On the Decay Properties of Solutions to The Nonstationary Navier– Stokes Equations in R^3

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Abstract: In this paper, we study the asymptotic decay properties in both spatial and temperel variables for a class of weak and strong solutions, by constructing the weak and strong solutions in corresponding weighted spaces. It is shown that, for the strong solution, the rate of temperel decay depends on the rate of spatial decay of the initial data. Such a rates of decay are optimal.

1. Introduction

We consider the decay properties of solutions to the nonstationary Navier– Stokes equations in $R^3 \times [0, +\infty)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u = -\nabla p, & \text{in } R^3 \times (0, \infty), \\ \text{div} u = 0, & \text{in } R^3 \times (0, \infty), \\ u \longrightarrow 0, & \text{as } |x| \to +\infty, \\ u(x, 0) = a(x), & \text{in } R^3. \end{cases} \tag{1.1}$$

Here $u = u(x, t) = (u_1, u_2, u_3)$ and $p = p(x, t)$ denote the unknown velocity vector and the pressure of the fluid at point $(x, t) \in R^3 \times (0, \infty)$ respectively, while $\nu > 0$ is the viscosity and $a(x)$ is a given initial velocity vector field. For simplicity, let $\nu = 1$.

Since Leray [25] constructed the weak solutions for (1.1) in R^3, and posed the question about the decay properties of his weak solution, there is large literature discussing the decay properties of weak solutions and strong solutions to the nonstationary Navier-Stokes equations, cf.[1-4,7,8, 10,11, 14-20, 22,23, 26-34, 40,42]. The L^2 decay properties have been extensively studied and the decay

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rates, similar to that of heat equation, are also obtained. Their results show: for each $a \in \overset{o}{J}{}^2(R^3)$, subspace of $L^2(R^3)$ consisting of all solenoidal vector fields, there exists a weak solution u such that

$$\lim_{t\to\infty} \|u(t)\|_2 = 0. \tag{1.2}$$

If $a \in L^r(R^3)(1 \le r < 2)$, then

$$\|u(t)\|_2 \le Ct^{-(3/r-3/2)/2}. \tag{1.3}$$

Here and after, $\|\cdot\|_r$ denotes the norm in $L^r(R^3)$. Furthermore, if $a \in \overset{o}{J}{}^2(R^3) \cap L^1(R^3)$, then

$$\|u(t)\|_r \le C(1+t)^{-(3-3/r)/2} \tag{1.4}$$

for $1 < r \le 2$ and

$$\lim_{t\to\infty} \|u(t)\|_1 = 0. \tag{1.5}$$

If $\|e^{-tA}a\|_1 \le C(1+t)^{-\beta}$ for some $\beta > 0$ with A being the Stokes operator, then

$$\|u(t)\|_r \le C(1+t)^{-(3-3/r+\gamma)/2}, \qquad \gamma = \min\{1, 2\beta\}, \tag{1.6}$$

for $1 \le r \le 2$. See [2-4,19,26,27,29,30,32,33,41]. Schonbek gave the estimates of upper and the lower bound of decay rates for weak solutions in a series of works [32-34]. Moreover, the decay rates of high order derivatives have been studied in [22,23] for weak solutions after sufficient large time. Recent, Miyakawa studied the decay rates in Hardy space and established the decay results for weak solutions in some "L^r- like space" with $r < 1$. For details, see [30,31] and references therein. Therefore, the time-decay properties are well understood. However, there are few results on the spatial decay properties. Farwing and Sohr [11] showed a class of weighted ($|x|^\alpha$) weak solutions with second derivatives about spatial various and one order derivatives about time variable in $L^s(0,+\infty;L^q)$ for $1 < q < 3/2$, $1 < s < 2$ and $0 \le 3/q + 2/s - 4 \le \alpha < \min\{1/2, 3-3/q\}$, in the case of exterior domain. In [10], they also showed that there exists a class of weak solutions satisfying

$$\||x|^{\alpha/2}u\|_2^2 + \int_0^t \||x|^{\alpha/2}\nabla u\|_2^2 d\tau \le \begin{cases} C(a,f,\alpha) & \text{if } 0 \le \alpha < 1/2 & (1.7)_1 \\ C(a,f,\alpha',\alpha)t^{\alpha'/2-1/4} & \text{if } 1/2 \le \alpha < \alpha' < 1, & (1.7)_2 \\ C(a,f)(t^{1/4}+t^{1/2}) & \text{if } \alpha = 1. & (1.7)_3 \end{cases}$$

While in [15], a class of weak solutions

$$(1+|x|^2)^{1/4}u \in L^\infty(0,+\infty;L^p(R^3)) \tag{1.8}$$

was constructed for $6/5 \le p < 3/2$, which satisfies $(1.7)_3$ for $f = 0$.

The first purpose in this paper is to construct weak solutions in weighted spaces. We show, if $a \in L^1(R^3) \cap \overset{o}{J}{}^2(R^3)$ and $|x|a \in L^2(R^3)$, there exists a class of weak solution u satisfying

$$\|(1+|x|^2)^{1/2}u(t)\|_2^2 + \int_0^t \|(1+|x|^2)^{1/2}\nabla u(\tau)\|_2^2 d\tau \le C,$$

which improves the corresponding local weighted estimate as $(1.7)_3$ in [15]. By the interpolation inequality, it implies that u satisfies

$$(1+|x|)^\alpha u(t) \in L^\infty(0,+\infty;L^p(R^3))$$

for $1 < p \leq 2$ and $\alpha = 2(p-1)/p$, which also improves estimate (1.8). Furthermore, if $|x|^{3/2}a \in L^2(R^3)$, then we show that there exists a class of weak solutions satisfying

$$\|(1+|x|^2)^{\alpha/2}u(t)\|_2^2 + \int_0^t \|(1+|x|^2)^{\alpha/2}\nabla u(\tau)\|_2^2 d\tau \leq C(1+\log(1+t)), \quad \forall t \geq 0, \tag{1.9}$$

for $0 \leq \alpha \leq 3$. If $\|e^{-tA}a\|_1 \leq C(1+t)^{-\gamma}$ for some $\gamma > 0$, then the right hand side of (1.9) can be replaced by a constant independent of t. It should be noted that Schonbek and Schonbek [35] studied the decay properties of moment estimate $\||x|^{\alpha/2}u\|_2$ for smooth solutions.

The second purpose of this paper is to establish the estimates of decay rate of weighted norm for strong solutions in $L^p(R^3)$. For the existence and decay properties of strong solutions, there are many results, see [1, 5-9, 12-18, 20,21,28,40,42]. In particular, Heywood showed the existence and decay properties of strong solutions as $\|a\|_{H^1}$ is small in [17]. Miyakawa [28] improved Heywood's results by a different method. On the other hand, many results about the existence and decay properties of strong solution have been obtained in space $L^p(0,T;L^q(R^n))$ for $q \geq n \geq 3$ and $1/p + n/2q \leq 1/2$ in [1,6-9,12,14,16,18,20,21,42]. In particular, let $BC(0,T)(T>0)$ denote the set of bounded and continuous functions defined in $(0,T)$, Kato[20] first obtains the following decay rates for strong solutions in $L^p(R^n)$

$$t^{(1-n/q)/2}u \in BC([0,+\infty; L^q(R^n)) \quad \forall n \leq q \leq +\infty, \tag{1.10}$$

$$t^{(1-n/q)/2+1/2}\nabla u \in BC([0,+\infty; L^q(R^n)) \quad \forall n \leq q < +\infty \tag{1.11}$$

provided that $a \in L^n(R^n)$ and $\|a\|_n$ is small. In the case that $a \in L^n(R^n) \cap L^p(R^n)(1 < p < n)$, then

$$t^{(3/p-n/q)/2}u \in BC([1,+\infty; L^q(R^n)) \quad \forall p \leq q \leq +\infty, \tag{1.12}$$

$$t^{(n/p-n/q)/2+1/2}\nabla u \in BC([1,+\infty; L^q(R^n)) \quad \forall p \leq q < +\infty \tag{1.13}$$

provided the exponent of t in (1.12) and (1.13) are smaller than 1; otherwise, (1.12) and (1.13) are valid for any positive number less than 1 as the exponent of t are bigger than 1. By adding a correction term, Carpio [7] showed, for initial data a satisfying: 1) $\|a\|_n (n \geq 3)$ is small; 2) $a \in L^p(R^n) \cap L^n(R^n)(1 \leq p < n)$, the solution to the Navier-Stokes equations behaves like the solutions of the heat equations taking the same initial data. Thus Carpio [7] removes Kato's restriction on exponent of t and extends the decay estimate to reach the case $p = 1$. So this problem is also well understood. While for the spatial decay properties at large distance, a class of weighted strong solutions was constructed, which satisfies that $(1+|x|^2)u \in L^\infty(0,+\infty; L^2(R^3))$ and $t^{1/2}\nabla u \in L^\infty(0,+\infty; L^2(R^3))$, if $\|a\|_2 + \|a\|_p$ small for some $1 \leq p < 2$ in [15], and similar results for exterior domains in [16]. Recently, Takahashi[39] showed that if u is a smooth solution, then

$$t^\beta |x|^\alpha |u(x,t)| \leq C \quad \text{for } |x| + t \ \text{ large} \tag{1.14}$$

for all $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha + 2\beta < 3$. Meanwhile, Miyakawa [31] pointed out, if $a = \partial b/\partial x_1$, then

$$\sup_{t>0} |e^{-tA}a| \sim C|x|^{-1-n} \quad \text{and} \quad \sup_x |e^{-tA}a| \sim Ct^{-(n+1)/2},$$

as $|x| \to \infty$ and $t \to \infty$. Thus he conjectures that, if u is a smooth solution, then u should satisfy that

$$|u(x,t)| \sim C|x|^{-\alpha}t^{-\beta} \quad \text{as } |x| + t \to \infty, \tag{1.15}$$

for all $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha + 2\beta = n+1$.

In this paper, we show that, with the hypothesis that $\|a\|_1 + \|a\|_2$ is small, then there exists a unique strong solution u such that

$$t^{\beta}\|(1+|x|^2)^{\alpha/2}u(t)\|_p \leq C \quad \text{for } t \geq 0 \tag{1.16}$$

and

$$t^{\beta+1/2}\|(1+|x|^2)^{\alpha/2}\nabla u(t)\|_p \leq C \quad \text{for } t \geq 0 \tag{1.17}$$

with $\alpha = 3 - 3/p_0$, $\beta = (3/p_0 - 3/p)/2$ for $1 \leq p_0 \leq p \leq +\infty$ and $3 < p$. So $\alpha + 2\beta = 3 - 3/p$. Therefore, if $p = \infty$, our results improve (1.14). Moreover, the time decay rates of weighted norm of the solutions also improve the results of Kato's in the sense that there is no restriction on exponent of t and p_0 can be taken to be 1. If $a = \partial b/\partial x_i$ for some $i = 1,2,3$, we show, if $\|a\|_1 + \|a\|_2$ small, there exists a unique strong solutions u, which satisfies that

$$t^{\beta+1/2}\|(1+|x|^2)^{\alpha/2}u(t)\|_p \leq C \quad \text{for } t \geq 0. \tag{1.18}$$

In the special case $p = \infty$, our result (1.18) shows that Miyakawa's conjecture (1.15) is right. Finally, it should be noted that, in order to obtain the global strong solutions, we assume the smallness of the initial data, i.e., $\|a\|_1 + \|a\|_2 \leq \delta$, which different from any previous known small assumptions.

The paper organize as follows: In section 2, we state our main results. A new class of approximate solutions are constructed and then the integral representations are delivered in section 3. In section 4 and 5, we establish the main weighted estimates for weak and strong solutions.

We conclude this introduction by listing some notations used in the rest of the paper.

Let $L^p(R^3), 1 \leq p \leq +\infty$, represent the usual Lesbegue space of scalar functions as well as that of vector-valued functions with norm denoted by $\|\cdot\|_p$. Let $C^\infty_{0,\sigma}(R^3)$ denote the set of all C^∞ real vector-valued functions $\phi = (\phi_1, \phi_2, \phi_3)$ with compact support in R^3, such that $\mathrm{div}\phi = 0$. $\overset{o}{J}{}^p(R^3)$, $1 \leq p < \infty$, is the closure of $C^\infty_{0,\sigma}(R^3)$ with respect to $\|\cdot\|_p$. $H^m(R^3)$ denotes the usual Sobolev Space. Finally, given a Banach space X with norm $\|\cdot\|_X$, we denote by $L^p(0,T;X), 1 \leq p \leq +\infty$, the set of function $f(t)$ defined on $(0,T)$ with values in X such that $\int_0^T \|f(t)\|^p_X dt < +\infty$. Let P be the Helmholtz projection from $L^p(R^3)$) to $\overset{o}{J}{}^p(R^3)$. Then the Stokes operator A is defined by $A = -P\Delta$ with $D(A) = H^2(R^3) \cap \overset{o}{J}{}^2(R^3)$. Let $D^2 = \sum_{i,j}|\partial^2/\partial x_i\partial x_j|$ and $D^3 = \sum_{jik}|\partial^3/\partial x_i\partial x_j\partial x_k|$. At last, by symbol C, we denote a generic constant whose value is unessential to our aims, and it may change from line to line.

2. The Main Results

We first give the definitions of weak and strong solutions.

Definition 1 u is called a weak solution to the Cauchy problem (1.1) if
1) $u \in L^\infty(0,T;L^2(R^3) \cap L^2(0,T;H^1(R^3))$ for any $T > 0$,
2) u satisfies the equations (1.1) in the sense ofdistribution, i.e.,

$$\int_0^\infty \int_{R^3} (-\frac{\partial \phi}{\partial \tau}u + \nabla u \cdot \nabla \phi + (u\cdot\nabla)u\cdot\phi)dxd\tau = \int_{R^3} \phi(x,0)a(x)$$

for every $\phi \in C_{0,\sigma}^{\infty}(R \times R^3)$.

3) div$u = 0$ in the sense of distribution, i.e.,

$$\int_{R^3} u(x,t)\nabla\psi(x) = 0$$

for every $\psi \in C_0^{\infty}(R^3)$.

Definition 2 u is called a strong solution to the Cauchy problem (1.1) if $u \in L^{\infty}(0,T;L^p(R^3))$ for $3 < p \le +\infty$ and any $T > 0$, and 2) 3) in the Definition 1 hold for u.

The main results in this paper are described in the following theorems. Our first result concerns the existence and global estimates of weak solutions in a weighted L^2-space.

Theorem 1 Let $a \in L^1(R^3) \cap \overset{o}{J}{}^2(R^3)$ and $(1+|x|^2)^{1/2}a \in L^2(R^3)$. Then there exists a weak solution u in $L^{\infty}(0,+\infty;L^2(R^3))$ to (1.1) such that

$$\|u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 d\tau \le 4\|a\|_2^2 \tag{2.1}$$

and

$$\|(1+|x|^2)^{1/2}u\|_2^2 + \int_0^t \|(1+|x|^2)^{1/2}\nabla u\|_2^2 d\tau \le CA_1 \tag{2.2}$$

for any $t \ge 0$. Moreover, for $t \ge 0$,

$$\|u(t)\|_2 \le CN(1+t)^{-3/4} \tag{2.3}$$

and

$$\|u(t)\|_1 \le CB. \tag{2.4}$$

Here

$$\begin{aligned} A_1 =&\quad e^{C\|a\|_2^2}\Big(\|(1+|x|^2)^{1/2}a\|_2^2 + N^2\Big) \\ N =&\quad \|a\|_1 + \|a\|_1^2 + \|a\|_2 + \|a\|_2^2 \\ B =&\quad \|a\|_1 + \|a\|_2 N. \end{aligned}$$

In the case that the initial velocity field possesses higher order moments, we have the following estimates:

Theorem 2 Let $a \in L^1(R^3) \cap \overset{o}{J}{}^2(R^3)$ and $|x|^{3/2}a \in L^2(R^3)$, then there exists a weak solution u in $L^{\infty}(0,+\infty;L^2(R^3))$ to (1.1), which satisfies that

$$\int_{R^3}(1+|x|^2)^{3/2}u^2dx + \int_0^t\int_{R^3}(1+|x|^2)^{3/2}|\nabla u|^2dxd\tau \le C(A_2 + B^{2/3}N^{4/3}\log(1+t)) \tag{2.5}$$

for any $t \ge 0$, (2.1)-(2.4) are valid for u. Moreover, if $\|e^{-tA}a\|_1 \le C(1+t)^{-\gamma}$ for some $\gamma > 0$, then

$$\int_{R^3}(1+|x|^2)^{3/2}u^2dx + \int_0^t\int_{R^3}(1+|x|^2)^{3/2}|\nabla u|^2dxd\tau \le C(A_2 + B^{2/3}N^{4/3}) \tag{2.6}$$

for any $t \ge 0$ with $A_2 = A_1^{3/2}(\|a\|_2^{5/4}N^{3/4} + N\|a\|_2^{5/6})$.

Next, the weighted norm (both in time and in space) estimates of strong solutions are established in the following theorem.

Theorem 3 Let $a \in L^1(R^3) \cap \overset{o}{J}{}^2(R^3)$, $(1+|x|^2)^{1/2}a \in L^2(R^3)$ and $(1+|x|^2)^{\alpha/2}a \in L^{p_0}(R^3)$ for $1 \le p_0 \le +\infty$ and $\alpha = 3 - 3/p_0$. Then there exists a constant $\lambda > 0$, such that if $\|a\|_1 + \|a\|_2 \le \lambda$, then there exists a unique strong solution $u \in L^\infty(0, +\infty; L^p(R^3))$ for $3 < p \le +\infty$, which satisfies the estimate

$$\begin{aligned}&\|t^\beta(1+|x|^2)^{\alpha/2}u\|_p + \|t^{1/2+\beta}(1+|x|^2)^{\alpha/2}\nabla u\|_p \\ &\quad \le C\Big(\|(1+|x|^2)^{\alpha/2}a\|_{p_0} + N^2 + A_1^{5/2}B^{3/2} + B^{1/2}A_1N\Big) \qquad (2.7)\end{aligned}$$

for any $t \ge 0$ and $\beta = (3/p_0 - 3/p)/2$.

Finally, for a class of special initial data, the results in Theorem 3 can be improved as:

Theorem 4 Let $a \in L^1(R^3) \cap \overset{o}{J}{}^2(R^3)$ and $(1+|x|^2)^{1/2}a \in L^2(R^3)$. Let $a = \partial b/\partial x_i$ for some $i = 1, 2, 3$ with $b \in L^1(R^3)$ and $(1+|x|^2)^{\alpha/2}b \in L^{p_0}(R^3)$ for $1 \le p_0 \le +\infty$ and $\alpha = 3 - 3/p_0$. Then there exist a constant $\lambda_0 > 0$, such that if $\|a\|_1 + \|a\|_2 \le \lambda_0$, then there exists a unique strong solution $u \in L^\infty(0, +\infty; L^p(R^3))$ for $3 < p \le +\infty$, which satisfies the estimate

$$\|t^{1/2+\beta}(1+|x|^2)^{\alpha/2}u\|_p \le C\Big(\|(1+|x|^2)^{\alpha/2}b\|_{p_0} + \|b\|_1 + N^2 + A_1^{5/2}B^{3/2} + B^{1/2}A_1N\Big) \qquad (2.8)$$

for any $t \ge 0$ and $\beta = (3/p_0 - 3/p)/2$.

Remarks:

1. By interpolation inequality, (2.3) and (2.4) implies that the solution u, obtained in Theorem 1 and 2, satisfies decay property (1.4), which for weak solutions have already been obtained in [3,19] etc. However, (1.5) is not a consequence.

2. The weak solution u, obtained in Theorem 1, also satisfies that

$$(1+|x|)^\alpha u \in L^\infty(0, +\infty; L^p(R^3))$$

for $1 < p \le 2$ and $\alpha = 2(p-1)/p$, which also improve the corresponding results in [15,10,11].

3. Schonbek and Schonbek [35] studied the decay properties of the moment estimate $\||x|^\alpha u\|_2$ for $0 \le \alpha \le 3/2$, when u is a smooth solution. While we get estimates (2.5) and (2.6) for weak solution. Moreover, the weak solution u, obtained in Theorem 2, satisfies that, for any $0 \le \alpha \le 3$,

$$\int_{R^3} |x|^\alpha |u|^2 dx + \int_0^t \int_{R^3} |x|^\alpha |\nabla u|^2 dx d\tau$$

can be dominated by the terms at right hand of (2.5) or (2.6), respectively.

4. In Theorem 2, the assumption $\|e^{-tA}a\|_1 \le C(1+t)^{-\gamma}$ holds for some $\gamma > 0$, if $a = A^\gamma b$ for some $b \in L^1(R^3)$.

5. In Theorem 3, $\alpha + 2\beta = 3 - 3/p$. When $p = +\infty$, our result improve Takahashi's result. Moreover, We obtain the decay rate similar to that of Kato, for weighted norm. At same time, we remove the restriction on the exponent of t and extend the decay rate to reach to the case $p_0 = 1$. By adding a correction term, Carpio [7] have showed the solution to the Navier- Stokes equations behaves like the solution of the heat equation, with same initial data, in $L^q(R^n)(n \ge 3)$ for $q \ge p$, as initial data a satisfying 1) $\|a\|_n$ is small, 2) $a \in L^p(R^n) \cap L^n(R^n)$ for $1 \le p < n$. But our results are different from that in [7].

6. Taking $p = \infty$ in Theorem 4, then estimate (2.8) yields (1.15), which gives an assertive answer to Miyakawa's conjecture.

7. Let $p_0 = 1$ and $p = \infty$ in Theorem 4, then

$$\|u(t)\|_\infty = O(t^{-2}),$$

which has been proved by Miyakawa under different assumptions on initial data a. See Theorem 1.9 i) in [31].

Applying the weighted estimates obtained in section 4 and 5, the proof of Theorem 1-4 are standard. So we will only deduce the necessary weighted estimates, and omit the details of the procedure of the proof.

3. The Approximation Solutions and Their Integral Representations

In this section, we construct a sequence of approximate solutions by using the linearized Navier-Stokes equations in R^3, and derive the integral representations of the approximate solutions. First, let $a \in \overset{o}{J}{}^p(R^3) \cap \overset{o}{J}{}^q(R^3)(1 \le p, q \le +\infty)$. We select $a^k \in C^\infty_{0,\sigma}(R^3)$, such that

$$a^k \longrightarrow a \quad \text{in } \overset{o}{J}{}^p(R^3) \cap \overset{o}{J}{}^q(R^3) \text{ strongly}$$

and

$$\|a^k\|_p \le 2\|a\|_p, \qquad \|a^k\|_q \le 2\|a\|_q. \tag{3.1}$$

The approximate solutions are defined as follows: let (u^0, p^0) solve the the Cauchy problem of the Stokes equations

$$\begin{cases} \dfrac{\partial u^0}{\partial t} - \Delta u^0 = -\nabla p^0, & \text{in } R^3 \times (0,\infty), \\ \text{div} u^0 = 0, & \text{in } R^3 \times (0,\infty), \\ u^0 \longrightarrow 0, & \text{as } |x| \to +\infty, \\ u^0(x,0) = a^0(x), & \text{in } R^3 \end{cases} \tag{3.2}$$

and $(u^k, p^k)(k \ge 1)$ solve the Cauchy problem for the linearized Navier- Stokes equations

$$\begin{cases} \dfrac{\partial u^k}{\partial t} - \Delta u^k + (u^{k-1} \cdot \nabla) u^k = -\nabla p^k, & \text{in } R^3 \times (0,\infty), \\ \text{div} u^k = 0, & \text{in } R^3 \times (0,\infty), \\ u^k \longrightarrow 0, & \text{as } |x| \to +\infty, \\ u^k(x,0) = a^k(x), & \text{in } R^3 \end{cases} \tag{3.3}$$

for $k \ge 1$. It is well known (cf. [24]) that there exists a unique solution $u^k (k \ge 0)$ to (3.2) and (3.3) satisfying

$$\frac{\partial u^k}{\partial t}, \ \frac{\partial u^k}{\partial x_i}, \ \frac{\partial^2 u^k}{\partial x_i \partial x_j}, \ \frac{\partial p^k}{\partial x_i} \in L^2(0,T; L^2(R^3)) \tag{3.4}$$

for $i, j = 1, 2, 3$, $k \ge 0$ and any $T > 0$.

In order to derive an integral expression for u^k, one can use the singular integral expression of the projection operator $P : L^2(R^3) \longrightarrow \overset{o}{J}{}^2(R^3)$, that is:

$$P\phi = \phi + \frac{1}{4\pi}\nabla\text{div}\int_{R^3}\frac{\phi(y)}{|x-y|}dy \tag{3.5}$$

for any $\phi \in L^2(R^3)$(cf. [24]). Applying the fundamental solution of the heat equation, on can rewrite the solution to the Cauchy problem for the Stokes equations

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = -\nabla p + f, \\ \text{div}v = 0, \\ v(x,0) = 0 \end{cases}$$

as

$$v_i = \int_0^t \int_{R^3} V^i(x-y, t-\tau)\cdot f(y,\tau)dyd\tau, \qquad i = 1,2,3, \tag{3.6}$$

where

$$\begin{cases} V^i(x,t) = \Gamma(x,t)e^i + \dfrac{1}{4\pi}\nabla\dfrac{\partial}{\partial x_i}\displaystyle\int_{R^3}\frac{\Gamma(x-z,t)}{|z|}dz = P(\Gamma e^i) \\ \Gamma(x,t) = (4\pi t)^{-3/2}e^{-|x|^2/4t} \end{cases} \tag{3.7}$$

and e^i is the unite vector along x_i- axis. It is easy to see that

$$V^i(x,t) = \text{curl}(\text{curl}\omega^i) = -\Delta\omega^i + \nabla\text{div}\omega^i, \quad i = 1,2,3$$

with

$$\begin{cases} \omega^i(x,t) = \dfrac{1}{4\pi}\displaystyle\int_{R^3}\frac{\Gamma(x-z,t)}{|z|}dze^i, \\ \bar{\theta}(x,t) = \dfrac{1}{4\pi}\displaystyle\int_{R^3}\frac{\Gamma(x-z,t)}{|z|}dz. \end{cases} \tag{3.8}$$

For the detailed derivation of (3.7) and (3.8), see Ladyzhenskaya[24].

For simplicity of writting, we drop the right upper label k of the solution u^k of (3.3) and use b to denote u^{k-1}. Let y and τ denote the variables in equations (3.3). We multiply both sides of (3.3) by $V^i(x-y,t-\tau)$, then integrate for $y \in R^3$ and $\tau \in [0, t-\varepsilon]$ for arbitrary $0 < \varepsilon < t$, to get that

$$\int_0^{t-\varepsilon}\int_{R^3}(\frac{\partial u}{\partial \tau} - \Delta u)(y,\tau)V^i(x-y,t-\tau)dyd\tau$$
$$= \int_0^{t-\varepsilon}\int_{R^3}(-\nabla_y p - (b\cdot\nabla)u)(y,\tau)V^i(x-y,t-\tau)dyd\tau.$$

Since $(-\partial/\partial\tau - \Delta_y)V^i = 0$ and $V^i = P(\Gamma e^i)$, it follows that

$$\int_{R^3} u(y,t-\varepsilon)V^i(x-y,\varepsilon)dy - \int_{R^3} a(y)V^i(x-y,t)dy = -\int_0^{t-\varepsilon}\int_{R^3}(b\cdot\nabla)u(y,\tau)V^i(x-y,t-\tau)dyd\tau.$$

Since u is divergence free, so it follows, by the structure of $V^i(x-y,t-\tau)$, that

$$\lim_{\varepsilon\to 0}\int_{R^3} u(y,t-\varepsilon)V^i(x-y,\varepsilon)dy = u_i(x,t)$$

here u_i denotes the i-th component of vector u. Thus,

$$u_i = -\int_0^t \int_{R^3} (b\cdot\nabla)u(y,\tau)V^i(x-y,t-\tau)dyd\tau + \int_{R^3} a(y)V^i(x-y,t)dy.$$

Substituting (3.7) into above equation, we get that

$$\begin{aligned} u_i = & -\int_0^t \int_{R^3} (b\cdot\nabla)u(y,\tau)\Gamma(x-y,t-\tau)e^i dyd\tau \\ & -\int_0^t \int_{R^3} (b\cdot\nabla)u(y,\tau)\nabla\frac{\partial}{\partial y_i}\bar{\theta}(x-y,t-\tau)dyd\tau + \int_{R^3} a(y)\Gamma(x-y,t)e^i dy. \end{aligned} \tag{3.9}$$

By integration by parts, we arrive at the desired integral representation form:

$$\begin{aligned} u_i = & \int_0^t \int_{R^3} \sum_j b_j u_i(y,\tau)\frac{\partial}{\partial y_j}\Gamma(x-y,t-\tau)dyd\tau \\ & + \int_0^t \int_{R^3} \sum_{l,k=1}^3 b_l u_k(y,\tau)\frac{\partial^3}{\partial y_i\partial y_l\partial y_k}\bar{\theta}(x-y,t-\tau)dyd\tau + \int_{R^3} a(y)\Gamma(x-y,t)e^i dy. \end{aligned} \tag{3.10}$$

Let

$$\begin{cases} I_1^k = \int_{R^3} |a^k|(y)\Gamma(x-y,t)dy, \\ I_2^k = \int_0^{t/2}\int_{R^3} |u^{k-1}||u^k|(y,\tau)\big(|\nabla\Gamma| + |D^3\bar{\theta}|\big)(x-y,t-\tau)dyd\tau, \\ I_3^k = \int_{t/2}^t\int_{R^3} |u^{k-1}||u^k|(y,\tau)\big(|\nabla\Gamma| + |D^3\bar{\theta}|\big)(x-y,t-\tau)dyd\tau. \end{cases} \tag{3.11}$$

Thus

$$|u^k(x,t)| \le C(I_1^k + I_2^k + I_3^k). \tag{3.12}$$

For Γ and $\bar{\theta}$, direct calculations show that

$$\begin{cases} |D^m\Gamma(x,t)| \le C_m(|x|^2+t)^{-(m+3)/2}, \\ |D^m\bar{\theta}(x,t)| \le C_m(|x|^2+t)^{-(m+1)/2}. \end{cases} \tag{3.13}$$

for $m \in N$.

4. Weighted Estimates for the Approximate Solutions I

In this section, we establish some a priori estimates for the approximate solutions constructed in section 3, which result in Theorem 1 and 2 by standard compactness argument. First, standard enery estimates yield that

Lemma 4.1 Let $a \in \overset{o}{J}{}^2(R^3)$. Then the estimates

$$\begin{cases} \|u^k(t)\|_2 \le 2\|a\|_2 \quad \forall t>0 \\ \int_0^\infty \|\nabla u^k(s)\|_2^2 ds \le 4\|a\|_2^2 \end{cases} \tag{4.1}$$

hold uniformly for $k \geq 0$.

Next, it follows, from (3.10)-(3.13), (4.1) and standard convolution estimates, that

Lemma 4.2 Let $a \in L^1(R^3) \cap \overset{o}{J}{}^2(R^3)$. Then, we have

$$\|u^k(t)\|_2 \leq CNt^{-3/4} \tag{4.2}$$

and

$$\|u^k(t)\|_1 \leq CB \tag{4.3}$$

holds uniformly for $k \geq 0$ and $t > 0$. Furthermore, if $\|e^{-tA}a\|_1 \leq C(1+t)^{-\gamma}$ for some $\gamma > 0$, then

$$\|u^k(t)\|_2 \leq Ct^{-3/4-\gamma_1/2}, \tag{4.4}$$

for $\gamma_1 = \min\{1, 2\gamma\}$. Where $N = \|a\|_1 + \|a\|_1^2 + \|a\|_2 + \|a\|_2^2$ and $B = \|a\|_1 + \|a\|_2 N$.

We now turn to the main weighted norm estimates in Theorem 1 and 2.

Lemma 4.3 Let $a \in L^1(R^3)$ and $(1+|x|^2)^{1/2}a \in L^2(R^3)$. Then

$$\|(1+|x|^2)^{1/2}u^k\|_2^2 + \int_0^t \|(1+|x|^2)^{1/2}|\nabla u^k|\|_2^2 dx d\tau \leq CA_1 \tag{4.5}$$

for any $k \geq 0$ and $t \geq 0$ with $A_1 = e^{C\|a\|_2^2}(\|(1+|x|^2)^{1/2}a\|_2^2 + N^2)$.

Proof Taking the divergence of the first equations of (3.3) yields that

$$-\Delta p^k = \sum_{ij=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}(u_i^{k-1}u_j^k). \tag{4.6}$$

Then standard Calderón-Zygmund estimate gives that

$$\|p^k\|_r \leq C\|u^{k-1}\|_{2r}\|u^k\|_{2r}$$

for $1 < r < +\infty$.

If $(1+|x|^2)^{1/2}u^{k-1} \in L^\infty_{loc}(0,\infty; L^2(R^3))$, we can show that $(1+|x|^2)^{1/2}u^k \in L^\infty_{loc}(0,\infty; L^2(R^3))$, by (3.10)- (3.13). By induction,

$$\int_{R^3}(1+|x|^2)|u^k|^2 dx \quad \text{and} \quad \int_0^t\int_{R^3}(1+|x|^2)|\nabla u^k|^2 dx d\tau$$

are well defined. Multiplying the first equation in (3.3) by $(1+|x|^2)u^k$ and integrating over R^3, one obtains that

$$\begin{aligned}\frac{d}{dt}\|(1+|x|^2)^{1/2}u^k\|_2^2 + \int_{R^3}(1+|x|^2)|\nabla u^k|^2 dx &\leq C\|u^k\|_2^2 + C\|(1+|x|^2)^{1/2}u^k\|_2\|u^{k-1}\|_4\|u^k\|_4 \\ &\leq 2\|u^k\|_2^2 + C\|(1+|x|^2)^{1/2}u^k\|_2\|u^{k-1}\|_2^{1/4}\|u^k\|_2^{1/4}\|\nabla u^{k-1}\|_2^{3/4}\|\nabla u^k\|_2^{3/4} \\ \leq \|(1+|x|^2)^{1/2}u^k\|_2^2\|\nabla u^{k-1}\|_2\|\nabla u^k\|_2 &+ C\|u^k\|_2^2 + C\|u^{k-1}\|_2\|u^k\|_2 + \|\nabla u^{k-1}\|_2\|\nabla u^k\|_2.\end{aligned} \tag{4.7}$$

Now (4.5) follows from (4.1), (4.2) and (4.7), by Gronwall's inequality. □

The higher order moment are estimated as follows:

Lemma 4.4 Let $a \in L^1(R^3) \cap \overset{o}{J}{}^2(R^3)$ and $|x|^{3/2}a \in L^2(R^3)$. Then

$$\int_{R^3} (1+|x|^2)^{3/2}|u^k|^2 dx + \int_0^t \int_{R^3} (1+|x|^2)^{3/2}|\nabla u^k|^2 dx d\tau \le CA_2 + B^{3/2}N^{4/3}\log(1+t) \quad (4.8)$$

hold uniformly for $k \ge 0$ and $t \ge 0$ with $A_2 = A_1^{3/2}(\|a\|_2^{5/4}N^{3/4} + \|a\|_2^{5/6}N)$.

Moreover, if $\|e^{-tA}a\|_1 \le C(1+t)^{-\gamma}$, then

$$\int_{R^3} (1+|x|^2)^{3/2}|u^k|^2 dx + \int_0^t \int_{R^3} (1+|x|^2)^{3/2}|\nabla u^k|^2 dx d\tau \le CA_2 + N^2(3/2) \quad (4.9)$$

hold uniformly for $k \ge 0$ and $t \ge 0$.

Proof Applying estimate (3.13) and the inequality

$$(1+|x|^2)^{\alpha/2} \le 2^{\alpha/2}((1+|y|^2)^{\alpha/2} + |x-y|^{\alpha}) \quad \text{for } \alpha \ge 0, \quad (4.10)$$

one can show by using (3.10)-(3.12) and longthy calculation that

$$(1+|x|^2)^{3/4}u^k \in L^{\infty}_{loc}(0,+\infty; L^2(R^3)) \text{ and } (1+|x|^2)^{3/4}|\nabla u^k| \in L^2_{loc}(0,+\infty; L^2(R^3))$$

as long as $(1+|x|^2)^{3/4}u^{k-1} \in L^{\infty}_{loc}(0,+\infty; L^2(R^3))$. By induction,

$$\int_{R^3} (1+|x|^2)^{3/2}|u^k|^2 dx \quad \text{and} \quad \int_0^t \int_{R^3} (1+|x|^2)^{3/2}|\nabla u^k|^2 dx d\tau$$

are well-defined.

We now multiply both sides of (3.3) by $(1+|x|^2)^{3/2}u^k$ and integrate over R^3 to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{R^3} (1+|x|^2)^{3/2}|u^k|^2 dx + \int_{R^3} (1+|x|^2)^{3/2}|\nabla u^k|^2 dx \\ \le\ & 3\int_{R^3} (1+|x|^2)|u^k||\nabla u^k| dx + 3\int_{R^3} (1+|x|^2)|u^{k-1}||u^k|^2 dx + 3\int_{R^3} (1+|x|^2)|u^k||p^k| dx. \end{aligned} \quad (4.11)$$

Employing the weighted estimates on singular integral operators (cf. [36]), we deduce from (4.6) that

$$\|(1+|x|^2)^{1/2}p^k\|_2 \le C\|(1+|x|^2)^{1/2}|u^{k-1}||u^k|\|_2.$$

Thus

$$\begin{aligned} &\int_{R^3} (1+|x|^2)|p^k||u^k| dx \le \|(1+|x|^2)^{1/2}u^k\|_2 \|(1+|x|^2)^{1/2}p^k\|_2 \\ \le\ & C\|(1+|x|^2)^{1/2}u^k\|_2 \|(1+|x|^2)^{1/2}|u^{k-1}||u^k|\|_2 \\ \le\ & C\|(1+|x|^2)^{1/2}u^k\|_2 \|(1+|x|^2)^{3/4}u^k\|_6^{2/3} \|u^k\|_{24/7}^{1/3} \|u^{k-1}\|_{24/7} \\ \le\ & CA_1\|\nabla\{(1+|x|^2)^{3/4}u^k\}\|_2^{2/3} \|u^k\|_{24/7}^{1/3} \|u^{k-1}\|_{24/7} \\ \le\ & \frac{1}{4}\|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 + CA_1^{3/2}\|u^k\|_{24/7}^{1/2}\|u^{k-1}\|_{24/7}^{3/2} \\ & \qquad + CA_1\|(1+|x|^2)^{1/4}u^k\|_2 \|u^k\|_{24/7}^{1/3}\|u^{k-1}\|_{24/7} \\ \le\ & \frac{1}{4}\|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 + CA_1^{3/2}\|u^k\|_2^{3/16}\|u^{k-1}\|_2^{9/16}\|\nabla u^k\|_2^{5/16}\|\nabla u^{k-1}\|_2^{15/16} \\ & \qquad + CA_1^{3/2}\|u^k\|_2^{5/8}\|u^{k-1}\|_2^{3/8}\|\nabla u^k\|_2^{5/24}\|\nabla u^{k-1}\|_2^{5/8}, \end{aligned}$$

where one has used Lemma 4.3 and the inequality

$$\|u\|_{24/7} \le \|u\|_2^{3/8} \|\nabla u\|_2^{5/8}.$$

By (4.1) and (4.2), we obtain that

$$\begin{aligned}\int_0^t \int_{R^3} (1+|x|^2)|p^k||u^k| dx d\tau \le\ & \frac{1}{4}\int_0^t \|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 d\tau \\ & + C A_1^{3/2}\Big(N^{3/4}\|a\|_2^{5/4} + N\|a\|_2^{5/6}\Big).\end{aligned} \tag{4.12}$$

Similarly,

$$\begin{aligned}\int_0^t \int_{R^3} (1+|x|^2)|u^{k-1}||u^k|^2 dx d\tau \le\ & \frac{1}{4}\int_0^t \|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 d\tau \\ & + C A_1^{3/2}\Big(N^{3/4}\|a\|_2^{5/4} + N\|a\|_2^{5/6}\Big).\end{aligned} \tag{4.13}$$

Finally, we estimate the first term on the right hand side of (4.11). By Hölder inequality, we get

$$\begin{aligned}\int_{R^3} (1+|x|^2)|u^k||\nabla u^k| dx &= \int_{R^3} (1+|x|^2)^{3/4}|\nabla u^k|(1+|x|^2)^{1/4}|u^k| dx \\ &\le \|(1+|x|^2)^{3/4}|\nabla u^k|\|_2 \|(1+|x|^2)^{1/4}u^k\|_2 \\ &\le \frac{1}{8}\|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 + C\|(1+|x|^2)^{1/4}u^k\|_2^2.\end{aligned}$$

Since

$$\begin{aligned}\|(1+|x|^2)^{1/4}u^k\|_2^2 &= \int_{R^3} (1+|x|^2)^{1/2}|u^k|^{2/3}|u^k|^{4/3} dx \\ &\le \|(1+|x|^2)^{3/4}u^k\|_6^{2/3}\|u^k\|_{3/2}^{4/3} \\ &\le C\|\nabla\{(1+|x|^2)^{3/4}u^k\}\|_2^{2/3}\|u^k\|_{3/2}^{4/3} \\ &\le \varepsilon\|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 + \frac{1}{2}\|(1+|x|^2)^{1/4}u^k\|_2^2 + C\|u^k\|_{3/2}^2,\end{aligned}$$

for some $\varepsilon > 0$. Thus,

$$\|(1+|x|^2)^{1/4}u^k\|_2^2 \le 2\varepsilon\|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 + C\|u^k\|_1^{2/3}\|u^k\|_2^{4/3}.$$

Taking $2\varepsilon = 1/8$ yields

$$\int_{R^3} (1+|x|^2)|u^k||\nabla u^k| dx \le \frac{1}{4}\|(1+|x|^2)^{3/4}|\nabla u^k|\|_2^2 + CB^{2/3}\|u^k\|_2^{4/3}. \tag{4.14}$$

Substituting (4.1)-(4.14) into (4.11), we obtain estimates (4.8) and (4.9), by Lemma 4.2. □

5. Weighted Estimates for Approximate Solutions II

In this section, we establish the decay rates estimates of weighted norms for approximate solutions in $L^p(R^3)(p > 3)$, which are used to show the existence of corresponding strong solutions. To this end, we first recall some basic estimates on Γ and $\bar{\theta}$. Applying the inequality $\tau^\alpha e^{-C\tau} \le C^\alpha e^{-1}$ for $\alpha > 0$, one can verify directly that

$$\begin{cases} \||x|^\alpha \Gamma\|_p \le Ct^{\alpha/2-(3-3/p)/2}, \\ \||x|^\alpha \nabla\Gamma\|_p \le Ct^{(\alpha-1)/2-(3-3/p)/2} \end{cases} \tag{5.1}$$

for $1 \le p \le +\infty$ and $\alpha \ge 0$. By the weighted estimates about the singular integrals (cf.[36-38]) and (3.8), one has that

$$\begin{cases} \||x|^\alpha D^2\bar{\theta}\|_p \le C\||x|^\alpha \Gamma\|_p \le Ct^{\alpha/2-(3/2)(1-1/p)}, \\ \||x|^\alpha D^3\bar{\theta}\|_p \le C\||x|^\alpha |\nabla\Gamma|\|_p \le Ct^{(\alpha-1)/2-(3/2)(1-1/p)}, \end{cases} \tag{5.2}$$

for $1 < p < +\infty$ and $-1/p < \alpha < 3 - 3/p$.

Now we can deduce the main estimates needed for Theorem 3 and 4.

Lemma 5.1 Let $a \in L^1(R^3) \cap \overset{o}{J}{}^2(R^3)$, $(1+|x|^2)^{1/2} a \in L^2(R^3)$ and $(1+|x|^2)^{\alpha/2} a \in L^{p_0}(R^3)$ for $1 \le p_0 \le +\infty$ and $\alpha = 3 - 3/p_0$. Then there exists a constant $\lambda > 0$, such that if $N \le \lambda$, then, for $\beta = (3/p_0 - 3/p)/2$, the estimate

$$\|t^\beta (1+|x|^2)^{\alpha/2} u^k\|_p \le C\Big(\|(1+|x|^2)^{\alpha/2} a\|_{p_0} + N^2 + A_1^{5/2} + B^{1/2} A_1 N\Big) \tag{5.3}$$

holds uniformly for $k \ge 0$ and for $3 < p \le +\infty$.

Proof By (3.12) and Minkowski inequality, we obtain that

$$\|(1+|x|^2)^{\alpha/2} u^k\|_p \le C \sum_{i=1}^{3} \|(1+|x|^2)^{\alpha/2} I_i^k\|_p. \tag{5.4}$$

By the Minkowski inequality, (4.10) and the basic L^p- estimates for convolutions, we have, for $3 < p \le +\infty$, that

$$\begin{aligned} \|(1+|x|^2)^{\alpha/2} I_1^k\|_p &\le C\| \int_{R^3} (1+|y|^2)^{\alpha/2} |a(y)| \Gamma(x-y,t) dy\|_p \\ &\qquad + C\| \int_{R^3} |a(y)||x-y|^\alpha \Gamma(x-y,t) dy\|_p \\ &\le C\|(1+|x|^2)^{\alpha/2} a\|_{p_0} t^{-(3/2)(1/p_0-1/p)} + C\|a\|_1 t^{-3/2+3/(2p)+\alpha/2} \\ &\le C(\|a\|_1 + \|(1+|x|^2)^{\alpha/2} a\|_{p_0}) t^{-(3/2)(1/p_0-1/p)} \end{aligned} \tag{5.5}$$

where one has used estimates (5.1) (if $p = \infty$, we use (3.13) to estimate the second term of the first step in (5.5)).

By (4.10) and the Minkowski inequality, one can get that for $3 < p \le +\infty$

$$\begin{aligned} \|(1+|x|^2)^{\alpha/2} I_2^k\|_p &\le C\| \int_0^{t/2} \int_{R^3} (1+|y|^2)^{\alpha/2} |u^{k-1}||u^k|(|\nabla\Gamma| + |D^3\bar{\theta}|)(x-y,t-\tau) dy d\tau\|_p \\ &\quad + C\| \int_0^{t/2} \int_{R^3} |u^{k-1}||u^k||x-y|^\alpha (|\nabla\Gamma| + |D^3\bar{\theta}|)(x-y,t-\tau) dy d\tau\|_p \\ &\triangleq C(\|I_{21}^k\|_p + \|I_{22}^k\|_p). \end{aligned} \tag{5.6}$$

Similarly,

$$\begin{aligned}\|(1+|x|^2)^{\alpha/2}I_3^k\|_p &\le C\|\int_{t/2}^t\int_{R^3}(1+|y|^2)^{\alpha/2}|u^{k-1}||u^k|(|\nabla\Gamma|+|D^3\bar\theta|)(x-y,t-\tau)dyd\tau\|_p\\ &\quad+C\|\int_{t/2}^t\int_{R^3}|u^{k-1}||u^k||x-y|^\alpha(|\nabla\Gamma|+|D^3\bar\theta|)(x-y,t-\tau)dyd\tau\|_p\\ &\triangleq C(\|I_{31}^k\|_p+\|I_{32}^k\|_p).\end{aligned} \tag{5.7}$$

Let $J_p^k \triangleq \|(1+|x|^2)^{\alpha/2}u^k\|_p$. In order to estimate I_2^k and I_3^k, we discuss three separated cases: 1) $p=+\infty$ and $1\le p_0\le+\infty$; 2) $1\le p_0<p<\infty$ and $p>3$; 3) $3<p_0=p<\infty$.

Case I: $p=+\infty$ and $1\le p_0\le+\infty$

In order to establish the uniform estimates on $t^\beta\|(1+|x|^2)^{\alpha/2}u^k\|_\infty$ with $\beta=3/(2p_0)$, the singular factor $t^{-\beta}$ will appear in the integral later. So it seems necessary to treat two cases: $p_0>3/2$ and $1\le p_0\le 3/2$. Similarly, to establish the uniform estimates on $t^{1/2+\beta}\|(1+|x|^2)^{\alpha/2}\nabla u^k\|_\infty$, the singular factor $t^{-1/2-\beta}$ will appear in this procedure, one needs to distingush three cases: $p_0>3$, $3/2<p_0\le 3$ and $1\le p_0\le 3$.

First, it follows from (5.1), (5.2) and Lemma 4.2, that

$$\begin{aligned}\|I_{21}^k\|_\infty &\le C\int_0^{t/2}\|u^{k-1}\|_2J_\infty^k\||\nabla\Gamma|+|D^3\bar\theta|\|_2d\tau\\ &\le C\int_0^{t/2}\|u^k\|_2J_\infty^{k-1}(t-\tau)^{-5/4}d\tau\\ &\le CN\int_0^{t/2}J_\infty^{k-1}(1+\tau)^{-3/4}(t-\tau)^{-5/4}d\tau\end{aligned} \tag{5.8}$$

which yields the desired estimate for $p_o>3$. If $1\le p_0\le 3/2$, then $\alpha\le 1$ and $1\le\beta\le 3/2$. By (3.13), we get, with the help of Lemma 4.2 and 4.3, that

$$\begin{aligned}\|I_{21}^k\|_\infty &\le C\int_0^{t/2}\|u^k\|_{3/2}\|(1+|y|^2)^{\alpha/2}u^{k-1}\|_2^{2/3}(J_\infty^{k-1})^{1/3}(t-\tau)^{-2}d\tau\\ &\le CA_1^{2/3}\int_0^{t/2}\|u^k\|_1^{1/3}\|u^k\|_2^{2/3}(J_\infty^{k-1})^{1/3}(t-\tau)^{-2}d\tau\\ &\le CA_1^{2/3}B^{1/3}N^{2/3}\int_0^{t/2}(J_\infty^{k-1})^{1/3}(1+\tau)^{-1/2}(t-\tau)^{-2}d\tau.\end{aligned} \tag{5.9}$$

If $3/2<p_0\le 3$, then $\alpha-1\in(0,1]$ and $1/2\le\beta\le 1$. By (3.13), we obtain, with the aid of Lemma 4.2 and 4.3, that

$$\|I_{21}^k\|_\infty\le CA_1^{5/3}B\int_0^{t/2}(J_\infty^{k-1})^{1/3}(t-\tau)^{-2}d\tau \tag{5.10}$$

here we have used the inequality

$$\|(1+|x|^2)^{\gamma_0/2}u\|_p\le\|u\|_q\|(1+|x|^2)^{1/2}u\|_2 \tag{5.11}$$

for $1\le q<p<2$ and $\gamma_0=2(p-q)/(p(2-q))$, which follows from the Hölder inequality.

The $\|I_{22}^k\|_\infty$ can be estimated as

$$\begin{aligned}\|I_{22}^k\|_\infty &\le C\|\int_0^{t/2}|u^{k-1}||u^k||x-y|^\alpha(|x-y|^2+(t-\tau))^{-2}dyd\tau\|_\infty\\ &\le C\int_0^{t/2}\|u^{k-1}\|_2\|u^k\|_2(t-s)^{-2+\alpha/2}d\tau\\ &\le CN^2\int_0^{t/2}(1+\tau)^{-3/2}(t-\tau)^{-1/2-3/(2p_0)}d\tau\\ &\le CN^2t^{-1-3/(2p_0)}, \end{aligned}\tag{5.12}$$

where one has used (3.13).

Next, we estimate I_3^k. By (5.1) and (5.2), we have

$$\begin{aligned}\|I_{31}^k\|_\infty &\le C\int_{t/2}^t\|(1+|x|^2)^{\alpha/2}|u^{k-1}||u^k|\|_4(t-\tau)^{-7/8}d\tau\\ &\le C\int_{t/2}^t J_\infty^{k-1}(J_\infty^k)^{1/2}\|u^k\|_2^{1/2}(t-\tau)^{-7/8}d\tau\\ &\le CN^{1/2}\int_{t/2}^t J_\infty^{k-1}(J_\infty^k)^{1/2}(1+\tau)^{-3/8}(t-\tau)^{-7/8}d\tau.\end{aligned}\tag{5.13}$$

Note that in the estimate of I_{32}^k, the singular factor $(t-\tau)^{-2+\alpha/2}$ will appear in the integral with $\alpha=3-3/p_0$. In order to control this singularity, we will separate two cases: $p_0>3$ and $1\le p_0\le 3$.

If $p_0>3$, it follows from the estimate (3.13) that

$$\|I_{32}^k\|_\infty\le C\int_{t/2}^t\|u^{k-1}\|_2\|u^k\|_2(t-\tau)^{-1/2-3/(2p_0)}d\tau\le CN^2t^{-1-\beta}.\tag{5.14}$$

If $1\le p_0\le 3$, using (5.1) and (5.2), one can get that

$$\|I_{32}^k\|_\infty\le C\int_{t/2}^t\||u^{k-1}||u^k|\|_{(3p_0+1)/(3p_0-2)}(t-\tau)^{-1/2-3/(2p_0(3p_0+1))}d\tau.\tag{5.15}$$

If $5/3\le p_0\le 3$, then $(3p_0+1)/(3p_0-2)\le 2$. By Lemma 4.2, one gets that

$$\begin{aligned}\||u^{k-1}||u^k|\|_{(3p_0+1)/(3p_0-2)} &\le J_\infty^{k-1}\|u^k\|_1^{(3p_0-5)/(3p_0+1)}\|u^k\|_2^{6/(3p_0+1)}\\ &\le J_\infty^{k-1}B^{(3p_0-5)/(3p_0+1)}N^{6/(3p_0+1)}(1+\tau)^{-9/(6p_0+2)}.\end{aligned}\tag{5.16}$$

If $1\le p_0<5/3$, then $(3p_0+1)/(3p_0-2)>2$. Then,

$$\begin{aligned}&\||u^{k-1}||u^k|\|_{(3p_0+1)/(3p_0-2)}\le J_\infty^{k-1}(J_\infty^k)^{(5-3p_0)/(3p_0+1)}\|u^k\|_2^{2(3p_0-2)/(3p_0+1)}\\ &\le J_\infty^{k-1}(J_\infty^k)^{(5-3p_0)/(3p_0+1)}N^{2(3p_0-2)/(3p_0+1)}(1+\tau)^{-3(3p_0-2)/(2(3p_0+1))}.\end{aligned}\tag{5.17}$$

Case II: $1\le p_0<p<+\infty$ and $3<p$.

This can be achieved in a similar way as case I with slight modification. Indeed, we deal with three cases for p_0. For $p_0 > 3$, then, (5.1), (5.2) and Lemma 4.2 implies that

$$\begin{aligned}\|I_{21}^k\|_p &\leq C\int_0^{t/2}\|u^k\|_2 J_p^{k-1}\||\nabla\Gamma| + |D^3\bar{\theta}|\|_2 d\tau\\ &\leq C\int_0^{t/2}\|u^k\|_2 J_p^{k-1}(t-\tau)^{-5/4}d\tau\\ &\leq CN\int_0^{t/2} J_p^{k-1}(1+\tau)^{-3/4}(t-\tau)^{-5/4}d\tau. \end{aligned}\tag{5.18}$$

If $1 \leq p_0 \leq 3/2$, then $\alpha \leq 1$. It now follows from (5.1), (5.2) and Lemma 4.2-3, that

$$\begin{aligned}\|I_{21}^k\|_p &\leq C\int_0^{t/2}\|u^k\|_{4p/(3p-2)}\|(1+|y|^2)^{\alpha/2}u^{k-1}\|_2^{1/2}(J_p^{k-1})^{1/2}(t-\tau)^{-2+3/(2p)}d\tau\\ &\leq CA_1^{1/2}\int_0^{t/2}\|u^k\|_1^{(p-2)/(2p)}\|u^k\|_2^{(p+2)/(2p)}(J_p^{k-1})^{1/2}(t-\tau)^{-2+3/(2p)}d\tau\\ &\leq CA_1^{1/2}B^{(p-2)/(2p)}N^{(p+2)/(2p)}\int_0^{t/2}(J_p^{k-1})^{1/2}(1+\tau)^{-3/8-3/(4p)}(t-\tau)^{-2+3/(2p)}d\tau \end{aligned}\tag{5.19}$$

If $3/2 < p_0 \leq 3$, then $\alpha - 1 \in (0, 1]$. By (5.1) and (5.2), we obtain, with the help of Lemma 4.2-4.3 and inequality (5.11), that

$$\begin{aligned}\|I_{21}^k\|_p &\leq C\int_0^{t/2}\|(1+|y|^2)^{1/2}u^{k-1}\|_2^{1/2}\|(1+|y|^2)^{(\alpha-1)/4}u^k\|_{4p/(3p-2)}\\ &\qquad\qquad\times(J_p^{k-1})^{1/2}(t-\tau)^{-2+3/(2p)}d\tau\\ &\leq CA_1^{1/2}\int_0^{t/2}\|u^k\|_1\|(1+|x|^2)^{1/2}u^k\|_2(J_p^{k-1})^{1/2}(t-\tau)^{-2+3/(2p)}d\tau\\ &\leq CA_1^{3/2}B\int_0^{t/2}(J_p^{k-1})^{1/2}(t-\tau)^{-2+3/(2p)}d\tau. \end{aligned}\tag{5.20}$$

Applying (5.1) and (5.1), one can estimate the $\|I_{22}^k\|_p$ as

$$\|I_{22}^k\|_p \leq C\int_0^{t/2}\|u^{k-1}\|_2\|u^k\|_2(t-s)^{-2+\alpha/2+3/(2p)}d\tau \leq CN^2t^{-1-\beta}.\tag{5.21}$$

Now we estimate I_3^k. Let $r = 6(p-1)/(p+1)$. By (5.1) and (5.2), we have

$$\begin{aligned}\|I_{31}^k\|_p &\leq C\int_{t/2}^t\|(1+|x|^2)^{\alpha/2}|u^{k-1}||u^k|\|_{pr/(p+r)}(t-\tau)^{-1/2-3/(2r)}d\tau\\ &\leq C\int_{t/2}^t J_p^{k-1}(J_p^k)^{p(r-2)/(r(p-2))}\|u^k\|_2^{2(p-r)/(r(p-2))}(t-\tau)^{-1/2-3/(2r)}d\tau\\ &\leq CN^{(p-3)/(3(p-1))}\int_{t/2}^t J_p^{k-1}(J_p^k)^{2p/(3(p-1))}\\ &\qquad\qquad\times(1+\tau)^{-(p-3)/(4(p-1))}(t-\tau)^{-1/2-3/(2r)}d\tau. \end{aligned}\tag{5.22}$$

For the estimate on I_{32}^k, We also consider two cases. If $p_0 \geq 3$, we have, by (5.1) and (5.2), that

$$\|I_{32}^k\|_p \leq C\int_{t/2}^t\|u^{k-1}\|_2\|u^k\|_2(t-\tau)^{-1/2-\beta}d\tau \leq CN^2t^{-1-\beta}.\tag{5.23}$$

By the Young inequality and estimates (5.1) and (5.2), we get that

$$\|I_{32}^k\|_p \leq C \int_{t/2}^t \||u^{k-1}||u^k|\|_l (t-\tau)^{-1/2-3/(2p_0(3p_0+1))} d\tau$$

for $1/l = 1/p + (3p_0 - 2)/(3p_0 + 1)$. If $5/3 \leq p_0 \leq 3$, by Lemma 4.2,

$$\begin{aligned}
\|I_{32}^k\|_p &\leq C \int_{t/2}^t J_p^{k-1} \|u^k\|_{(3p_0+1)/(3p_0-2)} (t-\tau)^{-1/2-3/(2p_0(3p_0+1))} d\tau \\
&\leq C \int_{t/2}^t J_p^{k-1} \|u^k\|_1^{(3p_0-5)/(3p_0+1)} \|u^k\|_2^{6/(3p_0+1)} (t-\tau)^{-1/2-3/(2p_0(3p_0+1))} d\tau \\
&\leq CB^{(3p_0-5)/(3p_0+1)} N^{6/(3p_0+1)} \int_{t/2}^t J_p^{k-1} (1+\tau)^{-9/(2(3p_0+1))} \\
&\qquad \times (t-\tau)^{-1/2-3/(2p_0(3p_0+1))} d\tau. \qquad (5.24)
\end{aligned}$$

If $1 \leq p_0 < 5/3$, then

$$\begin{aligned}
\|I_{32}^k\|_p &\leq C \int_{t/2}^t J_p^{k-1} \|u^k\|_{(3p_0+1)/(3p_0-2)} (t-\tau)^{-1/2-3/(2p_0(3p_0+1))} d\tau \\
&\leq C \int_{t/2}^t J_p^{k-1} (J_p^k)^{(5-3p_0)/(3p_0+1))} \|u^k\|_2^{2(3p_0-2)/(3p_0+1)} \\
&\qquad \times (t-\tau)^{-1/2-3/(2p_0(3p_0+1))} d\tau \\
&\leq CN^{2(3p_0-2)/(3p_0+1)} \int_{t/2}^t J_p^{k-1} (J_p^k)^{(5-3p_0)/(3p_0+1)} \\
&\qquad \times (1+\tau)^{-3(3p_0-2)/(2(3p_0+1))} (t-\tau)^{-1/2-3/(2p_0(3p_0+1))} d\tau. \qquad (5.25)
\end{aligned}$$

Case III $3 < p_0 = p < +\infty$

The estimates of $\|I_{21}^k\|_p$ and $\|I_{31}^k\|_p$ in this case are same as (5.18) and (5.22). So we only give the estimate of $\|I_{22}^k\|_p$ and $\|I_{32}^k$. Applying (3.13) and the theory on singular integral operator (cf. [38]), we get that

$$\begin{aligned}
\|I_{22}^k\|_p + \|I_{32}^k\|_p &\leq C \| \int_0^t |u^{k-1}||u^k||x-y|^\alpha (|x-y|^2 + (t-\tau))^{-2} dy d\tau \|_p \\
&\leq C \| \int_0^t |u^{k-1}||u^k||x-y|^{-1-3/p} dy d\tau \|_p \\
&\leq C \int_0^t \||u^{k-1}||u^k|\|_{3/2} d\tau \\
&\leq C\|a\|_2 N t^{-1/4}. \qquad (5.26)
\end{aligned}$$

Summarizing above estimates, it is obvious that, if $t^\beta J_p^{k-1} \leq C$, then $t^\beta J_p^k \leq C$, by the

Gronwall inequality. Therefore, it holds that $t^\beta J_p^k \in L^\infty_{loc}(0,+\infty)$, by induction. Thus

$$
\begin{aligned}
t^\beta J_\infty^k \le\ & C\Big(\|a\|_1 + \|(1+|x|^2)^{\alpha/2}a\|_{p_0} + N^2(1+t)^{-1} \\
& \qquad + N^{1/2}(1+t)^{-\beta/2-1/4}\max_{0\le t\le\infty}(t^\beta J_\infty^{k-1})(t^\beta J_\infty^k)^{1/2}\Big) \\
& + \begin{cases}
C\Big(N(1+t)^{-1}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1}) + N^2(1+t)^{-1}\Big), & \text{as } p_0>3; \\
C\Big(A_1^{5/3}B(1+t)^{-1+2\beta/3}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{1/3} & \\
\quad + B^{(3p_0-5)/(3p_0+1)}N^{6/(3p_0+1)}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})\Big), & \text{as } 5/3\le p_0\le 3; \\
C\Big(A_1^{5/3}B(1+t)^{-1+2\beta/3}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{1/3} & \\
\quad + N^{2(3p_0-2)/(3p_0+1)}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})(t^\beta J_\infty^{k})^{(5-3p_0)/(3p_0+1)}\Big), & \text{as } 3/2<p_0<5/3; \\
C\Big(B^{1/3}A_1^{2/3}N^{2/3}(1+t)^{-3/2+2\beta/3}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{1/3} & \\
\quad + N^{2(3p_0-2)/(3p_0+1)}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})(t^\beta J_\infty^{k})^{(5-3p_0)/(3p_0+1)}\Big), & \text{as } 1\le p_0\le 3/2.
\end{cases}
\end{aligned}
$$

By the Young inequality, we deduce that

$$
\begin{aligned}
\max_{t\in[0,\infty]}(t^\beta J_\infty^k) \le\ & C\Big(\|a\|_1 + \|(1+|x|^2)^{\alpha/2}a\|_{p_0} + N^2 + N\max_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^2\Big) \\
& + \begin{cases}
C\Big(N\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1}) + N^2\Big), & \text{as } p_0>3; \\
C\Big(A_1^{5/3}B\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{1/3} & \\
\quad + B^{(3p_0-5)/(3p_0+1)}N^{6/(3p_0+1)}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})\Big), & \text{as } 5/3\le p_0\le 3; \\
C\Big(A_1^{5/3}B\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{1/3} & \\
\quad + N\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{(3p_0+1)/(2(3p_0-2))}\Big), & \text{as } 3/2<p_0<5/3; \\
C\Big(B^{1/3}A_1^{2/3}N^{2/3}\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{1/3} & \\
\quad + N\max\limits_{t\in[0,\infty]}(t^\beta J_\infty^{k-1})^{(3p_0+1)/(2(3p_0-2))}\Big), & \text{as } 1\le p_0\le 3/2.
\end{cases}
\end{aligned}
$$

Therefore, if N is suitable small, then

$$
\max(t^\beta J_\infty^k) \le C\Big(\|a\|_1 + \|(1+|x|^2)^{\alpha/2}a\|_{p_0} + N^2 + A_1^{5/2}B^{3/2} + B^{1/2}A_1N\Big),
$$

which yields the desired estimate in the case $p=\infty$. The argument for $3<p<\infty$ is similar. □

By a longthy but similar calculation, one can show that

Lemma 5.2 Assume the conditions in Lemma 5.1 satisfied. Then if $N\le\lambda$, the estimates, for $t\ge 0$,

$$
t^{1/2+\beta}\|\nabla u^k\|_p \le C\Big(\|a\|_1 + \|(1+|x|^2)^{\alpha/2}a\|_{p_0} + N^2 + A_1^{5/2}B^{3/2} + B^{1/2}A_1N\Big)
$$

hold uniformly for $k \geq 0$ and $3 < p \leq +\infty$.

If $a = (\partial b)/(\partial x_i)$ for some $i = 1, 2, 3$, one can shows, by similar discussion, that

Lemma 5.3 Assume the conditions in Lemma 5.1 hold. If $a = (\partial b)/(\partial x_i)$ for some $i = 1, 2, 3$ with $b \in L^1)(R^3)$ and $(1 + |x|^2)^{\alpha/2} \in L^{p_0}(R^3)$, then there exists a constant $\lambda_0 > 0$ such that if $N \leq \lambda_0$, the estimate

$$t^{1/2+\beta}\|u^k\|_p \leq C\Big(\|b\|_1 + \|(1+|x|^2)^{\alpha/2}b\|_{p_0} + N^2 + A_1^{5/2}B^{3/2} + B^{1/2}A_1N\Big), \quad \text{for } t \in [0, \infty)$$

holds uniformly for $k \geq 0$ and $3 < p \leq +\infty$.

Acknowledgments: This work was done when HE was visiting the Institute of Mathematical Sciences at the Chinese University of Hong Kong. The authors express their sincere thanks to Prof. Zhimin Chen for helpful discussions. The research of He is supported in part by National Youth Fundation of China, and Zheng Ge Ru Fundation. The research of Xin is supported in part by the Zheng Ge Ru Fundation, NSF grant DMS-96-00137, and DOE Grant De-FG02-88ER-25053.

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