Convergence of A Galerkin Method for 2-D Discontinuous Euler Flows

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Abstract: We prove the convergence of a discontinuous Galerkin Method approximating the 2-D incompressible Euler equations with discontinuous initial vorticity: $\omega_0 \in L^2(\Omega)$. Furthermore, when $\omega_0 \in L^\infty(\Omega)$, the whole sequence is shown to be strongly convergent. This is the first convergence result in numerical approximations of this general class of discontinuous flows. Some important flows such as vortex patches belong to this class.

§1. Introduction. Numerical simulation of 2-D discontinuous incompressible flows is of considerable interests in both theoretical analysis and applications. It is believed that the Lagrangian methods such as vortex methods [5,9], or the ones based on contour

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dynamics [1,15] give preferable treatments for such flows especially for inviscid inter-facial flows. However, the convergence of such methods poses great difficulties. Past efforts concentrate on either special flows (see [4,12,13]), or require heavy machinery (such as large derivation [14]) and yield much weak convergence results [2,3,14]. However, for more complicated flows (such as a flow mixing), such front-tracking methods are impossible to implement. Thus, grid-based methods such as finite difference and finite elements are called for. Yet, the convergence of such methods is unknown as we know of [10, 12]. Recently, a discontinuous Galerkin method was proposed in [12] which has the main advantages that the energy is conserved even for upwind type numerical fluxes, and amusingly, the numerical enstropy is non-increasing in time. The main observation of this paper is to point out that the boundness of energy and enstropy are sufficient condition for strong convergence for a class of discontinuous initial data $\omega_0 \in L^2$ including vortex patches. In particularly, our results imply that the discontinuous Galerkin methods in [12] do converge for such flows.

§2. A Discontinuous Galerkin Method. The 2-D incompressible Euler equation in vorticity stream-function formulation reads:

$$\begin{aligned} \partial_t \omega + (\nabla^\perp \psi \cdot \nabla)\,\omega &= 0\,, \quad \nabla^\perp = (-\partial_y, \partial_x) \\ \Delta \psi &= \omega \end{aligned} \tag{2.1a}$$

with no-flow boundary condition

$$\psi = 0\,, \quad \text{on } \partial\Omega \tag{2.1b}$$

and initial condition

$$\omega|_{t=0} = \omega_0(x) \in L^2(\Omega) \tag{2.1c}$$

where $\Omega \subset R^2$ is a simply-connected domain with a C^2 boundary, or piecewise C^2 boundary with convex corners. Assume that Ω is equipped with a quasi-uniform triangulation $\mathcal{T}_h = \{K\}$ consisting of polygons K of maximum size (diameter) h. Denote $\Omega_h = \cup K$. The vorticity ω is approximated by ω_h in a discontinuous finite element space $V_h^k = \left\{ v : v\,|_K \in P^k(K),\ \forall K \in \mathcal{T}_h \right\}$, while the stream function ψ is approximated by ψ_h in a continuous one $W_{0,h}^k = V_h^k \cap C_0(\Omega_h)$. Here $P^k(K)$ denotes the set of all polynomials of degree at most k on the cell K. In the following, we will also use the notations that $\langle \cdot \rangle$ stands for the standard integration over the whole domain Ω_h, while an integral over a sub-domain K is denoted by $\langle \cdot \rangle_K$. The semi-discrete discontinuous Galerkin method in [12] can be described by looking for $\omega_h \in V_h^k$ and $\psi_h \in W_{0,h}^k$ such that

$$\langle \partial_t \omega_h v_h \rangle_K - \langle \omega_h \boldsymbol{u}_h \cdot \nabla v_h \rangle_K + \sum_{e \in \partial K} \langle \boldsymbol{u}_h \cdot \boldsymbol{n} \, \widehat{\omega_h} \, v_h^- \rangle_e = 0, \quad \forall v_h \in V_h^k, \tag{2.2a}$$

$$-\langle \nabla \psi_h \cdot \nabla \varphi_h \rangle_{\Omega_h} = \langle \omega_h, \varphi_h \rangle_{\Omega_h}, \quad \forall \varphi_h \in W_{0,h}^k. \tag{2.2b}$$

where e is a cell boundary and $\boldsymbol{n}$ is its unit out-normal. We now explain the notations used in (2.2). First the velocity field is given by $\boldsymbol{u}_h = \nabla^\perp \psi_h$. Note that even through both ω_h and test function v_h may be discontinuous across the cell boundaries, yet the velocity field possesses continuous normal component across each cell boundary due to the definition of the finite element space $W_{0,h}^k$. Thus the numerical flux in (2.2a) can be defined as follows: Denote by v_h^- (v_h^+) the value of v_h from the inside (outside) of the element K, then the *upwind flux* is set to be

$$\widehat{\omega_h} = \begin{cases} \omega_h^- & \text{if } \boldsymbol{u}_h \cdot \boldsymbol{n} \geq 0, \\ \omega_h^+ & \text{if } \boldsymbol{u}_h \cdot \boldsymbol{n} < 0. \end{cases} \tag{2.3}$$

It should be remarked that for smooth flows, we could use a central flux defined by

$$\widehat{\omega_h} = \tfrac{1}{2} \left(\omega_h^+ + \omega_h^- \right). \tag{2.3$'$}$$

However, the up-wind fluxes (2.3) are preferred since the main concerns here are discontinuous flows.

The first important property of this scheme is the conservation of (no numerical dissipation in) energy

$$\|\nabla^\perp \psi_h(\cdot, t)\|_{L^2(\Omega_h)} = \|\nabla^\perp \psi_h(\cdot, 0)\|_{L^2(\Omega_h)} \tag{2.4}$$

for the upwind flux (2.3), which can be verified directly by taking $v_h = \psi_h$ in (2.2a), summing up the resulting equations over all K in the triangulation, and using (2.2b) and the continuity of the normal velocity across the cell boundaries. Next, taking $v_h = \omega_h$, integrating by parts for the second term in (2.2a), and summing up for all K and estimating the terms involving cell boundaries by using (2.3) and the continuity of the normal component of the velocity field across the cell boundaries, one can show that the enstropy decays in the sense that

$$\|\omega_h(\cdot,t)\|_{L^2(\Omega_h)} \le \|\omega_h(\cdot,0)\|_{L^2(\Omega_h)} \le \|\omega_0\|_{L^2(\Omega_h)} \tag{2.5}$$

where the initial data $\omega_h(\cdot,0)$ is taken as the L^2 projection of ω_0 and hence is uniformly bounded in L^2. Furthermore, taking $\varphi_h = \psi_h$ in (2.2b), one derives the fact that

$$\|\nabla\psi_h\|^2_{L^2(\Omega_h)} = -\langle \omega_h, \psi_h\rangle_{\Omega_h}. \tag{2.6}$$

Our main observation in this paper is the fact that these three simple properties, (2.4)-(2.6), yield a strong convergence. To prove and state such a result, one needs some time-regularity estimate first.

§3. Time Regularity Estimate. In this section, we will prove the following lemma about the time regularity for the approximate solutions constructed by the discontinuous Galerkin method.

Lemma 1. (time regularity) It holds that

$$\|\partial_t\,\omega_h\|_{L^\infty([0,T),W^{-2,q}(\Omega))} + \|\partial_t\,\psi_h\|_{L^\infty([0,T),L^q(\Omega))} \le C \quad \text{for any } 1 \le q < 2 \tag{3.1}$$

where ω_h and ψ_h denote respectively their natural extension or restriction from Ω_h to Ω.

Proof: We first show that

$$\|\partial_t\,\omega_h\|_{L^\infty([0,T),L^2(\Omega_h))} \le \frac{C}{h^2} \tag{3.2}$$

For any smooth function $v \in C_0^\infty(\Omega)$ with zero extension to the outside, let v_h be the L^2 projection of v in the space of V_h^k. Now taking v_h as a test function in (2.2a), one gets

$$\langle \partial_t\,\omega_h\, v\rangle_{\Omega_h} = \langle \partial_t\,\omega_h\, v_h\rangle_{\Omega_h} = \sum_K \langle \omega_h\, \boldsymbol{u}_h \cdot \nabla v_h\rangle_K - \sum_K \sum_{e\in\partial K} \langle \boldsymbol{u}_h \cdot \boldsymbol{n}\, \widehat{\omega_h}\, v_h^-\rangle_e \equiv I_1 + I_2 \tag{3.3}$$

I_1 can be estimated directly as

$$|I_1| \le \|\omega_h\|_{L^\infty} \sum_k \|\boldsymbol{u}_h\|_{L^2(K)} \|\nabla v_h\|_{L^2(K)} \le C\|\omega_h\|_{L^\infty(\Omega_h)} \|\boldsymbol{u}_h\|_{L^2(\Omega_h)} \|\nabla v_h\|_{L^2(\Omega_h)}$$

Using the inverse inequality, the L^2 estimate for $\boldsymbol{u}_h$ and ω_h in (2.4) and (2.5), and the fact that v_h is the L^2 projection of v, one has

$$|I_1| \le \frac{C}{h^2} \|\omega_h\|_{L^2(\Omega_h)} \|\boldsymbol{u}_h\|_{L^2(\Omega_h)} \|v_h\|_{L^2(\Omega_h)} \le \frac{C}{h^2} \|v\|_{L^2(\Omega_h)} \tag{3.4}$$

Next, we proceed to estimate I_2. Since $\boldsymbol{u}_h \cdot \boldsymbol{n}$ is continuous and $\widehat{\omega_h}$ take same value from both sides of a cell boundary, the contribution of the two sides will gives the jump of v_h. Therefore, one can obtain

$$|I_2| \le \sum_K \sum_{e \in \partial K} \langle |\boldsymbol{u}_h \cdot \boldsymbol{n}\, \widehat{\omega_h} (v_h^+ - v_h^-)| \rangle_e \tag{3.5}$$

Thanks to the quasi-uniformly regularity in the triangulation, one can show

$$\sum_K \sum_{e \in \partial K} \|\omega_h\|_{L^2(e)} \|v_h\|_{L^2(e)} \le \frac{C}{h} \sum_K \|\omega_h\|_{L^2(K)} \|v_h\|_{L^2(K)} \tag{3.6}$$

Hence,

$$|I_2| \le \frac{C}{h^2} \|v_h\|_{L^2(\Omega_h)} \le \frac{C}{h^2} \|v\|_{L^2(\Omega_h)} \tag{3.7}$$

Combining (3.4) with (3.7) shows

$$|\langle \partial_t\, \omega_h\, v \rangle_{\Omega_h}| \le \frac{C}{h^2} \|v\|_{L^2(\Omega_h)} \tag{3.8}$$

Which yields the desired estimate (3.2). Since the natural extension of $\partial_t \omega_h$ to Ω from Ω_h gives equivalent norms. Hence we have also shown

$$\|\partial_t\, \omega_h\|_{L^\infty([0,T), L^2(\Omega))} \le \frac{C}{h^2} \tag{3.2$'$}$$

Next, let $\mathcal{I}_h v$ be the piecewise linear interpolation of v in V_h^k. Then one can decompose I_1 in (3.3) as

$$I_1 = \sum_K \langle \omega_h\, \boldsymbol{u}_h \cdot \nabla \mathcal{I}_h v \rangle_K + \sum_K \langle \omega_h\, \boldsymbol{u}_h \cdot \nabla (v_h - \mathcal{I}_h v) \rangle_K \equiv I_{11} + I_{12} \tag{3.9}$$

I_{11} is bounded by

$$|I_{11}| \le \sum_K \|\omega_h\|_{L^2(K)} \|\boldsymbol{u}_h\|_{L^2(K)} \|\nabla \mathcal{I}_h v\|_{L^\infty} \le \|\omega_h\|_{L^2(\Omega_h)} \|\boldsymbol{u}_h\|_{L^2(\Omega_h)} \|\nabla \mathcal{I}_h v\|_{L^\infty(\Omega_h)} \tag{3.10}$$

Due to the inequality

$$\|\nabla \mathcal{I}_h v\|_{L^\infty(\Omega_h)} \le C\|v\|_{W^{1,\infty}(\Omega)}$$

one obtains from (2.4), (2.5), and (3.10) that

$$|I_{11}| \le C\|v\|_{W^{1,\infty}(\Omega)} \tag{3.11}$$

One can estimate I_{12} similarly. Indeed,

$$|I_{12}| \le \sum_K \|\omega_h\|_{L^2(K)} \|\boldsymbol{u}_h\|_{L^2(K)} \|\nabla(v_h - \mathcal{I}_h v)\|_{L^\infty(\Omega_h)} \le C\|\nabla(v_h - \mathcal{I}_h v)\|_{L^\infty(\Omega_h)}$$

Using the inverse inequality

$$\|\nabla(v_h - \mathcal{I}_h v)\|_{L^\infty(\Omega_h)} \le \frac{C}{h^2}\|v_h - \mathcal{I}_h v\|_{L^2(\Omega_h)}$$

and

$$\|v_h - \mathcal{I}_h v\|_{L^2(\Omega_h)} \le \|v - \mathcal{I}_h v\|_{L^2(\Omega_h)} \tag{3.12}$$

which holds true since v_h is the L^2 projection of v, one gets

$$|I_{12}| \le \frac{C}{h^2}\|v - \mathcal{I}_h v\|_{L^2(\Omega_h)} \tag{3.13}$$

This, together with the standard estimate for the interpolation, leads to

$$|I_{12}| \le C\|v\|_{H^2(\Omega)} \tag{3.14}$$

It follows from (3.11) and (3.14) that we have obtained an estimate on I_1 in (3.3) independent of h. Now we can also derive an h-independent estimate on I_2 in (3.3) as follows. Noting that $\mathcal{I}_h v$ is continuous at the cell boundary, one can insert it into the right hand side of (3.5) to obtain

$$|I_2| \le \sum_K \sum_{e\in\partial K} \langle |\boldsymbol{u}_h \cdot \boldsymbol{n}\, \widehat{\omega_h}(v_h - \mathcal{I}_h v)| \rangle_e \le C\|\boldsymbol{u}_h\|_{L^\infty} \sum_K \sum_{e\in\partial K} \|\omega_h\|_{L^2(e)} \|v_h - \mathcal{I}_h v\|_{L^2(e)} \tag{3.15}$$

As in (3.6), one has

$$|I_2| \leq \frac{C}{h^2}\|v_h - \mathcal{I}_h v\|_{L^2(\Omega_h)} \leq \frac{C}{h^2}\|v - \mathcal{I}_h v\|_{L^2(\Omega_h)} \leq C\|v\|_{H^2(\Omega)} \tag{3.16}$$

Collecting all the estimates (3.11), (3.14), and (3.16), we arrive at

$$|\langle \partial_t\, \omega_h, v\rangle_{\Omega_h}| \leq C(\|v\|_{W^{1,\infty}(\Omega)} + \|v\|_{H^2}(\Omega)) \leq C\|v\|_{W^{2,p}(\Omega)} \quad \text{for any } p > 2\,. \tag{3.17}$$

To complete the proof of the first part of (3.1), we need to estimate the difference for the left hand term between Ω and Ω_h. First,

$$|\langle \partial_t\, \omega_h, v\rangle_{\Omega\backslash\Omega_h}| \leq \|\partial_t \omega_h\|_{L^2(\Omega\backslash\Omega_h)}\|v\|_{L^\infty(\Omega\backslash\Omega_h)}\sqrt{|\Omega\backslash\Omega_h|}$$

Note the simple estimate $|\Omega\backslash\Omega_h| \leq Ch^2$, and

$$\|v\|_{L^\infty(\Omega\backslash\Omega_h)} \leq Ch^2\|v\|_{W^{1,\infty}(\Omega)}$$

since v vanishes on the boundary $\partial\Omega$, we can obtain from these and (3.2') that

$$|\langle \partial_t\, \omega_h\, v\rangle_{\Omega\backslash\Omega_h}| \leq Ch\|v\|_{W^{1,\infty}(\Omega)} \leq Ch\|v\|_{W^{2,q}(\Omega)} \tag{3.18}$$

for any $1 \leq q < 2$. Consequently,

$$\|\partial_t\, \omega_h\|_{L^\infty([0,T),W^{-2,q}(\Omega))} \leq C \quad \text{for any } 1 \leq q < 2 \tag{3.19}$$

This proves the first part of (3.1).

Next, we prove the second part of inequality in (3.1). For $f \in L^p(\Omega)$ with zero extension to the outside, we let ϕ solve the following problem

$$-\Delta\phi = f \quad \text{in } \Omega \qquad \phi|_{\partial\Omega} = 0 \tag{3.20}$$

and let $\phi_h \in W^k_{0,h}$ be the finite element solution:

$$\langle \nabla\phi_h, \nabla\varphi_h\rangle_{\Omega_h} = \langle f, \varphi_h\rangle_{\Omega_h} \qquad \text{for any } \varphi_h \in W^k_{0,h} \tag{3.21}$$

Since $\partial_t\psi_h \in W^k_{0,h}$, we have

$$\langle \partial_t\psi_h, f\rangle_{\Omega_h} = \langle \nabla\partial_t\psi_h, \nabla\phi_h\rangle_{\Omega_h} = -\langle \partial_t\omega_h, \phi_h\rangle_{\Omega_h} \tag{3.22}$$

where we have used (3.21) and (2.2b) after taking a time derivative. Rewrite (3.22) as

$$\langle \partial_t \psi_h, f\rangle_{\Omega_h} = -\langle \partial_t \omega_h, \phi\rangle_{\Omega_h} - \langle \partial_t \omega_h, (\phi_h - \phi)\rangle_{\Omega_h} \tag{3.23}$$

Let $p > 2$ be the dual number of q: $1/p + 1/q = 1$, one gets

(3.24)

$$\begin{aligned} |\langle \partial_t \psi_h, f\rangle_{\Omega_h}| &\leq \|\partial_t \omega_h\|_{L^\infty([0,T),W^{-2,q}(\Omega))} \|\phi\|_{W^{2,p}(\Omega)} + \|\partial_t \omega_h\|_{L^\infty([0,T),L^2(\Omega_h))} \|\phi_h - \phi\|_{L^2(\Omega_h)} \\ &\leq C\|\phi\|_{W^{2,p}(\Omega)} + \frac{C}{h^2}\|\phi_h - \phi\|_{L^2(\Omega_h)} \end{aligned}$$

Since the domain is either C^2 or piecewise C^2 with convex corners, the following the elliptic regularity and L^2 estimate is true:

$$\|\phi\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)} \tag{3.25}$$

and

$$\|\phi_h - \phi\|_{L^2(\Omega_h)} \leq Ch^2 \|\phi\|_{H^2(\Omega)} \tag{3.26}$$

Thus, (3.24-26) imply that

$$|\langle \partial_t \psi_h, f\rangle_{\Omega_h}| \leq C\|f\|_{L^p(\Omega)}$$

Or

$$\|\partial_t \psi_h\|_{L^\infty([0,T),L^q(\Omega_h))} \leq C\,, \qquad \text{for any } 1 \leq q < 2 \tag{3.27}$$

Since natural extension of ψ_h to Ω from Ω_h gives equivalent norms, we therefore have shown

$$\|\partial_t \psi_h\|_{L^\infty([0,T),L^q(\Omega))} \leq C\,, \qquad \text{for any } 1 \leq q < 2 \tag{3.27}$$

This gives the second part of (3.1). The proof of the lemma is completed.

§4. A Uniqueness Theorem. In this section, we generalize Yudovich's uniqueness theorem to show that weak solutions with corresponding vorticity in L^∞ are unique in the wider class where the vorticities are in L^2. This generalization will be used in next section to obtain a stronger convergence theorem. The precise statement is given by:

Theorem 1: Assume that $\omega_0 \in L^\infty$. Then the weak solution

$$\omega \in L^\infty([0,T), L^\infty(\Omega)) \cap \text{Lip}\ ([0,T), W^{-2,r}(\Omega)) \tag{4.1}$$

to (2.1) is unique in the space $L^\infty([0,T), L^q(\Omega)) \cap \text{Lip}\ ([0,T), W^{-2,r}(\Omega))$ where $q > 4/3$ and $1 \leq r < 2$.

Proof: Suppose that the initial boundary value problem (2.1) for the 2D Euler equations has two weak solutions with same initial vorticity $\omega_0 \in L^\infty$ and the following regularities:

$$\omega_1 \in L^\infty([0,T), L^\infty(\Omega)) \cap \text{Lip}\ ([0,T), W^{-2,r}(\Omega)) \tag{4.2}$$

and

$$\omega_2 \in L^\infty([0,T), L^q(\Omega)) \cap \text{Lip}\ ([0,T), W^{-2,r}(\Omega)) \tag{4.3}$$

$q > 4/3$ and $1 \leq r < 2$. It suffices to show that $\omega_1 = \omega_2$.

Denote by ψ_1 and ψ_2 the stream functions in (2.1) corresponding to ω_1 and ω_2 respectively. Then by the elliptic regularity, we have

$$\psi_1 \in L^\infty([0,T), W^{2,p}(\Omega)) \cap \text{Lip}\ ([0,T), L^r(\Omega)) \tag{4.4}$$

and

$$\psi_2 \in L^\infty([0,T), W^{2,q}(\Omega)) \cap \text{Lip}\ ([0,T), L^r(\Omega)) \tag{4.5}$$

We denote the correspond velocities by $\boldsymbol{u}_1 = \nabla^\perp \psi_1$ and $\boldsymbol{u}_2 = \nabla^\perp \psi_2$. Then one can rewrite the Euler equations (2.1a) in a distribution sense:

$$\nabla^\perp(\partial_t \boldsymbol{u}_1 + \boldsymbol{u}_1 \nabla \boldsymbol{u}_1) = 0 \quad \in \mathcal{D}' \tag{4.6}$$

and

$$\nabla^\perp(\partial_t \boldsymbol{u}_2 + \boldsymbol{u}_2 \nabla \boldsymbol{u}_2) = 0 \quad \in \mathcal{D}' \tag{4.7}$$

Set

$$\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2, \qquad \psi = \psi_1 - \psi_2 \tag{4.8}$$

and subtract (4.7) from (4.6) to get

$$\nabla^{\perp}(\partial_t \boldsymbol{u} + \boldsymbol{u}_2 \nabla \boldsymbol{u} + \boldsymbol{u} \nabla \boldsymbol{u}_1) = 0 \quad \in \mathcal{D}' \tag{4.9}$$

It follows from (4.4) and (4.5) that

$$\psi \in L^{\infty}([0,T), W^{2,q}(\Omega)) \cap \text{Lip}\ ([0,T), L^r(\Omega)) \tag{4.10}$$

Therefore, one can take ψ as a test function in (4.9) to obtain integration by parts, and use fact that $\boldsymbol{u} = \nabla \perp \psi$ and ψ vanishes on the boundary, we have

$$\int_{\Omega} \boldsymbol{u}(\partial_t \boldsymbol{u} + \boldsymbol{u}_2 \nabla \boldsymbol{u} + \boldsymbol{u} \nabla \boldsymbol{u}_1)\, d\boldsymbol{x} = 0 \tag{4.11}$$

where one has used the fact that $\boldsymbol{u} = \nabla \perp \psi$ and ψ vanishes on the boundray. Define

$$E(t) = \int_{\Omega} |\boldsymbol{u}|^2 \, d\boldsymbol{x} \tag{4.12}$$

Due to the regularity assumption (4.2-4.3), one can show that

$$\frac{d}{dt} E(t) = 2 \int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{u}_t \, d\boldsymbol{x} \tag{4.13}$$

Using the fact

$$\int_{\Omega} \boldsymbol{u}\, \boldsymbol{u}_2 \nabla \boldsymbol{u}\, d\boldsymbol{x} = 0 \tag{4.14}$$

and the equation (4.9), we have

$$\frac{d}{dt} E(t) \leq 2 \int_{\Omega} |\boldsymbol{u}|^2 |\nabla \boldsymbol{u}_1|\, d\boldsymbol{x} \tag{4.15}$$

Using the classical potential estimate

$$\|\nabla u_1(\cdot,t)\|_{L^p} \leq Cp \|\omega_1(\cdot,t)\|_{L^\infty} \tag{4.16}$$

for all $1 < p < \infty$, where C is an constant independent on p, u_1 and ω_1, together with the fact that

$$\|\omega_1(\cdot,t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} \tag{4.18}$$

one shows by the Hölder inequality that

$$\frac{d}{dt}E(t) \leq \left(\int_\Omega |\boldsymbol{u}|^{2p/(p-1)}\, d\boldsymbol{x}\right)^{(p-1)/p} \|\nabla \boldsymbol{u}_1\|_{L^p} \leq Cp\left(\int_\Omega |\boldsymbol{u}|^{2p/(p-1)}\, d\boldsymbol{x}\right)^{(p-1)/p}$$

where C is independent on p. The right hand side above can be further estimated as

$$\int_\Omega |\boldsymbol{u}|^{2p/(p-1)}\, d\boldsymbol{x} = \int_\Omega (|\boldsymbol{u}|^2)^{(p-2)/(p-1)}(|\boldsymbol{u}|^4)^{1/(p-1)} = \|\boldsymbol{u}\|_{L^2}^{2(p-2)/(p-1)}\|\boldsymbol{u}\|_{L^4}^{4/(p-1)}$$

On the other hand,

$$\|\boldsymbol{u}\|_{L^4} \leq C\|\boldsymbol{u}\|_{W^{1,q}}^{q/(4(q-1))}\|\boldsymbol{u}\|_{L^2}^{(3q-4)/(4(q-1))}$$

Since $\|\boldsymbol{u}\|_{W^{1,q}}$ is bounded, we thus have shown that

$$\frac{d}{dt}E(t) \leq CpE(t)^{1-(5q-4)/(4p(q-1))} \tag{4.19}$$

Therefor

$$\frac{d}{dt}\left(E(t)^{(5q-4)/(4p(q-1))}\right) \leq C \tag{4.20}$$

Now one can conclude that $E(t) \equiv 0$. Indeed, taking an interval $[0, T^*]$ with the property that $CT^* \leq (\frac{1}{2})$, one obtains from (4.20) and $E(0) = 0$ that

$$E(t) \leq (\frac{1}{2})^{4p(q-1))/(5q-4)} \to 0 \quad \text{as p tends to infinity} \tag{4.21}$$

So $E(t) \equiv 0$ for $t \in [0, T^*]$. Repeating these arguments we conclude that $E(t) = 0$ for all $t < T$. This completes the proof.

§5. Main Convergence Theorem. Finally, we are able to state and prove our main convergence theorem.

Theorem 2: Let $\Omega \subset R^2$ be a simply connected domain with C^2 boundary (or piecewise smooth C^2 boundary with convex corners), and equipped with a quasi-uniform triangulation. Suppose that the initial vorticity ω_0 belongs to $L^2(\Omega)$. Let $(\omega_h, \psi_h) \in V_h^k \times W_{0,h}^k$ be the approximate solutions generated by the discontinuous Galerkin method (2.2). Then there exists a convergent subsequence of (ω_h, ψ_h) (for which we will still use the same notation for simplicity) such that

$$\omega_h \rightharpoonup \omega \quad \text{(star weakly) in } L^\infty([0,T), L^2(\Omega)) \cap \text{Lip}\ ([0,T), W^{-2,q}(\Omega)) \tag{5.1}$$

for any $1 \leq q < 2$ and

$$\psi_h \to \psi \quad \text{(strongly) in } L^2([0,T), H^1(\Omega)) \tag{5.2}$$

and the limiting functions ω, ψ have the properties that

$$\omega \in L^\infty([0,T), L^2(\Omega)) \cap \text{Lip } ([0,T), W^{-2,q}(\Omega)) \tag{5.3}$$

and

$$\psi \in L^2([0,T), H_0^1(\Omega_0)) \cap \text{Lip } ([0,T), L^q(\Omega)) \tag{5.4}$$

for any $1 \leq q < 2$, and for any $\phi \in C_0^\infty((0,t) \times \Omega)$

$$\int_0^T \int_\Omega (\omega \phi_t + \omega \nabla^\perp \psi \cdot \nabla \phi) \, d\boldsymbol{x} dt = 0 \tag{5.5}$$

$$\Delta \psi = \omega \qquad \text{in } \mathcal{D}' \tag{5.6}$$

In other words, (ω, ψ) is a weak solution to the Euler equation (2.1) with initial data ω_0.

Furthermore, if the initial data $\omega_0 \in L^\infty(\Omega)$, then the whole sequence of (ω_h, ψ_h) will converge to the unique solution of the Euler equations and the limiting vorticity ω is bounded in $L^\infty([0,T), L^\infty(\Omega))$.

Proof: As in the Lemma 1, we extend ω_h and ψ_h to Ω form Ω_h naturally. First, (2.5) and (3.1) show that there is a subsequence of ω_h (for which we still use the same notation) such that

$$\omega_h \rightharpoonup \omega \quad \text{(star weakly) in } L^\infty([0,T), L^2(\Omega)) \cap \text{Lip } ([0,T), W^{-2,q}(\Omega)) \tag{5.7}$$

and the limiting function ω satisfies (5.3).

Next it follows from (2.5), (2.6), and the Poincare inequality that

$$\|\psi_h\|_{L^\infty([0,T), H^1)} \leq C. \tag{5.8}$$

This, together with (3.1), shows that there there is a subsequence of ψ_h such that

$$\psi_h \rightharpoonup \psi \quad \text{(star weakly) in } L^\infty([0,T), H^1(\Omega)) \cap \text{Lip } ([0,T), L^q(\Omega)) \tag{5.9}$$

and

$$\psi \in L^\infty([0,T), H^1(\Omega)) \cap \text{Lip}\ ([0,T), L^q(\Omega))$$

Since the distance between $\partial\Omega_h$ to $\partial\Omega$ is of order $O(h^2)$ and ψ_h vanishes on $\partial\Omega_h$, we have

$$\|\psi_h\|_{L^\infty(\partial\Omega)} \le Ch^2 \|\nabla\psi_h\|_{L^\infty(\Omega_h)} \le Ch\|\nabla\psi_h\|_{L^2(\Omega_h)} \le Ch \to 0 \tag{5.10}$$

Hence ψ satisfies (5.4).

We now show that ψ_h converges strongly. First, It follows from (5.8), (3.1) and the Lions-Aubin lemma that

$$\psi_h \to \psi \quad \text{strongly in } L^2([0,T) \times \Omega)\,, \tag{5.11}$$

which, together with (2.6) and (5.7), yields

$$\int_0^T \|\nabla\psi_h\|_{L^2}^2\, dt = -\int_0^T \langle \omega_h\, \psi_h\rangle\, dt \to -\int_0^T \langle \omega\, \psi\rangle\, dt. \tag{5.12}$$

To obtain the strong convergence that

$$\nabla\psi_h \to \nabla\psi \quad \text{strongly in } L^2([0,T) \times \Omega)\,, \tag{5.13}$$

it suffices to show that

$$\int_0^T \|\nabla\psi_h\|_{L^2}^2\, dt \to \int_0^T \|\nabla\psi\|_{L^2}^2\, dt,$$

which is a direct consequence of

$$-\int_0^T \langle \omega\, \psi\rangle\, dt = \int_0^T \|\nabla\psi\|_{L^2}^2\, dt. \tag{5.14}$$

that will be verified below. Indeed, for any $\phi \in C_0^\infty((0,T) \times \Omega)$, taking $\varphi_h = \mathcal{I}_h\phi$ in (2.2b) yields

$$-\int_0^T \langle \nabla\psi_h \cdot \nabla\mathcal{I}_h\phi\rangle\, dt = \int_0^T \langle \omega_h\, \mathcal{I}_h\phi\rangle\, dt \tag{5.15}$$

where $\mathcal{I}_h$ is the interpolation operator in $W_{0,h}^k$. Using the strong convergence [6],

$$\mathcal{I}_h\phi \to \phi \quad \text{strongly in } L^\infty([0,T), L^2(\Omega)) \text{ and } L^\infty([0,T), H^1(\Omega))\,, \tag{5.16}$$

and weak convergences (5.7) and (5.9), one shows from (5.15) that

$$-\int_0^T \langle \nabla\psi \cdot \nabla\phi \rangle \, dt = \int_0^T \langle \omega \, \phi \rangle \, dt \tag{5.17}$$

This immediately gives (5.14) since $C_0^\infty((0,T) \times \Omega)$ is dense in $L^2([0,T), H_0^1(\Omega))$.

As a sequence of (5.13), one concludes that

$$\boldsymbol{u}_h \to \boldsymbol{u} \quad \text{strongly in } L^2([0,T) \times \Omega) \tag{5.18}$$

where $\boldsymbol{u} = \nabla^\perp \psi$

Now we show that the limit functions (ω, ψ) are indeed a weak solution to the Euler equations. To this end, one can take v_h to be $\mathcal{I}_h \phi$ for any $\phi \in C_0^\infty((0,t) \times \Omega)$ in (2.2a), sum over all the cells, and integrate in time to obtain

$$\int_0^T (\langle \partial_t \, \omega_h \, \mathcal{I}_h \phi \rangle - \langle \omega_h \, \boldsymbol{u}_h \cdot \nabla \mathcal{I}_h \phi \rangle) \, dt = 0, \tag{5.19}$$

where we have used the facts that the upwind fluxes $\widehat{\omega_h}$ are the same for the adjacent elements, both $\mathcal{I}_h \phi$ and the normal component of the velocity field are continuous across the interior cell boundaries, and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on the exterior cell boundaries. Since ϕ is compactly supported, and $\partial_t \mathcal{I}_h \phi = \mathcal{I}_h \partial_t \phi$, we can integrate by part to get

$$\int_0^T (\langle \omega_h \, \mathcal{I}_h \partial_t \phi \rangle + \langle \omega_h \, \boldsymbol{u}_h \cdot \nabla \mathcal{I}_h \phi \rangle) \, dt = 0, \tag{5.20}$$

Using the weak convergence of ω_h in (5.7) and strong convergence of $\mathcal{I}_h \phi$, one shows that

$$\int_0^T \langle \omega_h \, \mathcal{I}_h \partial_t \phi \rangle \, dt \to \int_0^T \langle \omega \, \partial_t \phi \rangle \, dt$$

Similarly, it follows from the weak convergence of ω_h in (5.7), strong convergence of $\boldsymbol{u}_h$ in (5.18) and strong convergence of $\nabla \phi_h$ that

$$\int_0^T \langle \omega_h \, \boldsymbol{u}_h \cdot \nabla \mathcal{I}_h \phi \rangle \, dt \to \int_0^T \langle \omega \, \boldsymbol{u} \cdot \nabla \phi \rangle \, dt.$$

Hence

$$\int_0^T \langle \omega \, \partial_t \phi \rangle \, dt + \int_0^T \langle \omega \, \boldsymbol{u} \cdot \nabla \phi \rangle \, dt = 0. \tag{5.21}$$

This gives (5.5). Finally, (5.6) follows from (2.2b) by taking $\varphi_h = \mathcal{I}_h\phi$. Thus we have proved that (ψ, ω) is a weak solution to the Euler equations (2.1).

In the case that the initial data $\omega_0 \in L^\infty(\Omega)$, then the Cauchy problem for the Euler equations has a solutions $\omega \in L^\infty([0,T) \times \Omega)$, and from Theorem 1 we know that this solution is unique in the class of (5.3) and (5.4). Therefor every convergent subsequence has the same limit. As a consequence, the whole sequence of (ω_h, ψ_h) converges to the unique solution. This completes the proof of the theorem.

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