On Complete Noncompact Kähler Manifolds with Positive Bisectional Curvature

Bing-Long Chen and Xi-Ping Zhu

Department of Mathematics,
Zhongshan University,
Guangzhou 510275, P. R. China

The Institute of Mathematical Sciences,
The Chinese University of Hong Kong
Shatin, N.T., Hong Kong

Abstract

We prove that a complete noncompact Kähler manifold M^n of positive bisectional curvature satisfying suitable growth conditions is biholomorphic to a pseudoconvex domain of $\mathbf{C}^n$ and we show that the manifold is topologically $\mathbf{R}^{2n}$. In particular, when M^n is a Kähler surface of positive bisectional curvature satisfying certain natural geometric growth conditions, it is biholomorphic to $\mathbf{C}^2$.

1. Introduction

This paper is concerned with complete noncompact Kähler manifolds with positive bisectional curvature. Let us first recall some results on real Riemannian manifolds with positive sectional curvature. Gromoll and Meyer [GM] proved that if M is a complete noncompact Riemannian manifold with everywhere positive sectional curvature, then M is diffeomorphic to the Euclidean space. Later on Greene and Wu [GW] observed that this result readily follows from the fact that a complete noncompact Riemannian manifold of positive sectional curvature has a strictly convex exhaustion function. And the Busemann function of Cheeger-Gromoll [CG] directly gives a strictly convex exhaustion function on a positive curved manifold. A complex manifold is said to be Stein if it is holomorphically convex and its global holomorphic functions separate points and give local coordinates at every point. It is well-known that a complex Stein manifold can be holomorphically embedded in some Euclidean space. A result of Grauert [Gr] says that a complex manifold which admits a smooth strictly plurisubharmonic exhaustion function is Stein. Nonetheless, in the case of positive bisectional curvature,the Busemann function does not immediately give rise to a plurisubharmonic exhaustion function because one does not have the geometric comparison theorem for geodesic distances as in the case of positive sectional curvature (the Toponogov's comparison theorem). This consideration motivated the following conjecture which was formulated by Siu [Si].

Conjecture I: A complete noncompact Kähler manifold of positive holomorphic bisectional curvature is a Stein manifold.

Some results concerning this conjecture were obtained. In [MSY], among other things, Mok, Siu and Yau proved the following theorem:

Theorem (Mok-Siu-Yau [MSY]) Let M be a complete noncompact Kähler manifold of nonnegative bisectional curvature of complex dimension $n \geq 2$. If the Ricci curvatures are positive and for a fixed base point x_0,there exist positive constants C_1, C_2 such that

$$
\begin{array}{ll}
(i) & Vol(B(x_0,r)) \geq C_1 r^{2n} \ , \qquad 0 \leq r < +\infty \ , \\
(ii) & \frac{C_2^{-1}}{1+d(x_0,x)^2} \leq R(x) \leq \frac{C_2}{1+d(x_0,x)^2} \ , \quad x \in M \ ,
\end{array}
$$

where $B(x_0, r)$ denotes the geodesic ball of radius r and centered at x_0 , $R(x)$ denotes the scalar curvature and $d(x_0, x)$ denotes the distance between x_0 and x. Then M is a Stein manifold.

The method used in Mok-Siu-Yau's paper is the study of the Poincaré-Lelong equation on complete noncompact Kähler manifolds. Their result was improved by Mok [Mo2], where the two-side-bound assumption (ii) was replaced by the one-side-bound assumption:

$$(ii)' \qquad R(x) \leq \frac{C_2}{1+d(x_0,x)^2} , \qquad x \in M .$$

In this paper we further improve their result in the following theorem:

Theorem 1.1 Let M be a complete noncompact Kähler manifold of nonnegative holomorphic bisectional curvature of complex dimension $n \geq 2$. Suppose for a fixed base point x_0 there exist positive constants C_1, C_2 and $0 < \varepsilon < 1$ such that

$$(i) \quad Vol(B(x_0,r)) \geq C_1 r^{2n} , \quad 0 \leq r < +\infty ,$$
$$(ii) \qquad R(x) \leq \tfrac{C_2}{1+d(x_0,x)^{1+\varepsilon}} , \qquad x \in M .$$

Then M is a Stein manifold.

Our method is the study of the following Ricci flow equation on M :

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t) ,$$

where $g_{ij}(t)$ is a family of metrics, and $R_{ij}(t)$ denotes the Ricci curvature of $g_{ij}(t)$. In [Sh4], Shi established some dedicate decay estimates for the volume element and the curvatures of the evolving metric $g_{ij}(t)$. Based on Shi's estimates we will show the injectivity radius of the evolving metric $g_{ij}(t)$ is getting bigger and bigger and any geodesic ball with radius less than half of the injectivity radius is almost pseudoconvex. By a perturbation argument , we will be able to construct a sequence of pseudoconvex domains of M such that arbitrary two of these domains form a Runge pair . Then by appealing a theorem of Markoe [Ma] (see also Siu [Si]), we can deduce that M is Stein.

The Gromoll and Meyer's theorem is a uniformization theorem in Riemannian geometry category. The analogue in Kähler geometry is the following well-known conjecture:

Conjecture II: A complete noncompact Kähler manifold of positive holomorphic bisectional curvature of complex dimension n is biholomorphic to $\mathbf{C}^n$.

The first result concerning this conjecture is the following isometrically embedding theorem of Mok, Siu and Yau [MSY] and Mok [Mo2].

Theorem (Mok-Siu-Yau [MSY], Mok [Mo2]) Let M be a complete noncompact Kähler manifold of nonnegative holomorphic bisectional curvature of complex dimension $n \geq 2$. Suppose for a fixed base point x_0 :

$$\begin{aligned} &(i) \quad Vol(B(x_0,r)) \geq C_1 r^{2n} \ , \quad 0 \leq r < +\infty \ , \\ &(ii) \quad R(x) \leq \frac{C_2}{1+d(x_0,x)^{2+\varepsilon}} \ , \quad x \in M \ , \end{aligned}$$

for some $C_1, C_2 > 0$ and for any arbitrarily small positive constant ε. Then M is isometrically biholomorphic to $\mathbf{C}^n$ with the standard flat metric.

Also in his paper [Mo2], Mok used some algebraic geometrical techniques to control the holomorphic functions of polynomial growth on M and obtained the following holomorphic embedding theorem.

Theorem (Mok [Mo2]) Let M be a complete noncompact Kähler manifold of positive holomorphic bisectional curvature of complex dimension $n \geq 2$. Suppose for a fixed base point x_0:

$$\begin{aligned} &(i) \quad Vol(B(x_0,r)) \geq C_1 r^{2n} \ , \quad 0 \leq r < +\infty \ , \\ &(ii)' \quad R(x) \leq \frac{C_2}{1+d(x_0,x)^{2}} \ , \quad x \in M \ , \end{aligned}$$

for some positive constants C_1 and C_2.Then M is biholomorphic to an affine algebraic variety. Moreover in case of $n = 2$, if the Kähler manifold M is actually of positive Riemannian sectional curvature, then M is biholomorphic to $\mathbf{C}^2$.

This result was also improved in [Mo3] and [To]. Instead of the bisectional curvature, only the positivity of the Ricci curvature was assumed there.

In [Sh2], Shi announced that under the same assumptions as in the above Mok's theorem, the Kähler manifold M is biholomorphic to $\mathbf{C}^n$. This is actually the main theorem of Shi's Ph.D. thesis [Sh3]. Here we are unwilling to point out that there exists a gap in the proof [Sh3]. Shi used the Ricci flow to construct a flat Kähler metric on M. But the flat metric may be incomplete. Thus one can only get the biholomorphic embedding of M into a domain of $\mathbf{C}^n$.

In this paper ,with the help of the resolutions of generalised Poincaré conjecture and a gluing argument of Shi [Sh4], we will get the following result.

Theorem 1.2 Under the same assumptions of Theorem1.1, we have

(1) M is diffeomorphic to $\mathbf{R}^{2n}$ if $n > 2$, M is homeomorphic to $\mathbf{R}^4$ if $n = 2$, and

(2) M is biholomorphic to a pseudoconvex domain in $\mathbf{C}^n$.

The part (2) of this result is analogous to the main theorem of Shi in [Sh4], where the positivity of Riemannian sectional curvature was assumed.

The combination of part (1) of Theorem 1.2 with a theorem of Ramanujam [R] immediately gives an improvement of the above Mok's holomorphically embeddding theorem in dimension $n = 2$. More precisely, we have:

Corollary 1.3 Let M be a complete noncompact Kähler surface with positive holomorphic bisectional curvature . Suppose for a fixed base point x_0,there exist constants C_1, C_2 suth that

$$(i) \quad Vol(B(x_0,r)) \geq C_1 r^4 \ , \quad 0 \leq r < +\infty \ ,$$
$$(ii)' \quad R(x) \leq \frac{C_2}{1+d(x_0,x)^2} \ , \quad x \in M \ .$$

Then M is biholomorphic to $\mathbf{C}^2$.

The isometrically embedding throrem of Mok, Siu and Yau can be interpreted as a gap phenomenon which shows that the metrics of positive holomorphic bisectional curvature can not be too close to the flat one.Their method is to solve the Poincaré-Lelong equation on complete noncompact Kähler manifolds where the maximal volume growth condition (i) was assumed. Thus it is interesting to investigate the gap phenomena on manifolds without maximal volume growth assumption.

Recently Ni [N] got some results in this direction by improving the argument of Mok, Siu and Yau [MSY]. In section 4 of this paper , we use Shi′s a priori estimate and the differential Harnack inequality of Cao [Ca] for the Ricci flow to prove a general version of Mok-Siu-Yau′s gap theorem.

Finally in section 5 we will give a remark on the flat metric constructed by Shi [Sh3].

Acknowledgement We are grateful to Professor L.F. Tam for many helpful discussions. The work is supported by the Foundation for Outstanding Young Scholars of China, Zheng Ge Ru Foundation and "The Institute of Mathematical Sciences, The Chinese University of Hong Kong".

2. The Ricci Flow and A Priori Estimates

Let $(M, \tilde{g}_{ij})$ be a complete noncompact Kähler manifold of complex dimension $n \geq 2$ with bounded and nonnegative holomorphic bisectional curvature . The Ricci flow is the following evolution equation for the metric

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) \ , \quad on \quad M \times [0,T] \ , \\ g_{ij}(x,0) = \tilde{g}_{ij}(x) \ , \qquad\qquad x \in M. \end{cases} \tag{2.1}$$

Suppose $\{z^1, z^2, \cdots, z^n\}$ is the local holomorphic coordinate system on M and $z^k = x^k + \sqrt{-1}x^{k+n}, x^k, x^{k+n} \in R,\ k = 1, 2, ...n$. Then $\{x^1, x^2, \cdots, x^{2n}\}$

is the local real coordinate system on M. We use $i,\ j,\ k,\ l$ to denote the indices corresponding to the real vectors and real covectors, $\alpha,\ \beta,\ \gamma,\ \delta,\cdots$ the indices corresponding to the holomorphic vectors and covectors. Then the above Ricci flow equation (2.1) can be written in holomorphic coordinates as

$$\begin{cases} \dfrac{\partial}{\partial t} g_{\alpha\overline{\beta}}(x,t) = -R_{\alpha\overline{\beta}}(x,t)\ , \quad on \quad M\times[0,T]\ , \\ \\ g_{\alpha\bar{\beta}}(x,0) = \tilde{g}_{\alpha\bar{\beta}}(x)\ , \qquad\qquad x\in M. \end{cases} \tag{2.1$'$}$$

¿From [Sh1] we know that the Ricci flow (2.1) has a maximal solution $g_{ij}(\cdot,t)$ on $[0,t_{\max})$ with $t_{\max}>0$ and the curvature of $g_{ij}(\cdot,t)$ becomes unbounded as t tends to $t_{\max}$ if $t_{\max}<+\infty$. Since the maximum principle has been well understood for noncompact manifolds, the preserving kählerity and nonnegativity of holomorphic bisectional curvature of the Ricci flow (2.1) were essentially obtained by Hamilton [Ha1] and Mok [Mo1] (see also Shi [Sh4]).

Define a function $F(x,t)$ on $M\times[0,t_{\max})$ as follows

$$F(x,t) = \log\frac{\det(g_{\alpha\bar{\beta}}(x,t))}{\det(g_{\alpha\bar{\beta}}(x,0))}\ . \tag{2.2}$$

It can be easily obtained from (2.1) that

$$\frac{\partial F(x,t)}{\partial t} = -R(x,t)\ . \tag{2.3}$$

Since the holomorphic bisectional curvature of $g_{\alpha\bar{\beta}}(\cdot,t)$ is nonnegative, it follows that $F(\cdot,t)$ is non-increasing in t and $F(\cdot,0)=0$. And by the equation (2.1) we know that

$$g_{\alpha\bar{\beta}}(\cdot,t)\le g_{\alpha\bar{\beta}}(\cdot,0)\ , \qquad on \quad M\ . \tag{2.4}$$

Then by definition , we have

$$\begin{aligned} e^{F(x,t)}R(x,t) &= g^{\alpha\bar{\beta}}(x,t)R_{\alpha\bar{\beta}}(x,t)\cdot\frac{\det(g_{\alpha\bar{\beta}}(x,t))}{\det(g_{\alpha\bar{\beta}}(x,0))} \\ &\leq g^{\alpha\bar{\beta}}(x,0)R_{\alpha\bar{\beta}}(x,t) \\ &= g^{\alpha\bar{\beta}}(x,0)(R_{\alpha\bar{\beta}}(x,t)-R_{\alpha\bar{\beta}}(x,0))+R(x,0) \\ &= -g^{\alpha\bar{\beta}}(x,0)\frac{\partial^2 F(x,t)}{\partial z^\alpha \partial \bar{z}^\beta}+R(x,0) \\ &= -\Delta_0 F(x,t)+R(x,0)\ , \end{aligned}$$

where Δ_0 is the Laplace operator with respect to the initial metric $g_{ij}(\cdot,0)$.

Combining with (2.3) we obtain

$$e^{F(x,t)}\frac{\partial F(x,t)}{\partial t} \geq \Delta_0 F(x,t) - R(x,0)\ , \tag{2.5}$$

and

$$\Delta_0 F(x,t) \leq R(x,0)\ , \qquad on \quad M\times[0,t_{\max})\ . \tag{2.6}$$

In the PDE jargon, if the scalar curvature $R(x,0)$ of the initial metric satisfies suitable growth conditions, the differential inequalities (2.5) and (2.6) will give two opposite estimates of F by its average. Shi [Sh4] observed that the combination of these two opposite estimates will give the following a priori estimate for the function F.

Lemma 2.1 (Shi [Sh4]) Suppose $(M,\tilde{g}_{\alpha\bar{\beta}})$ is a complete noncompact Kähler manifold of complex dimension $n\geq 2$ with bounded and nonnegative holomorphic bisectional curvature. Suppose there exist positive constants C_1, C_2 and $0<\varepsilon<1$ such that for any x_0,

$$(i) \qquad R(x,0)\leq C_1\ , \qquad x\in M\ ,$$

$$(ii) \qquad \frac{1}{Vol(B_0(x_0,r))}\int_{B_0(x_0,r)} R(x,0)dV_0 \leq \frac{C_2}{1+r^{1+\varepsilon}}\ , \quad 0\leq r<+\infty\ ,$$

where $B_0(x_0, r)$ is the geodesic ball of radius r and centered at x_0 with respect to the metric $\tilde{g}_{\alpha\bar{\beta}}(x)$. Then the solution of $(2.1)'$ satisfies the estimate:

$$F(x,t) \geq -C(t+1)^{\frac{1-\varepsilon}{1+\varepsilon}}\ , \qquad on\ M \times [0, t_{\max})\ ,$$

where $0 < C < +\infty$ is a constant depending only on n, ε, $C_1$and C_2.

Proof. Since we will use the two opposite estimates later in this paper, for convenience, we sketch the proof here.

Without loss of generality , by replacing M by $M \times \mathbf{C}^2$ if necessary, we may assume that the dimension of M is ≥ 4 and

$$C_3 \left(\frac{r_2}{r_1}\right)^4 \leq \frac{Vol(B_0(x,r_2))}{Vol(B_0(x,r_1))} \leq \left(\frac{r_2}{r_1}\right)^{2n}\ , \qquad 0 \leq r_1 \leq r_2 < +\infty\ , \tag{2.7}$$

by the standard volume comparison, where C_3 is a positive constant depending only on n.

Denote $\tilde{\nabla}$ to be the covariant derivatives with respect to the initial metric $\tilde{g}_{\alpha\bar{\beta}}(x)$. Since the holomorphic bisectional curvature of $\tilde{g}_{\alpha\bar{\beta}}(x)$ is nonnegative, we know that the Ricci curvature is also nonnegative. Then by Theorem 1.4.2 of Schoen and Yau [SY] and a simple scaling argument, there exists a constant $C(n) > 0$ depending only on n such that for any fixed point $x_0 \in M$ and any number $0 < a < +\infty$, there exists a smooth function $\varphi(x) \in C^\infty(M)$ satisfying

$$\begin{cases} e^{-C(n)\left(1+\frac{d_0(x,x_0)}{a}\right)} \leq \varphi(x) \leq \quad e^{-\left(1+\frac{d_0(x,x_0)}{a}\right)}\ , \\ \left|\tilde{\nabla}\varphi(x)\right|_0 \leq \frac{C(n)}{a}\varphi(x)\ , \\ \left|\Delta_0\varphi(x)\right|_0 \leq \frac{C(n)}{a^2}\varphi(x)\ , \end{cases} \tag{2.8}$$

for $x \in M$, where $d_0(x, x_0)$ is the distance between x and x_0 with respect to the metric $\tilde{g}_{\alpha\bar{\beta}}(x)$ and $|\cdot|_0$ is the corresponding norm.

By the volume growth (2.7), the manifold $(M, \tilde{g}_{\alpha\bar{\beta}})$ is parabolic. Let $G_0(x,y)$ be the positive Green's function on $(M,\ \tilde{g}_{\alpha\bar{\beta}})$. From Li-Yau's estimate for Green's function and (2.7) we know that for $x, y \in M$,

$$\begin{cases} \dfrac{C_4^{-1} d_0(x,y)^2}{Vol(B_0(x, d_0(x,y)))} \le G_0(x,y) \le \dfrac{C_4 d_0(x,y)^2}{Vol(B_0(x, d_0(x,y)))} \ , \\ \left| \tilde{\nabla} G_0(x,y) \right|_0 \le \frac{C_5 d_0(x,y)}{Vol(B_0(x, d_0(x,y)))} \ , \end{cases} \tag{2.9}$$

for some positive constants C_4, C_5 depending only on n.

Denote

$$\Omega_\alpha = \{ y \in M | G_0(x,y) > \alpha \} \qquad for \quad any \quad \alpha > 0 \ .$$

By the differential inequality (2.6), we have for any $x_0 \in M$ and any $t \in [0, t_{\max})$,

$$\begin{aligned} F(x_0,t) &= \int_{\Omega_\alpha} (\alpha - G_0(x_0,y)) \, \Delta_0 F(y,t) dV_0(y) - \int_{\partial \Omega_\alpha} F(y,t) \frac{\partial G_0(x_0,y)}{\partial \nu} d\sigma(y) \\ &\ge - \int_{\Omega_\alpha} G_0(x_0,y) R(y,0) dV_0(y) - \int_{\partial \Omega_\alpha} F(y,t) \frac{\partial G_0(x_0,y)}{\partial \nu} d\sigma(y) \ . \end{aligned} \tag{2.10}$$

Let $r(\alpha) \ge 1$ be a number such that

$$\frac{r(\alpha)^2}{Vol(B_0(x_0, r(\alpha)))} = \alpha \ . \tag{2.11}$$

Then by (2.7), (2.9) and the assumption (ii) , we have

$$C_6^{-1} r(\alpha) \le d_0(x_0,y) \le C_6 r(\alpha) \ , \qquad for \quad any \quad y \in \partial \Omega_\alpha \ , \tag{2.12}$$

$$- \int_{\partial \Omega_\alpha} F(y,t) \frac{\partial G_0(x_0,y)}{\partial \nu} d\sigma(y) \ge \frac{C_5 r(\alpha)}{Vol(B_0(x_0, r(\alpha)))} \int_{\partial \Omega_\alpha} F(y,t) d\sigma(y) \ , \tag{2.13}$$

and

$$\begin{aligned} - \int_{\Omega_\alpha} G_0(x_0,y) R(y,0) dV_0(y) &\ge - \int_{B(x_0, C_6 r(\alpha))} G_0(x_0,y) R(y,0) dV_0(y) \\ &\ge -C_7 (r(\alpha))^{1-\varepsilon} \ , \end{aligned} \tag{2.14}$$

where C_6, C_7 are positive constants depending only on C_1, C_2 and n .

Substituting (2.13), (2.14) into (2.10) and integrating from $\frac{\alpha}{2}$ to α , one readily get

$$F(x_0,t) \geq -C_8 r(\alpha)^{1-\varepsilon} + \frac{C_8}{Vol(B_0(x_0,r(\alpha)))} \int_{B_0(x_0,C_6 r(\alpha))} F(y,t)dV_0(y) \ ,$$

for some positive constant depending only on ε, C_1, C_2 and n .

Then by (2.12) and (2.7) , we obtain that for any $a > 0$,

$$F(x_0,t) \geq -C_9 a^{1-\varepsilon} + \frac{C_9}{Vol(B_0(x_0,a))} \int_{B_0(x_0,a)} F(y,t)dV_0(y) \ , \tag{2.15}$$

where C_9 is a positive constant depending only on ε, C_1, C_2 and n.

On the other hand , by multiplying the differential inequality (2.5) by the function $\varphi(x)$ in (2.8) and integrating by parts, we get

$$\begin{aligned} \frac{\partial}{\partial t}\int_M \varphi(x)e^{F(x,t)}dV_0 &\geq \int_M (\Delta_0 F(x,t) - R(x,0))\varphi(x)dV_0 \\ &\geq \frac{C(n)}{a^2}\int_M F(x,t)\varphi(x)dV_0 - \int_M R(x,0)\varphi(x)dV_0 \ . \end{aligned}$$

By (2.8) and the assumption (i) (ii), it is easy to see that

$$\int_M R(x,0)\varphi(x)dV_0 \leq \frac{C_{11}}{a^{1+\varepsilon}}Vol(B_0(x_0,a))$$

for some positive constant depending only on n . Then we have

$$\int_M \varphi(x)(1-e^{F(x,t)})dV_0 \leq \frac{C_{11}t}{a^{1+\varepsilon}}Vol(B_0(x_0,a)) + \frac{C(n)t}{a^2}\int_M (-F(x,t))\varphi(x)dV_0. \tag{2.16}$$

Denote

$$F_{\min}(t) = \inf\{F(x,t) | x \in M\} \ .$$

It is clear that

$$\int_M \varphi(x)(1-e^{F(x,t)})dV_0 \geq \frac{1}{2(1-F_{\min}(t))}\int_M (-F(x,t))\varphi(x)dV_0 \ . \tag{2.17}$$

Therefore we get from (2.16) and (2.17),

$$\frac{1}{Vol(B_0(x_0,a))}\int_{B_0(x_0,a)} F(x,t)dV_0 \geq -C_{12}t(1-F_{\min}(t))\left(\frac{1}{a^{1+\varepsilon}}-\frac{F_{\min}(t)}{a^2}\right) , \tag{2.18}$$

for any $a > 0$ and $t \in [0, t_{\max})$, where C_{12} is a positive constant depending only on ε, C_1, C_2 and n.

Combining (2.15) and (2.18) , we deduce that

$$F_{\min}(t) \geq -C_9 a^{1-\varepsilon} - C_9 C_{12}(1-F_{\min}(t))\left(\frac{t}{a^{1+\varepsilon}}-\frac{t}{a^2}F_{\min}(t)\right) , \tag{2.19}$$

for any $a > 0$ and $t \in [0, t_{\max})$. By taking $a = C_{13}(t+1)^{\frac{1}{2}}(-F_{\min}(t))^{\frac{1}{2}}$ with C_{13} large enough , we then get the desired estimate

$$F_{\min}(t) \geq -C(t+1)^{\frac{1-\varepsilon}{1+\varepsilon}} , \qquad for\ t \in [0, t_{\max}) .$$

Q.E.C.

Next we are going to prove that the maximal volume growth condition is preserved under the Ricci flow (2.1) (or $(2.1)'$) . More precisely, we have

Lemma 2.2 Suppose $(M, \tilde{g}_{\alpha\bar{\beta}})$ is a complete noncompact Kähler manifold of complex dimension $n \geq 2$ with bounded and nonnegative bisectional curvature. Suppose for a fixed base point x_0 there exist positive constant C_1, C_2 and $0 < \varepsilon < 1$ such that

$$(i) \quad Vol(B(x_0,r)) \geq C_1 r^{2n} , \quad 0 \leq r < +\infty ,$$

$$(ii) \qquad R(x) \leq \frac{C_2}{1+d(x_0,x)^{1+\varepsilon}} , \qquad x \in M .$$

Let $g_{\alpha\bar{\beta}}(\cdot,t)$ be the solution of the Ricci flow $(2.1)'$ with $\tilde{g}_{\alpha\bar{\beta}}$ as initial metric, and let $Vol_t(B_t(x_0,r))$ be the volume of the geodesic ball of radius r and centered at x_0 with respect to the metric $g_{\alpha\bar{\beta}}(\cdot,t)$. Then

$$Vol_t(B_t(x_0,r)) \geq C_1 r^{2n} \tag{2.20}$$

for all $t \in [0, t_{\max})$ and $0 \le r < +\infty$.

Proof. We have already noticed that the Ricci curvature of the solution $g_{\alpha\bar{\beta}}(\cdot, t)$ is nonnegative . That says the metric is shrinking under the flow $(2.1)'$. Thus we have

$$\begin{aligned} Vol_t(B_t(x_0, r)) &\ge Vol_t(B_0(x_0, r)) \\ &= \int_{B_0(x_0,r)} e^{F(x,t)} dV_0 \\ &= Vol(B_0(x_0, r)) + \int_{B_0(x_0,r)} (e^{F(x,t)} - 1) dV_0 \end{aligned} \tag{2.21}$$

Clearly the assumptions of Lemma 2.2 are stronger then those of Lemma 2.1. Then we can use (2.8) and (2.16) to deduce

$$\begin{aligned} \int_{B_0(x_0,r)} (e^{F(x,t)} - 1) dV_0 &\ge C_3 \int_M (e^{F(x,t)} - 1)\varphi(x) dV_0 \\ &\ge C_4 t \left[\frac{F_{\min}(t)}{r^2} - \frac{1}{r^{1+\varepsilon}} \right] \cdot Vol(B_0(x_0, r)) \end{aligned}$$

for some positive constants C_3, C_4 depending on C_1, C_2 and n.

Substituting the above inequality into the right hand side of (2.21) and dividing by r^{2n}, we have

$$\lim_{r \to +\infty} \frac{Vol_t(B_t(x_0, r))}{r^{2n}} \ge \lim_{r \to +\infty} \frac{Vol(B_0(x_0, r))}{r^{2n}} \ge C_1 \ .$$

Hence the standard volume comparison theorem implies (2.20). *Q.E.C.*

3. The Proof of Theorem 1.1 and 1.2

In this section we will prove Theorem 1.1 and 1.2 simutaneously.

First, let us recall the local injectivity radius estimate of Cheeger, Gromov and Taylor [CGT], which says that for any complete Riemannian manifold N of dimension m with $\lambda \le$ sectional curvatures of $N \le \Lambda$, let r be a positive constant and $r < \frac{\pi}{4\sqrt{\Lambda}}$ if $\Lambda > 0$, the injetivity radius of N at a point P can be bounded from below as follows

$$inj_N(P) \geq r \frac{Vol(B(P,r))}{Vol(B(P,r)) + V^{2m}(2r)} \ , \tag{3.1}$$

where $V^m(2r)$ denotes the volume of a ball of radius $2r$ in the $m-$dimensional model space V^m with constant sectional curvature λ.

In particular , it implies that for a complete Riemannian manifold N of dimension m with the sectional curvature bounded between -1 and 1 , the injectivity radius at a point P can be estimated as follows

$$inj_N(P) \geq \frac{1}{2} \frac{Vol(B(P,\frac{1}{2}))}{Vol(B(P,\frac{1}{2})) + V} \tag{3.2}$$

for some positive constant V depending only on m. Further, if in addition N satisfies the maximal volume growth condition

$$Vol(B(x_0,r)) \geq C_1 r^m \ , \qquad 0 \leq r \leq +\infty \ ,$$

then (3.2) gives

$$inj_N(P) \geq C > 0 \tag{3.3}$$

for some positive constant C depending only on C_1 and m.

Consider $(M, \tilde{g}_{\alpha\bar{\beta}})$ to be a complete noncompact Kähler manifold with complex dimension $n \geq 2$ and satisfying the assumptions of Theorem 1.1 (or Theorem 1.2). Let $g_{\alpha\bar{\beta}}(\cdot,t)$ be the maximal solution of the Ricci flow $(2.1)'$. Lemma 2.1 tells us that

$$\frac{\det(g_{\alpha\bar{\beta}}(x,t))}{\det(g_{\alpha\bar{\beta}}(x,0))} \geq e^{-C(t+1)^{\frac{1-\varepsilon}{1+\varepsilon}}}, \quad \text{on } M \times [0, t_{\max}),$$

which together with (2.4) implies

$$g_{\alpha\bar{\beta}}(x,0) \geq g_{\alpha\bar{\beta}}(x,t) \geq e^{-C(t+1)^{\frac{1-\varepsilon}{1+\varepsilon}}} g_{\alpha\bar{\beta}}(x,0), \quad \text{on } M \times [0, t_{\max}).$$

Since the Ricci flow equation $(2.1)'$ is the parabolic version of the complex Monge-Ampère eqution on the Kähler manifold, the above inequality is corresponding to the second order estimate for the Monge-Ampère eqution. It is well known that the third order and higher order estimates for Monge-Ampère were developed by Calabi and Yau. Similarly, by adapting the Calabi and Yau's arguments, Shi proved in [Sh4] that the derivative and higher order estimates for $g_{\alpha\bar{\beta}}(x,t)$ are uniformly bounded on any finite time interval. In particular, this implies that the solution $g_{\alpha\bar{\beta}}(\cdot,t)$ exists for all $t \in [0,+\infty)$. On the other hand, by applying the differential Harnack inequality of Cao [Ca], we know that $tR(\cdot,t)$ is nondecreasing in time. It then follows from (2.3) that for $x \in M$ and $t \in [0,+\infty)$,

$$\begin{aligned} -F(x,2t) &= \int_0^{2t} R(x,s)ds \\ &\geq \int_t^{2t} R(x,s)ds \\ &\geq tR(x,t)\int_t^{2t} \frac{1}{s}ds \\ &= (\log 2)\cdot tR(x,t)\ . \end{aligned} \tag{3.4}$$

Combining (3.4) with Lemma 2.1, we get

$$R(x,t) \leq C_3(t+1)^{\frac{-2\varepsilon}{1+\varepsilon}}\ , \tag{3.5}$$

for some positive constant C_3 depending only on ε, C_1, C_2 and n.

Moreover by using the derivative estimates of Shi (see [Sh1] or Theorem 7.1 of Hamilton [Ha1]), we have

$$|\nabla^p R_{ijkl}(x,t)| \leq C(\varepsilon, C_1, C_2, n, p)(t+1)^{-\frac{\varepsilon}{1+\varepsilon}(p+2)}\ , \tag{3.6}$$

for $x \in M, t \geq 1$ and any integer $p \geq 0$.

We have shown in Lemma 2.2 that the maximal volume growth condition is preserved under the flow $(2.1)'$. By a standard scaling argument , it follows from (3.3) and (3.5) that the injectivity radius of M with respect to the metric $g_{\alpha\bar{\beta}}(\cdot,t)$ has the following estimate

$$inj(M, g_{\alpha\bar{\beta}}(\cdot,t)) \geq C_4(t+1)^{\frac{\varepsilon}{1+\varepsilon}} \ , \tag{3.7}$$

where C_4 is a positive constant depending only on $\varepsilon,\ C_1,\ C_2$ and n.

Since the Ricci curvature of $g_{\alpha\bar{\beta}}(\cdot,t)$ is nonnegative for all $x \in M$ and $t \geq 0$, we know from the equation $(2.1)'$ that the ball $B_t\left(x_0, \frac{C_4}{2}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}\right)$, of radius $\frac{C_4}{2}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}$ with respect to the metric $g_{\alpha\bar{\beta}}(\cdot,t)$, contains the ball $B_0\left(x_0, \frac{C_4}{2}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}\right)$, of the same radius with respect to the metric $\tilde{g}_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}}(\cdot,0)$.Thus we deduce from (3.7) that

$$\pi_p(M) = 0 \qquad and \qquad \pi_q(M,\infty) = 0 \tag{3.8}$$

for any $p \geq 1, 1 \leq q \leq 2n-2$, where $\pi_q(M,\infty)$ is the $q-$th homotopy group of M at infinity.

Then by the resolutions of generalised Poincaré conjecture (see [F], [Sm]), we know that M is homeomorphic to $\mathbf{R}^{2n}$. Notice Gompf's result says that among the Euclidean spaces only $\mathbf{R}^4$ has exotic differential structures. So for $n > 2$ the homeomorphisms can be maken to be diffeomorphisms. This gives the proof of the part (1) of Theorem 1.2 .

Also the injectivity radius estimate (3.7) tells us the exponential map provides a diffeomorphism between the balls of M and the Euclidean space. In the following we want to modify the exponential maps to become biholomorphims.

Let $T_{x_0}M$ denote the real tangent space of M at x_0, J_M denote the complex structure of $T_{x_0}M$. For fixed t, choose a standard orthonormal basis $\{e_1, J_Me_1, \cdots, e_n, J_Me_n\}$ of $T_{x_0}M$. Since the metric $g_{\alpha\bar{\beta}}(\cdot,t)$ is a family of smooth Kähler metrics, we may assume the basis is smooth in time.

For any $v \in T_{x_0}M$,one can write

$$v = x_1e_1 + y_1J_Me_1 + ... + x_ne_n + y_nJ_Me_n \ .$$

We now construct a real linear isomorphism $L : T_{x_0}M \to \mathbf{C}^n$ defined by

$$L(v) = (z_1, z_2, ...z_n) \in \mathbf{C}^n \ ,$$

where $z_i = x_i + \sqrt{-1}y_i$, $i = 1, 2, \cdots, n$. It is also clear that L varies smoothly in t. Equip $\mathbf{C}^n$ with the standard flat Kähler metric. Let $\hat{\nabla}$ denote the covariant derivative and $\hat{B}(0, r)$ denote the ball of radius r with respect to this standard metric.

Let us use $\exp{}^t_{x_0}$ to denote the exponential map with respect to the metric $g_{\alpha\bar{\beta}}(\cdot, t)$. By (3.7), the map $\varphi_t = \exp{}^t_{x_0} \circ L^{-1}$ is a diffeomorphism from $\hat{B}\left(0, \frac{C_4}{2}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}\right)$ to the geodesic ball $B_t\left(x_0, \frac{C_4}{2}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}\right)$ of M with repect to the metric $g_{\alpha\bar{\beta}}(\cdot, t)$, and the map is nonsingular on $\hat{B}\left(0, \frac{C_4}{2}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}\right)$ .Let us write the solution $g_{\alpha\bar{\beta}}(\cdot, t)$ in real coordinates as $g_{ij}(\cdot, t)$. We consider the pull back metric

$$\varphi_t^*(g_{ij}(\cdot, t)) = g^*_{ij}(\cdot, t)dx^i dx^j \ , \qquad on \quad \hat{B}\left(0, \frac{C_4}{2}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}\right) \ . \tag{3.9}$$

Clearly we can also write the pull back metric in the complex coordinates of $\mathbf{C}^n$ as

$$\varphi_t^*(g_{ij}(\cdot, t)) = g^*_{AB}(\cdot, t)dz^A dz^B \ , \qquad on \quad \hat{B}(0, \frac{C_4}{2}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}) \ , \tag{3.10}$$

where $A, B = \alpha$ or $\bar{\alpha}(\alpha = 1, 2, ...n)$. Since φ_t is not holomorphic in general , the metric $g^*_{AB}(\cdot, t)$ is not Hermitian with respect to the standard complex structure of $\mathbf{C}^n$

Notice that $g^*_{AB}(\cdot, t)$ is just the representation of the metric $g_{\alpha\bar{\beta}}(\cdot, t)$ in geodesic coordinates .The following lemma due to Hamilton (Theorem 4.10 in [Ha2]) is useful to estimate $g^*_{AB}(\cdot, t)$.

Lemma 3.1 Suppose the metric $g_{ij}dx^i dx^j$ is in geodesic coordinates. Suppose the Riemannian curvature R_m is bounded between $-B_0$ and B_0. Then there exist positive constants c, C_0 depending only on the dimension such that for any $|x| \le \frac{c}{\sqrt{B_0}}$, the following holds

$$|g_{ij} - \delta_{ij}| \le C_0 B_0 |x|^2 \ .$$

Furthermore, if in addition $|\nabla R_m| \le B_0$ and $|\nabla^2 R_m| \le B_0$, then

$$\left|\frac{\partial}{\partial x^j} g_{kl}\right| \le C_0 B_0 |x| \qquad and \qquad \left|\frac{\partial^2}{\partial x^i \partial x^j} g_{kl}\right| \le C_0 B_0$$

for any $|x| \le \frac{c}{\sqrt{B_0}}$.

Applying this lemma to the evolving metric $g_{\alpha\bar\beta}(\cdot, t)$, we can find positive constants $\bar c, C_5$ depending only on ε, C_1, C_2 and n such that for any $z \in \hat B\left(0, \bar c(t+1)^{\frac{\varepsilon}{1+\varepsilon}}\right)$ and $t \in [1, +\infty)$,

$$\begin{aligned}
\left|g^*_{\alpha\bar\beta}(z,t) - \delta_{\alpha\beta}\right|_t &\le C_5 |z|^2 (t+1)^{-\frac{2\varepsilon}{1+\varepsilon}} \ , && (3.11)\\
\left|g^*_{\bar\alpha\beta}(z,t) - \delta_{\alpha\beta}\right|_t &\le C_5 |z|^2 (t+1)^{-\frac{2\varepsilon}{1+\varepsilon}} \ , && (3.12)\\
\left|g^*_{\alpha\beta}(z,t)\right|_t &\le C_5 |z|^2 (t+1)^{-\frac{2\varepsilon}{1+\varepsilon}} \ , && (3.13)\\
\left|g^*_{\bar\alpha\bar\beta}(z,t)\right|_t &\le C_5 |z|^2 (t+1)^{-\frac{2\varepsilon}{1+\varepsilon}} \ , && (3.14)\\
\left|\hat\nabla g^*_{AB}(z,t)\right|_t &\le C_5 |z| (t+1)^{-\frac{2\varepsilon}{1+\varepsilon}} \ , && (3.15)\\
\left|\hat\nabla\hat\nabla g^*_{AB}(z,t)\right|_t &\le C_5 (t+1)^{-\frac{2\varepsilon}{1+\varepsilon}} \ , && (3.16)
\end{aligned}$$

where $|\cdot|_t$ is the norm with respect to the metric $g^*_{ij}(z,t)$.

By a further restriction on the small constant $\bar c$, we may assume that in the real coordinates

$$\frac{1}{2}\ g^*_{ij}(z,t) \le \delta_{ij} \le 2 g^*_{ij}(z,t) \ . \qquad (3.17)$$

Let $\varphi_t^* J_M$ and $\bar\partial^t = \varphi_t^*(\bar\partial)$ be the pull back complex structure and $\bar\partial$-operator of M on $\hat B\left(0, \bar c(t+1)^{\frac{\varepsilon}{1+\varepsilon}}\right)$. It is obvious that φ_t is holomorphic with respect to the pull back complex structure $\varphi_t^* J_M$ .But the functions z^α $(\alpha = 1, 2, ...n)$ are not holomorphic with respect to $\varphi_t^* J_M$ in general. This just indicates the difference of the pull back complex structure $\varphi_t^* J_M$ with the standard complex srtucture $J_{\mathbf{C}^n}$ on $\hat{B}\left(0, \bar{c}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}\right)$. However we can estimate the difference as follows.

Let us denote $\{x^1, ..., x^n, x^{n+1}, ...x^{2n}\} = \{x^1, ..., x^n, y^1, ...y^n\}$ as real coordinates for $\hat{B}(0, \bar{c}(t+1)^{\frac{\varepsilon}{1+\varepsilon}})$. We then write

$$\varphi_t^* J_M = J_j^i(x,t)\frac{\partial}{\partial x^i} \otimes dx^j \qquad and \qquad J_{\mathbf{C}^n} = \hat{J}_j^i \frac{\partial}{\partial x^i} \otimes dx^j \ .$$

By definition,$J_j^i(x,t)$ is just the representation of the complex structure J_M in the normal coordinate at x_0,and $J_j^i(0,t) = \hat{J}_j^i(0)$. Denote ∇^t and $\Gamma_{ij}^{t_k}$ by the covariant derivative and Christoffel symboles with respect to the pull back metric $g_{ij}^*(\cdot, t)$of $g_{\alpha\bar{\beta}}(\cdot, t)$ on $\hat{B}\left(0, \bar{c}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}\right)$.

Set

$$H_k^j = \hat{J}_k^j - J_k^j \ .$$

By the kahlerity of J_M, we have

$$x^i \nabla_i^t H_k^j = x^i \Gamma_{ip}^{t_j} \hat{J}_k^p - x^i \Gamma_{ik}^{t_p} \hat{J}_p^j \ . \tag{3.18}$$

Since the metric $g_{ij}^*(\cdot, t)$ is actually the representation of $g_{\alpha\bar{\beta}}(\cdot, t)$ in geodesic coordinates, it follows from the Gauss Lemma (see also Lemma 4.1 of [Ha2]) that

$$g_{ij}^* x^i = \delta_{ij} x^i \ , \qquad on \quad \hat{B}(0, \bar{c}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}) \ . \tag{3.19}$$

As in [Ha2], we introduce the symmetric tensor

$$A_{ij} = \frac{1}{2} x^k \frac{\partial}{\partial x^k} g_{ij}^* \ .$$

By (3.19), we get

$$x^k \frac{\partial}{\partial x^i} g_{jk}^* = \delta_{ij} - g_{ij}^* = x^k \frac{\partial}{\partial x^j} g_{ik}^* \ ,$$

and hence from the formula for $\Gamma^{t_k}_{ij}$, it follows

$$x^j\Gamma^{t_i}_{jk} = g^{*il}A_{kl} \ . \tag{3.20}$$

Combining (3.17), (3.18) and (3.20), we get

$$\left|x^i\nabla^t_i H^j_k\right|_t \leq C_6\left|A_{pq}\right|_t \ , \qquad on \quad \hat{B}\left(0,\bar{c}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}\right) \ , \tag{3.21}$$

where C_6 is a positive constant depending only on n.

The following lemma due to Hamilton [Ha2] gives an estimate for $|A_{pq}|$.

Lemma 3.2 There exist constants $c > 0$ and $C_0 < \infty$ such that if the metric g_{ij} is in geodesic coordinates with $|R_m| \leq B_0$ in the ball of radius $r \leq \frac{c}{\sqrt{B_0}}$, then

$$|A_{ij}| \leq C_0 B_0 r^2 \ .$$

Proof. This is Theorem 4.5 of [Ha2]. *Q.E.C.*

In our context, this lemma implies

$$\left|x^i\nabla^t_i H^j_k\right|_t \leq C_7\left|x\right|^2(t+1)^{-\frac{2\varepsilon}{1+\varepsilon}} \ , \qquad on \quad \hat{B}(0,\bar{c}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}) \ , \tag{3.22}$$

for some positive contant C_7 depending only on ε, C_1, C_2 and n.

Set

$$M(r) = \sup_{\{|x|\leq r\}}\left|H^j_k\right|_t \ .$$

Suppose in the ball $|x| \leq r$, $\left|H^j_k\right|_t$ achieves its maximum at x_0. Then by (3.22) and the fact that $H^j_k(0) = 0$, we have

$$\begin{aligned} M(r)^2 &= \int_0^1 \frac{\partial}{\partial s}\left|H^j_k(sx^1_0,\cdots,sx^{2n}_0)\right|^2_t ds \\ &= 2\int_0^1 \langle H^j_k(sx^1_0,\cdots,sx^{2n}_0), x^i_0\nabla^t_i H^j_k(sx^1_0,\cdots,sx^{2n}_0)\rangle_t ds \\ &\leq M(r)\cdot C_7 r^2(t+1)^{-\frac{2\varepsilon}{1+\varepsilon}} \ . \end{aligned}$$

Therefore we get the following estimate for the difference of the two complex structures $\varphi_t^* J_M$ and $J_{\mathbf{C}^n}$,

$$M(r) \leq C_7 r^2 (t+1)^{-\frac{2\varepsilon}{1+\varepsilon}} \qquad on \quad \hat{B}(0, \bar{c}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}) \ . \tag{3.23}$$

Since $z^\alpha(\alpha = 1, 2, ...n)$ are holomorphic with respect to $J_{\mathbf{C}^n}$, we get from (3.17) and (3.23) that

$$\begin{aligned}
\left|\bar{\partial}^t z^\alpha\right|_t &\leq \sum_i \left|\frac{\partial}{\partial x^i} z^\alpha + \sqrt{-1}\varphi_t^* J_M(\frac{\partial}{\partial x^i}) z^\alpha\right|_t \\
&\leq \sum_i \left|\frac{\partial}{\partial x^i} z^\alpha + \sqrt{-1} J_{\mathbf{C}^n}(\frac{\partial}{\partial x^i}) z^\alpha\right|_t + \sum_i \left|(\varphi_t^* J_M - J_{\mathbf{C}^n})(\frac{\partial}{\partial x^i}) z^\alpha\right|_t \\
&= \sum_i \left|(\varphi_t^* J_M - J_{\mathbf{C}^n})(\frac{\partial}{\partial x^i}) z^\alpha\right|_t \\
&\leq C_8 \left|z\right|^2 (t+1)^{-\frac{2\varepsilon}{1+\varepsilon}} \ ,
\end{aligned} \tag{3.24}$$

for some positive constant C_8 depending only on ε, C_1, C_2 and n.

For any fixed $t \geq 1$, we define

$$r(t) = (\frac{\bar{c}}{2})^{\frac{1}{2}} (t+1)^{\frac{\varepsilon}{2(1+\varepsilon)}}$$

and consider the $\bar{\partial}-$equation

$$\bar{\partial}\xi^\alpha(z, t) = \bar{\partial} z^\alpha, \qquad z \in \hat{B}(0, r(t)), \quad \alpha = 1, 2, \cdots, n \ . \tag{3.25}$$

By (3.24), we know

$$\left|\bar{\partial}^t z^\alpha\right|_t \leq C_8 (\frac{\bar{c}}{2})(t+1)^{-\frac{\varepsilon}{1+\varepsilon}} \ , \qquad on \quad \hat{B}(0, r(t)) \ , \quad \alpha = 1, 2, \cdots, n \ . \tag{3.26}$$

When the positive constant $\bar{c}$ is chosen small enough, the estimates (3.11)-(3.16) imply the metric $g_{ij}^*(\cdot, t)$ can be arbitrarily close to the standard Kähler metric on $B\left(0, \bar{c}(t+1)^{\frac{\varepsilon}{1+\varepsilon}}\right)$. Thus the sphere $\partial\hat{B}(0, r(t))$ must be strictly convex with respect to the metric $g_{ij}^*(\cdot, t)$ (i.e. the second fundamental form of $\partial\hat{B}(0, r(t))$ with respect to metric $g_{ij}^*(\cdot, t)$ is strictly positive).

Then by using L^2 estimate theory for $\bar{\partial}-$operator (see the book of Hörmander [Hö]), from (3.6), (3.11)-(3.17) and (3.26) we know that (3.25) has smooth solutions $\{\xi^\alpha(z,t)|\alpha=1,2,...n\}$ such that for $\alpha=1,2,\cdots,n$,

$$|\xi^\alpha(z,t)| \leq \frac{C_9}{r(t)} \tag{3.27}$$

and

$$\left|\hat{\nabla}\xi^\alpha(z,t)\right| \leq \frac{C_9}{r(t)^2} \tag{3.28}$$

on $\hat{B}(0,r(t))$, where C_9 is a positive constant depending only on ε, C_1, C_2 and n.

We now define a new map $\Phi_t=(\Phi_t^1,\Phi_t^2,...\Phi_t^n):\hat{B}(0,r(t))\to \mathbf{C}^n$ by

$$\Phi_t^\alpha = z^\alpha - \xi^\alpha(z,t) \ , \qquad \alpha=1,2,\cdots,n \ .$$

The equation (3.25) says that Φ_t is holomorphic from $\hat{B}(0,r(t))$ equipped with the pull back complex structure $\varphi_t^* J_M$ to $\mathbf{C}^n$ equipped with standard complex structure. When t is large enough , the estimate (3.27) and (3.28) imply Φ_t is a diffeomorphism from $\hat{B}(0,r(t))$ to $\Phi_t(\hat{B}(0,r(t)))(\subset\mathbf{C}^n)$ and

$$\hat{B}\left(0,\frac{1}{2}r(t)\right) \subset \Phi_t(\hat{B}(0,r(t))) \subset \hat{B}(0,2r(t)) \ . \tag{3.29}$$

Thus the map $\Phi_t\circ\varphi_t^{-1}$ is a holomorphic and injective map from $B_t(x_0,r(t))(\subset M)$ to $\mathbf{C}^n$ (equipped with the standard complex structure). It follows direcly from (3.29) that the image of the ball $B_t(x_0,r(t))$ of radius $r(t)$ with respect to the metric $g_{\alpha\bar{\beta}}(\cdot,t)$ contains the Euclean ball $\hat{B}\left(0,\frac{r(t)}{2}\right)(\subset\mathbf{C}^n)$.

Denote

$$\Omega_t = \left(\Phi_t\circ\varphi_t^{-1}\right)^{-1}\left(\hat{B}\left(0,\frac{r(t)}{2}\right)\right) \ .$$

As noted before, the Ricci flow (2.1) is shrinking, so by the virtue of (3.29) , for any t,

$$\Omega_t \supset B_t\left(x_0,\frac{r(t)}{4}\right) \supset B_0\left(x_0,\frac{r(t)}{4}\right) \ .$$

Hence there exists a sequence of $t_k \to +\infty$, as $k \to \infty$, such that

$$M = \bigcup_{k=1}^{\infty} \Omega_{t_k} \qquad and \qquad \Omega_{t_1} \subset \quad \Omega_{t_2} \subset \cdots .$$

Since each $\Omega_{t_k}(k = 1, 2, ...)$ is biholomorphic to the unit ball of $\mathbf{C}^n$, $(\Omega_{t_k}, \Omega_{t_l})$ is Runge pair for any k, l. Then we can appeal to a theorem of Markoe [Ma] (see also Siu [Si]) to conclude that M is a Stein manifold. This completes the proof of Theorem1.1.

Note that $\Omega_{t_k}(k = 1, 2, ...)$ are a sequence of exhaustion domains of M such that each of them is biholomorphic to the unit ball of $\mathbf{C}^n$. Then by the Steiness of M, one can repeat the gluing argument of Shi in Section 9 of [Sh4] to construct a biholomorphic map from M to a pseudoconvex domain of $\mathbf{C}^n$. In his paper [Sh4], Shi assumed the positivity of Riemannian sectional curvature of M to ensure the Steiness of the manifold. Thus we get the proof for the part (2) of Theorem1.2 .

Therefore we have completed the proof of Theorem 1.1 and Theorem1.2.

4. A Gap Theorem

In this section we are interested in the question that how much the curvature could have near the infinities for complete noncompact Kähler manifolds with nonnegative bisectional curvature. The curvature behavior of a complete Riemannian surface is quite arbitrary. For example, it is easy to construct complete metrics on $\mathbf{R}^2$ from surface of revolution such that their curvatures are zero outside some compact set, nonnegative everywhere, and positive somewhere. It is surprising that the corresponding situation can not occur for higher dimentions. The isometrically embedding theorem of Mok, Siu and Yau stated before implies that there cannot too less positive bisectional curvature near the infinity for a (complex) $n-$dimensional Kähler manifold with $n \geq 2$. Recently, Ni [N] extended the Mok-Siu-Yau's result to some nonmaximal volume growth Kähler manifolds. Here we give a further generalization as follows:

Theorem 4.1 Suppose M is a complete noncompact Kähler manifold of complex dimension $n \geq 2$ with bounded and nonnegative holomorphic bisectional curvature. Suppose there exists a positive function $\varepsilon : \mathbf{R} \to \mathbf{R}$ with $\lim_{r\to+\infty} \varepsilon(r) = 0$, such that for any x_0,

$$\frac{1}{Vol(B(x_0,r))}\int_{B(x_0,r)} R(x)dv \leq \frac{\varepsilon(r)}{r^2} \ .$$

Then M^n is isometrically biholomorphic to a flat complete Kähler manifold.

Proof. Suppose the metric in the theorem is $\tilde{g}_{\alpha\bar{\beta}}(\cdot)$. Consider the Ricci flow

$$\begin{cases} \dfrac{\partial}{\partial t} g_{\alpha\overline{\beta}}(x,t) = -R_{\alpha\overline{\beta}}(x,t) \ , & x \in M \ , \ t > 0 \ , \\ g_{\alpha\bar{\beta}}(x,0) = \tilde{g}_{\alpha\bar{\beta}}(x) \ , & x \in M \ . \end{cases} \tag{4.1}$$

Let $F(x,t)$ be the nonpositive function defined in (2.2). In Section 2 we have known that $F(x,t)$ satisfies the following two differential inequalities

$$e^{F(x,t)}\frac{\partial F(x,t)}{\partial t} \geq \Delta_0 F(x,t) - R(x,0) \ , \tag{4.2}$$

and

$$\Delta_0 F(x,t) \leq R(x,0) \ , \qquad x \in M \ , \quad t \geq 0 \ , \tag{4.3}$$

where $\triangle_0$ is the Laplace operator with respect to the initial metric $\tilde{g}_{\alpha\bar{\beta}}$.

Exactly as in the proof of (2.18), we can get from (4.2) that for any $x_0 \in M$ and any $a > 0$,

$$\frac{1}{Vol(B_0(x_0,a))}\int_{B_0(x_0,a)} F(x,t)dV_0 \geq -\frac{C_1 t}{a^2}(1 - F_{\min}(t))^2 \ , \tag{4.4}$$

where C_1 is a positive constant depending only on n and the function $\varepsilon(r)$.

While by using (4.3), a slight modification of the proof for (2.15) gives the estimate

$$F(x_0,t) \geq -\tilde{\varepsilon}(a)\log(2+a) + \frac{C_2}{Vol(B_0(x_0,a))}\int_{B_0(x_0,a)} F(x,t)dV_0 \qquad (4.5)$$

for any $a > 0$, $x_0 \in M$ and $t > 0$. Here $\tilde{\varepsilon}(r)$ is some continuous positive function with $\lim\limits_{r\to+\infty}\tilde{\varepsilon}(r) = 0$ and C_2 is a positive constant depending only on n and the function $\varepsilon(r)$.

Choose $x_0 \in M$ such that $F(x_0,t) \leq \frac{1}{2}F_{\min}(t)$. We get from (4.4) and (4.5) that for any $a > 0$,

$$F_{\min}(t) \geq -2\tilde{\varepsilon}(a)\log(a+2) - \frac{2C_1C_2t}{a^2}(1-F_{\min}(t))^2 \ . \qquad (4.6)$$

By taking $a = (2+t)(1-F_{\min}(t))$, we get

$$F_{\min}(t) + \log(1-F_{\min}(t)) \geq -C_3\left[\tilde{\tilde{\varepsilon}}(t)\log(2+t)+1\right] \qquad (4.7)$$

where C_3 is a positive constant depending only on n and $\varepsilon(r)$, $\tilde{\tilde{\varepsilon}}(t)$ is a positive function with $\lim\limits_{r\to+\infty}\tilde{\tilde{\varepsilon}}(r) = 0$.

In particular, it follows from (4.7)

$$\lim_{t\to+\infty}\frac{F_{\min}(t)}{\log(2+t)} = 0 \ . \qquad (4.8)$$

On the other hand , by the differential Harnack inequality of Cao [Ca] as before, we have

$$\begin{aligned} -F(x,2t) &= \int_0^{2t} R(x,s)ds \\ &\geq tR(x,t)\int_t^{2t}\frac{1}{s}ds \\ &= (\log 2)\cdot tR(x,t) \ . \end{aligned} \qquad (4.9)$$

The combination of (4.8) and (4.9) implies that the solution $g_{\alpha\bar{\beta}}(\cdot,t)$ of the Ricci flow (4.1) exists for all time $t > 0$.By using the differential Harnack inequality again, we get for $t > 1$,

$$\begin{aligned} -F(x,t) &= \int_0^t R(x,s)ds \\ &\geq \int_{\sqrt{t}}^t R(x,s)ds \\ &\geq \sqrt{t}R\left(x,\sqrt{t}\right)\int_{\sqrt{t}}^t \frac{1}{s}ds \\ &= \left(\frac{1}{2}\log t\right)\cdot\left(\sqrt{t}R\left(x,\sqrt{t}\right)\right) \ , \end{aligned} \tag{4.10}$$

which together with (4.8) implies

$$\lim_{t\to\infty}\sqrt{t}R\left(\cdot,\sqrt{t}\right) = 0 \ , \tag{4.11}$$

While $\sqrt{t}R\left(\cdot,\sqrt{t}\right)$ is nondecreasing in time by the differential Harnack inequality, therefore we conclude that

$$R(x,t) = 0 \qquad on \quad M\times[0,+\infty) \ .$$

This completes the proof of the theorem. *Q.E.C.*

5. A Remark on Shi's Flat Metric

In [Sh3], Shi studied the following Ricci flow on Kähler manifolds,

$$\begin{cases} \dfrac{\partial}{\partial t}g_{\alpha\overline{\beta}}(x,t) = -R_{\alpha\overline{\beta}}(x,t) \ , & x\in M \ , \ t>0 \ , \\ g_{\alpha\bar{\beta}}(x,0) = \tilde{g}_{\alpha\bar{\beta}}(x) \ , & x\in M \ . \end{cases} \tag{5.1}$$

where $(M,\tilde{g}_{\alpha\bar{\beta}})$ is a complete noncompact Kähler manifold of complex dimension n with positive holomorphic bisectional curvature.

He introduced the following Harnack quantity

$$Q(x,t) = \left\{1 + g^{\alpha\bar{\delta}}(x,t)g^{\gamma\bar{\beta}}(x,t)g^{\xi\bar{\varsigma}}(x,0)\tilde{\nabla}_\xi g_{\alpha\bar{\beta}}\tilde{\nabla}_{\bar{\varsigma}}g_{\gamma\bar{\delta}}(x,t)\right\}^{\frac{1}{2}} \ ,$$

where $\tilde{\nabla}$ is the covariant derivative with respect to the initial metric $\tilde{g}_{\alpha\bar{\beta}}$. By a direct computation, one can get

$$\frac{\partial Q}{\partial t} \leq C(n) \left| \nabla_\gamma R_{\alpha\bar{\beta}} \right|_t \ , \tag{5.2}$$

where $|\cdot|_t$ is the norm with respect to the evolving metric $g_{\alpha\bar{\beta}}(\cdot, t)$. As seen in the previous sections, under the assumptions that for a fixed $x_0 \in M$,

$$\begin{array}{ll} (i) & Vol(B_0(x_0, r)) \geq C_1 r^{2n} \ , \quad 0 \leq r < +\infty \ , \\ (ii) & \quad R(x,0) \leq \frac{C_2}{1+d(x_0,x)^2} \ , \qquad x \in M \ , \end{array}$$

the Ricci flow (5.1) exists for all time $t \geq 0$. Furthermore Shi [Sh3] proved the quantity $Q(x,t)$ is uniformly bounded on $M \times [0, +\infty)$.

Let $S(M)$ be the sphere bundle with respect to the initial metric $\tilde{g}_{\alpha\bar{\beta}}$. Since the holomorphic bisectional curvature is positive , the holonomy group is transitive on $S(M)$. For any two elements $(x,v),(y,w) \in S(M)$, suppose $\gamma : [0,1] \to M$ is a piecewise smooth curve such that $\gamma(0) = x, \gamma(1) = y$ and $V(s)$ is a parallel vector field along γ such that $V(0) = v, V(1) = w$. At the smooth point $\gamma(s)$ of γ, let

$$\theta(s) = \frac{\gamma'(s)}{|\gamma'(s)|}$$

be the unit tangent vector of γ at $\gamma(s)$.

Since Q is bounded , it follows from the definition of Q that

$$\begin{aligned} \left| \tilde{\nabla}_{\theta(s)} \log g_{V(s)\bar{V}(s)}(\gamma(s), t) \right|_0^2 &= \frac{1}{[g_{V(s)\bar{V}(s)}(\gamma(s),t)]^2} \left[\tilde{\nabla}_{\theta(s)} g_{V(s)\bar{V}(s)}(\gamma(s), t) \right]^2 \\ &\leq const \ . \end{aligned}$$

Integrating from $\gamma(0)$ to $\gamma(1)$ along γ, we get

$$\left| \log \frac{g_{w\bar{w}}(y,t)}{g_{v\bar{v}}(x,t)} \right| \leq const \cdot |\gamma| \ , \tag{5.3}$$

where $|\gamma|$ is the length of γ with respect to the initial metric $\tilde{g}_{\alpha\bar{\beta}}$.

We now fix a point $(x_0, v_0) \in S(M)$ and define

$$U(t) = g_{v_0\bar{v}_0}(x_0, t) \ , \qquad 0 \le t < +\infty$$

and

$$\hat{g}_{\alpha\bar{\beta}}(x,t) = \frac{1}{U(t)} \ g_{\alpha\bar{\beta}}(x,t) \ , \qquad on \quad M \times [0,\infty) \ .$$

Since Ricci flow (5.1) is shrinking, we have $U(t) \le 1$ for $t \in [0, +\infty)$. Thus the estimates (3.5),(3.6) on the curvature of $\hat{g}_{\alpha\bar{\beta}}(x,t)$ and its derivatives still hold . Combining with (5.3) one can choose a sequence of times $t_k \to +\infty$ such that

$$\hat{g}_{\alpha\bar{\beta}}(\cdot, t_k) \to \hat{g}_{\alpha\bar{\beta}}(x, \infty) \ ,$$

in the C^∞_{loc} topology of M, where $\hat{g}_{\alpha\bar{\beta}}(x,\infty)$ is a smooth flat metric on M (by the decay estimate(3.5)). Clearly one cannot ensure the completeness for the limit metric $\hat{g}_{\alpha\bar{\beta}}(x,\infty)$.

¿From the above argument, one can see that once we have a bound on the Harnack quantity Q, then we can construct a flat Kähler metric on the manifold. However, in order to get a bound on Q, the quadratic decay assumption (ii) can be weakened to the following condition

$$(ii)' \qquad R(x,0) \le \frac{C_2}{1 + d_0(x,x_0)^{\left(\frac{3}{2}+\varepsilon\right)}} \ , \qquad x \in M \ ,$$

where ε is an arbitrarily small positive constant .

In fact by the estimate (3.5) in Section3, we have

$$\left|\nabla_\gamma R_{\alpha\bar{\beta}}\right|_t \le C_3(t+1)^{-\frac{3+6\varepsilon}{3+2\varepsilon}} \ , \qquad on \quad M \times [0, +\infty) \ . \tag{5.4}$$

Since $Q(\cdot, 0) = 0$, we deduce from (5.2) and (5.4) that

$$Q(x,t) \le \int_0^\infty C(n) \left|\nabla_\gamma R_{\alpha\bar{\beta}}\right|_t dt \le C_4 \ ,$$

where C_4 is some positive constant depending only on ε, C_1, C_2 and n.

Hence we obtain the following result.

Proposition 5.1 Let M be a complete noncompact Kähler manifold of complex dimension n with positive holomorphic bisectional curvature. Suppose the maximal volume growth condition (i) and $\frac{3}{2}-$decay assumption (ii)$'$ hold for M. Then with respect to the complex structure we can construct a flat Kähler metric on M .

References

[Ca] H.D. Cao, *On Harnack's inequalities for the Kähler-Ricci flow*, Invent. Math. **109** (1992) 247-263.

[CG] J.Cheeger and D.Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann.of Math. **96** (1972), 413-443.

[CGT] J.Cheeger, M.Gromov, and M.Taylor, *Finite propagation speed,kernal estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifold*, J. Diff. Geom. **17** (1982), 15-53.

[F] M.H.Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), 357-453.

[Gr] H.Grauert, *On Levi's problem and the embedding of real-analytic manifolds*, Ann. of Math. **68** (1958),460-472.

[GW] R.E.Green and H.Wu, C^∞ *convex functions and manifolds of positive curvature*, Acta Math. **137** (1976),209-245.

[GM] D.Gromoll and W.Meyer, *On complete open manifolds of positive curvature*, Ann. of Math. **90** (1969),75-90.

[Ha1] R.S.Hamilton, *Formation of singularities in the Ricci flow*, surveys in Diff. Geom.Vol.**2**, Boston, International Press 1995,7-136.

[Ha2] R.S.Hamilton, *A compactness property for solutions of the Ricci flow*, Amer.J.Math. **117** (1995) 545-572.

[Hö] L.Hörmander, *An Introduction to Complex Analysis in Several Variables*, Princeton, N.J.:Van Nostrand,1966.

[Ma] A.Markoe, *Runge families and increasing unions of Sein spaces*, Bull. Amer. Math. Soc. **82** (1976),787-788.

[Mo1] N.Mok, *The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature*, J. Diff. Geom. **27** (1988), 179-214.

[Mo2] N.Mok, *An embedding theorem of complete Kähler manifolds of positive bisectional curvature onto affine algebraic varieties*, Bull. Soc. Math. France. **112** (1984),197-258.

[Mo3] N.Mok, *An embedding theorem of complex Kähler manifolds of positive Ricci curvature onto quasi-projective varieties*, Math. Ann. **286** (1990), No.1-3, 373-408.

[MSY] N.Mok, Y.T.Siu and S.T.Yau, *The Poincaré-Lelong equation on complete Kähler manifolds*, Comp.Math., Vol.**4**., Fasc.1-3(1981), 183-218.

[N] L.Ni, *Vanishing theorems on complete Kähler manifolds and their applications*, J.Diff.Geom. **50** (1998) 89-122.

[R] C.D.Ramanujam, *A topological characterization of the affine plane as an algebraic variety*, Ann.of Math. **94** (1971),69-88.

[SY] R.Schoen and S.T.Yau, *Lectures on differential geometry*, in conference proceedings and Lecture Notes in Geometry and Topology, Volume **1**, International Press Publications, 1994.

[Sh1] W.X.Shi, *Deforming the metric on complete Riemannian manifolds*, J. Diff. Geom. **30** (1989), 223-301.

[Sh2] W.X.Shi, *Complete noncompact Kähler manifolds with positive bisectional curvature*, Bull. Amer.Math.Soc. **23** (1990),437-440.

[Sh3] W.X.Shi, *Ricci deformation of the metric on complete noncompact Kähler manifolds*, Ph.D. Thesis; Harvard University(1990).

[Sh4] W.X.Shi, *Ricci flow and the uniformization on complete noncompact Kähler manifolds*, J.Diff.Geom. **45** (1997),94-220.

[Si] Y.T.Siu, *Pseudoconvexity and the problem of Levi*, Bull. Amer. Math. Soc. **84** (1978), 481-512.

[Sm] S.Smale, *Generalised Poincaré's conjecture in dimension greater than four*, Ann. of Math. **74** (1961),391-466.

[To] To Wing-Keung, *Quasi-projective embeddings of noncompact complete Kähler manifolds of positive Ricci Curvature and satisfying certain topological condictions*, Duke Math. J. **63** (1991), No.3, 745-789.