Formation and construction of shock wave for quasilinear hyperbolic system and its application to inviscid compressible flow*

Shuxing Chen

(Institute of Mathematics, Fudan University, Shanghai, 200433, P.R.China)

(The Institute of Mathematical Sciences, CUHK, Shatin,N.T.,HongKong)

Zhouping Xin

(Courant Institute of Mathematics Science, New York University, NY10012,U.S.A)

(Department of Mathematics and IMS, CUHK, Shatin,N.T.,HongKong)

Huicheng Yin

(Department of Mathematics, Nanjing University, Nanjing 210093, P.R.China)

(The Institute of Mathematical Sciences, CUHK, Shatin,N.T.,HongKong)

Abstract

This paper concerns the initial formation and construction of shock waves for 3×3 quasilinear strictly hyperbolic system in one space dimension with small initial data. The system is assumed to be genuinely nonlinear with respect to a characteristic family. For such system, it is well-known that if the given smooth initial data satisfies certain nondegenerate condition, then the corresponding classical solution will blow up in finite time. Near the blowup point, we construct a weak entropy solution which is not uniformly Lipschitz continuous on two sides of the shock curve. This concrete construction yields detailed and precise estimates on the solution in the neighbourhood of the blowup point. This result is also applied to the Euler system describing the inviscid compressible flow in one space dimension.

Keywords: Lifespan, blowup system, conservation law system, shock wave

Mathematics Subject Classification: 35L70, 35L65

§1. Introduction

In this paper, we discuss the development of singularities of solution to the following 3×3 quasilinear strictly hyperbolic conservation law system with the small initial data:

$$\begin{cases} \partial_t u + (f(u))_x = 0 \\ u(x,0) = \varepsilon u_0(x) \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)^T$, $f(u) = (f_1(u), f_2(u), f_3(u))^T$, $f_i(u)$ is smooth, $\varepsilon > 0$ is small enough, $u_0(x) = (u_1^0(x), u_2^0(x), u_3^0(x))^T$, $u_i^0(x) \in C_0^\infty(\mathbb{R})$. The system (1.1) is assumed to be

* This research was supported by the Zheng Ge Ru Foundation when Chen Shuxing and Yin Huicheng were visiting the Institute of Mathematical Sciences, Chinese University of HongKong. Chen was supported in part by the Tianyuan Foundation of China and Doctoral Programme Foundation of High Education Institution of China. Xin is supported in part by NSF grants DMS-96-00137 and DMS-99-71978, DOE Grant DE-FG02-88ER-25053, and Hong Kong RGC Earmarked Research Grant CUHK-4219-99P. Yin was supported in part by the Tianyuan Foundation of China and Laboratary of Mathematics for Nonlinear Sciences at Fudan University.

genuinely nonlinear for at least one eigenvalue. From the results in [1],[2] and [3], we know that the lifespan T_ε of classical solution to (1.1) satisfies:

$$\lim_{\varepsilon\to 0}\varepsilon T_\varepsilon=\tau_0=-\frac{1}{\min\limits_{1\le j\le 3}M_j} \tag{1.2}$$

here $M_j=\min\limits_x h_j(x)$ and $h_j(x)$ can be determined by the coefficients and the initial data of (1.1) (see §2). On the other hand, it has been proved that the system (1.1) admits a unique global weak solution in the BV spaces(see [4], [5] and [6]). However, to better understand the physical process of development of singularities from smooth flow and the evolution of singularities starting from the blowup point, we are motivated to give a precise description of the location of shocks, as well as the estimates of solution and its derivatives near the blowup point.

The above problem is simply called as formation and construction of shock in this paper. For scalar equations, this problem has been completely solved early(see [7], [8], [9], [10] and so on). It is well known that in this case the formation of shock is caused by the squeeze of characteristics. For 2×2 p-system the same fact is also true. In [11] M.P.Lebaud gives an positive answer when the solution is a simple wave before the appearance of shock. For such a system with general smooth data, if only one Riemann invariant blows up and the blowup point is formed by the normal squeeze of only one family of characteristics, while another family of characteristics don't squeeze at the same point, the problem of formation and construction of shock wave was also completed in [12]. It is natural to study such problems for the Euler system describing the compressible flow and the physical shocks. However, even in one space dimension, the complete Euler system for gas dynamics is a 3×3 system. Therefore, in order to study the process of formation of shock waves, we need to study the case of 3×3 system.

In our study we benifit from S.Alinhac's result on the analysis of the mechanism of blowup of solutions to nonlinear hyperbolic system as well as the result on the extension of solution to the blowup system of (1.1) across the blowup time (see [13]). Based on this result we can concentrate our effort to the construction of solution with shock front after the blowup time. As in [11] and [12], one of the main difficulties is that the derivatives of solution blow up at the blowup point and their ratios of blowup are of $\frac{1}{T_\varepsilon-t}$ although the solution itself is continuous, hence here problem is different from the usual Riemann problem on the hyperbolic conservation laws because the Riemann problem generally has the discontinuous and piecewise smooth initial data. However, in constrast to the case of the 2×2 p-system considered in [11] and [12], where the existence of Riemann invariant coordinates plays the crucial role in their analysis, new ideas are needed for 3×3 hyperbolic system, which generally cannot be diagonalized by the Riemann invariants. In particular, we must find a new form of (1.1) such that its solution is more singular along one direction than other directions, this will be guaranteed by a new transformation given in §2. Furthermore, one needs to choose a good iterative scheme to construct the weak entropy solution which isn't uniformally Lipschitzian on two sides of shock curve. Moreover, in order to prove the convergence of iterative scheme we must pay more attention to the eigenvalue of (1.1) whose expression is more complex than the case for p-system.

Our paper is organized as follows. In §2, we first trasform the system (1.1) to the form which is suited to our further discussion, and then prove that the solution of blowup system of (1.1) can be extended across the blowup time. Meanwhile, we will give a precise description of result on the formation and construction of shock. In §3, by constructing an iterative sequence of approximate solutions near the blowup point, we prove the existence of solution with a shock starting from the blowup point. Finally, we apply for the above result to the compressible Euler equations with the initial data which is a small perturbation from the constant state. Consequently, the location of shock front as well as the solution near the blowup point are obtained.

§2. Reduction and main results

Let us first introduce some notations. We assume that three eigenvalues of matrix $f'(u)$ in (1.1) are different from each other, satisfying $\lambda_1(u) < \lambda_2(u) < \lambda_3(u)$. The corresponding right eigenvectors and left eigenvectors are $r_1(u), r_2(u), r_3(u)$ and $l_1(u), l_2(u), l_3(u)$ respectively. The system (1.1) is assumed to be genuinely nonlinear for at least one eigenvalue, that is, $\nabla_u \lambda_i(u) r_i(u) \neq 0$ for some i. Define

$$h_j(x) = \frac{\nabla_u \lambda_j(0) r_j(0)}{l_j(0) r_j(0)} l_j(0) u_0'(x), \qquad 1 \leq j \leq 3$$

and let $M_j = \min_x h_j(x)$, then (1.2) holds according to [2]. For simplicity, we assume $M_2 < \infty$ and $M_2 < inf\{M_1, M_3\}$ in the sequel. In fact, if $M_1 < inf\{M_2.M_3\}$ or $M_3 < inf\{M_1, M_2\}$, the discussion will be simpler.

Lemma 2.1. Under the assumptions of strict hyperbolicity and genuine nonlinearity, the system (1.1) can be reduced to the following form by an invertible transformation in the neighborhood of origin:

$$\begin{cases} \partial_t w + B(w)\partial_x w = 0 \\ w(0,x) = \varepsilon w^1(x) + \varepsilon^2 w^2(x) + \ldots \end{cases} \tag{2.1}$$

here $B(w) = \begin{pmatrix} b_{11}(w) & 0 & b_{13}(w) \\ b_{21}(w) & \lambda_2(w) & b_{23}(w) \\ b_{31}(w) & 0 & b_{33}(w) \end{pmatrix}$ and $B(0)$ is a diagonal matrix $diag\{\lambda_1(0), \lambda_2(0), \lambda_3(0)\}$.

Proof. The proof is just the process of reduction. The whole reduction consists of three steps. First, one reduces the matrix $f'(u)$ to a matrix whose elements a_{12} and a_{32} vanish. The second step is to make the coefficient matrix be diagonal at the origin. Finally, we transform the result into one for which each equation in the system contains only directional derivatives along same directions.

Choose two Riemann invariants $\alpha_1(u) \in C^\infty$ and $\alpha_3(u) \in C^\infty$ corresponding to $\lambda_2(u)$, that is, $\alpha_1(u)$ and $\alpha_3(u)$ satisfy $\nabla_u \alpha_1(u) r_2(u) = 0$ and $\nabla_u \alpha_3(u) r_2(u) = 0$, and choose a smooth function $\alpha_2(u)$ such that the Jacobian matrix $J = \left(\frac{\partial(\alpha_1,\alpha_2,\alpha_3)}{\partial(u_1,u_2,u_3)} \right)$ is invertible for small $|u|$. Introduce the transform:

$$\begin{cases} \alpha_1 = \alpha_1(u) - \alpha_1(0) \\ \alpha_2 = \alpha_2(u) - \alpha_2(0) \\ \alpha_3 = \alpha_3(u) - \alpha_3(0) \end{cases} \tag{2.2}$$

the system (1.1) can be changed into the form:

$$\partial_t \alpha + A(\alpha)\partial_x \alpha = 0 \tag{2.3}$$

here the 3×3 matrix $A(\alpha) = (a_{ij}(\alpha)) = J(\partial_{u_i} f_j)J^{-1}$. Denote the second right eigenvector of $A(\alpha)$ by $\tilde{r}_2(\alpha)$. Since $0 = \nabla_u \alpha_1(u) r_2(u) = \nabla_\alpha \alpha_1 \tilde{r}_2(\alpha)$ and $0 = \nabla_u \alpha_3(u) r_2(u) = \nabla_\alpha \alpha_3 \tilde{r}_2(\alpha)$, then we can choose $\tilde{r}_2(\alpha) = (0,1,0)^T$. It follows from $A(\alpha)\tilde{r}_2(\alpha) = \lambda_2(\alpha)\tilde{r}_2(\alpha)$ that $a_{12} = a_{32} = 0$, and (2.3) can be written as:

$$\begin{cases} \partial_t \alpha_1 + a_{11}(\alpha)\partial_x \alpha_1 + \qquad a_{13}(\alpha)\partial_x \alpha_3 = 0 \\ \partial_t \alpha_2 + a_{21}(\alpha)\partial_x \alpha_1 + a_{22}(\alpha)\partial_x \alpha_2 + a_{23}(\alpha)\partial_x \alpha_3 = 0 \\ \partial_t \alpha_3 + a_{31}(\alpha)\partial_x \alpha_1 + \qquad a_{33}(\alpha)\partial_x \alpha_3 = 0 \end{cases} \tag{2.4}$$

Since $A(\alpha)$ has three distinct real eigenvalues $\lambda_1(\alpha) < \lambda_2(\alpha) < \lambda_3(\alpha)$, then $a_{22}(\alpha) = \lambda_2(\alpha)$, and $\bar{B}(\alpha) = \begin{pmatrix} a_{11}(\alpha) & a_{13}(\alpha) \\ a_{31}(\alpha) & a_{33}(\alpha) \end{pmatrix}$ has two eigenvalues $\lambda_1(\alpha)$ and $\lambda_3(\alpha)$. So there exists a 2×2 invertible number matrix $C = (c_{ij})_{i,j=1}^2$ such that $C^{-1}\bar{B}(0)C = \begin{pmatrix} \lambda_1(0) & 0 \\ 0 & \lambda_3(0) \end{pmatrix}$. Set $\begin{pmatrix} w_1 \\ w_3 \end{pmatrix} = C \begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix}$, then (2.4) shows

$$\partial_t \begin{pmatrix} w_1 \\ w_3 \end{pmatrix} + \tilde{B}(w_1, \alpha_2, w_3)\partial_x \begin{pmatrix} w_1 \\ w_3 \end{pmatrix} = 0$$

where $\tilde{B}(w_1, \alpha_2, w_3) = C\bar{B}(\alpha)C^{-1}$ and $\tilde{B}(0,0,0) = \begin{pmatrix} \lambda_1(0) & 0 \\ 0 & \lambda_3(0) \end{pmatrix}$.

Hence (2.4) has the form:

$$\begin{cases} \partial_t w_1 + \tilde{a}_{11}(w_1, \alpha_2, w_3)\partial_x w_1 + \qquad \tilde{a}_{13}(w_1, \alpha_2, w_3)\partial_x w_3 = 0 \\ \partial_t \alpha_2 + \tilde{a}_{21}(w_1, \alpha_2, w_3)\partial_x w_1 + \tilde{a}_{22}(w_1, \alpha_2, w_3)\partial_x \alpha_2 + \tilde{a}_{23}(w_1, \alpha_2, w_3)\partial_x w_3 = 0 \\ \partial_t w_3 + \tilde{a}_{31}(w_1, \alpha_2, w_3)\partial_x w_1 + \qquad \tilde{a}_{33}(w_1, \alpha_2, w_3)\partial_x w_3 = 0 \end{cases} \tag{2.5}$$

where $\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{13} \\ \tilde{a}_{31} & \tilde{a}_{33} \end{pmatrix}$ at $(0,0,0)$ is a 2×2 diagonal matrix.

Set $\alpha_2 = w_2 + m_1 w_1 + m_2 w_3$, where $m_1, m_2 \in \mathbb{R}$ are constants to be determined. Then

$$\begin{aligned} &\partial_t \alpha_2 + \tilde{a}_{21}\partial_x w_1 + \tilde{a}_{22}\partial_x \alpha_2 + \tilde{a}_{23}\partial_x w_3 \\ &= \partial_t w_2 + [\tilde{a}_{21} + m_1(\tilde{a}_{22} - \tilde{a}_{11}) - m_2 \tilde{a}_{31}]\partial_x w_1 + [\tilde{a}_{23} + m_2(\tilde{a}_{22} - \tilde{a}_{33}) \\ &- m_1 \tilde{a}_{13}]\partial_x w_3 + \tilde{a}_{22}\partial_x w_2 \end{aligned}$$

Note $\tilde{a}_{22}(0) = \lambda_2(0)$, and let the following equalities hold

$$\begin{cases} (\lambda_2(0) - \tilde{a}_{11}(0))m_1 - \tilde{a}_{31}(0)m_2 = -\tilde{a}_{21}(0) \\ -\tilde{a}_{13}(0)m_1 + (\lambda_2(0) - \tilde{a}_{33}(0))m_2 = -\tilde{a}_{23}(0) \end{cases} \tag{2.6}$$

In fact, since $\begin{pmatrix} \tilde{a}_{11}(0) & \tilde{a}_{31}(0) \\ \tilde{a}_{13}(0) & \tilde{a}_{33}(0) \end{pmatrix}$ has two distinct eigenvalues $\lambda_1(0)$ and $\lambda_3(0)$, then (2.6) has a unique solution (m_1, m_2). Hence the form (2.1) is obtained.

Three left eigenvectors of $B(w)$ are $l_i(w) = (l_{i1}(w), l_{i2}(w), l_{i3}(w))$, $i = 1, 2, 3$, which are unit vectors at $w = 0$. Because the solution w is small, we know $l_{ii}(w) \neq 0$ for $i = 1, 2, 3$. Since the second right eigenvector of $B(w)$ is $r_2(w) = (0, 1, 0)^T$, and $l_1(w)r_2(w) = l_3(w)r_2(w) = 0$, then we have $l_{12}(w) = 0$ and $l_{32}(w) = 0$ for all small w.

¿From (2.1), we easily get: $l_i(w) \begin{pmatrix} \partial_t w_1 + \lambda_i(w)\partial_x w_1 \\ \partial_t w_2 + \lambda_i(w)\partial_x w_2 \\ \partial_t w_3 + \lambda_i(w)\partial_x w_3 \end{pmatrix} = 0$ for $i = 1, 2, 3$. Hence we have

$$\begin{cases} l_{11}(w)(\partial_t w_1 + \lambda_1 \partial_x w_1) + \qquad l_{13}(w)(\partial_t w_3 + \lambda_1 \partial_x w_3) = 0 \\ l_{22}(w)(\partial_t w_2 + \lambda_2 \partial_x w_2) + l_{21}(w)(\partial_t w_1 + \lambda_2 \partial_x w_1) + l_{23}(w)(\partial_t w_3 + \lambda_2 \partial_x w_3) = 0 \\ l_{33}(w)(\partial_t w_3 + \lambda_3 \partial_x w_3) + \qquad l_{31}(w)(\partial_t w_1 + \lambda_3 \partial_x w_1) = 0 \\ w(0, x) = \varepsilon w^1(x) + \varepsilon^2 w^2(x) + ... \end{cases} \tag{2.7}$$

We emphasize that in each equation of (2.7) the differentiation is taken along the same direction, moreover its coefficient matrix at $w = 0$ is diagonal.

Since $M_2 < inf\{M_1, M_3\}$, it follows from [1,2,3] that $l_2(u)\partial_x u$ blows up at T_ε but $l_1(u)\partial_x u$ and $l_3(u)\partial_x u$ don't blow up at T_ε. Then one checks easily that $l_2(w)\partial_x w$ blows up at T_ε while $l_1(w)\partial_x w$ and $l_3(w)\partial_x w$ remain bounded at T_ε. Therefore, from the expressions of l_1, l_2, l_3, we know that only $\partial_x w_2$ blows up at T_ε but $\partial_x w_1$ and $\partial_x w_3$ remain bounded.

To study the structure of solution near the blowup point, we recall the definition of blowup system of hyperbolic system introduced in [13]. Take a transform $x = \varphi(t, y), t = t$ and denote $\bar{w}(t, y) = w(t, \varphi(t, y))$, the blowup system for (2.1) is defined as follows:

$$\begin{cases} \partial_t \varphi = \lambda_2(\bar{w}) \\ l_2(\bar{w})\partial_t \bar{w} = 0 \\ l_k(\bar{w})[\partial_y \varphi \partial_t \bar{w} + (\lambda_k - \lambda_2)(\bar{w})\partial_y \bar{w}] = 0, k = 1, 3 \\ \varphi(\frac{\tau_0}{2\varepsilon}, y) = y, \bar{w}(\frac{\tau_0}{2\varepsilon}, y) = w(\frac{\tau_0}{2\varepsilon}, y) \end{cases} \tag{2.8}$$

where τ_0 is defined in (1.2).

Lemma 2.2. Assume that $h_2(x)$ has a unique strictly negative quadratic minimum point. Then the blowup system (2.8) has a unique smooth solution for small ε and $t \leq \frac{2\tau_0}{\varepsilon}$. The solution satisfies

$|\partial^\alpha_{t,y}\varphi(t, y)| \leq C_\alpha, |\partial^\alpha_{t,y}\bar{w}(t, y)| \leq C_\alpha \varepsilon$ for $|\alpha| \geq 0$ and $t \leq \frac{2\tau_0}{\varepsilon}$,
$\partial_y \varphi(t, y) \geq 0$ in $t \leq T_\varepsilon$.

Moreover, there is a unique point $(T_\varepsilon, y_\varepsilon)$ such that

$$\partial_y \varphi(T_\varepsilon, y_\varepsilon) = 0, \partial_y^2 \varphi(T_\varepsilon, y_\varepsilon) = 0, \partial_y^3 \varphi(T_\varepsilon, y_\varepsilon) > 0, \partial_{ty}^2 \varphi(T_\varepsilon, y_\varepsilon) < 0$$

Proof. Without loss of generality, we can assume $\lambda_2(0) = 0$.

We would like to apply the Theorem 3 in [13]. To this end, we first simplify system (2.1) as follows.

Let the right eigenvectors of $B(w)$ be written as $r_1(w) = (r_{11}(w), r_{12}(w), r_{13}(w))^T$, $r_2(w) = (0,1,0)^T$, $r_3(w) = (r_{31}(w), r_{32}(w), r_{33}(w))^T$. Obviously, $r_{11}(w) \neq 0$ and $r_{33}(w) \neq 0$ for small $|w|$.

Set

$$\left\{\begin{array}{l} \frac{dw_1^{(1)}}{dm_1} = r_{11}(w_1^{(1)}, w_2^{(1)}, w_3^{(1)}) \\ \frac{dw_2^{(1)}}{dm_1} = r_{12}(w_1^{(1)}, w_2^{(1)}, w_3^{(1)}) \\ \frac{dw_3^{(1)}}{dm_1} = r_{13}(w_1^{(1)}, w_2^{(1)}, w_3^{(1)}) \\ w_i^{(1)}(m_1)|_{m_1=0} = 0, i = 1,2,3 \end{array}\right. \tag{2.9}$$

and $w_1^{(2)}(m_1, m_2) = w_1^{(1)}(m_1), w_2^{(2)}(m_1, m_2) = m_2 + w_2^{(1)}(m_1), w_3^{(2)}(m_1, m_2) = w_3^{(1)}(m_1)$, and

$$\left\{\begin{array}{l} \frac{dw_1^{(3)}}{dm_3} = r_{31}(w_1^{(3)}, w_2^{(3)}, w_3^{(3)}) \\ \frac{dw_2^{(3)}}{dm_3} = r_{32}(w_1^{(3)}, w_2^{(3)}, w_3^{(3)}) \\ \frac{dw_3^{(3)}}{dm_3} = r_{33}(w_1^{(3)}, w_2^{(3)}, w_3^{(3)}) \\ w_i^{(3)}(m_1, m_2, m_3)|_{m_3=0} = w_i^{(2)}(m_1, m_2), i = 1,2,3 \end{array}\right. \tag{2.10}$$

We obtain a diffeomorphism H in a small neighbourhood of the origin:

$$m = (m_1, m_2, m_3) \to w = (w_1, w_2, w_3) \tag{2.11}$$

here $w_i = w_i^{(3)}(m_1, m_2, m_3)$ are difined as in (2.10). Under transform (2.11), the system (2.1) takes the form as follows:

$$\left\{\begin{array}{l} \partial_t m + D(m)\partial_x m = 0 \\ m(0,x) = \varepsilon m^1(x) + \varepsilon^2 m^2(x) + ... \end{array}\right. \tag{2.12}$$

where three right eigenvectors $r_1(m)$, $r_2(m)$ and $r_3(m)$ of $D(m)$ satisfy $r_1(m_1, 0, 0) = (1,0,0)^T$, $r_2(0, m_2, 0) = (0,1,0)^T$, $r_3(0,0,m_3) = (0,0,1)^T$ respectively. It turns out that the solution m of (2.12) is 3-simple and 1-simple respectively in the right and left of support of m_2.

The blowup system of (2.12) corresponding to $\lambda_2(m)$ is:

$$\left\{\begin{array}{l} \partial_t \tilde{\varphi} = \lambda_2(\tilde{m}) \\ l_2(\tilde{m})\partial_t \tilde{m} = 0 \\ l_k(\tilde{m})[\partial_y \tilde{\varphi}\partial_t \tilde{m} + (\lambda_k - \lambda_2)(\tilde{m})\partial_y \tilde{m}] = 0, k = 1,3 \\ \tilde{\varphi}(\frac{\tau_0}{2\varepsilon}, y) = y, \tilde{m}(\frac{\tau_0}{2\varepsilon}, y) = m(\frac{\tau_0}{2\varepsilon}, y) \end{array}\right. \tag{2.13}$$

Furthermore, assume $\tau = \varepsilon t$, $\bar{\varphi}(\tau, y) = \tilde{\varphi}(\frac{\tau}{\varepsilon}, y)$, $\bar{m}(\tau, y) = \tilde{m}(\frac{\tau}{\varepsilon}, \bar{\varphi}(\tau, y))$, from (2.13) we get

$$\left\{\begin{array}{l} \varepsilon\partial_\tau \bar{\varphi} = \lambda_2(\bar{m}) \\ l_2(\bar{m})\partial_\tau \bar{m} = 0 \\ l_k(\bar{m})[\varepsilon\partial_y \bar{\varphi}\partial_\tau \bar{m} + (\lambda_k - \lambda_2)(\bar{m})\partial_y \bar{m}] = 0, k = 1,3 \\ \bar{\varphi}(\frac{\tau_0}{2}, y) = y, \bar{m}(\frac{\tau_0}{2}, y) = m(\frac{\tau_0}{2\varepsilon}, y) \end{array}\right. \tag{2.14}$$

Since the system (2.12) satisfies all the conditions of Theorem 3 in [13], it follows that (2.14) has a unique smooth solution $(\bar{\varphi}, \bar{m})$ for small ε and $\frac{\tau_0}{2} \le \tau \le 2\tau_0$, furthermore there exists a unique point $(\tau_\varepsilon = \frac{T_\varepsilon}{\varepsilon}, y_\varepsilon)$ such that

$$\partial_y \bar{\varphi}(\tau_\varepsilon, y_\varepsilon) = 0, \partial_y^2 \bar{\varphi}(\tau_\varepsilon, y_\varepsilon) = 0, \partial_y^3 \bar{\varphi}(\tau_\varepsilon, y_\varepsilon) > 0, \partial_{y\tau}^2 \bar{\varphi}(\tau_\varepsilon, y_\varepsilon) < 0 \tag{2.15}$$

Moreover $\partial_y \bar{\varphi}(\tau, y) \ge 0$ for $\tau \le \tau_\varepsilon$. In terms of the original coordinates (t, y) and the unkown functions in (2.8), one has

$$\varphi(t, y) = \tilde{\varphi}(t, y), \bar{w}(t, y) = H(\tilde{m}(t, y))$$

We obtain the conclusion of Lemma 2.2.

It follows from Lemma 2.1 that the unique blowup point of (2.1) at T_ε is $(T_\varepsilon, x_\varepsilon = \varphi(T_\varepsilon, y_\varepsilon)$. Now our main results can be stated as:

Theorem 2.1. For the system (2.1), suppose $\partial_2 \lambda_2(0) \partial_x w_2^1(x)$ has a unique strictly negative quadratic minimum point, then for small ε (2.1) admits a weak entropy solution with a shock $x = \phi(t)$ starting from the blowup point $(T_\varepsilon, x_\varepsilon)$ in $[T_\varepsilon, T_\varepsilon + 1]$. Moreover, in the neighbourhood of $(T_\varepsilon, x_\varepsilon)$, the following estimates hold:

$$\begin{aligned}
&\phi(t) = x_\varepsilon + \lambda_2(w(T_\varepsilon, x_\varepsilon))(t - T_\varepsilon) + O((t - T_\varepsilon)^2) \\
&w_2(t, x) = w_2(T_\varepsilon, x_\varepsilon) + O((t - T_\varepsilon)^3 + (x - x_\varepsilon - \lambda_2(w(T_\varepsilon, x_\varepsilon))(t - T_\varepsilon))^2)^{\frac{1}{6}} \\
&w_i(t, x) = w_i(T_\varepsilon, x_\varepsilon) + O((t - T_\varepsilon)^3 + (x - x_\varepsilon - \lambda_2(w(T_\varepsilon, x_\varepsilon))(t - T_\varepsilon))^2)^{\frac{1}{3}}, i = 1, 3
\end{aligned}$$

Therefore, returning to the system (1.1) we have near $(T_\varepsilon, x_\varepsilon)$:

$$u_i(t, x) = u_i(T_\varepsilon, x_\varepsilon) + O((t - T_\varepsilon)^3 + (x - x_\varepsilon - \lambda_2(u(T_\varepsilon, x_\varepsilon))(t - T_\varepsilon))^2)^{\frac{1}{6}}, i = 1, 2, 3$$

here "O'' stands for a uniformly bounded quantity independent of ε.

Remark 2.1. By [1], [2] and [13], under the assumptions of Theorem 2.1 we know the solution of (1.1) or (2.1) doesn't blow up away from the small neighbourhood of x_ε for $t \in [T_\varepsilon, T_\varepsilon + 1]$. Hence in order to complete the construction of shock wave in $t \in [T_\varepsilon, T_\varepsilon + 1]$, we only study that problem in the neighbourhood Ω of $(T_\varepsilon, x_\varepsilon)$, here $\Omega = \{(t, x) : T_\varepsilon < t \le T_\varepsilon + 1, x_\varepsilon - K(T_\varepsilon + 1 - t) \le x \le x_\varepsilon + K(T_\varepsilon + 1 - t)\}$ and $K = 2max\{|\lambda_1(0)|, |\lambda_2(0)|, |\lambda_3(0)|\}$.

§3. The proof of Theorem 2.1

As remarked in the last section, we only need to analyze the problem in the neighbourhood Ω of $(T_\varepsilon, x_\varepsilon)$. Following similar ideas in [11] and [12] (but with more careful treatments of the uniform bounds independent of small ε), we can prove the following three lemmas which describe some subtle properties of solution of the blowup system.

Lemma 3.1. 1) For $t \in (T_\varepsilon, T_\varepsilon + 1]$ and in the small neighbourhood of y_ε, $\partial_y \varphi(t, y) = 0$ has two distinct real roots $\eta_-^\varepsilon(t)$ and $\eta_+^\varepsilon(t)$, moreover $\eta_+^\varepsilon(t) < y_\varepsilon < \eta_-^\varepsilon(t)$ and $\eta_-^\varepsilon(t)$, $\eta_+^\varepsilon(t) \in C^\infty(T_\varepsilon, T_\varepsilon + 1]$.

2) Set $x_-^\varepsilon(t) = \varphi(t, \eta_-^\varepsilon(t))$ and $x_+^\varepsilon(t) = \varphi(t, \eta_+^\varepsilon(t))$, then

$x = \varphi(t, y)$ has three real roots $y_-^\varepsilon(t, x) < y_c^\varepsilon(t, x) < y_+^\varepsilon(t, x)$ if $x \in (x_+^\varepsilon(t), x_-^\varepsilon(t))$.

$x = \varphi(t, y)$ has a unique real root $y_+^\varepsilon(t, x)$ if $x \ge x_-^\varepsilon(t)$.

$x = \varphi(t, y)$ has a unique real root $y_-^\varepsilon(t, x)$ if $x \le x_+^\varepsilon(t)$.

3) Denote $\Omega_+ = \{(t,x) \in \Omega : T_\varepsilon < t \le T_\varepsilon + 1, x > x_+^\varepsilon(t)\}$ and $\Omega_- = \{(t,x) \in \Omega : T_\varepsilon < t \le T_\varepsilon + 1, x < x_-^\varepsilon(t)\}$, then $y_\pm^\varepsilon(t,x) \in C^\infty(\Omega_\pm)$ and $y_\pm^\varepsilon(t,x) \in C(\bar{\Omega}_\pm)$.

Lemma 3.2. We write $w_{i,\pm}^0(t,x) = \bar{w}_i(t, y_\pm^\varepsilon(t,x))$ in $\Omega_\pm$ and $w_\pm^0(t,x) = (w_{1,\pm}^0, w_{2,\pm}^0, w_{3,\pm}^0)$, denote the second eigenvalue of matrix $(\int_0^1 (\partial_{u_i} f_j)(\theta u(w_+^0) + (1-\theta)u(w_-^0))d\theta)_{i,j=1}^3$ by $\tilde{\lambda}_2$, then the solution of the initial data problem

$$\left\{ \begin{array}{l} \frac{d\phi^0(t)}{dt} = \tilde{\lambda}_2(\int_0^1 (\partial_{u_i} f_j)(\theta u(\bar{w}(t, y_+^\varepsilon(t, \phi^0(t)))) + (1-\theta)u(\bar{w}(t, y_-^\varepsilon(t, \phi^0(t)))))d\theta) \\ \phi^0(T_\varepsilon) = x_\varepsilon \end{array} \right. \tag{3.1}$$

satisfies $\phi^0(t) \in C^\infty[T_\varepsilon, T_\varepsilon + 1]$ and $x_+^\varepsilon < \phi^0(t) < x_-^\varepsilon(t)$, and

$$\phi^0(t) = x_\varepsilon + \lambda_2(u(T_\varepsilon, x_\varepsilon))(t - T_\varepsilon) + O((t-T_\varepsilon)^2), t \in [T_\varepsilon, T_\varepsilon + 1]$$

Lemma 3.3. Denoting $d_\varepsilon = (t-T_\varepsilon)^3 + (x - x_\varepsilon - \lambda_2(u(T_\varepsilon, x_\varepsilon))(t-T_\varepsilon))^2$, then

$$|y_\pm^\varepsilon(t,x) - y_\varepsilon| < C d_\varepsilon^{\frac{1}{6}}, |\partial_x y_\pm^\varepsilon(t,x)| \le C d_\varepsilon^{-\frac{1}{3}}$$
$$|\partial_\ell y_\pm^\varepsilon(t,x)| \le C d_\varepsilon^{-\frac{1}{6}}, |\partial_x^2 y_\pm^\varepsilon(t,x)| \le C d_\varepsilon^{-\frac{5}{6}}$$

where ℓ is the direction of second charecteristics passing $(T_\varepsilon, x_\varepsilon)$.

We emphasize that in the above lemmas both the constant C and the estimates "O" are independent of ε.

Define the function

$$w_i^0(t,x) = \left\{ \begin{array}{l} w_{i,+}^0(t,x), x > \phi^0(t) \\ w_{i,-}^0(t,x), x < \phi^0(t) \end{array} \right.$$

in Ω. Obviously, $w^0(t,x)$ is the solution of (2.1) in $\Omega_\pm$ respectively. But it isn't a weak solution of (2.1) because it doesn't satisfy the Rankine-Hugoniot condition along the curve γ: $x = \phi^0(t)$. We will use an iterative scheme to construct the shock starting from the point $(T_\varepsilon, x_\varepsilon)$ for the system (2.1) by modifying the location of curve γ as well as the solution on both sides of γ. In the forthcoming iteration, $(w^0(t,x), \phi^0(t))$ will be chosen as the first approximation of iterative scheme.

Lemma 3.4. In the domain $\Omega \backslash \gamma$, we have

1) $w_2^0(t,x)$ satisfies the estimates:

$$\left\{ \begin{array}{l} |w_2^0(t,x) - w_2^0(T_\varepsilon, x_\varepsilon)| \le C\varepsilon d_\varepsilon^{\frac{1}{6}} \\ |\partial_\ell w_2^0(t,x)| \le C\varepsilon d_\varepsilon^{-\frac{1}{6}} \\ |\partial_x w_2^0(t,x)| \le C\varepsilon d_\varepsilon^{-\frac{1}{3}} \\ |\partial_x^2 w_2^0(t,x)| \le C\varepsilon d_\varepsilon^{-\frac{5}{6}} \end{array} \right. \tag{3.2}$$

2) $w_i^0(t,x)$ $(i = 1,3)$ satisfies the estimates:

$$\left\{ \begin{array}{l} |w_i^0(t,x) - w_i^0(T_\varepsilon, x_\varepsilon)| \le C\varepsilon d_\varepsilon^{\frac{1}{3}} \\ |\partial_t w_i^0(t,x)| \le C\varepsilon \\ |\partial_x w_i^0(t,x)| \le C\varepsilon \\ |\partial_x^2 w_i^0(t,x)| \le C\varepsilon d_\varepsilon^{-\frac{1}{2}} \end{array} \right. \tag{3.3}$$

Proof. It is enough to prove the lemma in the domain Ω_+.

1) Thanks to Lemma 2.2, and one has

$$\begin{aligned}
w_2^0(t,x) - w_2^0(T_\varepsilon, x_\varepsilon) &= \bar{w}_2(t, y_+^\varepsilon(t,x)) - \bar{w}_2(T_\varepsilon, y_\varepsilon)\\
&= \partial_t \bar{w}_2(T_\varepsilon, y_\varepsilon)(t-T_\varepsilon) + \partial_y \bar{w}_2(T_\varepsilon, y_\varepsilon)(y_+^\varepsilon(t,x) - y_\varepsilon) + O(\varepsilon(t-T_\varepsilon)^2 + \varepsilon(y_+^\varepsilon - y_\varepsilon)^2)\\
\partial_\ell w_2^0(t,x) &= \partial_t \bar{w}_2(t, y_+^\varepsilon(t,x)) + \partial_y \bar{w}_2(t, y_+^\varepsilon(t,x))\partial_\ell y_+^\varepsilon\\
\partial_x w_2^0(t,x) &= \partial_y \bar{w}_2(t, y_+^\varepsilon(t,x))\partial_x y_+^\varepsilon\\
\partial_x^2 w_2^0(t,x) &= \partial_y^2 \bar{w}_2(t, y_+^\varepsilon(t,x))(\partial_x y_+^\varepsilon)^2 + \partial_y \bar{w}_2(t, y_+^\varepsilon(t,x))\partial_x^2 y_+^\varepsilon
\end{aligned}$$

Hence (3.2) follows from Lemma 2.2 and Lemma 3.3.

2) It follows from the blowup system (2.8) and the property that $\begin{pmatrix} l_{11} & l_{13} \\ l_{31} & l_{33} \end{pmatrix}$ is invertible for small $|\bar{w}|$ that

$$\left\{\begin{array}{l}
\partial_y \varphi \partial_t \bar{w}_1 + (\lambda_1 - \lambda_2)(\bar{w})\partial_y \bar{w}_1 = 0\\
\partial_y \varphi \partial_t \bar{w}_3 + (\lambda_3 - \lambda_2)(\bar{w})\partial_y \bar{w}_3 = 0
\end{array}\right. \tag{3.4}$$

and $\partial_y \varphi(T_\varepsilon, y_\varepsilon) = 0$ implies $\partial_y \bar{w}_1(T_\varepsilon, y_\varepsilon) = \partial_y \bar{w}_3(T_\varepsilon, y_\varepsilon) = 0$.

Differentiate with respect to y on two sides of the first equation in (3.4), we get:

$$\partial_y^2 \varphi \partial_t \bar{w}_1 + \partial_y \varphi \partial_{ty}^2 \bar{w}_1 + (\sum_{j=1}^{3} (\partial_{\bar{w}_j}(\lambda_1 - \lambda_2))(\bar{w})\partial_y \bar{w}_j)\partial_y \bar{w}_1 + (\lambda_1 - \lambda_2)(\bar{w})\partial_y^2 \bar{w}_1 = 0$$

Set $(t,y) = (T_\varepsilon, y_\varepsilon)$ in the above equality, due to $\partial_y \varphi(T_\varepsilon, y_\varepsilon) = \partial_y^2 \varphi(T_\varepsilon, y_\varepsilon) = \partial_y \bar{w}_1(T_\varepsilon, y_\varepsilon) = 0$, one obtains $\partial_y^2 \bar{w}_1(T_\varepsilon, y_\varepsilon) = 0$. Hence

$$\begin{aligned}
\partial_y \bar{w}_1(t, y_+^\varepsilon) &= \partial_{ty}^2 \bar{w}_1(T_\varepsilon, y_\varepsilon)(t - T_\varepsilon) + O(\varepsilon(t-T_\varepsilon)^2 + \varepsilon(y_+^\varepsilon - y_\varepsilon)^2)\\
\partial_y^2 \bar{w}_1(t, y_+^\varepsilon) &= \partial_t \partial_y^2 \bar{w}_1(T_\varepsilon, y_\varepsilon)(t-T_\varepsilon) + \partial_y^3 \bar{w}_1(T_\varepsilon, y_\varepsilon)(y_+^\varepsilon - y_\varepsilon)\\
&\quad + O(\varepsilon(t-T_\varepsilon)^2 + \varepsilon(y_+^\varepsilon - y_\varepsilon)^2)
\end{aligned}$$

Then by Lemma 2.2 and Lemma 3.3 one has that

$$|\partial_y \bar{w}_1(t, y_+^\varepsilon)| \le C\varepsilon d_\varepsilon^{\frac{1}{3}}, |\partial_y^2 \bar{w}_1(t, y_+^\varepsilon)| \le C\varepsilon d_\varepsilon^{\frac{1}{3}} \tag{3.5}$$

Additionally,

$$\begin{aligned}
w_1^0(t,x) - w_1^0(T_\varepsilon, x_\varepsilon) &= \partial_t \bar{w}_1(T_\varepsilon, y_\varepsilon)(t-T_\varepsilon) + \partial_y \bar{w}_1(T_\varepsilon, y_\varepsilon)(y_+^\varepsilon - y_\varepsilon) + O(\varepsilon(t-T_\varepsilon)^2 +\\
&\quad + \varepsilon(t-T_\varepsilon)(y_+^\varepsilon - y_\varepsilon)^2 + \varepsilon(y_+^\varepsilon - y_\varepsilon)^3)\\
\partial_t w_1^0(t,x) &= \partial_t \bar{w}_1(t, y_+^\varepsilon(t,x)) + \partial_y \bar{w}_1(t, y_+^\varepsilon(t,x))\partial_t y_+^\varepsilon\\
\partial_x w_1^0(t,x) &= \partial_y \bar{w}_1(t, y_+^\varepsilon(t,x))\partial_x y_+^\varepsilon\\
\partial_x^2 w_1^0(t,x) &= \partial_y^2 \bar{w}_1(t, y_+^\varepsilon(t,x))(\partial_x y_+^\varepsilon)^2 + \partial_y \bar{w}_1(t, y_+^\varepsilon(t,x))\partial_x^2 y_+^\varepsilon
\end{aligned}$$

Combine this with (3.5) and Lemma 3.3, we show that the estimates on $w_1^0(t,x)$ in (3.3) hold. The estimates on $w_3^0(t,x)$ are completely similar.

Denoting the jump of $w_i^0(t,x)$ on γ by $[w_i^0]$, which equals $w_i^0(t,\phi^0(t)+0)-w_i^0(t,\phi^0(t)-0)$, we have

Lemma 3.5. The jump of w_i^0 satisfies the estimates

$$|[w_2^0]|\le C_0\varepsilon(t-T_\varepsilon)^{\frac{1}{2}}, |[w_i^0]|\le C_0\varepsilon(t-T_\varepsilon)^{\frac{3}{2}}, i=1,3$$

Proof. By using the estimates of $\phi^0(t)$ on γ, we have $d_\varepsilon=(t-T_\varepsilon)^3+(\phi^0(t)-x_\varepsilon-\lambda_2(u(T_\varepsilon,x_\varepsilon))(t-T_\varepsilon))^2\sim(t-T_\varepsilon)^3$. Therefore Lemma3.4. 1) implies

$$|[w_2^0]|\le|w_2^0(t,\phi^0(t)+0)-w_2^0(T_\varepsilon,x_\varepsilon)|+|w_2^0(t,\phi^0(t)-0)-w_2^0(T_\varepsilon,x_\varepsilon)|\le C_0\varepsilon(t-T_\varepsilon)^{\frac{1}{2}}$$

Now we state Lemma 3.5 holds for $i=1,3$.

Since $w_i^0(t,x)-w_i^0(T_\varepsilon,x_\varepsilon)=\partial_t\bar{w}_i(T_\varepsilon,y_\varepsilon)(t-T_\varepsilon)+O(\varepsilon(t-T_\varepsilon)^2+\varepsilon(t-T_\varepsilon)(y_+^\varepsilon-y_\varepsilon)^2+\varepsilon(y_+^\varepsilon-y_\varepsilon)^3)$ in Ω_+ and $w_i^0(t,x)-w_i^0(T_\varepsilon,x_\varepsilon)=\partial_t\bar{w}_i(T_\varepsilon,y_\varepsilon)(t-T_\varepsilon)+O(\varepsilon(t-T_\varepsilon)^2+\varepsilon(t-T_\varepsilon)(y_-^\varepsilon-y_\varepsilon)^2+\varepsilon(y_-^\varepsilon-y_\varepsilon)^3)$ in Ω_-, then $|[w_i^0]|=|w_i^0(t,\phi(t)+0)-w_i^0(T_\varepsilon,x_\varepsilon)-\{w_i^0(t,\phi(t)-0)-w_i^0(T_\varepsilon,x_\varepsilon)\}|=|O(\varepsilon d_\varepsilon^{\frac{1}{2}})|\le C_0\varepsilon(t-T_\varepsilon)^{\frac{3}{2}}$, $i=1,3$.

Denote the unknown shock curve by $x=\phi(t)$. Then the slope of shock $\sigma(t)=\phi'(t)$ must satisfy the Rankine-Hugoniot condition:

$$\begin{cases}\sigma[u_1]=[f_1(u)]\\ \sigma[u_2]=[f_2(u)]\\ \sigma[u_3]=[f_3(u)]\end{cases}\tag{3.6}$$

and the entropy condition.

In terms of the transform in section 2, the relation (3.6) is equivelent to the following condition:

$$\begin{cases}\sigma[u_1(w)]=[f_1(u(w))]\\ \sigma[u_2(w)]=[f_2(u(w))]\\ \sigma[u_3(w)]=[f_3(u(w))]\end{cases}\tag{3.7}$$

and the entropy condition for the shock wave can be written as:

$$\lambda_1(w_-(t))<\sigma<\lambda_2(w_-(t)), \lambda_2(w_+(t))<\sigma<\lambda_3(w_+(t))\tag{3.8}$$

here $w_\pm(t)=(w_{1,\pm}(t),w_{2,\pm}(t),w_3(t,\pm))=(w_1(t,\phi(t)\pm0),w_2(t,\phi(t)\pm0),w_3(t,\phi(t)\pm0))$.

Now we claim that for small ε, $(w_{1,-}(t),w_{3,+}(t))$ can be uniquely determined from $(w_{1,+}(t),w_{2,-}(t),w_{2,+}(t),\ w_{3,-}(t),\sigma(t))$ by two of three equalities in (3.7).

Indeed, note that Lemma 2.1 implies $(\frac{\partial u}{\partial w}(0))^{-1}(f'(0)-\lambda_2(0)I)(\frac{\partial u}{\partial w}(0))=diag\{\lambda_1(0)-\lambda_2(0),0,\lambda_3(0)-\lambda_2(0)$, that is,

$$(f'(0)-\lambda_2(0)I)\begin{pmatrix}\frac{\partial u_1}{\partial w_1}(0) & \frac{\partial u_1}{\partial w_3}(0)\\ \frac{\partial u_2}{\partial w_1}(0) & \frac{\partial u_2}{\partial w_3}(0)\\ \frac{\partial u_3}{\partial w_1}(0) & \frac{\partial u_3}{\partial w_3}(0)\end{pmatrix}=\begin{pmatrix}(\lambda_1(0)-\lambda_2(0))\frac{\partial u_1}{\partial w_1}(0) & (\lambda_3(0)-\lambda_2(0))\frac{\partial u_1}{\partial w_3}(0)\\ (\lambda_1(0)-\lambda_2(0))\frac{\partial u_2}{\partial w_1}(0) & (\lambda_3(0)-\lambda_2(0))\frac{\partial u_2}{\partial w_3}(0)\\ (\lambda_1(0)-\lambda_2(0))\frac{\partial u_3}{\partial w_1}(0) & (\lambda_3(0)-\lambda_2(0))\frac{\partial u_3}{\partial w_3}(0)\end{pmatrix}$$

Hence the matrix $\Big((f'(u(w))) - \sigma I\Big)\begin{pmatrix} \frac{\partial u_1(w)}{\partial w_1} & \frac{\partial u_1(w)}{\partial w_3} \\ \frac{\partial u_2(w)}{\partial w_1} & \frac{\partial u_2(w)}{\partial w_3} \\ \frac{\partial u_3(w)}{\partial w_1} & \frac{\partial u_3(w)}{\partial w_3} \end{pmatrix}$ has the rank 2 at the point $(w^0_{1,-}(T_\varepsilon, x_\varepsilon), w^0_{1,+}(T_\varepsilon, x_\varepsilon), w^0_{2,-}(T_\varepsilon, x_\varepsilon),\ w^0_{2,+}(T_\varepsilon, x_\varepsilon), w^0_{3,-}(T_\varepsilon, x_\varepsilon), w^0_{3,+}(T_\varepsilon, x_\varepsilon), \lambda_2(0))$ for small ε due to the entropy condition. Without loss of generality we assume

$$\begin{pmatrix} (\partial_{u_1} f_1)(u(w)) - \sigma & (\partial_{u_2} f_1)(u(w)) & (\partial_{u_3} f_1)(u(w)) \\ (\partial_{u_1} f_3)(u(w)) & (\partial_{u_2} f_3)(u(w)) & (\partial_{u_3} f_3)(u(w)) - \sigma \end{pmatrix}\begin{pmatrix} \frac{\partial u_1(w)}{\partial w_1} & \frac{\partial u_1(w)}{\partial w_3} \\ \frac{\partial u_2(w)}{\partial w_1} & \frac{\partial u_2(w)}{\partial w_3} \\ \frac{\partial u_3(w)}{\partial w_1} & \frac{\partial u_3(w)}{\partial w_3} \end{pmatrix}$$

has rank 2. By the implicit function theorem we know $(w_{1,-}(t), w_{3,+}(t))$ can be determined by the two equalities $[f_1(u(w))] = \sigma[u_1(w)]$ and $[f_3(u(w))] = \sigma[u_3(w)]$. Consequently, (3.7) and (3.8) are equivalent to:

$$\begin{cases} [f_1(u(w))] - \sigma[u_1(w)] = 0 \\ [f_3(u(w))] - \sigma[u_3(w)] = 0 \\ \sigma = \tilde{\lambda}_2(\int_0^1 (\partial_{u_i} f_j)(\theta u(w_+(t)) + (1-\theta)u(w_-(t)))d\theta) \end{cases} \tag{3.9}$$

With the above preparations, we are now in the position to construct the weak entropy solution of (2.1) by using an approximate procedure. To avoid the difficulty caused by the unknown shock curve, which may change its location in the process of iteration, we introduce a coordinate transform to fix the shock location on the $t-$axis:

$$\begin{cases} z = x - \phi(t) \\ t = t \end{cases}$$

Under the new coordinates, the blowup point becomes $(T_\varepsilon, 0)$ and the system (2.7) can be changed into the following form by dividing $l_{ii}(w) \neq 0$:

$$\begin{cases} \partial_t w_1 + (\lambda_1 - \sigma(t))\partial_z w_1 + p_{11}(w)(\partial_t w_3 + (\lambda_1 - \sigma(t))\partial_z w_3) = 0 \\ \partial_t w_2 + (\lambda_2 - \sigma(t))\partial_z w_2 + p_{21}(w)(\partial_t w_1 + (\lambda_2 - \sigma(t))\partial_z w_1) + p_{23}(w)(\partial_t w_3 \\ \qquad +(\lambda_2 - \sigma(t))\partial_z w_3) = 0 \\ \partial_t w_3 + (\lambda_3 - \sigma(t))\partial_z w_3 + p_{31}(w)(\partial_t w_1 + (\lambda_3 - \sigma(t))\partial_z w_1) = 0 \\ w_i(t,z)|_{t=T_\varepsilon} = w_i^0(T_\varepsilon, z + x_\varepsilon), i = 1,2,3 \end{cases} \tag{3.10}$$

here $p_{11}(w), p_{21}(w), p_{23}(w)$ and $p_{31}(w)$ are smooth for small $|w|$, moreover $p_{11}(0) = p_{21}(0) = p_{23}(0) = p_{31}(0) = 0$.

Denoting $\tilde{\Omega}_- = \{(t,z) : T_\varepsilon \le t \le T_\varepsilon + 1, -K(T_\varepsilon + 1 - t) \le z < 0\}$ and $\tilde{\Omega}_+ = \{(t,z) : T_\varepsilon \le t \le T_\varepsilon + 1, 0 < z \le K(T_\varepsilon + 1 - t)\}$, with $K = 2max\{|\lambda_1(0)|, |\lambda_2(0)|, |\lambda_3(0)|\}$. Obviously, for small ε, $\tilde{\Omega}_- \cup \tilde{\Omega}_+$ lies in the determinate region of $\{(T_\varepsilon, z) : -K \le z \le K\}$. In order to construct the weak entropy solution of (1.1) and prove Theorem 2.1, we will solve (3.10) on the domain $\tilde{\Omega}_- \cup \tilde{\Omega}_+$ by an approximate procedure. To this end, we first rewrite explicitly the system (3.10) into the following system:

$$\begin{cases} \partial_t w_{1,+} + (\lambda_1(w_+) - \sigma(t))\partial_z w_{1,+} + p_{11}(w_+)(\partial_t w_{3,+} + (\lambda_1(w_+) - \sigma(t))\partial_z w_{3,+}) = 0 \\ \partial_t w_{2,\pm} + (\lambda_2(w_\pm) - \sigma(t))\partial_z w_{2,\pm} + p_{21}(w_\pm)(\partial_t w_{1,\pm} + (\lambda_2(w_\pm) - \sigma(t))\partial_z w_{1,\pm}) \\ \qquad + p_{23}(w_\pm)(\partial_t w_{3,\pm} + (\lambda_2(w_\pm) - \sigma(t))\partial_z w_{3,\pm}) = 0 \\ \partial_t w_{3,-} + (\lambda_3(w_-) - \sigma(t))\partial_z w_{3,-} + p_{31}(w_-)(\partial_t w_{1,-} + (\lambda_3(w_-) - \sigma(t))\partial_z w_{1,-}) = 0 \\ \partial_t w_{1,-} + (\lambda_1(w_-) - \sigma(t))\partial_z w_{1,-} + p_{11}(w_-)(\partial_t w_{3,-} + (\lambda_1(w_-) - \sigma(t))\partial_z w_{3,-}) = 0 \\ \partial_t w_{3,+} + (\lambda_3(w_+) - \sigma(t))\partial_z w_{3,+} + p_{31}(w_+)(\partial_t w_{1,+} + (\lambda_3(w_+) - \sigma(t))\partial_z w_{1,+}) = 0 \\ \sigma(t) = \tilde{\lambda}_2(\int_0^1 (\partial_{u_i} f_j)(\theta u(w_+(t,0+)) + (1-\theta)u(w_-(t,0-)))d\theta) \\ w_{i,\pm}(t,z)|_{t=T_\varepsilon} = w^0_{i,\pm}(T_\varepsilon, z + x_\varepsilon), i = 1,2,3 \\ w_{1,-}(t,z)|_{z=0} = w_{1,-}(t,0-), w_{3,+}(t,z)|_{z=0} = w_{3,+}(t,0+) \end{cases} \tag{3.11}$$

here $w_{i,\pm}$ is defined in $\tilde{\Omega}_\pm$ and the boundary values $w_{1,-}(t,0-)$ and $w_{3,+}(t,0+)$ are determined by (3.9). We now solve the initial-boundary value problem by the following iterative scheme:

$$\begin{cases} \partial_t w^{n+1}_{1,+} + (\lambda_1(w^n_+) - \sigma^n(t))\partial_z w^{n+1}_{1,+} + p_{11}(w^n_+)(\partial_t w^n_{3,+} + (\lambda_1(w^n_+) \\ \qquad - \sigma^n(t))\partial_z w^n_{3,+}) = 0 \\ \partial_t w^{n+1}_{2,\pm} + (\lambda_2(w^n_\pm) - \sigma^n(t))\partial_z w^{n+1}_{2,\pm} + p_{21}(w^n_\pm)(\partial_t w^n_{1,\pm} + (\lambda_2(w^n_\pm) - \sigma^n(t))\partial_z w^n_{1,\pm}) \\ \qquad + p_{23}(w^n_\pm)(\partial_t w^n_{3,\pm} + (\lambda_2(w^n_\pm) - \sigma^n(t))\partial_z w^n_{3,\pm}) = 0 \\ \partial_t w^{n+1}_{3,-} + (\lambda_3(w^n_-) - \sigma^n(t))\partial_z w^{n+1}_{3,-} + p_{31}(w^n_-)(\partial_t w^n_{1,-} + (\lambda_3(w^n_-) \\ \qquad - \sigma^n(t))\partial_z w^n_{1,-}) = 0 \\ w^{n+1}_{1,+}(t,z)|_{t=T_\varepsilon} = w^0_{1,+}(T_\varepsilon, z + x_\varepsilon), w^{n+1}_{2,\pm}(t,z)|_{t=T_\varepsilon} = w^0_{2,\pm}(T_\varepsilon, z + x_\varepsilon), \\ w^{n+1}_{3,-}(t,z)|_{t=T_\varepsilon} = w^0_{3,-}(T_\varepsilon, z + x_\varepsilon), \end{cases} \tag{3.12}$$

and

$$\begin{cases} \partial_t w^{n+1}_{1,-} + (\lambda_1(w^n_-) - \sigma^n(t))\partial_z w^{n+1}_{1,-} + p_{11}(w^n_-)(\partial_t w^n_{3,-} + (\lambda_1(w^n_-) \\ \qquad - \sigma^n(t))\partial_z w^n_{3,-}) = 0 \\ \partial_t w^{n+1}_{3,+} + (\lambda_3(w^n_+) - \sigma^n(t))\partial_z w^{n+1}_{3,+} + p_{31}(w^n_+)(\partial_t w^n_{1,+} + (\lambda_3(w^n_+) \\ \qquad - \sigma^n(t))\partial_z w^n_{1,+}) = 0 \\ \sigma^n(t) = \tilde{\lambda}_2(\int_0^1 (\partial_{u_i} f_j)(\theta u(w^n_+(t,0+)) + (1-\theta)u(w^n_-(t,0-)))d\theta) \\ w_{1,-}(t,z)^{n+1}|_{t=T_\varepsilon} = w^0_{1,-}(T_\varepsilon, z + x_\varepsilon), w_{3,+}(t,z)^{n+1}|_{t=T_\varepsilon} = w^0_{3,+}(T_\varepsilon, z + x_\varepsilon), \\ w^{n+1}_{1,-}(t,z)|_{z=0} = w^{n+1}_{1,-}(t,0-), w^{n+1}_{3,+}(t,z)|_{z=0} = w^{n+1}_{3,+}(t,0+) \end{cases} \tag{3.13}$$

where $w^{n+1}_{1,-}(t,0-)$ and $w^{n+1}_{3,+}(t,0+)$ are determined by the equalities:

$$\begin{cases} [f_1(u(w^{n+1}))] = \sigma^n[u_1(w^{n+1})] \\ [f_3(u(w^{n+1}))] = \sigma^n[u_3(w^{n+1})] \end{cases} \tag{3.14}$$

It should be clear from our claim that the problem (3.12)-(3.14) has a unique smooth solution. Thus our main task is to obtain some uniform bounds for the approximate solution sequences $w^n_\pm$ and $\sigma^n(t)$ and show their convergence, which leads to Theorem 2.1 directly.

Lemma 3.6.(Boundedness) For small ε, there exists a constant $M > C_0$ independent of ε, such that in $\tilde{\Omega}_-$ or $\tilde{\Omega}_+$

$$w_{\pm}^n \in C^1(\tilde{\Omega}_{\pm} \setminus (T_\varepsilon, 0)) \tag{3.15}$$

$$|w_{2,\pm}^n - w_{2,\pm}^0| \leq M\varepsilon(t - T_\varepsilon) \tag{3.16}$$

$$|\partial_z(w_{2,\pm}^n - w_{2,\pm}^0)| \leq M\varepsilon((t - T_\varepsilon)^3 + z^2)^{-\frac{1}{6}} \tag{3.17}$$

$$|\partial_t(w_{2,\pm}^n - w_{2,\pm}^0)| \leq M\varepsilon((t - T_\varepsilon)^3 + z^2)^{-\frac{1}{6}} \tag{3.18}$$

$$|w_{i,\pm}^n - w_{i,\pm}^0| \leq M\varepsilon(t - T_\varepsilon)^{\frac{3}{2}}, i = 1, 3 \tag{3.19}$$

$$|\partial_z(w_{i,\pm}^n - w_{i,\pm}^0)| \leq M\varepsilon(t - T_\varepsilon)^{\frac{1}{2}}, i = 1, 3, \tag{3.20}$$

$$|\partial_t(w_{i,\pm}^n - w_{i,\pm}^0)| \leq M\varepsilon(t - T_\varepsilon)^{\frac{1}{2}}, i = 1, 3, \tag{3.21}$$

hold for all n.

Proof. We will prove the conclusion by induction. Obviously, (3.15)-(3.21) hold for $n = 0$. Assume that these estimates hold for n, we will prove they are still valid for $n + 1$. This will be completed by the following six steps.

Step 1. The estimate of $\sigma^n(t)$

Since (3.15)-(3.21) are true for n, by using the expression of $\sigma^n(t)$ and the mean value theorem one has

$$|\sigma^n(t) - \sigma^0(t)| \leq C_M\varepsilon(t - T_\varepsilon)$$

in $[T_\varepsilon, T_\varepsilon + 1]$. here C_M is a constant depending only on M.

Step 2. Estimates of $w_{2,\pm}^{n+1}$, $w_{1,+}^{n+1}$ and $w_{3,-}^{n+1}$

We only give the estimate on $w_{2,+}^{n+1}$, the others are treated similarly.

Set $v(t, z) = w_{2,+}^{n+1} - w_{2,+}^0$, then $v(t, z)$ satisfies the equation:

$$\begin{cases} \partial_t v + (\lambda_2(w_+^n) - \sigma^n)\partial_z v = (\lambda_2(w_+^0) - \lambda_2(w_+^n) + \sigma^n - \sigma^0)\partial_z w_{2,+}^0 - \sum_{k=1,3} p_{2k}(w_+^n) \times \\ \{\partial_t(w_{k,+}^n - w_{k,+}^0) + (\lambda_2(w_+^n) - \sigma^n)\partial_z(w_{k,+}^n - w_{k,+}^0) - (\lambda_2(w_+^0) - \lambda_2(w_+^n) + \sigma^n \\ -\sigma^0)\partial_z w_{k,+}^0\} - \sum_{k=1,3} (p_{2k}(w_+^n) - p_{2k}(w_+^0))(\partial_t w_{k,+}^0 + (\lambda_2(w_+^0) - \sigma^0)\partial_z w_{k,+}^0) \\ v(T_\varepsilon, z) = 0 \end{cases} \tag{3.22}$$

Noting

$$\begin{aligned} p_{2k}(w_+^n)\{\partial_t(w_{k,+}^n - w_{k,+}^0) + (\lambda_2(w_+^n) - \sigma^n)\partial_z(w_{k,+}^n - w_{k,+}^0)\} = (\partial_t + (\lambda_2(w_+^n) \\ - \sigma^n)\partial_z)(p_{2k}(w_+^n)(w_{k,+}^n - w_{k,+}^0)) - \{\sum_{j=1}^3 (\partial_{w_j} p_{2k})(w_+^n)(\partial_t w_{j,+}^n + (\lambda_2(w_+^n) \\ - \sigma^n)\partial_z w_{j,+}^n)\}(w_{k,+}^n - w_{k,+}^0), k = 1, 3 \end{aligned}$$

So in view of the inductive hypothesis, $p_{21}(0) = p_{23}(0) = 0$ and Lemma 3.4, one can integrate along the characteristics directly to derive

$$\begin{aligned} |v(t, y)| &\leq |p_{21}(w_+^n)(w_{1,+}^n - w_{1,+}^0)| + |p_{23}(w_+^n)(w_{1,+}^n - w_{1,+}^0)| + C_M\varepsilon^2 \int_{T_\varepsilon}^t (1 + \sqrt{s - T_\varepsilon})ds \\ &\leq C_M\varepsilon^2(t - T_\varepsilon) \end{aligned}$$

here and below C_M denotes a generic constant depending only on M. Hence (3.16) holds for small ε.

Similarly, one can show that

$$|w_{1,\pm}^{n+1} - w_{1,\pm}^0| \leq C_M\varepsilon^2(t-T_\varepsilon)^{\frac{3}{2}}, |w_{3,\pm}^{n+1} - w_{3,\pm}^0| \leq C_M\varepsilon^2(t-T_\varepsilon)^{\frac{3}{2}}$$

Step 3. Estimates of $w_{1,-}^{n+1}$ and $w_{3,+}^{n+1}$

It suffices to estimate $w_{1,-}^{n+1}$. Set $v(t,z) = w_{1,-}^{n+1} - w_{1,-}^0$. Then $v(t,z)$ solves:

$$\begin{cases} \partial_t v + (\lambda_1(w_-^n) - \sigma^n)\partial_z v = (\lambda_1(w_-^0) - \lambda_1(w_-^n) + \sigma^n - \sigma^0)\partial_z w_{1,-}^0 - p_{11}(w_-^n)\{\partial_t(w_{3,-}^n \\ -w_{3,-}^0) + (\lambda_1(w_-^n) - \sigma^n)\partial_z(w_{3,-}^n - w_{3,-}^0) - (\lambda_1(w_-^0) - \lambda_1(w_-^n) + \sigma^n - \sigma^0)\partial_z w_{3,-}^0\} \\ -(p_{11}(w_-^n) - p_{11}(w_-^0))(\partial_t w_{3,-}^0 + (\lambda_1(w_-^0) - \sigma^0)\partial_z w_{3,-}^0) \\ v(T_\varepsilon, z) = 0, v(t,z)|_{z=0} = w_{1,-}^{n+1}(t,0-) - w_{1,-}^0(t,0-) \end{cases} \tag{3.23}$$

Let $\xi = \xi(t,z,s)$ be the back characteristics of (3.23) through the point (t,z) in the domain $\tilde{\Omega}_-$.

If the characteristics $\xi = \xi(t,z,s)$ intersects with $z-$axis, then in a similar way as in Step 2, we have $|v(t,z)| \leq C_M\varepsilon^2(t-T_\varepsilon)^{\frac{3}{2}}$. If the characteristics $\xi = \xi(t,z,s)$ intersects with $t-$axis at $(s,0)$ with $s > T_\varepsilon$, then we have to estimate the value of $w_{1,-}^{n+1}(t,0-)$. First, by using the inductive hypothesis and characteristics method as before, one gets that

$$|v(t,z)| \leq |w_{1,-}^{n+1}(s,0-) - w_{1,-}^0(s,0-)| + C_M\varepsilon^2(t-T_\varepsilon)^{\frac{3}{2}} \tag{3.24}$$

Next, we claim that

$$[w_1^{n+1}] = F_1(w_{1,+}^{n+1}(s,0+), w_{2,+}^{n+1}(s,0+), w_{2,-}^{n+1}(s,0-), w_{3,-}^{n+1}(s,0-))[w_2^{n+1}]^3 \tag{3.25}$$

here F_1 is a given smooth function of its arguments.

In fact, (3.7) can be rewritten as:

$$(f'(u(w_-(t,0-))) - \sigma I)\frac{\partial(u_1,u_2,u_3)}{\partial(w_1,w_2,w_3)}\bigg|_{w=w_-(t,0-)}\begin{pmatrix}[w_1]\\ [w_2]\\ [w_3]\end{pmatrix}$$

$$= \tilde{B}\begin{pmatrix}[w_1]^2 & [w_1][w_2] & [w_1][w_3]\\ [w_1][w_2] & [w_2]^2 & [w_2][w_3]\\ [w_1][w_3] & [w_2][w_3] & [w_3]^2\end{pmatrix} \tag{3.26}$$

here $\tilde{B} = (\tilde{b}_{ij}(w_-(t,0-), w_+(t,0+)))_{i,j=1}^3$ is a 3×3 matrix of smooth functions.

Since $\sigma = \lambda_2(w_-(t,0-)) + \sum_{i=1}^{3} F_i(w_-(t,0-))[w_i] + \sum_{i,j=1}^{3} F_{ij}(w_-(t,0-), w_+(t,0+))[w_i][w_j]$ and by Lemma 2.1

$$\left(\frac{\partial(u_1,u_2,u_3)}{\partial(w_1,w_2,w_3)}\right)^{-1}\bigg|_{w=w_-(t,0-)}(f'(u(w_-(t,0-))) - \sigma I\bigg)\frac{\partial(u_1,u_2,u_3)}{\partial(w_1,w_2,w_3)}\bigg|_{w=w_-(t,0-)}$$

$$= \begin{pmatrix} b_{11}(w_-(t,0-)) - \sigma & 0 & b_{13}(w_-(t,0-))\\ b_{21}(w_-(t,0-)) & \lambda_2(w_-(t,0-)) - \sigma & b_{23}(w_-(t,0-))\\ b_{31}(w_-(t,0-)) & 0 & b_{33}(w_-(t,0-)) - \sigma \end{pmatrix}$$

then multiplying $\left(\frac{\partial(u_1,u_2,u_3)}{\partial(w_1,w_2,w_3)}\right)^{-1}\Big|_{w=w_-(t,0-)}$ on two sides of (3.26) we can get

$$\begin{cases} [w_1] = \sum_{i,j=1}^{3} Q^1_{ij}(w_-(t,0-))[w_i][w_j] + \sum_{i,j,k=1}^{3} Q^1_{ijk}(w_-(t,0-),w_+(t,0+))[w_i][w_j][w_k] \\ [w_3] = \sum_{i,j=1}^{3} Q^3_{ij}(w_-(t,0-))[w_i][w_j] + \sum_{i,j,k=1}^{3} Q^3_{ijk}(w_-(t,0-),w_+(t,0+))[w_i][w_j][w_k] \end{cases} \tag{3.27}$$

here Q^1_{ij}, Q^3_{ij} and Q^1_{ijk}, Q^3_{ijk} are smooth.

Permute $w_-(t,0-)$ and $w_+(t,0+)$ in (3.27), one gets

$$\begin{cases} [w_1] = -\sum Q^1_{ij}(w_+(t,0+))[w_i][w_j] + \sum Q^1_{ijk}(w_+(t,0+),w_-(t,0-))[w_i][w_j][w_k] \\ [w_3] = -\sum Q^3_{ij}(w_+(t,0+))[w_i][w_j] + \sum Q^3_{ijk}(w_+(t,0+),w_-(t,0-))[w_i][w_j][w_k] \end{cases} \tag{3.28}$$

Summimg up (3.27) and (3.28), we have

$$\begin{cases} [w_1] = \sum_{i,j,k=1}^{3} \tilde{Q}^1_{ijk}(w_-(t,0-),w_+(t,0+))[w_i][w_j][w_k] \\ [w_3] = \sum_{i,j,k=1}^{3} \tilde{Q}^3_{ijk}(w_-(t,0-),w_+(t,0+))[w_i][w_j][w_k] \end{cases} \tag{3.29}$$

where $\tilde{Q}^1_{ijk}$ and $\tilde{Q}^3_{ijk}$ are smooth. Set $[w_1] = x_1[w_2]^3$ and $[w_3] = x_2[w_2]^3$, and note that $w_{1,-}(t,0-) = w_{1,+}(t,0+) - [w_1]$, $w_{3,+}(t,0+) = w_{3,-}(t,0-) + [w_3]$, then applying the implicit function theorem to (3.29) one can obtain for small $[w_2]$

$$\begin{cases} x_1 = F_1(w_{1,+}(t,0+), w_{2,+}(t,0+), w_{2,-}(t,0-), w_{3,-}(t,0-)) \\ x_3 = F_3(w_{1,+}(t,0+), w_{2,+}(t,0+), w_{2,-}(t,0-), w_{3,-}(t,0-)) \end{cases}$$

where the functions F_i are smooth.

Similar analysis yields (3.25).

Since

$$|w^{n+1}_{1,-}(s,0-) - w^0_{1,-}(s,0-)| \le |[w^{n+1}_1]| + |w^{n+1}_{1,+}(s,0+) - w^0_{1,+}(s,0+)| + |[w^0_1]|$$

and

$$|[w^{n+1}_2]| \le |w^{n+1}_{2,+}(s,0+) - w^0_{2,+}(s,0+)| + |w^{n+1}_{2,-}(s,0-) - w^0_{2,-}(s,0-)| + |[w^0_2]|$$

It follows from (3.24), (3.25) and Step 2 that for small ε

$$|v(t,z)| \le C_0\varepsilon(t-T_\varepsilon)^{\frac{3}{2}} + C_M\varepsilon^2(t-T_\varepsilon)^{\frac{3}{2}} \le M\varepsilon(t-T_\varepsilon)^{\frac{3}{2}} \tag{3.30}$$

Step 4. Estimates of $|\nabla(w^{n+1}_{2,\pm} - w^0_{2,\pm})|$

Set $v(t,z)=\partial_z(w_{2,+}^{n+1}-w_{2,+}^0)$, then $v(t,z)$ satisfies the following equation

$$\begin{cases} \partial_t v+(\lambda_2(w_+^n)-\sigma^n(t))\partial_z v+\partial_z(\lambda_2(w_+^n))v=-\sum_{k=1,3}p_{2k}(w_+^n)\{\partial_t\partial_z(w_{k,+}^n-w_{k,+}^0) \\ \quad +(\lambda_2(w_+^n)-\sigma^n)\partial_z\partial_z(w_{k,+}^n-w_{k,+}^0)+(\lambda_2(w_+^n)-\sigma^n-\lambda_2(w_+^0)+\sigma^0)\partial_z^2 w_{k,+}^0\} \\ \quad +(\lambda_2(w_+^0)-\lambda_2(w_+^n)+\sigma^n-\sigma^0)\partial_z^2 w_{2,+}^0-\sum_{j=1}^{3}\{(\partial_{w_j}\lambda_2)(w_+^n)\partial_z w_{j,+}^n \\ \quad -(\partial_{w_j}\lambda_2)(w_+^0)\partial_z w_{j,+}^0\}\partial_z w_{2,+}^0-\sum_{k=1,3}\sum_{j=1}^{3}\{(\partial_{w_j}p_{2k})(w_+^n)\partial_z w_{j,+}^n\{\partial_t(w_{k,+}^n-w_{k,+}^0) \\ \quad +(\lambda_2(w_+^n)-\sigma^n)\partial_z(w_{1,+}^n-w_{k,+}^0)-(\lambda_2(w_+^0)-\lambda_2(w_+^n)+\sigma^n-\sigma^0)\partial_z w_{k,+}^0 \\ \quad +((\partial_{w_j}p_{2k})(w_+^n)\partial_z w_{j,+}^n-(\partial_{w_j}p_{2k})(w_+^0)\partial_z w_{j,+}^0)(\partial_t w_{k,+}^0+(\lambda_2(w_+^0) \\ \quad -\sigma^0)\partial_z w_{k,+}^0)\}-\sum_{k=1,3}\sum_{j=1}^{3}p_{2k}(w_+^n)\{\partial_{w_j}\lambda_2(w_+^n)\partial_z w_{j,+}^n\partial_z w_{k,+}^n \\ \quad -\partial_{w_j}\lambda_2(w_+^0)\partial_z w_{j,+}^0\partial_z w_{k,+}^0\} \\ v(T_\varepsilon,z)=0 \end{cases} \tag{3.31}$$

Let $\xi^{n+1}=\xi^{n+1}(t,z,s)$ be the back characteristics of (3.31) through the point (t,z), that is, ξ^{n+1} satisfies the equation

$$\begin{cases} \frac{d\xi^{n+1}}{ds}=\lambda_2(w_+^n(s,\xi^{n+1}))-\sigma^n(s), \qquad T_\varepsilon\le s\le t \\ \xi^{n+1}|_{s=t}=z \end{cases}$$

Note that we can assume $\partial_{w_2}\lambda_2(w)>\frac{\partial_{w_2}\lambda_2(0)}{2}>0$ for small $|w|$ due to the genuine nonlinearity of (2.1) with respect to $\lambda_2(w)$. Furthermore, λ_2 and σ are related by the equation (3.44) below. Hence by following the arguments as for Lemma 8.1 and Lemma 8.3 of [11], one can prove that there exists a constant C independent of n and ε such that

$$(s-T_\varepsilon)^3+(\xi^{n+1})^2\ge C((t-T_\varepsilon)^3+z^2) \tag{3.32}$$

and

$$\int_{T_\varepsilon}^{t}|(\partial_z(\lambda_2(w_+^n)))(s,\xi^{n+1})|ds\le ln\frac{3}{2}+C_M\varepsilon\sqrt{t-T_\varepsilon}<1 \tag{3.33}$$

Integrating (3.31) along characteristics and using (3.32), (3.33), Lemma 3.4 and the inductive hypothesis, we have

$$\begin{aligned} |v(t,z)| &\le \sum_{k=1,3}|p_{2k}(w_+^n)\partial_z(w_{k,+}^n-w_{k,+}^0)(t,z)|+\int_{T_\varepsilon}^{t}|(\partial_z(\lambda_2(w_+^n)))(s,\xi^{n+1})||v(s,y)|ds \\ &+C_M\varepsilon^2\int_{T_\varepsilon}^{t}\{\frac{s-T_\varepsilon}{((t-T_\varepsilon)^3+z^2)^{\frac{5}{6}}}+\frac{\sqrt{s-T_\varepsilon}}{((t-T_\varepsilon)^3+z^2)^{\frac{1}{3}}}+\frac{1}{((t-T_\varepsilon)^3+z^2)^{\frac{1}{2}}}\}ds \\ &\le C_M\varepsilon^2((t-T_\varepsilon)^3+z^2)^{-\frac{1}{6}}+\int_{T_\varepsilon}^{t}|(\partial_z(\lambda_2(w_+^n)))(s,\xi^{n+1})||v(s,y)|ds \end{aligned}$$

This yields the desired estimate (3.17) due to (3.33) and Gronwall's inequality. Moreover in view of (3.23) we obtain (3.18).

Step 5. Estimates on $|\nabla(w_{1,+}^{n+1}-w_{1,+}^0)|$ **and** $|\nabla(w_{3,-}^{n+1}-w_{3,-}^0)|$

We only estimate $|\partial_z(w_{1,+}^{n+1}-w_{1,+}^0)|$ and $|\partial_t(w_{1,+}^{n+1}-w_{1,+}^0)|$.

Set $v(t,z)=\partial_z(w_{1,+}^{n+1}-w_{1,+}^0)$, then $v(t,z)$ satisfies the equation:

$$\begin{cases} \partial_t v+(\lambda_1(w_+^n)-\sigma^n(t))\partial_z v+\partial_z(\lambda_1(w_+^n))v+p_{11}(w_+^n)\{\partial_t\partial_z(w_{3,+}^n-w_{3,+}^0) \\ \qquad +(\lambda_1(w_+^n)-\sigma^n)\partial_z\partial_z(w_{3,+}^n-w_{3,+}^0)-(\lambda_1(w_+^0)-\lambda_1(w_+^n)+\sigma^n \\ \qquad -\sigma^0)\partial_z^2 w_{3,+}^0\}+\sum_{j=1}^3\{(\partial_{w_j}p_{11})(w_+^n)\partial_z w_{j,+}^n(\partial_t w_{3,+}^n+(\lambda_1(w_+^n)-\sigma^n)\partial_z w_{3,+}^n) \\ \qquad +p_{11}(w_+^n)(\partial_{w_j}\lambda_1)(w_+^n)\partial_z w_{j,+}^n\partial_z w_{3,+}^n-(\partial_{w_j}p_{11})(w_+^0)\partial_z w_{j,+}^0(\partial_t w_{3,+}^0+(\lambda_1(w_+^0) \\ \qquad -\sigma^0)\partial_z w_{3,+}^0)-p_{11}(w_+^0)(\partial_{w_j}\lambda_1)(w_+^0)\partial_z w_{j,+}^0\partial_z w_{3,+}^0\}+\sum_{j=1}^3\{((\partial_{w_j}\lambda_1)(w_+^n)\partial_z w_{j,+}^n \\ \qquad -(\partial_{w_j}\lambda_1)(w_+^0)\partial_z w_{j,+}^0)\partial_z w_{1,+}^0\}=0 \\ v(T_\varepsilon,z)=0 \end{cases} \tag{3.34}$$

Let $\xi_1^{n+1}=\xi_1^{n+1}(t,z,s)$ be the back characteristics through the point (t,z), for small ε we have

$$\xi_1^{n+1}\geq z+\frac{1}{2}|\lambda_1(0)-\lambda_2(0)|(t-s) \tag{3.35}$$

By integration along the characteristics one can get

$$\begin{aligned} |v(t,z)| &\leq |p_{11}(w_+^n)\partial_z(w_{3,+}^n-w_{3,+}^0)|+C_M\varepsilon\int_{T_\varepsilon}^t\frac{|v(s,\xi_1^{n+1})|}{((s-T_\varepsilon)^3+(\xi_1^{n+1})^2)^{\frac{1}{3}}}ds \\ &+C_M\varepsilon^2\int_{T_\varepsilon}^t\{\frac{\sqrt{s-T_\varepsilon}}{((s-T_\varepsilon)^3+(\xi_1^{n+1})^2)^{\frac{1}{3}}}+\frac{s-T_\varepsilon}{((s-T_\varepsilon)^3+(\xi_1^{n+1})^2)^{\frac{1}{2}}}\}ds \end{aligned}$$

By using (3.35) and $p_{11}(0)=0$, we obtain

$$|v(t,z)|\leq C_M\varepsilon^2\sqrt{t-T_\varepsilon}+C_M\varepsilon\int_{T_\varepsilon}^t\frac{|v(s,\xi_1^{n+1})|}{(t-s)^{\frac{2}{3}}}ds$$

Hence Gronwall's inequality implies

$$|v(t,y)|\leq C_M\varepsilon^2\sqrt{t-T_\varepsilon}$$

This in turn yields the desired estimate on $\partial_t(w_{1,-}^{n+1}-w_{1,-}^0)$ due to (3.34).

Step 6. Estimates on $|\nabla(w_{1,-}^{n+1}-w_{1,-}^0)|$ **and** $|\nabla(w_{3,+}^{n+1}-w_{3,+}^0)|$

We only compute $|\partial_t(w_{1,-}^{n+1}-w_{1,-}^0)|$ and $|\partial_z(w_{1,-}^{n+1}-w_{1,-}^0)|$.

Set $v(t,z)=\partial_t(w_{1,-}^{n+1}-w_{1,-}^0)$, then $v(t,z)$ satisfies the equation:

$$\begin{cases} \partial_t v+(\lambda_1(w_-^n)-\sigma^n(t))\partial_z v+\partial_t(\lambda_1(w_-^n)-\sigma^n(t))v+p_{11}(w_-^n)\{\partial_t\partial_t(w_{3,-}^n-w_{3,-}^0) \\ \quad +(\lambda_1(w_-^n)-\sigma^n)\partial_t\partial_z(w_{3,-}^n-w_{3,-}^0)+(\lambda_1(w_-^n)-\lambda_1(w_-^0)+\sigma^0-\sigma^n)\partial_{tz}^2 w_{3,-}^0\} \\ \quad +(p_{11}(w_-^n)-p_{11}(w_-^0))\{\partial_t^2 w_{3,-}^0+(\lambda_1(w_-^0)-\sigma^0)\partial_{tz}^2 w_{3,-}^0\}-(\lambda_1(w_-^0)-\lambda_1(w_-^n) \\ \quad +\sigma^n-\sigma^0)\partial_{tz}^2 w_{1,-}^0+\sum_{j=1}^3\{(\partial_{w_j}p_{11})(w_-^n)\partial_t w_{j,-}^n(\partial_t w_{3,-}^n+(\lambda_1(w_-^n)-\sigma^n)\partial_z w_{3,-}^n) \\ \quad +p_{11}(w_-^n)\partial_t(\lambda_1(w_-^n)-\sigma^n)\partial_z w_{3,-}^n-(\partial_{w_j}p_{11})(w_-^0)\partial_t w_{j,-}^0(\partial_t w_{3,-}^0+(\lambda_1(w_-^0) \\ \quad -\sigma^0)\partial_z w_{3,-}^0)-p_{11}(w_-^0)\partial_t(\lambda_1(w_-^0)-\sigma^0)\partial_z w_{3,-}^0\}=0 \\ v(T_\varepsilon,z)=0, v(t,z)|_{z=0}=(\partial_t(w_{1,-}^{n+1}-w_{1,-}^0))(t,0-) \end{cases} \tag{3.36}$$

It follows from Lemma 3.4 and inductive hypothesis that

$$\begin{aligned} &|w_{l,-}^n(t,z)-w_{l,+}^n(t,0+)|\le|w_{l,-}^n(t,z)-w_{l,-}^0(t,z)|+|w_{l,+}^n(t,0+)-w_{l,+}^0(t,0+)|+|w_{l,-}^0(t,z) \\ &\qquad -w_{l,-}^0(t,0-)|+|[w_l^0]|\le C_M\varepsilon((t-T_\varepsilon)^3+z^2)^{\frac16}, \\ &|w_{l,-}^n(t,z)-w_{l,-}^0(t,0-)|\le|w_{l,-}^n(t,z)-w_{l,-}^0(t,z)|+|w_{l,-}^0(t,z)-w_{l,-}^0(t,0-)| \\ &\qquad \le C_M\varepsilon((t-T_\varepsilon)^3+z^2)^{\frac16} \end{aligned}$$

Hence by the expressions of λ_2 and $\sigma^n(t)$, we have

$$|\lambda_2(w_-^n)-\sigma^n(t)|\le C_M\varepsilon((t-T_\varepsilon)^3+z^2)^{\frac16} \tag{3.37}$$

Note that $|\partial_t w_{2,-}^0(t,z)|\le|(\partial_t+\lambda_2(w_-^0)\partial_x)w_{2,-}^0(t,z)|+|(\lambda_2^0(w_-^0)-\lambda_2(w_-^n)+\lambda_2(w_-^n)-\sigma^n(t))\partial_x w_{2,-}^0(t,z)|$. Then Lemma 3.4, the inductive assumption and (3.37) imply

$$|\partial_t w_{2,-}^n(t,z)|\le\frac{C_M\varepsilon}{\sqrt{t-T_\varepsilon}}$$

Consequently

$$\int_{T_\varepsilon}^t|(\partial_t(\lambda_1(w_-^n)))(s,\xi(t,z,s))-(\partial_t\sigma^n)(s)|ds\le C_M\varepsilon\sqrt{t-T_\varepsilon} \tag{3.38}$$

Let $\xi=\xi(t,z,s)$ be the back characteristics of (3.36) through the point (t,z) in the domain $\tilde\Omega_-$. If $\xi=\xi(t,z,s)$ intersects with $z-$axis before it meets t-axis, then integrating (3.36) along characteristics and using the result in Step 1, inductive hypothesis and the fact $p_{11}(0)=0$, one can show

$$\begin{aligned} |v(t,z)|&\le|p_{11}(w_-^n)\partial_t(w_{3,-}^n-w_{3,-}^0)(t,z)|+\int_{T_\varepsilon}^t|(\partial_t(\lambda_1(w_-^n)))(s,\xi(t,z,s))-(\partial_t\sigma^n)(s)| \\ &\quad\times|v(s,\xi(t,z,s))|ds+C_M\varepsilon^2\int_{T_\varepsilon}^t\{1+\sqrt{s-T_\varepsilon}+\frac{s-T_\varepsilon}{\sqrt{(s-T_\varepsilon)^3+z^2}}+\frac{1}{\sqrt{s-T_\varepsilon}}\}ds \\ &\le C_M\varepsilon^2(t-T_\varepsilon)^{\frac12}+\int_{T_\varepsilon}^t|(\partial_t(\lambda_1(w_-^n)))(s,\xi(t,z,s))-(\partial_t\sigma^n)(s)||v(s,\xi(t,z,s))|ds \end{aligned}$$

This and (3.38) lead to

$$|v(t,z)| \le M\varepsilon(t-T_\varepsilon)^{\frac{1}{2}}$$

In the case that if $\xi=\xi(t,z,s)$ intersects with $t-$axis at $(s,0)$ with $s>T_\varepsilon$, then we need to deal with the boundary conditions on t-axis. Indeed, by integration along characteristics one gets:

$$\begin{aligned}
|v(t,z)| &\le |(\partial_s(w^{n+1}_{1,-}-w^0_{1,-}))(s,0-)| + |p_{11}(w^n_-)\partial_t(w^n_{3,-}-w^0_{3,-})(t,z)| \\
&+ \int_s^t |(\partial_t(\lambda_1(w^n_-)))(s,\xi(t,z,s)) - (\partial_t\sigma^n(s))||v(s,\xi(t,z,s))|ds \\
&+ C_M\varepsilon^2\int_s^t \{1+\sqrt{s-T_\varepsilon} + \frac{s-T_\varepsilon}{\sqrt{(s-T_\varepsilon)^3+z^2}} + \frac{1}{\sqrt{s-T_\varepsilon}}\}ds \\
&\le C_M\varepsilon^2(t-T_\varepsilon)^{\frac{1}{2}} + |(\partial_s(w^{n+1}_{1,-}-w^0_{1,-}))(s,0-)| \\
&+ \int_{T_\varepsilon}^t |(\partial_t(\lambda_1(w^n_-)))(s,\xi(t,z,s)) - (\partial_t\sigma^n)(s)||v(s,\xi(t,z,s))|ds
\end{aligned} \tag{3.39}$$

To estimate the additional boundary condition $|(\partial_s(w^{n+1}_{1,-}-w^0_{1,-}))(s,0-)|$, one notes that the claim (3.25) shows

$$\begin{aligned}
&|\partial_s(w^{n+1}_{1,-}(s,0-)-w^0_{1,-}(s,0-))| \le |\partial_s(w^{n+1}_{1,+}(s,0+)-w^0_{1,+}(s,0+))| + \sum{}' O(1)|\partial_s w^{n+1} \\
&-\partial_s w^0||[w^{n+1}_2]^3| + \sum{}' O(1)|[w^{n+1}_2]^3 - [w^0_2]^3||\partial_s w^0| + \sum{}' O(1)|w^{n+1} \\
&- w^0||[w^{n+1}_2]^2[\partial_s w^{n+1}_2]| + \sum{}' O(1)|[w^{n+1}_2]^2[\partial_s w^{n+1}_2] - [w^0_2]^2[\partial_s w^0_2]|
\end{aligned}$$

where $\sum'$ means summation over all the indices except $(1,-)$ and $(3,+)$.

Now the inductive hypothesis implies that

$$\begin{aligned}
&|[w^{n+1}_2]| \le |w^{n+1}_{2,+}(s,0+)-w^0_{2,+}(s,0+)| + |w^{n+1}_{2,-}(s,0-)-w^0_{2,-}(s,0+)| + |[w^0_2]| \\
&\quad \le C_M\varepsilon(s-T_\varepsilon)^{\frac{1}{2}} \\
&|[\partial_s w^{n+1}_2] - [\partial_s w^0_2]| \le \frac{C_M\varepsilon}{\sqrt{s-T_\varepsilon}} \\
&|[\partial_s w^{n+1}_2]| \le \frac{C_M\varepsilon}{s-T_\varepsilon}
\end{aligned}$$

So it follows from the above computation and that in Step 5 that $|\partial_s(w^{n+1}_{1,-}(s,0-) - w^0_{1,-}(s,0-))| \le C_M\varepsilon^2(s-T_\varepsilon)^{\frac{1}{2}}$ holds for small ε. Substituting this into (3.39) we obtain $|v(t,z)| \le C_M\varepsilon^2(t-T_\varepsilon)^{\frac{1}{2}}$ for small ε.

Finally, since $|\lambda_1(w^n_-)-\sigma^n| \ge \frac{1}{2}(\lambda_2(0)-\lambda_1(0)) > 0$ for small ε, then from equation (3.23), we know $|\partial_z(w^{n+1}_{1,-}-w^0_{1,-})(t,z)| \le C_M\varepsilon^2(t-T_\varepsilon)^{\frac{1}{2}}$.

Collecting all the estimates obtained above we conclude the lemma by induction.

Now we prove the convergence of the sequences $\{\sigma^n\}$ in $[T_\varepsilon, T_\varepsilon+1]$ and $\{w^n_{i,\pm}\}$ in $\tilde\Omega_\pm$ based on Lemma 3.6. The key estimates are given in the following lemma.

Lemma 3.7.(Convergence) For small ε, there exists a constant C_M independent of ε and n such that

$$\|\sigma^n - \sigma^{n-1}\|_{L^\infty([T_\varepsilon, T_\varepsilon+1])} \leq C_M \sum_{i=1}^{3} \|w^n_{i,\pm} - w^{n-1}_{i,\pm}\|_{L^\infty(\tilde{\Omega}_\pm)} \tag{3.40}$$

$$\begin{aligned} \|w^{n+1}_{2,\pm} - w^n_{2,\pm}\|_{L^\infty(\tilde{\Omega}_\pm)} + C_M \sum_{i=1,3} \|w^{n+1}_{i,\pm} - w^n_{i,\pm}\|_{L^\infty(\tilde{\Omega}_\pm)} &\leq (1-\varepsilon)\{\|w^n_{2,\pm} - w^{n-1}_{2,\pm}\|_{L^\infty(\tilde{\Omega}_\pm)} \\ + C_M \sum_{i=1,3} \|w^n_{i,\pm} - w^{n-1}_{i,\pm}\|_{L^\infty(\tilde{\Omega}_\pm)}\} \end{aligned} \tag{3.41}$$

here $\|w^{n+1}_{i,\pm} - w^n_{i,\pm}\|_{L^\infty(\tilde{\Omega}_\pm)} = \|w^{n+1}_{i,+} - w^n_{i,+}\|_{L^\infty(\tilde{\Omega}_+)} + \|w^{n+1}_{i,-} - w^n_{i,-}\|_{L^\infty(\tilde{\Omega}_-)}$

Proof. First, (3.40) follows from the expression of $\sigma^n(t)$ and Lemma 3.6.

In order to prove (3.41), we start with the estimate on $w^{n+1}_{2,+} - w^n_{2,+}$. Set $v(t,z) = w^{n+1}_{2,+} - w^n_{2,+}$, then $v(t,z)$ solves the following equation:

$$\begin{cases} \partial_t v + (\lambda_2(w^n_+) - \sigma^n)\partial_z v = (\lambda_2(w^{n-1}_+) - \lambda_2(w^n_+) + \sigma^n - \sigma^{n-1})\partial_z w^{n-1}_{2,+} \\ - \sum_{k=1,3} p_{2k}(w^n_+)\{\partial_t(w^n_{k,+} - w^{n-1}_{k,+}) + (\lambda_2(w^n_+) - \sigma^n)\partial_z(w^n_{k,+} - w^{n-1}_{k,+}) - (\lambda_2(w^{n-1}_+) \\ -\lambda_2(w^n_+) + \sigma^n - \sigma^{n-1})\partial_z w^n_{k,+}\} - \sum_{k=1,3} (p_{2k}(w^n_+) - p_{2k}(w^{n-1}_+))(\partial_t w^{n-1}_{k,+} + (\lambda_2(w^{n-1}_+) \\ -\sigma^{n-1})\partial_z w^{n-1}_{k,+}) \\ v(T_\varepsilon, z) = 0 \end{cases} \tag{3.42}$$

The most singular term on the right hand side of (3.42) is the term $(\lambda_2(w^{n-1}_+) - \lambda_2(w^n_+) + \sigma^n - \sigma^{n-1})\partial_z w^{n-1}_{2,+}$, because it contains the unbounded term $\partial_z w^{n-1}_{2,+}$ which isn't integrable along the characteristics for (3.42).

To estimate $(\lambda_2(w^{n-1}_+) - \lambda_2(w^n_+))\partial_z w^{n-1}_{2,+}$, we rewrite the term $(\partial_{w_k}\lambda_2)(w^n_+)\partial_z w^n_{2,+}$ as $\frac{(\partial_{w_k}\lambda_2)(w^n_+)}{(\partial_{w_2}\lambda_2)(w^n_+)}\{\partial_z(\lambda_2(w^n_+)) - \sum_{j=1,3}(\partial_{w_j}\lambda_2)(w^n_+)\partial_z w^n_{j,+}\}$ for $k = 1, 3$, which is valid due to the genuine nonlinearity that $\partial_{w_2}\lambda_2(w) \neq 0$ for small $|w|$. Note that $\partial_z w^n_{j,+} (j = 1, 3)$ and $\partial_z(\lambda_2(w^n_+))$ can be estimated thanks to Lemma 3.6 and (3.33). We now set

$$(\lambda_2(w^n_+) - \lambda_2(w^{n-1}_+))\partial_z w^{n-1}_{2,+} = \sum_{i=1}^{7} I_i$$

with

$$\begin{aligned} I_1 &= \sum_{j,k=1}^{3}\{\int_0^1\int_0^1 (\partial^2_{w_j w_k}\lambda_2)(\theta_1(\theta w^n_+ + (1-\theta)w^{n-1}_+) + (1-\theta_1)w^{n-1}_+)\theta d\theta d\theta_1 \\ &\quad \times (w^n_{k,+} - w^{n-1}_{k,+})(w^n_{j,+} - w^{n-1}_{j,+})\}\partial_z w^{n-1}_{2,+} \\ I_2 &= \sum_{j=1}^{3}\{(\partial_{w_j}\lambda_2)(w^{n-1}_+) - (\partial_{w_j}\lambda_2)(w^n_+)\}\partial_z w^{n-1}_{2,+}(w^n_{j,+} - w^{n-1}_{j,+}) \\ I_3 &= \sum_{j=1}^{3}(\partial_{w_j}\lambda_2)(w^n_+)(\partial_z w^{n-1}_{2,+} - \partial_z w^n_{2,+})(w^n_{j,+} - w^{n-1}_{j,+}) \end{aligned}$$

$$I_4 = \partial_z(\lambda_2(w_+^n))(w_{2,+}^n - w_{2,+}^{n-1})$$
$$I_5 = -\sum_{k=1,3}(\partial_{w_k}\lambda_2)(w_+^n)\partial_z w_{k,+}^n(w_{2,+}^n - w_{2,+}^{n-1})$$
$$I_6 = \sum_{k=1,3}\frac{(\partial_{w_k}\lambda_2)(w_+^n)}{(\partial_{w_2}\lambda_2)(w_+^n)}\partial_z(\lambda_2(w_+^n))(w_{k,+}^n - w_{k,+}^{n-1})$$
$$I_7 = -\sum_{k,j=1,3}\frac{(\partial_{w_k}\lambda_2)(w_+^n)}{(\partial_{w_2}\lambda_2)(w_+^n)}\{(\partial_{w_j}\lambda_2)(w_+^n)\partial_z w_{j,+}^n\}(w_{k,+}^n - w_{k,+}^{n-1})$$

Hence one can get

$$\begin{aligned}|(\lambda_2(w_+^n) - \lambda_2(w_+^{n-1}))\partial_z w_{2,+}^{n-1}| \le (|\partial_z(\lambda_2(w_+^n))| + \frac{C_M\varepsilon}{\sqrt{t-T_\varepsilon}})|w_{2,+}^n - w_{2,+}^{n-1}| \\ + C_M\sum_{i=1,3}|w_{i,\pm}^n - w_{i,\pm}^{n-1}| \qquad (3.43)\end{aligned}$$

Next, we decompose the term $(\sigma^n - \sigma^{n-1})\partial_z w_{2,+}^{n-1}$. Note that the relation between σ^n and $\lambda_2(w_\pm^n(t,0\pm))$ can be derived in a similar way as in [5] or [14] to obtain:

$$\sigma^n = \lambda_2(w_-(t,0-)) + \frac{1}{2}\sum_{k=1}^{3}(\partial_{w_k}\lambda_2)(w_-(t,0-))[w_k^n] + O([w^n]^2) \qquad (3.44)$$

Then we write $(\sigma^n - \sigma^{n-1})\partial_z w_{2,+}^{n-1}$ as $\sum_{i=1}^{8} J_i$, where

$$\begin{aligned}J_1 = \{&\frac{1}{2}\sum_{k=1}^{3}(\partial_{w_k}\lambda_2(w_-(t,0-)) - \partial_{w_k}\lambda_2(w_-^{n-1}))[w_k^n - w_k^{n-1}] \\ &+ O([w^n]^2) - O([w^{n-1}]^2)\}\partial_z w_{2,+}^{n-1}\end{aligned}$$
$$J_2 = \frac{1}{2}\sum_{k=1}^{3}(\partial_{w_k}\lambda_2(w_-^{n-1}) - \partial_{w_k}\lambda_2(w_-^n))[w_k^n - w_k^{n-1}]\partial_z w_{2,+}^{n-1}$$
$$J_3 = \frac{1}{2}\sum_{k=1}^{3}\partial_{w_k}\lambda_2(w_-^n)[w_k^n - w_k^{n-1}](\partial_z w_{2,+}^{n-1} - \partial_z w_{2,+}^n)$$
$$J_4 = \frac{1}{2}\sum_{k=1}^{3}(\partial_{w_k}\lambda_2(w_-^n) - \partial_{w_k}\lambda_2(w_+^n))[w_k^n - w_k^{n-1}]\partial_z w_{2,+}^n$$
$$J_5 = \frac{1}{2}\partial_z(\lambda_2(w_+^n))[w_{2,+}^n - w_{2,+}^{n-1}]$$
$$J_6 = -\frac{1}{2}\sum_{j=1,3}(\partial_{w_j}\lambda_2)(w_+^n)\partial_z w_{j,+}^n[w_{2,+}^n - w_{2,+}^{n-1}]$$
$$J_7 = \frac{1}{2}\sum_{j=1,3}\frac{(\partial_{w_j}\lambda_2)(w_+^n)}{(\partial_{w_2}\lambda_2)(w_+^n)}\partial_z(\lambda_2(w_+^n))[w_{j,+}^n - w_{j,+}^{n-1}]$$

$$J_8 = -\frac{1}{2}\sum_{k,j=1,3}\frac{(\partial_{w_j}\lambda_2)(w_+^n)}{(\partial_{w_2}\lambda_2)(w_+^n)}\{(\partial_{w_k}\lambda_2)(w_+^n)\partial_z w_{k,+}^n\}[w_{j,+}^n - w_{j,+}^{n-1}]$$

Hence it is easy to obtain

$$\begin{aligned}|(\sigma^n - \sigma^{n-1})\partial_z w_{2,+}^{n-1}| \leq & (\frac{1}{2}|\partial_z(\lambda_2(w_+^n))| + \frac{C_M\varepsilon}{\sqrt{t-T_\varepsilon}})|[w_{2,+}^n - w_{2,+}^{n-1}]| \\ & + C_M\sum_{i=1,3}|[w_{i,+}^n - w_{i,+}^{n-1}]| \end{aligned} \tag{3.45}$$

Based on the estimates (3.43) and (3.45), analyzing (3.42) in a similar way as in the proof of Lemma 3.6 (in particular, the Step 4), and noting that $|[w_k^n - w_k^{n-1}]| \leq \|w_k^n - w_k^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}$ in $J_1 - J_8$, we can show that

$$\begin{aligned}\|w_{2,+}^{n+1} - w_{2,+}^n\|_{L^\infty(\tilde\Omega_+)} \leq & (ln\frac{3}{2} + C_M\varepsilon\sqrt{t-T_\varepsilon})\|w_{2,+}^n - w_{2,+}^{n-1}\|_{L^\infty(\tilde\Omega_+)} + (\frac{1}{2}ln\frac{3}{2}+ \\ & + C_M\varepsilon\sqrt{t-T_\varepsilon})\|w_{2,\pm}^{n+1} - w_{2,\pm}^n\|_{L^\infty(\tilde\Omega_\pm)}) + C_M\sum_{i=1,3}\|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}\end{aligned}$$

Similar analysis shows that

$$\begin{aligned}\|w_{2,-}^{n+1} - w_{2,-}^n\|_{L^\infty(\tilde\Omega_-)} \leq & (ln\frac{3}{2} + C_M\varepsilon\sqrt{t-T_\varepsilon})\|w_{2,-}^n - w_{2,-}^{n-1}\|_{L^\infty(\tilde\Omega_-)} + (\frac{1}{2}ln\frac{3}{2}+ \\ & + C_M\varepsilon\sqrt{t-T_\varepsilon})\|w_{2,\pm}^{n} - w_{2,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}) + C_M\sum_{i=1,3}\|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}\end{aligned}$$

Summing up the above inequalities yields

$$\begin{aligned}\|w_{2,\pm}^{n+1} - w_{2,\pm}^n\|_{L^\infty(\tilde\Omega_\pm)} \leq & (2ln\frac{3}{2} + C_M\varepsilon\sqrt{t-T_\varepsilon})\|w_{2,\pm}^n - w_{2,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}+ \\ & + C_M\sum_{i=1,3}\|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)}\end{aligned} \tag{3.46}$$

We note that $2ln\frac{3}{2} + C_M\varepsilon\sqrt{t-T_\varepsilon} < 1$ provided ε is very small. Modifying the analysis in Step 3 of Lemma 3.6 we can also establish the estimate

$$\|w_{1,+}^{n+1} - w_{1,+}^n\|_{L^\infty(\tilde\Omega_+)} + \|w_{3,-}^{n+1} - w_{3,-}^n\|_{L^\infty(\tilde\Omega_-)} \leq C_M\varepsilon(t-T_\varepsilon)\sum_{i=1}^{3}\|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde\Omega_\pm)} \tag{3.47}$$

Finally, we estimate $w_{1,-}^{n+1} - w_{1,-}^n$. Set $v(t,z) = w_{1,-}^{n+1} - w_{1,-}^n$, then $v(t,z)$ satisfies the equation:

$$\begin{cases} \partial_t v + (\lambda_1(w_-^n) - \sigma^n)\partial_z v = (\lambda_1(w_-^{n-1}) - \lambda_1(w_-^n) + \sigma^n - \sigma^{n-1})\partial_z w_{1,-}^n - p_{11}(w_-^n)\times \\ \qquad \times\{\partial_t(w_{3,-}^n - w_{3,-}^{n-1}) + (\lambda_1(w_-^n) - \sigma^n)\partial_z(w_{3,-}^n - w_{3,-}^{n-1}) + (\lambda_1(w_-^{n-1}) \\ \qquad -\lambda_1(w_-^n) + \sigma^n - \sigma^{n-1})\partial_z w_{3,-}^{n-1}\} - (p_{11}(w_-^n) - p_{11}(w_-^{n-1}))(\partial_t w_{3,-}^{n-1} \\ \qquad +(\lambda_1(w_-^{n-1}) - \sigma^{n-1})\partial_z w_{3,-}^{n-1}) \\ v(T_\varepsilon, z) = 0, v(t,z)|_{z=0} = w_{1,-}^{n+1}(t,0-) - w_{1,-}^n(t,0-) \end{cases}$$

In the case that if the back characteristics $\xi = \xi(t,z,s)$ through the point (t,z) intersects with $z-$axis before it meets the t-axis, one shows easily that

$$|v(t,z)| \leq |p_{11}(w_-^n)(w_{3,-}^n - w_{3,-}^{n-1})| + C_M\varepsilon \sum_{i=1}^{3} \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)} \int_{T_\varepsilon}^{t} (1 + \frac{1}{\sqrt{s-T_\varepsilon}})ds$$
$$\leq C_M\varepsilon \sum_{i=1}^{3} \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)} \tag{3.48}$$

While if $\xi = \xi(t,z,s)$ intersects with t-axis at $(s,0)$ with $s \geq T_\varepsilon$, then

$$|v(t,z)| \leq |w_{1,-}^{n+1}(s,0-) - w_{1,-}^n(s,0-)| + C_M\varepsilon \sum_{i=1}^{3} \|w_{i,\pm}^{n+1} - w_{i,\pm}^n\|_{L^\infty(\tilde{\Omega}_\pm)} \tag{3.49}$$

By (3.25), Lemma 3.6 and the above estimates, we get as before that

$$|w_{1,-}^{n+1}(s,0-) - w_{1,-}^n(s,0-)| \leq |w_{1,+}^{n+1}(s,0+) - w_{1,+}^n(s,0+)| + C_M\varepsilon(|w_{1,-}^{n+1}(s,0-)$$
$$- w_{1,-}^n(s,0-)| + |w_{2,\pm}^{n+1}(s,0\pm) - w_{2,\pm}^n(s,0\pm)| + |w_{3,-}^{n+1}(s,0-) - w_{3,-}^n(s,0-)|)$$
$$\leq C_M\varepsilon \sum_{i=1}^{3} \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\Omega_\pm)}$$

Hence $|v(t,z)| \leq C_M\varepsilon \sum_{i=1}^{3} \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)}$ holds for all (t,z).

The estimate for $|w_{3,+}^{n+1}(t,z) - w_{3,+}^n(t,z)|$ is similar.

In summary, we have shown that

$$\begin{cases} \|w_{2,\pm}^{n+1} - w_{2,\pm}^n\|_{L^\infty(\tilde{\Omega}_\pm)} \leq (2ln\frac{3}{2} + C_M\varepsilon\sqrt{t-T_\varepsilon})\|w_{2,\pm}^n - w_{2,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)} \\ \qquad + C_M \sum_{i=1,3} \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)} \\ \sum_{i=1,3} \|w_{i,\pm}^{n+1} - w_{i,\pm}^n\|_{L^\infty(\tilde{\Omega}_\pm)} \leq C_M\varepsilon \sum_{i=1}^{3} \|w_{i,\pm}^n - w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)} \end{cases}$$

Consequently we have

$$\|w_{2,\pm}^{n+1} - w_{2,\pm}^n\|_{L^\infty(\tilde{\Omega}_\pm)} + (C_M+1)\sum_{i=1,3} \|w_{i,\pm}^{n+1} - w_{i,\pm}^n\|_{L^\infty(\tilde{\Omega}_\pm)} \leq (2ln\frac{3}{2} + C_M\varepsilon\sqrt{t-T_\varepsilon}$$
$$+ C_M(C_M+1)\varepsilon)\|w_{2,\pm}^n - w_{2,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)} + \frac{C_M + C_M(C_M+1)\varepsilon}{C_M+1} \sum_{i=1,3} (C_M+1)\|w_{i,\pm}^n$$
$$- w_{i,\pm}^{n-1}\|_{L^\infty(\tilde{\Omega}_\pm)}$$

Since $2ln\frac{3}{2} < 1$ and $\frac{C_M}{C_M+1} < 1$, and substituting C_M+1 by C_M in the above inequality, then Lemma 3.7 holds for small ε, this completes the proof of Lemma 3.7.

The proof of Theorem 2.1.

¿From Lemma 3.7, we know that there exist functions $\sigma(t) \in C[T_\varepsilon, T_\varepsilon+1]$ and $w_{i,\pm}(t,z) \in C(\tilde{\Omega}_\pm)$ such that $\sigma^n(t)$ converges to $\sigma(t)$ uniformly in $[T_\varepsilon, T_\varepsilon+1]$ and $w^n_{i,\pm}(t,z)$ converges to $w_{i,\pm}(t,z)$ uniformly on $\tilde{\Omega}_\pm$ respectively. Similarly, we can prove $\nabla_{t,z} w^n_{i,\pm}(t,z)$ converge to $\nabla_{t,z} w_{i,\pm}(t,z)$ uniformly in the any fixed closed subset of $\tilde{\Omega}_\pm$ respectively. Moreover, by Lemma 3.6 and Lemma 3.4, $w^n_{i,\pm}(t,z)$ are equicontinuous on z for the fixed $t \in (T_\varepsilon, T_\varepsilon+1)$ in $\tilde{\Omega}_\pm$ respectively. Hence $w_{i,\pm}(t, 0\pm)$ exist in $(T_\varepsilon, T_\varepsilon+1)$, furthermore it can be verified easily that the functions $w_i(t,z) = \begin{cases} w_{i,-}(t,z), z < \phi(t) \\ w_{i,+}(t,z), z > \phi(t) \end{cases}$ are the weak entropy solution of (2.1) due to the contraction of the approximate solution sequence. Finally, the estimates in Theorem 2.1 are the direct conclusions of Lemma 3.4 and Lemma 3.6 combining with the convergence of the sequence of approximate solutions, so the proof of Theorem 2.1 is complete.

§4. Application to compressible Euler equations

Now let's consider the compressible isentropic Euler equations with smooth initial data:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0 \\ \partial_t(\rho e + \frac{1}{2}\rho u^2) + \partial_x((\rho e + \frac{1}{2}\rho u^2 + p)u) = 0 \\ \rho|_{t=0} = \bar{\rho} + \varepsilon \rho_0(x), u|_{t=0} = \varepsilon u_0(x), S|_{t=0} = \bar{S} + \varepsilon S_0(x) \end{cases} \tag{4.1}$$

where $\bar{\rho} > 0$ and $\bar{S}$ are constants, $\varepsilon > 0$ is small enough, $\rho_0(x)$, $u_0(x)$ and $S_0(x)$ are smooth functions with compact support, $p = p(\rho, S)$ and $e = e(\rho, S)$ are smooth on their arguments. Moreover $\partial_\rho p(\rho, S) > 0$ and $\partial_S e(\rho, S) > 0$ for $\rho > 0$.

For smooth solutions, (4.1) is equivalent to problem:

$$\begin{cases} \partial_t \rho + u\partial_x \rho + \rho \partial_x u = 0 \\ \partial_t u + u \partial_x u + \frac{1}{\rho}\partial_x p = 0 \\ \partial_t S + u \partial_x S = 0 \\ \rho|_{t=0} = \bar{\rho} + \varepsilon \rho_0(x), u|_{t=0} = \varepsilon u_0(x), S|_{t=0} = \bar{S} + \varepsilon S_0(x) \end{cases} \tag{4.2}$$

Denoting $c^2(\rho, S) = \partial_\rho p(\rho, S)$, the system (4.2) has three distinct eigenvalues

$$\lambda_1 = u - c(\rho, S) < \lambda_2 = u < \lambda_3 = u + c(\rho, S)$$

The corresponding left eigenvectors are

$$l_1 = (-\frac{c}{\rho}, 1, 0), l_2 = (0, 0, 1), l_3 = (\frac{c}{\rho}, 1, 0)$$

The right eigenvectors are

$$r_1 = (1, -\frac{c}{\rho}, 0)^T, r_2 = (\partial_S p, 0, -c^2)^T, r_3 = (1, \frac{c}{\rho}, 0)^T$$

Meanwhile,

$$\begin{aligned}
&\nabla\lambda_1 r_1|_{\rho=\bar\rho,u=0,S=\bar S} = -\partial_\rho c(\bar\rho,\bar S) - \frac{\bar c}{\bar\rho} < 0\\
&\nabla\lambda_2 r_2|_{\rho=\bar\rho,u=0,S=\bar S} = 0\\
&\nabla\lambda_3 r_3|_{\rho=\bar\rho,u=0,S=\bar S} = \partial_\rho c(\bar\rho,\bar S) + \frac{\bar c}{\bar\rho} > 0
\end{aligned}$$

So the function $h_j(x)$ defined in (2.1) are

$$\begin{aligned}
&h_1(x) = \frac{\partial_\rho c(\bar\rho,\bar S)+\bar c}{2\bar c}(u_0'(x) - \frac{\bar c}{\bar\rho}\rho'(x))\\
&h_2(x) = 0\\
&h_3(x) = \frac{\partial_\rho c(\bar\rho,\bar S)+\bar c}{2\bar c}(u_0'(x) + \frac{\bar c}{\bar\rho}\rho'(x))
\end{aligned}\tag{4.3}$$

Correspondingly $M_1 = minh_1(x), M_2 = 0, M_3 = minh_3(x)$. Since the perturbation has compact support, then M_1 and M_3 are negative as long as $u_0(x)$ and $\rho_0(x)$ don't vanish identically. If $M_1 \neq M_3$, we can apply Theorem2.1 to the problem (4.1) and obtain the following conclusion.

Theorem 4.1. Assume $M_1 < M_3$, and $h_1(x)$ defined in (4.3) has a unique strictly negative quadratic minimum, then for small ε (4.1) admits an entropy weak solution, which is smooth in $[0, T_\varepsilon)$, continuous in $[0, T_\varepsilon]$ and has a unique shock $x = \phi(x)$ starting from the unique blowup point $(T_\varepsilon, x_\varepsilon)$ in $(T_\varepsilon, T_\varepsilon + 1)$. Moreover the entropy solution satifies near $(T_\varepsilon, x_\varepsilon)$ and in $(T_\varepsilon, T_\varepsilon + 1]$

$$\begin{aligned}
&\phi(t) = x_\varepsilon + \lambda_1(T_\varepsilon, x_\varepsilon)(t - T_\varepsilon) + O((t - T_\varepsilon)^2)\\
&\rho(t,x) = \rho(T_\varepsilon, x_\varepsilon) + O(d_\varepsilon^{\frac{1}{6}})\\
&u(t,x) = u(T_\varepsilon, x_\varepsilon) + O(d_\varepsilon^{\frac{1}{6}})\\
&S(t,x) = S(T_\varepsilon, x_\varepsilon) + O(d_\varepsilon^{\frac{1}{3}})
\end{aligned}$$

where $d_\varepsilon = (t - T_\varepsilon)^3 + (x - x_\varepsilon - \lambda_1(T_\varepsilon, x_\varepsilon)(t - T_\varepsilon))^2$.

Similar conclusion holds in the case of $M_1 > M_3$.

Remark 4.1. If $M_1 = M_3$, by using the same method we can obtain an entropy weak solution in $[T_\varepsilon, T_\varepsilon + \frac{C}{\varepsilon})$ with two shock waves starting from $(T_\varepsilon, x_\varepsilon)$ and $(T_\varepsilon', x_\varepsilon')$ respectively for sufficiently small ε, where C is an appropriate constant independent of ε and $T_\varepsilon \leq T_\varepsilon' < T_\varepsilon + \frac{C}{\varepsilon}$. This conclusion is based on such a fact, for small ε two strips formed by the first and third family of characteristics will separate and then the solution of (4.1) is a simple wave in each strip, where and only where the solution can blow up. The details can be found in [1], [2] and [3].

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