On The Behavior of Solutions to The Compressible Navier-Stokes Equations

Zhouping Xin*

Contents

1. Introduction

The motions of a d-dimensional compressible viscous, heat-conductive, isotropic Newtonian fluid can be governed by the following system of Navier-Stokes Equations [4,17]

$$(1) \qquad \begin{cases} \partial_t \rho + div(\rho u) = 0 \\ \partial_t(\rho u) + div(\rho u \otimes u) + \bigtriangledown p = div(T) \\ \partial_t(\rho E) + div(\rho u E + up) = div(uT) + k \bigtriangleup \theta \end{cases}$$

where $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+$, and for $d \geq 1$, $\rho = \rho(x, t)$,$u = (u_1, ..., u_d) \in \mathbb{R}^d$, p, e and θ denote the density, velocity, pressure, internal energy and absolute temperature respectively, $E = e + \frac{1}{2}|u|^2$ is the total energy, and T is the stress tensor given by $T = \mu(\bigtriangledown u + (\bigtriangledown u)^t) + \mu'(div\ u)I$ with I being the identify matrix, μ and μ' being the coefficient of viscosity and the second coefficient of viscosity respectively, k denotes the coefficient of the heat conduction. It is usually assumed that $\mu \geq 0$, $\mu' + \frac{2}{d}\mu \geq 0$, and $k \geq 0$. The relation between p, θ, ρ, and e is given by the equation of state and the second law of thermo dynamics. In the special case of polytropic fluids, the equations of state in given by

*This work is supported partially by Zheng Ge Ru Foundation, UGC Earmarked Grant CUHK–4129/99P, NSF Grants DMS-96-00137, and DMS-99-71978, and DOE Grant DE-FG02-88ER-25053.

$$p = R\rho\theta, \quad e = c\theta, \quad p = Ae^{\frac{S}{c}}\rho^\gamma \tag{2}$$

where $A > 0$ is a constant, $\gamma(> 1)$ is the ratio of specific heat, S is the entropy, and $C = R/(\gamma - 1)$.

For ideal inviscid fluids, $\mu' = \mu = k = 0$, then system (1) becomes the well-known compressible Euler equations:

$$\begin{cases} \partial_t\rho + div(\rho u) = 0 \\ \partial_t(\rho u) + div(\rho u \otimes u) + \bigtriangledown p = 0 \\ \partial_t(\rho E) + div(\rho u E + up) = 0 \end{cases} \tag{3}$$

which is one of the most important examples of systems of nonlinear hyperbolic conservation laws, and has been the main focus and the driving force for the mathematical theory of shock waves [18,34].

Tremendous progress has been made in solving systems (1) and (2) both theoretically and numerically, and in understanding and interpreting the behavior of solutions to these systems. This is particularly so in the case of one-space-dimension and for "small" solutions [7,5,8,12,16,21,28]. Two major issues have been investigated extensively. One is the global (in time) well-posedness of smooth solutions and their large time asymptotic behavior for (1) with fixed viscosity and heat conduction. This has been successfully treated in the case for small smooth non-vacuum data whose long time behavior is shown to be governed by the linear diffusion waves [16,28]. This theory has been generalized even for small discontinuous initial data in multi-dimensions [12]. The other issue is to understand the effects of the small scale dissipations on the behavior of large scale physical flows, i.e. to discribe the asymptotic relationship between the inviscid and viscous flows in the presence of discontinuities and physical boundaries for either small (but non-zero) dissipation or large time (with fixed dissipation). This is a very important issue not only for the understanding of the asymptotic behavior of the solutions to the Navier-Stokes system, but also crucial for the theory of the inviscid Euler equations, which is due to the fact that it is the dissipative mechanism which selects the entropy weak solutions to the Euler system from other nonphysical solutions. An almost complete theory on the nonlinear large time asymptotic stability toward basic viscous waves (viscous shock profile, rarefaction waves, contact discontinuities, and diffusive waves) has been developed for one-space-dimension [9,15,22-24,26,29,35,30]. And the asymptotic equivalence between the viscous system (1) and inviscid Euler equations (2) has also been understood in one-space-dimension for special flows [5,10,13,21,39]. More recently, the global (in time) existence of large amplitude weak solutions to either Cauchy problem or initial-boundary value problems of the isentropic compressible Navier-Stokes equations with some special equations of state has been established [19], and some important properties of these weak solutions, such as boundness of the total energy, the zero-mach number limit to the incompressible Navier-Stokes equations, and large time asymptotic behavior toward steady states etc, have been investigated [19,20]. For the corresponding inviscid Euler system (2), substantial progress has been made in the one-dimensional theory of shock wave for solutions of small total variation [1-3,8,18,25,34].

Despite these important achievements, there remain many significant challenging outstanding issues to be settled. One of these is the well-posedness and large time asymptotic behavior of smooth solutions of finite amplitudes to (1) in arbitrary space dimensions. Can a smooth solutions to the Navier-Stokes system (1) develop singularities in finite time? A related problem is the regularities of the weak solutions constructed by P. L. Lions [19]. It should be noted that in the case where the initial data is not periodic, the existence of weak solution [19] requires that the initial far fields must be vacuum, and then the weak solutions cannot be regular in general even for smooth initial due to the fact that the total pressure will decay sufficiently fast as will be seen in the next section (see also [36]). Another issue with considerable interest is the boundary layer theory for the compressible Navier-Stokes system (1). There have been extensive studies on the zero dissipation limit for the Navier Stokes system in the presence of the physical boundaries due to its importance in the applications to high speed flows [33,4,17]. In the case where the physical boundaries are uniformly non-characteristic, the multi-scale structure of the viscous solutions can be revealed by matched asymptotic analysis and the strong convergence of the Navier-Stokes solutions to the one of the Euler system away from boundaries can be obtained by nonlinear stability analysis of the boundary layer before shock formations provided the boundary layer is suitably weak [37,40,30,11,6]. For the well-known no-slip boundary conditions for (1), which is uniformly characteristic, the formal Prandtl's boundary layer theory has existed for a long time, yet its validity has been proved rigorously only in the analytical setting [32]. The rigorous justification of the Prandtl's boundary layer theory for the linearized flow will be given in section 3 (see also [38]).

2. An A-Priori Estimate on The Total Pressure And Some Applications

The main result of this section is the following a priori estimate on the total pressure for the viscous polytropic fluid. Consider the Cauchy problem for the Navier-Stokes system (1) with initial data:

$$(\rho, u, S)(x, t=0) = (\rho_0, u_0, S_0)(x) \tag{4}$$

Then one has that

Theorem 1 Assume that

$$m_2 = \int_{\mathbb{R}^d} |x|^2 \rho_0(x) dx < +\infty \tag{5}$$

Let (ρ, u, S) be a smooth solution to Navier-Stokes (1) with initial data (4), such that

$$(\rho, u, S - \bar{s}) \in C^1([0,T), H^m(\mathbb{R}^2)), \quad m \geq [\frac{d}{2}] + 2, \tag{6}$$

where $\bar{s}$ is a constant, and $0 < T \leq +\infty$. Then there exists a positive constant C_0 (independent of T) such that

$$\int_{\mathbb{R}^d} p(x,t)dx \le \begin{cases} \frac{\gamma-1}{2} m_2 t^{-2}, & \gamma \ge 1+\frac{2}{d} \\ \frac{\gamma-1}{2} C_0 (1+t)^{-(\gamma-1)d}, & 1<\gamma<1+\frac{2}{d} \end{cases} \tag{7}$$

for all $t \in [0,T)$, where

$$C_0 = m_2 - \int_{\mathbb{R}^d} (x \cdot u_0(x))\rho_0(x)dx + 2\int_{\mathbb{R}^d} \rho_0 E_0 dx < +\infty \quad .$$

A few remarks are in order:

Remark 1 The key assumption in Theorem 1 is (5), so that the initial far fields are in the vacuum. In this case, the fast decay in time of the total pressure given in (7) indicates that the pressure behaves dispersively instead of diffusively. Indeed, for the monotonic fluid in 3-dimension, $d=3, \gamma=\frac{5}{3}$, hence (7) becomes

$$\int_{\mathbb{R}^3} p(x,t)dx \le \frac{m_2}{3} t^{-2} \tag{8}$$

This shows that the pressure decays faster than any diffusive scales in 3-dimension.

Remark 2 The rigorous derivation of the a-priori estimate (7) is given by studying a differential inequality for the functional

$$I_\gamma(t) \equiv \begin{cases} \int_{\mathbb{R}^d} |x-u(x,t)(t+1)|^2 \rho(x,t)dx + \\ \frac{2}{\gamma-1}(t+1)^2 \int_{\mathbb{R}^d} p(x,t)dx, \quad 1<\gamma<1+\frac{2}{d}, \\ \\ \int_{\mathbb{R}^d} |x-u(x,t)t|^2 \rho(x,t)dx + \\ \frac{2}{\gamma-1} t^2 \int_{\mathbb{R}^d} p(x,t)dx, \quad 1+\frac{2}{d} \le \gamma < +\infty. \end{cases} \tag{9}$$

We refer the readers to [36] for details. However, the special case, (8), can be seen formally by a microscopic-kinetic description of the fluids as follows. Let $f(x,t,\xi)$ be the density function of particles at position x at time t with velocity ξ. Assume that $f(x,t,\xi)$ satisfies the Boltzmann type equation of the form.

$$\partial_t f + \xi \bigtriangledown_x f = \frac{1}{\varepsilon} Q(f,f)$$

where Q denotes the collision operator and ε is the mean free path. It is expected that as the mean free path, ε, tends to zero, f approaches a local Maxwellian, whose moments

$$\begin{aligned} \rho(x,t) &= \int_{\mathbb{R}^3} f(x,t,\xi)d\xi \quad , \quad (\rho u)(x,t,\xi) = \int_{\mathbb{R}^3} \xi f(x,t,\xi)d\xi \quad , \\ & (\rho E)(x,t) = \frac{1}{2}\int_{\mathbb{R}^3} |\xi|^2 f(x,t,\xi)d\xi \end{aligned}$$

are governed by the macroscopic Navier-Stokes system (1). On the other hand, for any reasonable description such that the principles of conservation of mass, momentum, and energy hold at the microscopic level, it is always true that $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x -$

$\xi t|^2 f(x,+,\xi)d\xi dx$ is a time invariant. Thus

$$\int_{\mathbb{R}^3} |x-u(x,+)t|^2\rho(x,t)dx + 2t^2\int_{\mathbb{R}^3}(\rho e)(x,t)dx = \int_{\mathbb{R}^3}|x|^2\rho_0(x)dx = m_2 \tag{10}$$

(10) implies the desired estimate (8) immediately since $p = \frac{2}{3}\rho e$ for the polytropic fluids.

We now give some simple applications of the a-priori estimate. We first recall the classical theory on the well-posedness of "small" as "non-vacuum" solutions to the Navier-Stokes equations [28,16,12].

Theorem (Classical Theory) Let $\bar\rho$ be a positive constant, $m = [\frac{d}{2}]+2$. Then there exists a suitably small constant δ such that if

$$\begin{cases} (p_0-\bar\rho, u_0, S_0) \in H^m(\mathbb{R}^d)\cap L^1(\mathbb{R}^d) \\ \|(\rho_0-\bar\rho, u_0, S_0)\|_{H^m} \le \delta \end{cases} \tag{11}$$

Then, these exists a unique global solution to the Navier-Stokes equations (1) with initial data (4) with the properties that

$$\begin{cases} (p-\bar\rho, u, S) \in C^1([0,\infty), \quad H^m(\mathbb{R}^d)) \\ \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C(1+t)^{-\beta}, \quad \beta=\beta(d) \end{cases} \tag{12}$$

where the rate of decay is exactly same as that of heat kernel.

It should be noted that for the classical theory, the key assumption is that the initial data is a "small" perturbation of a non-vacuum state $(\bar\rho,0,0)$. On the other hand, it is shown by P. L. Lions recently [19] that for some special isentropic fluids, a global weak solutions exists without smallness assumption provided that

$$\rho_0 \in L^q(\mathbb{R}^d), \quad d=1,2,3, \quad q=q(d,\gamma). \tag{13}$$

Note that (13) implies that the initial far fields must be in vacuum. Two natural questions arise:

- Can the classical theory hold true in the case that the initial far fields are in vacuum?
- What is the regularity of the weak solution obtained in [19]?

Applying the a priori estimate (7), we can easily derive some negative answers to both of these two questions in the case that

$$supp \quad \rho_0(x) \subseteq B_{R_0}, \quad 0 < R_0 < +\infty \tag{14}$$

Where B_{R_0} is the ball centered at origin with radius R_0, i.e., we assume that initial density has compact support. Indee, we have the following results [36]:

Corollary 1 Assume (14). Then there exists no non-trivial solution $(\rho, u, S) \in C^1([0,\infty) : H^m(\mathbb{R}^d))$ to the Cauchy problem for the Navier-Stokes system (1) with initial data (4) such that (12) holds.

Corollary 2 Assume (14), and

$$\mu > 0, \quad \mu' + \frac{2}{d}\mu > 0, \quad \mu \geq 0.$$

Then there exists no non-trivial solution $(\rho, u, S) \in C^1([0,\infty); H^m(\mathbb{R}^d))$ to (1) with initial data (4), which is bounded in time.

Furthermore, one can estimate the life span of the smooth solution as follows:

Corollary 3 Assume (11),

$$\mu > 0, \quad \mu' + \frac{2}{d}\mu > 0, \quad \text{and}$$

either $k = 0$, or the entropy will not increase in time.

Let $(u, \rho, S) \in C^1([0,T), H^m(\mathbb{R}^d))$ be a solution to the Navier-Stokes system (1). Then there exists a $T^* > 0$, $T^* < +\infty$, and T^* depends only on the initial data, d, γ, and R_0 such that

$$T \leq T^*,$$

in other words, every smooth solution blows up in finite time!

Remark 3 The theory and method also apply to the Euler system (2) to show that there exists no non-trivial solution in $C^1([0,\infty); H^m(\mathbb{R}^d))(m > [\frac{d}{2}] + max(1, \frac{4}{\gamma-1}))$ to (2) with compact initial density and velocity. This was shown before in [27] by a different method.

Remark 4 In contrast to the classical theory, there is no requirement on the amplitude of the initial data in corollaries 1-3.

Remark 5 These corollaries follow easily from the faster decay of the total pressure, (7), the conservation of mass, and the slow growth of the support of density, for details, see [36].

3. Boundary Layer Theory

We now consider the boundary layer problem for the compressible Navier-Stokes system. For simplicity in presentation, we will only discuss a 2-Dimensional viscous isentropic fluid. In this case, the Navier-Stokes system, (1), becomes

$$(15) \qquad \begin{cases} \partial_t \rho + div(\rho \vec{u}) = 0 \\ \partial_t(\rho \vec{u}) + div(\rho \vec{u} \otimes \vec{u}) + \nabla p = div \quad (T) \end{cases} \quad \text{on} \quad \Omega \times (0,t)$$

with

$$p = p(\rho), \quad p'(\rho) > 0, \quad \forall \rho > 0, \tag{16}$$

where Ω is a regionl is $\mathbb{R}^2$ with smooth boundary. We assume that

$$\mu = \varepsilon^2 \quad (\varepsilon > 0), \quad \mu' = 0(1)\varepsilon^2, \tag{17}$$

and ε is small.

We supplement (15) with initial data

$$(\rho, \vec{u})(x, t = 0) \equiv (\rho_0, \vec{u}_0)(x) \tag{18}$$

and boundary data

$$\vec{u}\ (x,t)|_{\partial\Omega \times |0,T)} = \vec{f}\ (x,t). \tag{19}$$

The associated initial-boundary value problem for the corresponding inviscid Euler equations is

$$\begin{cases} \partial_t \rho + div(\rho\ \vec{u}) = 0 \\ \partial_t(\rho\ \vec{u}) + div(\rho\ \vec{u} \otimes \vec{u}) + \bigtriangledown p = 0 \end{cases} \qquad \Omega \times (0,t) \tag{20}$$

$$(\rho, \vec{u})(x, t = 0) \equiv (\rho_0, \vec{u_0})(x) \quad , \tag{21}$$

$$\vec{u} \cdot \vec{n} = \vec{f} \cdot \vec{n} \quad \text{on} \quad \partial\Omega \times (0,t) \quad , \tag{22}$$

where $\vec{n}$ is the unit out normal of $\partial\Omega$.

We will discuss two distinct types of boundary conditions:

(A) Non-characteristic boundary conditions, i.e.

$$\vec{f}\ (x,t) \cdot \vec{n} \neq 0 \quad \text{on} \quad \partial\Omega \times (0,T) \quad . \tag{23}$$

In this case, the boundary condition (22) is uniformly non-characteristic for the Euler system (20).

(B) No-slip boundary condition, i.e.

$$\vec{f}\ (x,t) \equiv 0 \quad \text{on} \quad \partial\Omega \times (0,T) \quad . \tag{24}$$

In this case, the boundary condition (22) is uniformly-characteristic for the Euler system (20).

Due to the disparity of boundary conditions (19) and (22), the important pheonema, boundary layers, will appear for high Reynolds number flows. One would like to understand the asymptotic structures of solutions to the initial-boundary values problems, (15), (18)-(19), as $\varepsilon \to 0^+$, in particular, its relation to the solutions to the problem (20)-(22). Formally, one would expect (at least for short time) that

(A) for non-characteristic boundary, there appears a sharp changing gradient in region near the boundary of width of order $0(\varepsilon^2)$ (boundary layer), and away from the boundary layer, the flow is governed by the inviscid Euler equation. Furthermore, the boundary layer is governed by a system of ordinary differential equations.

(B) in the case of no-slip boundary condition, the width of the boundary layer is of order $0(\varepsilon)$, the flow near the boundary is governed by the Prandtl's boundary layer equations (which is a degenerate elliptic-parabolic system of partial differential equations) [33], and away from the boundary layer, the solution converges to inviscid one uniformly.

One of the main problems is to justify such formal theory rigorously. We will present some preliminary results here.

3.1. Non-Characteristic Boundary Condition. Under the assumption that the boundary is uniformly non-characteristic i.e., (23) holds. Then any "weak" boundary layers are nonlinearly stable and the visious solution converge uniformly to the corresponding inviscid solutions away from the boundary as $\varepsilon \to 0^+$ before shock formation.

To be precise, we consider the special case,

$$\Omega = \mathbb{R}^1_+ \times \mathbb{R} = \{(x_1, x_2) \in \mathbb{R}^2, \quad x_1 \geq 0\} \quad , \tag{25}$$

and the boundary condition (22) becomes

$$u_1(x_1 = 0, x_2, t) = f_1(x_2, t) \tag{26}$$

It can be shown that under suitable constraints on f_1, the initial-boundary value problem, (20),(21), and (26), has a unique smooth solution$(\rho^0, \vec{u}^{\,0})(x_1, x_3, t)$ with

$$(\rho^0 - \underline{\rho}, \vec{u}^{\,0}) \in C^l([0, T_*); H^{l+3}(\mathbb{R}^1_+ \times \mathbb{R}^1)) \tag{27}$$

where $\underline{\rho}(> 0)$ is a constant, $l \geq 1$, and $[0, T_\alpha)$ is the maximal interval of such a solution. Set

$$\delta(T) = max_{\{0 \leq t \leq T\}} \quad \|u_2^0(0, \cdot, t) - f_2(\cdot, t)\|_{L^\infty(\mathbb{R}^1)}, \forall T < T_* \quad . \tag{28}$$

Then, one has that (see [40])

Theorem 2 Let $(\rho^0, \vec{u}^{\,0})$ be the solution to the initial-boundary value problem (20), (21) and (26) such that (27) holds true with $l \geq 2$. Let $T \in (0, T_*)$ be fixed. Then there exist positive constants δ_0 and ε_0 independent of ε such that for

$$\delta(T) \leq \delta_0 \quad , \tag{29}$$

there exists a unique solution $(\rho^\varepsilon, \vec{u}^{\,\varepsilon})$ to the viscous initial-boundary value problem, (15), (18), and (19), such that

$$(\rho^\varepsilon - \underline{\rho}, \vec{u}^{\,\varepsilon}) \in C^1([0, T]; H^4(\mathbb{R}^1_+ \times \mathbb{R}^1)) \tag{30}$$

and

$$sup_{\{0 \leq t \leq T, x_1 \geq h > 0\}} \|(\rho^\varepsilon - \rho^O, \vec{u}^{\,\varepsilon} - \vec{u}^{\,0})(x_1, \cdot, t)\|_{L^\infty(\mathbb{R}^1)} \leq C_h \mu \tag{31}$$

provided that $\varepsilon \in (0, \varepsilon_0]$. Furthermore, for any given $n > 0$, there exists a smooth bounded function $(\tilde{\rho}, \tilde{u})(x, t, n, \varepsilon)$ such that

$$sup_{\{0 \leq t \leq T\}} \quad \|(\rho^\varepsilon, \vec{u}^{\,\varepsilon})(\cdot, t) - (\tilde{\rho}, \tilde{u})(\cdot, t)\|_{L^\infty(\Omega)} \leq C_n \varepsilon^{2n} \quad , \tag{32}$$

where

$$(\tilde{\rho}, \tilde{u})(x,t) \equiv (\rho^0, \vec{u}^{\,0}) \quad (x,t) + \sum_{i=1}^{n} \mu^{i-1} \vec{B_i} \left(\frac{x_1}{\mu}, x_2, t\right) + \sum_{i=1}^{n} \mu^i \vec{I_i} \,(x,t) \tag{33}$$

with B_i and I_i are smooth functions.

Remark 6 Due to the compatibility of the initial and boundary data, (29) holds for T small. So the asymptotic convergence for short time is a simple consequence of Theorem 2.

Remark 7 (32) and (33) give detailed asymptotic behavior of the solution to the Navier-Stokes system up to any given order, since the asymptotic ansatz $(\tilde{\rho}, \tilde{u})$ in (33) can be constructed explicitly by the method of matched asymptotic expansions [40]. Similar idea has been used for uniform parabolic systems in [37, 30, 11, 14].

Remark 8 The stability of strong boundary layer is still open.

We now turn to even more difficult case.

3.2. No-Slip boundary condition. The asymptotic behavior of the solutions to Navier-Stokes system for small viscosity in this case turns out to be one of most challenging problems in fluid dynamics. Very few rigorous results exist for the dynamical boundary layer behavior of the compressible Navier-Stokes system. To indicate the difficulties in treating this problem, we consider the special case that Ω is the half plane given in (25). Then the no-slip boundary condition becomes

$$\vec{u}\,(x_1 = 0, x_2, t) \equiv 0 \quad . \tag{34}$$

and the corresponding boundary condition for the Euler equation, (22) is

$$u_1(x_1 = 0, x_2, t) \equiv 0 \quad . \tag{35}$$

Applying Prandtl's ansatz

$$\begin{pmatrix} \vec{u} \\ P \end{pmatrix} (x_1, x_2, t) = \begin{pmatrix} \varepsilon u_1 \\ u_2 \\ p \end{pmatrix} \left(\frac{x_1}{\varepsilon}, x_2, \quad t\right) \quad , \tag{36}$$

one can derive the leading order governing system is the following Prandtl's type boundary layer equations [33]

$$\begin{cases} \partial_t u_2 + u_1 \partial_{y_1} u_2 + \partial_{x_2}(\frac{1}{2} u_2^2) = \partial_{y_1}^2 u \\ \\ \partial_{y_1} u_1 + \partial_{x_2} u_2 = 0(1) \end{cases} \tag{37}$$

where $y_1 = \frac{x_1}{\varepsilon}$, and $o(1)$ does not involve derivative of $\vec{u}$. (37) is a degenerate parabolic-elliptic system. Even short time well-posedness in a sobolev space is not known (except some special cases [31]). Even in the case that (37) can be solved in a class of smooth functions, the justification of the formal boundary layer theory is still very difficult due to the stiffness in the convection speed. Here we will present a justification of the Prandtl's boundary layer theory for linearized Navier-Stokes equations [38]. For the simplicity in presentation, we still consider the half plane problem so $\Omega = \mathbb{R}^1_+ \times \mathbb{R}^1$. Let (ρ', u_1', u_2') be a given smooth flow with the properties:

$$\begin{cases} (\rho', u_1', u_2')(x_1, x_2, t) \in C^l([0,T]; H^{l+2}(\Omega)) \quad , \\ \rho' \geq C_0 > 0 \quad \text{on} \quad \Omega \times (0,T) \quad , \\ u_1'(x_1 = 0, x_2, t) \equiv 0 \quad . \end{cases} \tag{38}$$

Set

$$V = \begin{pmatrix} p(\rho) \\ u_1 \\ u_2 \end{pmatrix} \tag{39}$$

Then the linearized initial-boundary value problem is

$$A_0(V')\partial_t V^\varepsilon + \sum_{j=1}^{2} A_j(V')\partial_j V^\varepsilon \quad = B(\varepsilon^2, C\varepsilon^2)V^\varepsilon \quad \text{on} \quad \Omega \times (0,T) \tag{40}$$

$$M^+ V^\varepsilon = 0 \quad , \qquad \text{on} \quad \partial\Omega \times [0,T] \tag{41}$$

$$V^\varepsilon(x,0) = (P_0, u_{10}, u_{20})^t \equiv V_0(x) \quad , \qquad x \in \Omega \tag{42}$$

where

$$A_0 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix} \quad , \quad A_j = \begin{pmatrix} \alpha u_j' & e_j^t \\ e_j & \beta u_j' I \end{pmatrix} \quad , \quad j = 1.2 \quad ,$$

$$B(\mu, \mu')V = \mu \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \triangle V + \mu' \begin{pmatrix} 0 \\ \nabla \end{pmatrix} (0, \nabla^t)V \quad , \tag{43}$$

$$\alpha = \frac{\rho_p'}{\rho'}, \quad e_1 = (1,0)^t, \quad e_2 = (0,1)^t, \quad \text{and} \quad \beta = \rho' \quad .$$

The corresponding linearized inviscid problem is

$$\begin{cases} A_\partial(V')\partial_t V^0 + \sum_{j=1}^{2} A_j(V')\partial_j V^0 = 0 \quad \Omega \times (0,T) \\ M^0 V^0 = 0 \qquad \text{on} \quad \partial\Omega \times [0,T] \\ \bar{V}^0(x, t=0) = V_0(x) \end{cases} \tag{44}$$

where

$$M^0 = (0,1,0) \quad . \tag{45}$$

Assume that the initial data V_0 is compatible with the boundary conditions up to high order [38]. Then the well-posedness of both problems (39)-(41) and (42) in the Sobolev space $C^l([0,T]; H^m(\Omega))$ can be established easily. Our goal is to give a pointwise estimate between V^ε and V^0. One has the following result [38].

Theorem 3 Assume V^0 is compatible with the boundary condition $M^+V = 0$. Let $V^0 \in C^l([0,T]; H^m(\Omega))$ be the unique solution to the initial-boundary value problem (42) for any T, $0 < T < \infty$. Then the unique solution V^ε to the viscous problem (39)-(41) admits the following estimate

$$\{|\rho^\varepsilon - \rho^0|(x_1, x_2, t) + \sum_{j=1}^{2} |u_j^\varepsilon - u_j^0|(x_1, x_2, t)\} \le C_h^\delta \varepsilon \tag{46}$$

holds uniformly for $t \in [0, T], x_1 \ge h\varepsilon^{1-\delta}$, and $x_2 \in \mathbb{R}^1$,
where $h > 0$ is any finite number, $0 < \delta < 1$, $0 < \varepsilon < 1$, and C_h^δ is a positive constant independent of ε. Furthermore, the following asymptotic expansion

$$V^\varepsilon(x, t) \sim V^0 + \sum_{i \ge 1} \varepsilon^i I^i(x_1, x_2, t) + \sum_{j \ge 0} \varepsilon^j B^j(\frac{x_1}{\varepsilon}, x_2, t) \tag{47}$$

can be constructed and justified up to any given order.

Remark 9 The main tools used in the proof of Theorem 2 are matched asymptotic analysis and weighted energy estimates. One of key elements is the analysis is to show that the linearized Prandtl's system is well-posed in a weighted Sobolev space [38]. However, these estimates are not enough to handle the nonlinear problem. Thus the boundary layer theory for the nonlinear Navier-Stokes system with no-slip boundary condition is far from being established.

References

[1] Bressan, A., Crasta, G., and Piccoli, B., Well-posedness of the Cauchy problem for nxn systems of conservation laws, preprint, 1997.

[2] Bressan, A., Liu, T. P. and Yang, T. L^1 - stability estimates for nxn systems of conservation laws, preprint, 1998.

[3] Chen, G. and Glimm, J., Global solution to the compressible Euler equations with geometrical structures, Comm. Math. Phys. 179 (1996), 153-193.

[4] Courant, R. and Friedrichs, K. O., Supersonic Flow And Shock Waves, Wiley-Interscience, 1948.

[5] Diperna, R., Convergence of the viscosity method for isentropic gas dynamics, Comm. Math. Phys., 91 (1983), 1-30.

[6] Gisclon, M. and Serre, D., Study of boundary conditions for a strictly hyperbolic system via parabolic approximation, C. R. Acad. Sci. Paris Ser. I Math. 319 (1994). No. 4, 377-382.

[7] Glimm, J., Solution in the large for nonlinear hyperbolic systems of equation, Comm. Pure Appl. Math. 18 (1965), 95-105.

[8] Glimm, J. and Lax, P. D., Decay of solutions of systems of nonlinear hyperbolic conservation laws, AMS Memoir 701 (1920).

[9] Goodman, J., Nonlinear asymptotic stability of viscous shock profiles for conservation laws, Arch. Rat. Mech. Anal. 95 (1986), 325-344.

[10] Goodman, J. and Xin, Z. P., Viscous limit to precewise smooth solutions of systems of conservation laws, Arch. Rat. Mech. Anal. 121 (1992), 235-265.

[11] Grenier, E., and Gues, O., On the inviscid limit of noncharacteristic nonlinear parabolic systems, preprint, 1998.

[12] Holf, D., Global solution of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, J. Diff. Equations, 120 (1995), 215-254.

[13] Holf, D., and Liu, T., The inviscid Limit for the Navier-Stokes equations of compressible isentropic flow with shock data, Indiana Univ. J. 38 (4) (1989), 861-915.

[14] Joseph, K. T., Boundary Layers in approximating solutions, Transac. AMS 314 (1989), 709-726.

[15] Kawashima, S. and Matsumura A., Asymptotic stability of traveling wave solutions of systems of one-dimensional gas motion, Comm. Math. Phys. 10 (1985), 97-127.

[16] Kawashima, S. and Nishida, T., The initial-value problems for the equations of viscous compressible and perfect compressible fluids, RIMS, Kokyuroku 428, Kyoto University, Nonlinear Functional Analysis, June 1981, 34-59.

[17] Landau, L. and Lifschitz, E., Fluid Mechanics, Addison-Wesley, New York , 1953.
[18] Lax, P. D., Hyperbolic Systems of Conservation Laws And The Mathematical Theory of Shock Waves, Conf. Board Math. Sci. 11, SIAM, 1973.
[19] Lions, P. L., Mathematical Topics In Fluid Mechanics, Vol. 2, Compressible Models, Lecture Series in Mathematics And Its Application, V. 6, Clarendon Press, Oxford, 1998.
[20] Lions, P. L. and Masmudi, N., Incompressible limit for a viscous compressible fluid, J. Math. Pure Appl. 77 (1998), 585-627.
[21] Lions, P. L. Perthame, B. and Souganidis, P. E., Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamic in Eulerian and Lagrangian coordinates, Comm. Pure Appl. Math., 49 (1996), 599-638.
[22] Liu, T. P., Pointwise convergence to shock waves for systems of visious conservation laws, preprint, to appear at Comm. Pure Applied Math.
[23] Liu, T. P. and Xin, Z. P., Pointwise decay to contact chscontinuities for systems of conservation laws, Asia J. Math., 1 (1997), 701-730.
[24] Liu, T. P. and Xin, Z. P. Nonlinear stability of rarefaction waves for compressible Navier-Stokes equations, Comm. Math. Phys. 118 (1988), 451-465.
[25] Liu, T. P. and Yang, T., L_1-well-posedness of Glrmm solution for systems of conservation laws, preprint, 1998.
[26] Liu, T. P. and Zeng, Y., Large time behavior of general systems of hyperbolic-parabolic conservation laws, AMS Memoir, 1997.
[27] Makino, T., Ukai, S. and Kawashima, S., Sur solutions a support de L'equation d'Euler Compressible, Japan J. Appl. Math. 3 (1986), 249-257.
[28] Matsumura, A. and Nishicla, T., The initial-value problem for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ. 20 (1980), 67-104.
[29] Matsumura, A. and Nishihara, K., On the stability of traveling waves of a one-dimensional model system for compressible viscous gas, Japan J. Appl. Math. 2 (1985), 17-25.
[30] Michaelson, D., Initial-boundary value problems for incomplete singular perturbations of hyperbolic systems, J. Analyse Math., 53 (1989), 1-138.
[31] Oleinik, O. A., The Prandtl system of equations in boundary layer theory, Dokl. Akad. Nank S.S.S.R., 150 4 (3) (1963), 583-586.
[32] Sammartino, M. and Caflish, R. E., Zero viscosity limit for analytic solutions of the Navier-Stokes equations on a half space, (I and II), preprint, 1997.
[33] Schlichting, H., Boundary-Layer Theory, 7th Edition, McGraw-Hill, 1987.
[34] Smoller, J. A., Shock Waves And Reaction-Diffusion Equations, Berlin-Heidallarg-New York: Springer-Verlay, New York, 1984.
[35] Szepessy, A., and Xin, Z. P., Nonlinear stability of viscous rarefaction craves, Arch. Rat. Mech. Anal. 122 (1993), 53-103.
[36] Xin, Z. P., On the blow-up of smooth solutions to the compressible Navier-Stokes equations with compact density, Comm. Pure Appl. Math., 51 (1998), 229-240.
[37] Xin, Z. P., Viscous boundary layers and their stability (I), T. Partial Diff. Eqs., 11 (1998), 497-541.
[38] Xin, Z. P. and Yanagisawa, T., Viscosity limit of the linearized Navier-Stokes equations for compressible viscous fluid in the half plan, Comm. Pure Appl. Math., Vol VII (1999), 497-541.
[39] Xin, Z. P., Zero dissipation limit to rarefaction waves for the one-dimensional Navier-Stokes equations for compressible isentropic gas, Comm. Pure Appl. Math. 86 (1993), 1499-1533.
[40] Xin Z. P., Zero dissipation limit for the compressible Navier-Stokes equations with non-characteristic boundary conditions, preprint, 2000.

Courant Institute, NYU, USA, and, IMS and Department of Mathematics, The Chinese University of Hong Kong, HK