbb{I}_d$.

Since $G_0^{-1} = \big(\det(G_0)\big)^{-1}(\operatorname{adj}G_0)^\top$, so

$$|G_0^{-1}|_\infty \le C\kappa^{-d}|G_0|_\infty^{d-1}.$$

Then for t large enough, one has $|t^{-1}G_0^{-1}(x_0)| < 1$ for all x_0, and

$$\begin{aligned}K(t,x_0) =&\frac{(t+1)^2}{t}(\mathbb{I}_d + t^{-1}G_0^{-1})^{-1} - (1+t)\mathbb{I}_d\\=&\frac{(t+1)^2}{t}\Big(\mathbb{I}_d - \frac{G_0^{-1}}{t} + O\Big(\frac{1}{t^2}\Big)\Big) - (1+t)\mathbb{I}_d\\=&\frac{(t+1)}{t}\mathbb{I}_d - \frac{(t+1)^2}{t^2}G_0^{-1} + O\Big(\frac{1}{t}\Big).\end{aligned}$$

Remark 10.1. Let λ be an eigenvalue of G_0. It follows from (10.13) that, any eigenvalue λ_G of G has the following form

$$\lambda_G = \frac{1}{t}\Big(1 - \big(1+t\lambda\big)^{-1}\Big) = \frac{\lambda}{1+t\lambda} > 0.$$

(II) Now we show that the C^1 solution $\widehat{u}(t,x)$ obtained above indeed exists in the space Ξ_m globally in time. To this end, one needs to show that $\nabla_x^k G(t,x) \in L^2(\mathbb{R}^d)$ $(1 \leq k \leq m)$ for any $t \in [0,\infty)$. Actually, it follows from $\widehat{u}(t,x) = u_0(x - t\widehat{u}(t,x))$ that

$$G(t,x) = \big(\mathbb{I}_d + tG_0(x - t\widehat{u}(t,x))\big)^{-1}G_0(x - t\widehat{u}(t,x)).$$

Then according to the initial assumptions on u_0, by induction, it is easy to show that $\nabla_x^k G(t,x)$ $(1 \leq k \leq m)$ is well defined in $[0,\infty)\times\mathbb{R}^3$ pointwisely. What we only need to do is to give the estimates (2) in Proposition 2.1. However, the information of $\nabla_x^k G(t,x)$ cannot be given directly from the problem (1.19). Thus we consider $\nabla_x^k G(t, X(t;x_0))$ first, and then obtain the desired estimates via a change of variables.

For the higher order derivatives, let $H(t,x_0) = G(t,X(t;x_0))$. Starting from the right hand side of the first line in (10.13), by induction, one can get easily that, for $1 \leq k \leq m$:

$$\nabla_{x_0}^k H(t,x_0) = (\mathbb{I}_d + tG_0)^{-1}\Lambda_k(\mathbb{I}_d + tG_0)^{-1}, \tag{10.16}$$

where Λ_k is a sum of products of $t(\mathbb{I}_d + tG_0)^{-1}$ and $\nabla^j G_0,\ j \in \{0,1,...,k\}$, appearing β_j times with $\sum_j j\beta_j = k$.

On the one hand, starting from the left hand side of the first line in (10.13), one gets by induction that, for $1 \leq k \leq m$:

$$\nabla_{x_0}^k H(t,x_0) = \sum_{j=1}^{k}\nabla_x^j G(t,X(t;x_0))\Big(\sum_{1\leq k_i\leq k}\nabla_{x_0}^{k_1}X \otimes ... \otimes \nabla_{x_0}^{k_j}X\Big) \tag{10.17}$$

with $\sum_{i=1}^{j} k_i = k$, and

$$\nabla_{x_0}X = \mathbb{I}_d + tG_0(x_0), \quad \nabla_{x_0}^l X = t\nabla^{l-1}G_0(x_0), \quad \text{for} \quad l \geq 2.$$

On the other hand, one can also show that for all $1 \leq j \leq m$ by induction: **IH(j)**.

$(\mathbf{IH(j)}_1)$ $\nabla_x^j G(t, X(t;x_0))$ is a sum of terms which are products in a certain order of: $(\mathbb{I}_d + tG_0)^{-1}$, $t\mathbb{I}_d$, $\mathbb{I}_d + tG_0$ or $\nabla^l G_0$ appearing β_l times, with $\sum_l l\beta_l = j$;

$(\mathbf{IH(j)}_2)$ the L^∞-norm of the terms with t is bounded by a constant times $(1+t)^{-(j+2)}$, and

$$|\nabla_x^j G(t, X(t;\cdot))|_2 \leq C_j(1+t)^{-(j+2)}, \quad \text{with} \quad C_j = C(\kappa, j, \|u_0\|_{\Xi_m}).$$

Actually, when $j=1$, one has

$$\nabla_x G(t, X(t;x_0)) = (\mathbb{I}_d + tG_0)^{-1}\nabla G_0(x_0)(\mathbb{I}_d + tG_0)^{-2}, \tag{10.18}$$

which, along with the estimates obtained in **(I)**, yields **IH(1)**.

Now suppose that **IH(k-1)** holds. For $\nabla_x^k G(t, X(t;x_0))$, it follows from (10.17) that

$$\begin{aligned}\nabla_x^k G(t, X(t;x_0)) =& \Big(\nabla_{x_0}^k H(t,x_0) - \sum_{j=1}^{k-1} \nabla_x^j G(t, X(t;x_0))\Big(\sum_{1\leq k_i\leq k-1} \nabla_{x_0}^{k_1} X \otimes ... \otimes \nabla_{x_0}^{k_j} X\Big)\Big)\\ &\odot \Big((\mathbb{I}_d + tG_0)^{-1}\Big)^{\otimes k}.\end{aligned}$$

In the right-hand side term, one has the norm of the following terms to estimate:

- $J_1 = (\mathbb{I}_d + tG_0)^{-1}\Lambda_k(\mathbb{I}_d + tG_0)^{-k-1}$;
- $J_2 = \nabla_x^j G(\mathbb{I}_d + tG_0)^{j-s}\, \amalg_{k_i\neq 1} t\nabla^{k_i-1}G_0(\mathbb{I}_d + tG_0)^{-k}$, $\sum_{k_i\neq 1}(k_i - 1) = k - j$,

where one has used (10.16), and s is the number of $k_j \neq 1$.

First, according to the definition of Λ_k in the formula (10.16) and (10.14), J_1 has the desired form in $(\mathbf{IH(j)}_1)$, and the L^∞-norm of the terms with t is bounded by a constant times $(1+t)^{-(k+2)}$.

Second, it follows from **IH(j)** for $j \leq k-1$ that J_2 is a product of $(\nabla^j G_0)^{\beta_j}$ with $\sum_j j\beta_j = k$ and of $t\mathbb{I}_d$, $\mathbb{I}_d + tG_0$, $(\mathbb{I}_d + tG_0)^{-1}$, such that the L^∞-norm of the terms with t is bounded by a constant times $(1+t)^{-(k+2)}$.

For the desired L^2 estimates, it follows from the initial assumptions on u_0, $m > 1 + \frac{d}{2}$ and the Gagliardo-Nirenberg inequality that one can find an upper bound in L^2-norm for $\amalg_{1\leq j\leq k}\big(\nabla^j G_0\big)^{\beta_j}$ with $\sum_j j\beta_j = k$. Similarly, under the help of **IH(j)** for $j \leq k-1$, one can also obtained an upper bound of $|\nabla_x^j G\, \amalg_{k_i\neq 1} \nabla^{k_i-1}G_0|_2$ with $\sum_{k_i\neq 1}(k_i-1) = k-j$. Then one gets:

$$\begin{aligned}|J_1|_2 \leq C(1+t)^{-k-2}|\Lambda_k|_2 \leq C(1+t\kappa)^{-k-2}|\amalg\big(\nabla^j G_0\big)^{\beta_j}|_2 \leq& C_k(1+t)^{-k-2},\\ |J_2|_2 \leq C(1+t)^{-k+j}|\nabla_x^j G\, \amalg_{k_i\neq 1} \nabla^{k_i-1}G_0|_2 \leq& C_k(1+t)^{-k-2},\end{aligned}$$

where the positive constant $C_k = C(\kappa, k, \|u_0\|_{\Xi_m})$. To conclude, one obtains the upper bound in **IH(k)** which depends on κ, $|G_0|_\infty$, and $|\nabla^k G_0|_2$ for $1 \leq k \leq m$.

Finally, it needs to make a change of variables to obtain:

$$\begin{aligned} |\nabla_x^j G(t,\cdot)|_2 =& \Big(\int_{\mathbb{R}^d} |\nabla_x^j G(t,x)|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \\ =& \Big(\int_{\mathbb{R}^d} |\nabla_x^j G(t,X(t;x_0))|^2 |\det(\mathbb{I}_d + tG_0)| \mathrm{d}x_0\Big)^{\frac{1}{2}}, \end{aligned} \tag{10.19}$$

which, along with **IH(j)**, yields that

$$|\nabla_x^j G(t,\cdot)|_2 \le C_j(1+t)^{\frac{d}{2}-(j+2)}.$$

Therefore, the conclusion (2) in Proposition 2.1 is true for $l \in \mathbb{N},\ 2 \le l \le m+1$ since $\nabla_x G = \nabla^2 \widehat{u}$. Then by interpolation one obtains the result for all $l \in \mathbb{R},\ 2 \le l \le m+1$.

Until now, one has established the global well-posedness of the solution $\widehat{u}$ in the space Ξ to the problem (1.19).

(III) Since $m-1 > \frac{d}{2}$, one gets that $\nabla^2 u_0 \in L^\infty$. Thus $\nabla^2 \widehat{u} \in L^\infty$. Then the estimates (3) in Proposition 2.1 follows directly from (10.18) and the estimates obtained in **(I)**, i.e.,

$$|\nabla_x G(t, X(t;x_0))|_\infty = O((1+t)^{-3}).$$

(IV) If $u_0(0) = 0$, then it is obvious that $\widehat{u}(t,0) = 0$ for any $t \ge 0$, and

$$\widehat{u}(t,x) = \widehat{u}(t,0) + \nabla \widehat{u}(t,ax)\cdot x = \nabla \widehat{u}(t,ax)\cdot x, \quad \text{for any} \quad (t,x) \in [0,T]\times \mathbb{R}^3,$$

where $a \in [0,1]$ is some constant. Thus (2.2) is proved. □

10.3. Proof of Proposition 2.2

Proof. Multiplying the equation in (2.4) by $(1-a)Z^{-a}$, one has

$$(1-a)\frac{dZ}{dt\cdot Z^a} + \frac{(1-a)b}{1+t}Z^{1-a} = C_1(1-a)(1+t)^{D_1} + C_2(1-a)(1+t)^{D_2}Z^{1-a},$$

which, by denoting $\mathcal{K} = Z^{1-a}$, yields that

$$\frac{d\mathcal{K}}{dt} + \frac{(1-a)b}{1+t}\mathcal{K} = C_2(1-a)(1+t)^{D_2}\mathcal{K} + C_1(1-a)(1+t)^{D_1}. \tag{10.20}$$

Next we solve (10.20) by using the method of variation of constant. For this purpose, we first need to consider the following homogeneous equation:

$$\frac{d\mathcal{L}}{dt} + \frac{(1-a)b}{1+t}\mathcal{L} = C_2(1-a)(1+t)^{D_2}\mathcal{L}. \tag{10.21}$$

Dividing (10.21) by $\mathcal{L}$ yields that

$$\frac{d\ln\mathcal{L}}{dt}+\frac{(1-a)b}{1+t}=C_2(1-a)(1+t)^{D_2}.$$

Integrating the above equation over $[0,t]$ shows

$$\ln\frac{\mathcal{L}}{\mathcal{L}(0)}=\ln(1+t)^{(a-1)b}+\frac{C_2(1-a)}{D_2+1}\left((1+t)^{D_2+1}-1\right),$$

which implies that

$$\mathcal{L}=\mathcal{L}(0)(1+t)^{(a-1)b}\exp\left(\frac{C_2(1-a)}{D_2+1}\left((1+t)^{D_2+1}-1\right)\right). \tag{10.22}$$

Second, we suppose that (10.20) has a solution of the form:

$$\mathcal{K}=\mathcal{J}(t)(1+t)^{(a-1)b}\exp\left(\frac{C_2(1-a)}{D_2+1}\left((1+t)^{D_2+1}-1\right)\right), \tag{10.23}$$

where $\mathcal{J}(t)$ is a function of t to be determined later.

It follows from direct calculations that

$$\begin{aligned}\frac{d\mathcal{K}}{dt}=&\frac{d\mathcal{J}(t)}{dt}(1+t)^{(a-1)b}\exp\left(\frac{C_2(1-a)}{D_2+1}\left((1+t)^{D_2+1}-1\right)\right)\\&+\mathcal{J}(t)(a-1)b(1+t)^{(a-1)b-1}\exp\left(\frac{C_2(1-a)}{D_2+1}\left((1+t)^{D_2+1}-1\right)\right)\\&+\mathcal{J}(t)(1+t)^{(a-1)b}\exp\left(\frac{C_2(1-a)}{D_2+1}\left((1+t)^{D_2+1}-1\right)\right)\left(C_2(1-a)(1+t)^{D_2}\right)\\=&\frac{d\mathcal{J}(t)}{dt}\cdot\frac{\mathcal{K}}{\mathcal{J}(t)}+\frac{(a-1)b}{1+t}\mathcal{K}+C_2(1-a)(1+t)^{D_2}\mathcal{K}.\end{aligned}$$

which, along with (10.20) and $\mathcal{K}(0)=Z_0^{1-a}$, yields that

$$\begin{cases}\dfrac{d\mathcal{J}(t)}{dt}=C_1(1-a)(1+t)^{M}\exp\left(\dfrac{C_2(a-1)}{D_2+1}\left((1+t)^{D_2+1}-1\right)\right),\\ J(0)=Z_0^{1-a},\end{cases}$$

where $M=D_1-(a-1)b<-1$.

Then it is easy to obtain that

$$\mathcal{J}(t)=Z_0^{1-a}-(a-1)C_1\int_0^t(1+s)^{M}\exp\left(\frac{(a-1)C_2}{D_2+1}\left((1+s)^{D_2+1}-1\right)\right)\mathrm{d}s. \tag{10.24}$$

It follows from (10.23)-(10.24) and $Z(t) = \mathcal{K}^{\frac{1}{1-a}}$ that the Cauchy problem (2.4) can be solved by (2.5). Moreover, according to the assumption (2.3), $Z(t)$ is globally well-defined for $t \geq 0$ if and only if

$$0 < Z_0 < \frac{1}{\left((a-1)C_1 \int_0^t (1+s)^M \exp\left(\frac{(a-1)C_2}{D_2+1}\left((1+s)^{D_2+1}-1\right)\right) \mathrm{d}s\right)^{\frac{1}{a-1}}}. \tag{10.25}$$

Therefore, by choosing Z_0 small enough, one can obtain the global existence of the solution to the Cauchy problem (2.4). □

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