# GLOBAL SHOCK WAVES FOR THE SUPERSONIC FLOW PAST A PERTURBED CONE

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**Abstract**

We prove the global existence of a shock wave for the stationary supersonic gas flow past an infinite curved and symmetric cone. The flow is governed by the potential equation, as well as the boundary conditions on the shock and the surface of the body. It is shown that the solution to this problem exists globally in the whole space with a pointed shock attached at the tip of the cone and tends to a self-similar solution under some suitable conditions. Our analysis is based on a global uniform weighted energy estimate for the linearized problem. Combining this with the local existence result of Chen-Li [1] we establish the global existence and decay rate of the solution to the nonlinear problem.

**Keywords:** Compressible Euler equations, supersonic flow, shock wave, global existence, decay rate

**Mathematical Subject Classification:** 35L70, 35L65, 35L67, 76N15

## §1. Introduction

In this paper we are concerned with the global existence of solution to the supersonic flow past a pointed body. Such a problem is a fundamental one in gas dynamics. It is also one of the basic models in studying the theory of weak solution to the quasilinear hyperbolic equations in multidimensional space. There exist extensive literatures in the study of supersonic flow past a pointed body by either physical experiments or numerical simulations. The rigorous mathematical analysis starts with the work of Courant and Friedrichs in [3], where they show that if a supersonic flow hits a circular cone with axis being parallel to the velocity of the upstream flow and the vertex angle being less than a critical value, then there appears a circular conical shock attached at the tip of the cone, and the flow field between the shock front and the surface of the body can be determined by solving a boundary value problem of an

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ordinary differential equation. Recently, the local existence of supersonic flow past a pointed body has been established by S.Chen and D.Li in the symmetric case [1], and by S.Chen in general nonsymmetric case [2]. In addition, W.C.Lien and T.P.Liu also obtained the global existence of a weak solution and long time asymptotic behaviour in the symmetric case under suitable conditions on Mach number, vertex angle and the shock strength by using the Glimm's scheme in [7]. Our main interest is on the structure of the global solution of such a problem. The goal of this paper is to establish the global existence of a shock as observed in physical experiments and numerical computations. This is achieved in the symmetric case by combining the local existence of the shock and the global uniform weighted energy estimates developed in this paper. Moreover, our method can be used to treat the general case of multidimensional perturbed cone. This result will be given in the near future.

Let us first give a brief description of our main result. The stationary inviscid flow is governed by the steady Euler system. Under the assumptions that the flow is isentropic and irrotational the system can be written as

$$\begin{cases} \sum_{j=1}^{3} \partial_j(\rho u_j) = 0, \\ \sum_{j=1}^{3} \partial_j(\rho u_i u_j) + \partial_i P = 0, i = 1,2,3 \end{cases} \tag{1.1}$$

where $\rho$, $u = (u_1, u_2, u_3)$ and $P$ stand for the density, the velocity, and the pressure respectively. For the polytropic gas, $P(\rho) = A\rho^\gamma$, here $A > 0$ and $1 < \gamma < 3$, $\gamma$ is the adiabatic exponent.

Suppose that there is a uniform supersonic flow $(u_1, u_2, u_3) = (0, 0, q_0)$ with constant density $\rho_0 > 0$ which comes from infinity. The flow hits a pointed body, whose surface is denoted by $m(x_1, x_2, x_3) = 0$. As we indicated above, if the vertex angle of the tangential cone of the pointed body is less than a given value, then there will be a pointed shock attached at the tip of pointed body. Denote by $\mu(x_1, x_2, x_3) = 0$ the equation of the shock front. Then on the surface of the body and the shock front the following boundary conditions should be satisfied. Namely,

$$u_1\partial_1 m + u_2\partial_2 m + u_3\partial_3 m = 0 \tag{1.2}$$

on $m(x_1, x_2, x_3) = 0$, and the Rankine-Hugoniot conditions become

$$\begin{cases} [\rho u_1]\partial_1\mu + [\rho u_2]\partial_2\mu + [\rho u_3]\partial_3\mu = 0, \\ \sum_{j=1}^{3}[\rho u_i u_j]\partial_j\mu + P\partial_i\mu = 0, i = 1,2,3 \end{cases} \tag{1.3}$$

on the shock front.

Since the flow is irrotational, one can deduce from (1.1) that

$$\frac{1}{2}\partial_i(|u|^2) + \partial_i h(\rho) = 0, \tag{1.4}$$

where $|u|^2 = u_1^2 + u_2^2 + u_3^2$, $h(\rho)$ is the specific enthalpy satisfying $h'(\rho) = \frac{P'(\rho)}{\rho} > 0$. For the polytropic gas,

$$P(\rho) = A\rho^\gamma, \gamma > 1, h(\rho) = \frac{A\gamma}{\gamma - 1}\rho^{\gamma-1}. \tag{1.5}$$

The integration of (1.4) is the Bernoulli's law

$$\frac{1}{2}|u|^2 + h(\rho) = \frac{1}{2}q_0^2 + h(\rho_0). \tag{1.6}$$

Since the flow is irrotational, we can introduce a potential $\Phi$ satisfying $u = \nabla\Phi$. Then by implicit function theorem we have

$$\rho = h^{-1}(\frac{1}{2}q_0^2 + h(\rho_0) - \frac{1}{2}|\nabla\Phi|^2) \equiv H(\nabla\Phi). \tag{1.7}$$

Consequently, the system (1.3) can be reduced to the following second order equation

$$\begin{aligned}&((\partial_1\Phi)^2 - c^2)\partial_{11}^2\Phi + ((\partial_2\Phi)^2 - c^2)\partial_{22}^2\Phi + ((\partial_3\Phi)^2 - c^2)\partial_{33}^2\Phi + 2\partial_1\Phi\partial_2\Phi\partial_{12}^2\Phi \\ &+ 2\partial_1\Phi\partial_3\Phi\partial_{13}^2\Phi + 2\partial_2\Phi\partial_3\Phi\partial_{23}^2\Phi = 0.\end{aligned} \tag{1.8}$$

where $c^2(\rho) = P'(\rho) = \dfrac{H(\nabla\Phi)}{H'(\nabla\Phi)}$. It is easy to verify that (1.8) is strictly hyperbolic with respect to $x_3$ if $\partial_3\Phi > c$ . (1.2) and (1.3) yield the boundary conditions for $\Phi$. On the surface of the body $m(x_1, x_2, x_3) = 0$, $\Phi$ satisfies

$$\partial_1\Phi\partial_1 m + \partial_2\Phi\partial_2 m + \partial_3\Phi\partial_3 m = 0, \tag{1.9}$$

while on the shock front $\mu(x_1, x_2, x_3) = 0$, $\Phi$ is continuous and satisfies

$$[\partial_1\Phi H(\nabla\Phi)]\partial_1\mu + [\partial_2\Phi H(\nabla\Phi)]\partial_2\mu + [\partial_3\Phi H(\nabla\Phi)]\partial_3\mu = 0. \tag{1.10}$$

This is also called the Rankine-Hugoniot condition.

Due to the geometry of the pointed body, it is convenient to work in the polar coordinates $(r, z)$, where $r = \sqrt{x_1^2 + x_2^2}, x_3 = z$. Assume that the tip of the pointed body locates at the origin, the equation of the pointed body is $r = \sigma(z)$ with $\sigma(0) = 0$ and the equation of the shock front is $r = \chi(z)$ with $\chi(0) = 0$. Set $\Phi = q_0 z + \varphi$. Then the equation (1.8) can be written as

$$((q_0 + \partial_z\varphi)^2 - c^2)\partial_{zz}^2\varphi + ((\partial_r\varphi)^2 - c^2)\partial_{rr}^2\varphi + 2\partial_r\varphi(q_0 + \partial_z\varphi)\partial_{rz}^2\varphi - \frac{c^2}{r}\partial_r\varphi = 0 \tag{1.11}$$

Meanwhile, the boundary conditions can be rewritten as

$$-(q_0 + \partial_z\varphi)\sigma'(z) + \partial_r\varphi = 0, \quad on \quad r = \sigma(z) \tag{1.12}$$

$$-[(q_0 + \partial_z\varphi)H]\chi'(z) + [\partial_r\varphi H] = 0, \quad on \quad r = \chi(z) \tag{1.13}$$

Moreover, the potential $\varphi(r, z)$ is continuous on the shock, so it should satisfy

$$\varphi(\chi(z), z) = 0 \tag{1.14}$$

The main conclusion in this paper can be summarized as:

**Theorem 1.1.** Assume that a curved and symmetric cone is given by $r = \sigma(z)$, which satisfies

$$\sigma(0) = 0, \quad \sigma'(0) = b_0, \quad \sigma^{(k)}(0) = 0 \qquad (2 \le k \le k_1), \tag{1.15}$$

$$|z^k \frac{d^k}{dz^k}(\sigma(z) - b_0 z)| \le \varepsilon_0 \qquad for \qquad 0 \le k \le k_2, z > 0, \tag{1.16}$$

where $k_1, k_2$ are suitable integers. Suppose that a supersonic polytropic flow parallel to the $z$-axis comes from infinity with velocity $q_0$, density $\rho_0 > 0$ satisfying $q_0 > c_0 = \sqrt{A\gamma}\rho_0^{\frac{\gamma-1}{2}}$. Additionally, $b_0 > 0$ is assumed to be very small and less than the critical value determined by $q_0$ and $\rho_0$. Then for the large $q_0$ and sufficiently small $\varepsilon_0$ depending on $q_0, \rho_0, b_0, \gamma, k_1$ and $k_2$, the problem (1.11)-(1.14) admits a global weak entropy solution with a pointed shock front attached at the origin. Moreover, the location of the shock front and the flow field between the shock and the surface of the body tend to the corresponding ones for the flow past the unperturbed circular cone $r = b_0 z$ with the rate $z^{-1/4}$.

**Remark 1.1.** It should be emphasized that there is no other discontinuities in our solution besides the main curved shock. This is in sharp contrast with the previous result [7]. The condition (1.15) comes from the local existence theorem ( see [1] or [2] ), and the condition (1.16) especially gives a restriction on the surface of body for large $z$. Since the perturbation of the surface of body is sufficiently small, any possible compression of the flow will be absorbed by the main shock. This is the mechanism to provent the formation of any new shock inside the flow field caused by the perturbation of the body. Thus our result demonstrats that the self-similar solution with a strong shock is structurally stable in a global sense. Such a phenomenon is contrast to the formation of singularity of smooth solution to the Cauchy problem of two or three dimensional compressible Euler equations (see [9], [11], [12] and the references therein).

**Remark 1.2.** Our result can be extended to the general case of miltidimensional perturbed cone, that is, the self-similar solution with a strong shock is still globally stable for the multidimensional stationary supersonic flow which is isentropic and irrotational. It will be given in a future publication.

To prove the theorem, we need to establish some global uniform weighted energy estimates for the linearized problem of (1.11)-(1.14). Based on such estimates we can use the standard continuity method for hyperbolic systems to obtain the existence and the asymptotic behaviour of the solution to the perturbed nonlinear problem ( for instance, see [4] or [6]). The key element in the analysis to obtain the weighted energy estimates is to look for suitable multiplier. Finding such suitable multiplier is much more involved due to the following reasons. First, in order to obtain the global existence, one needs to establish a global estimate independent of $z$ for the potential function and its derivatives on the boundary as well as in its interior of a domain. This yields stringent constraints on the multiplier and needs us to give a very delicate computation. Second, since our background solution is self-similar on a fixed domain and strongly depends on the position of boundary of the cone we must solve a system of ordinary differential inequalities with very complicated coefficients to obtain the multiplier. Here we should note that the method in [4] can not be adapted to our case, because the boundary condition on the surface of body is the Neumann type rather than the Dirichlet type which is artificial in our physical problem. Furthermore, it should be noted that the arbitrary closeness of the boundary plays a key role in the analysis of [4], which is not the case for our problem. We overcome all these difficulties to solve the system of ordinary differential inequalities by using the facts that the Mach number is large and the vertex angle of cone is small.

Our paper is organized as follows. In §2, we derive some basic estimates for the background self-similar solution, which are needed for the construction of multiplier. In §3, we reformulate the problem (1.11)-(1.14) by decomposing its solution as a sum of the background solution with a small perturbation. In §4, we establish the weighted energy estimate for the linearized problem, where the precise form of the appropriate multiplier is given . Based on this energy estimate Theorem 1.1 is proved in §5 for the special case when the body is a circular cone but initial data is perturbed. Finally, in §6 we indicate that the conclusion obtained above is also valid to the general case with some modifications. Some complicated computations and useful facts are given in the Appendix.

Notations:

$O(b_0^j)$ $(j \geq 1)$: a bounded quantity, which means that there exists a generic constant $M_0$ such that $|O(b_0^j)| \leq M_0 b_0^j$, where $M_0$ depends only on $\gamma$.

$O(q_0^{-\nu})$ $(\nu > 0)$: a bounded quantity, which means that there is a generic constant $M_1$ depending only on $b_0$ and $\gamma$ such that $|O(q_0^{-\nu})| \leq M_1 q_0^{-\nu}$.

$O(\varepsilon_0)$: a bounded quantity, which means that there exists a generic constant $M_2$ such that $|O(\varepsilon_0)| \leq M_2 \varepsilon_0$, where $M_2$ depends only on $b_0$, $q_0$ and $\gamma$.

## §2. The analysis on the self-similar solution

In this section we first discuss the solution to the case when the pointed body is a circular cone, whose equation is $r = b_0 z$. Such a solution can be obtained by using the method in [3] with replacing the shock polar by its similarity defined for the potential flow in [5]. The solution will be called as the background solution in this paper. The actual solution of the nonlinear problem discussed in this paper is a small perturbation of the background solution. Due to the requirement of proving our main theorem we need to have more information on the background solution, particularly, the estimates when the Mach number is sufficiently large. The required information will be given by several lemmas in this section.

As indicated in [3], if $b_0$ is less than a critical value $b^*$, the background solution is symmetric and self-similar, so that it can be obtained by solving a boundary value problem of ordinary differential equation. In this case the shock front is also a circular cone with equation $r = s_0 z$, while the solution between the shock front and the surface of the cone has the form: $\rho = \rho(\frac{r}{z})$, $u_1 = U(\frac{r}{z})\frac{x_1}{r}$, $u_2 = U(\frac{r}{z})\frac{x_2}{r}$, $u_3 = q_0 + W(\frac{r}{z})$, where $U$ and $W$ represent the radial and axial components of velocity respectively. In what follows, we denote $s = \frac{r}{z}$, $U_+ = \lim_{s\to s_0-0} U(s)$, $W_+ = \lim_{s\to s_0-0} W(s)$, $\rho_+ = \lim_{s\to s_0-0} \rho(s)$. Ahead of the shock, the flow is constant with the density $\rho_0$ and the velocity $(u_r, u_z) = (0, q_0)$, and behind the shock the flow is characterized by $\rho$ and $(u_r, u_z) = (U, q_0 + W)$. In the following Lemma 2.1 it will be proved that $c^2(\rho)(1+s^2) - (s(q_0+W) - U)^2 \neq 0$ for $b_0 \le s \le s_0$. Consequently, the system (1.1) can be reduced to

$$\begin{cases} \rho'(s) = -\dfrac{\rho U(s(q_0+W)-U)}{s(c^2(\rho)(1+s^2)-(s(q_0+W)-U)^2)} \\ U'(s) = -\dfrac{c^2(\rho)U}{s(c^2(\rho)(1+s^2)-(s(q_0+W)-U)^2)} \\ W'(s) = \dfrac{c^2(\rho)U}{c^2(\rho)(1+s^2)-(s(q_0+W)-U)^2} \end{cases} \qquad b_0 \le s \le s_0. \tag{2.1}$$

The parameters of the flow satisfy the boundary condition

$$U = b_0(q_0 + W) \qquad on \qquad s = b_0. \tag{2.2}$$

Moreover, they also satisfy the Rankine-Hugoniot conditions and entropy conditions on the shock $s = s_0$, namely

$$\begin{cases} [\rho U] - s_0[\rho(q_0+W)] = 0, \\ [\frac{1}{2}U^2 + \frac{1}{2}(q_0+W)^2 + h(\rho)] = 0, \\ [q_0 + W] + s_0[U] = 0, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \lambda_1(U_+, W_+) < s_0 < \lambda_2(U_+, W_+), \\ \dfrac{c(\rho_0)}{\sqrt{q_0^2 - c^2(\rho_0)}} < s_0, \end{cases} \tag{2.4}$$

where $\lambda_{1,2}(U,W) = \frac{U(q_0+W)\mp c(\rho)\sqrt{U^2+(q_0+W)^2-c^2(\rho)}}{(q_0+W)^2-c^2(\rho)}$. It is shown that the boundary value problem (2.1)-(2.4) can be solved by the shooting method. Corresponding to the given data of the incoming flow $(\rho_0, q_0)$ one can draw an apple curve on $(u_z, u_r)$ plane. The apple curve takes $(q_0, 0)$ as its intersection, and plays the similar role like the shock polar in the study of oblique shock wave. For any given small $b_0$, we can determine the solution by using the intersection of the apple curve and the ray starting from the origin with the slope $b_0$ (the details can see [3] or [5]).

**Lemma 2.1.** Denoting $\lambda_k(s) = \lambda_k(U(s), W(s))$, $k = 1, 2$. If $q_0 + W > c(\rho)$, then we have
(i) $U > 0$ and $c^2(\rho)(1+s^2) - (s(q_0+W) - U)^2 \ge c(\rho_+)(1+b_0^2)(c(\rho_+) - \frac{s_0(q_0+W_+)-U_+}{\sqrt{1+s_0^2}}) > 0$.

(ii) $\lambda_2(s) > s_0$.

**Proof.** (i) From (2.3) we have

$$\begin{cases} U_+ = \frac{s_0 q_0(\rho_+ - \rho_0)}{(1+s_0^2)\rho_+}, \\ W_+ = -\frac{s_0^2 q_0(\rho_+ - \rho_0)}{(1+s_0^2)\rho_+}, \\ h(\rho_+) - h(\rho_0) - \frac{s_0^2 q_0^2(\rho_+^2 - \rho_0^2)}{2(1+s_0^2)\rho_+^2} = 0. \end{cases} \tag{2.5}$$

Obviously, $U_+ > 0$ due to the entropy condition. Noticing that $\frac{s_0(q_0+W_+)-U_+}{\sqrt{1+s_0^2}}$ is equal to the component of the velocity normal to the shock front, we have

$$s(q_0 + W(s)) - U(s) > 0, c(\rho(s)) > \frac{s(q_0 + W(s)) - U(s)}{\sqrt{1+s^2}} \tag{2.6}$$

for $s = s_0$ ( the inequalities can also be verified directly ). Hence by the continuity of $\rho(s), U(s), W(s)$, (2.6) is also valid in $s_0 - \delta_0 \le s \le s_0$ with small $\delta_0$, and then (2.1) makes sense in this interval. Due to (2.1) $\rho'(s) < 0, U'(s) < 0$ and $W'(s) > 0$ in $s_0 - \delta_0 \le s \le s_0$, we know that the function $c(\rho(s)) - \frac{s(q_0+W(s))-U(s)}{\sqrt{1+s^2}}$ is a decreasing function of $s$. Then we can conclude in $s_0 - \delta_0 \le s \le s_0$: $U(s) \ge U_+$, $\rho(s) \ge \rho_+$, and

$$\begin{aligned} & c^2(\rho(s))(1+s^2) - (s(q_0 + W(s)) - U(s))^2 \\ &= (1+s^2)(c(\rho(s)) - \frac{s(q_0 + W(s)) - U(s)}{\sqrt{1+s^2}})(c(\rho(s)) + \frac{s(q_0 + W(s)) - U(s)}{\sqrt{1+s^2}}) \\ &\ge c(\rho_+)(1+b_0^2)(c(\rho_+) - \frac{s_0(q_0 + W_+) - U_+}{\sqrt{1+s_0^2}}) > 0. \end{aligned} \tag{2.7}$$

The solution of the system (2.1) will blow up only in the case when the denominator tends to zero. However, (2.7) means that the denominator is bounded away from zero as long as the solution of (2.1) exists. Therefore, (2.7) holds in the whole interval $[b_0, s_0]$, and the solution of (2.1) exists there, which satisfies

$$U'(s) < 0, W'(s) > 0, \rho'(s) < 0.$$

Moreover by a direct computation, we know that $c(\rho(s)) - \frac{s(q_0+V(s))-U(s)}{\sqrt{1+s^2}}$ is a decreasing function in $b_0 \le s \le s_0$. Hence we complete the proof of (i).

(ii) Since $\lambda_2'(s) = \frac{\partial \lambda_2}{\partial U}U'(s) + \frac{\partial \lambda_2}{\partial W}W'(s) + \frac{\partial \lambda_2}{\partial c}c'(\rho)\rho'(s)$, and

$$\begin{aligned} \frac{\partial \lambda_2}{\partial U} &= \frac{1}{(q_0+W)^2 - c^2}(q_0 + W + \frac{cU}{\sqrt{U^2 + (q_0+W)^2 - c^2}}) > 0, \\ \frac{\partial \lambda_2}{\partial W} &= -\frac{1}{((q_0+W)^2 - c^2)^2}\left\{U((q_0+W)^2 - c^2) + \frac{2cU^2(q_0+W) + c(q_0+W)((q_0+W)^2 - c^2)}{\sqrt{U^2 + (q_0+W)^2 - c^2}}\right\} < 0, \\ \frac{\partial \lambda_2}{\partial c} &= \frac{2cU(q_0+W)}{((q_0+W)^2 - c^2)^2} + \frac{\sqrt{U^2 + (q_0+W)^2 - c^2}}{(q_0+W)^2 - c^2} + \frac{2c^2U^2 + c^2((q_0+W)^2 - c^2)}{((q_0+W)^2 - c^2)\sqrt{U^2 + (q_0+W)^2 - c^2}} > 0, \end{aligned}$$

then by $U'(s) < 0, W'(s) > 0$ and $\rho'(s) < 0$, we obtain $\lambda_2'(s) > 0$ for $b_0 \le s \le s_0$. In light of the entropy condition (2.4), one gets $\lambda_2(s) \ge \lambda_2(s_0) > s_0$.

In this paper we are mainly concerned with the case when the Mach number is large, so next we estimate the solution of (2.1) by using the expression of the power of $q_0$ for large $q_0$. Such estimates will play an important role in the discussion of the following sections.

**Lemma 2.2.** If $q_0$ is large, that is, the Mach number of coming flow is large, and $1 < \gamma < 3$, $0 < b_0 < \sqrt{2} - 1$, then

(i) $s_0 = b_0 + O(q_0^{-\frac{2}{\gamma-1}})$,

(ii) $0 \le s(q_0 + W) - U \le O(q_0^{\frac{\gamma-3}{\gamma-1}})$,

(iii) $U(s) = \frac{b_0 q_0}{1+b_0^2} + O(q_0^{\frac{\gamma-3}{\gamma-1}})$,

(iv) $q_0 + W(s) = \frac{q_0}{1+b_0^2} + O(q_0^{\frac{\gamma-3}{\gamma-1}})$,

(v) $q_0 + W(s) - c(\rho(s)) \ge \frac{q_0(1-b_0^2(1+b_0^2))}{(1+b_0^2)(1+b_0\sqrt{1+b_0^2})} + O(\frac{1}{q_0}) + O(q_0^{\frac{\gamma-3}{\gamma-1}}) > 0$.

**Proof.** (i) From the third equation in (2.5), we have

$$\frac{A\gamma}{\gamma-1}(\rho_+^{\gamma-1} - \rho_0^{\gamma-1}) = \frac{s_0^2 q_0^2}{2(1+s_0^2)}(1 - (\frac{\rho_0}{\rho_+})^2)$$

Denoting $\alpha = \frac{\rho_+}{\rho_0}$, we have

$$\alpha^{\gamma-1} = 1 + \frac{s_0^2 \rho_0^{1-\gamma}(\gamma-1)q_0^2}{2A\gamma(1+s_0^2)}(1 - \frac{1}{\alpha^2}),$$

or

$$\alpha^2 \frac{\alpha^{\gamma-1} - 1}{\alpha^2 - 1} = \frac{s_0^2 \rho_0^{1-\gamma}(\gamma-1)q_0^2}{2A\gamma(1+s_0^2)}. \tag{2.8}$$

The left hand side of the above equality is bounded if $\alpha$ is bounded. Therefore, for large $q_0$ we obviously have $\alpha > 2$ and $1 - \frac{1}{\alpha^2} > \frac{3}{4}$. The fact implies $\alpha = \left(\frac{s_0^2\rho_0^{1-\gamma}(\gamma-1)}{2A\gamma(1+s_0^2)}\right)^{\frac{1}{\gamma-1}} q_0^{\frac{2}{\gamma-1}}(1 + O(\frac{1}{q_0^2}))$, hence

$$U_+ = \frac{s_0 q_0}{1+s_0^2}(1 - \frac{1}{\alpha}) = \frac{s_0 q_0}{1+s_0^2} + O(q_0^{\frac{\gamma-3}{\gamma-1}}), \tag{2.9}$$

$$q_0 + W_+ = \frac{q_0(s_0^2\rho_0 + \rho_+)}{(1+s_0^2)\rho_+} = \frac{q_0}{1+s_0^2} + O(q_0^{\frac{\gamma-3}{\gamma-1}}). \tag{2.10}$$

Furthermore, from

$$U_+ \le U(s) \le U(b_0) = b_0(q_0 + W(b_0)) \le b_0(q_0 + W_+), \tag{2.11}$$

we have

$$\frac{s_0 q_0}{1+s_0^2} + O(q_0^{\frac{\gamma-3}{\gamma-1}}) \le \frac{b_0 q_0}{1+s_0^2} + O(q_0^{\frac{\gamma-3}{\gamma-1}}), \tag{2.12}$$

which leads to $s_0 \le b_0 + O(q_0^{-\frac{2}{\gamma-1}})$.

(ii) From the argument in Lemma 2.1 we know $(s(q_0 + W(s)) - U(s))$ is an increasing function. Therefore,

$$0 \le s(q_0 + W) - U \le s_0(q_0 + W_+) - U_+ = \frac{s_0 q_0}{\alpha} = O(q_0^{\frac{\gamma-3}{\gamma-1}}) \tag{2.13}$$

(iii) comes from (2.9),(2.11).

(iv) comes from (2.10),(2.11) and the monotonicity of $W(s)$.

(v) According to the Bernoulli's law we have $\frac{1}{2}(U^2+(q_0+W)^2)+\frac{c^2(\rho)}{\gamma-1}=\frac{1}{2}q_0^2+h(\rho_0)$. Then $c^2(\rho)=\frac{(\gamma-1)b_0^2q_0^2}{2(1+b_0^2)}(1+O(\frac{1}{q_0^2})+O(q_0^{-\frac{2}{\gamma-1}}))$. Hence

$$
\begin{aligned}
q_0+W-c(\rho) &= \frac{q_0}{1+b_0^2}-\frac{\sqrt{\gamma-1}b_0q_0}{\sqrt{2(1+b_0^2)}}(1+O(\frac{1}{q_0^2})+O(q_0^{-\frac{2}{\gamma-1}}))+O(q_0^{\frac{\gamma-3}{\gamma-1}}) \\
&= \frac{q_0}{(1+b_0^2)}(1-b_0\sqrt{\frac{\gamma-1}{2}}\sqrt{1+b_0^2})+O(\frac{1}{q_0})+O(q_0^{\frac{\gamma-3}{\gamma-1}}) \\
&\geq \frac{q_0(1-b_0^2(1+b_0^2))}{(1+b_0^2)(1+b_0\sqrt{1+b_0^2})}+O(\frac{1}{q_0})+O(q_0^{\frac{\gamma-3}{\gamma-1}})>0
\end{aligned}
$$

Here we already used the conditions $1<\gamma<3$ and the smallness of $b_0$ in the last inequality.

**Lemma 2.3.** Under the assumptions of Lemma 2.2, we have

(i) $\lambda_1(s)<s$,

(ii) $U'(s)=-\frac{q_0}{(1+b_0^2)^2}+O(\frac{1}{q_0})+O(q_0^{\frac{\gamma-3}{\gamma-1}})$,

(iii) $W'(s)=\frac{b_0q_0}{(1+b_0^2)^2}+O(\frac{1}{q_0})+O(q_0^{\frac{\gamma-3}{\gamma-1}})$,

(iv) $|\rho'(s)|\leq C$,

for $b_0\leq s\leq s_0$, where $C$ is a constant independent of $q_0$.

**Proof.** (i) By using the expression of $\lambda_1$ and the conclusions of Lemma 2.2, we have

$$
\begin{aligned}
s-\lambda_1(s) &= b_0-\frac{\frac{b_0}{(1+b_0^2)^2}-\sqrt{\frac{(\gamma-1)b_0^2}{2(1+b_0^2)}}\sqrt{\frac{1}{1+b_0^2}-\frac{(\gamma-1)b_0^2}{2(1+b_0^2)}}}{\frac{1}{(1+b_0^2)^2}-\frac{(\gamma-1)b_0^2}{2(1+b_0^2)}}+O(q_0^{-\frac{2}{\gamma-1}}) \\
&= \frac{\sqrt{\gamma-1}b_0(1+b_0^2)}{\sqrt{2-(\gamma-1)b_0^2}+\sqrt{\gamma-1}b_0^2}+O(q_0^{-\frac{2}{\gamma-1}})>0.
\end{aligned}
$$

(ii) and (iii) can be directly derived from Lemma 2.2, here we omit the details.

(iv) Since $c^2(\rho)=A\gamma\rho^{\gamma-1}$, then $\rho=\left(\frac{(\gamma-1)b_0^2}{2A\gamma(1+b_0^2)}\right)^{\frac{1}{\gamma-1}}q_0^{\frac{2}{\gamma-1}}+O(\frac{1}{q_0^2})+O(q_0^{-\frac{2}{\gamma-1}})$. Hence by the expression of $\rho'(s)$ in (2.1) and the conclusion of Lemma 2.2 we get the boundedness of $\rho'(s)$.

## §3. The reformulation of the main problem

Since the local existence of solution to (1.11)-(1.14) has been established in [1] and [2], without loss of generality, we can study the global existence by solving an initial boundary value problem with initial data on $z=z_0$ for some $z_0$. To illustrate the main idea to obtain the global energy estimates on the linearized problem we will assume that the boundary is simply $r=b_0z$ ( but the initial data on $z=z_0$ are not the same as the background solution ). The general case will be treated in §6, where we will show the contribution of the perturbation of the boundary. Indeed, by introducing a coordinate transformation the general case can be reduced to the case discussed here with some modifications of the coefficients of the equation, while such modifications will not break down the main arguments.

According to the result in [1], the initial data on $z=z_0$ can be regarded as a small perturbation of the background solution given in §2 with the amplitude of order $O(\varepsilon_0)$. Moreover, the initial data also satisfy the compability conditions at the intersection points of $z=z_0$ with the shock front and the surface of the body. Later on, for convenience we simply assume $z_0=1$.

Since the denominator of the system (2.1) is positive in $[b_0,s_0]$, we can extend $\rho,U,W$, as well as the potential $\varphi$, to $[s_0,s_0+\eta_0]$ for small $\eta_0$ satisfying $0<\eta_0\leq q_0^{-\frac{2}{\gamma-1}}(s_0-b_0)$. Later on we will denote the extension of $\rho, U$ , $W$ and $\varphi$ in the domain $\{(r,z) : z \geq 1, b_0 z \leq r \leq (s_0+\eta_0)z\}$ by $\hat{\rho}, \hat{U}, \hat{W}$ and $\hat{\varphi}$ respectively.

Set $\dot{\varphi} = \varphi - \hat{\varphi}$, through a direct computation the equation (1.11) can be reduced to:

$$\begin{aligned}
&\partial_z^2 \dot{\varphi} + 2P_1(\frac{r}{z})\partial_{zr}^2\dot{\varphi} + P_2(\frac{r}{z})\partial_r^2\dot{\varphi} + P_3(r,z)\partial_z\dot{\varphi} + P_4(r,z)\partial_r\dot{\varphi} \\
&\quad = f_{11}(\frac{r}{z}, \nabla_{r,z}\dot{\varphi})\partial_{zz}^2\dot{\varphi} + f_{12}(\frac{r}{z}, \nabla_{r,z}\dot{\varphi})\partial_{rz}^2\dot{\varphi} + f_{22}(\frac{r}{z}, \nabla_{r,z}\dot{\varphi})\partial_{rr}^2\dot{\varphi} \\
&\quad + \frac{1}{r} f_0(\frac{r}{z}, \nabla_{r,z}\dot{\varphi}), \qquad z \geq 1, \qquad b_0 z \leq r \leq \chi(z)
\end{aligned} \tag{3.1}$$

where $f_{ij}(s,0,0) = 0$, $f_0(s,0,0) = \nabla_q f_0(s,q_1,q_2)|_{q=0} = 0$. Moreover

$$\begin{aligned}
P_1(s) &= \frac{(q_0 + \hat{W}(s))\hat{U}(s)}{(q_0 + \hat{W}(s))^2 - \hat{c}^2(s)}, \\
P_2(s) &= \frac{\hat{U}^2(s) - \hat{c}^2(s)}{(q_0 + \hat{W}(s))^2 - \hat{c}^2(s)}, \\
P_3(r,z) &= \frac{1}{r((q_0 + \hat{W}(s))^2 - \hat{c}^2(s))}\{-2(\hat{c}(s)\hat{c}'(s) + 1)(q_0 + \hat{W}(s))\hat{W}'(s)s^2 \\
&\quad + 2\hat{c}(s)\hat{c}'(s)(q_0 + \hat{W}(s))\hat{U}'(s)s + 2\hat{c}(s)\hat{c}'(s)(q_0 + \hat{W}(s))\hat{U}(s) - 2s^2\hat{U}(s)\hat{U}'(s)\} \equiv \frac{1}{r}\tilde{P}_3(s), \\
P_4(r,z) &= \frac{1}{r((q_0 + \hat{W}(s))^2 - \hat{c}^2(s))}\{-2s^2\hat{c}(s)\hat{c}'(s)\hat{U}(s)\hat{W}'(s) + 2s(1 + \hat{c}(s)\hat{c}'(s))\hat{U}(s)\hat{U}'(s) \\
&\quad - \hat{c}^2(s) + 2\hat{c}(s)\hat{c}'(s)\hat{U}^2(s) - 2s^2(q_0 + \hat{W}(s))\hat{U}'(s)\} \equiv \frac{1}{r}\tilde{P}_4(s).
\end{aligned}$$

with

$$\hat{c}(s) = c(\hat{\rho}(s)), \qquad \hat{c}'(s) = \hat{c}'(\hat{\rho}(s))\hat{\rho}'(s)$$

The boundary conditions are also reduced to the new forms. On the boundary $r = b_0 z$, we have

$$\partial_r \dot{\varphi} = b_0 \partial_z \dot{\varphi}. \tag{3.2}$$

On the boundary $r = \chi(z)$, we first write (1.13) as

$$H(\nabla\varphi)((\partial_r\varphi)^2 + (\partial_z\varphi)^2 + q_0\partial_z\varphi) - \rho_0 q_0 \partial_z\varphi = 0 \qquad on \qquad r = \chi(z) \tag{3.3}$$

Using $\varphi = \hat{\varphi} + \dot{\varphi}$, and introducing the notation $\xi(z) = \frac{\chi(z) - s_0 z}{z}$, which describes the perturbation of the slope of the shock front, the above equality can be rewritten as

$$B_1\partial_r\dot{\varphi} + B_2\partial_z\dot{\varphi} + B_3\xi = \kappa_0(\xi, \nabla_{r,z}\dot{\varphi}) \qquad on \qquad r = \chi(z) \tag{3.4}$$

where

$$\begin{aligned}
B_1 &= -\frac{\rho_+}{c_+^2}(U_+^2 + W_+^2 + q_0 W_+)U_+ + 2\rho_+ U_+, \\
B_2 &= -\frac{\rho_+}{c_+^2}(U_+^2 + W_+^2 + q_0 W_+)(q_0 + W_+) + 2\rho_+ W_+ + (\rho_+ - \rho_0)q_0, \\
B_3 &= \rho_+(2U_+\hat{U}'(s_0) + 2W_+\hat{W}'(s_0) + q_0\hat{W}'(s_0)) + \hat{\rho}'(s_0)(U_+^2 + W_+^2 + q_0 W_+) - \rho_0 q_0 \hat{W}'(s_0),
\end{aligned}$$

and

$$\kappa_0(\xi, \nabla_{r,z}\dot{\varphi}) \le C(|(\xi, \nabla_{r,z}\dot{\varphi})|^2).$$

Later on the function $\kappa_j(\xi, \nabla_{r,z}\dot{\varphi})$ or the notation $O_2(\xi, \nabla\dot{\varphi})$ will be used to denote any quantity dominated by $C|(\xi, \nabla_{r,z}\dot{\varphi})|^2$, here the generic constant C doesn't depend on $\varepsilon_0$.

By using Lemma 2.2 and Lemma 2.3, we have the following estimates for large $q_0$.

**Lemma 3.1.**

$$B_1 = \frac{2b_0}{1+b_0^2}\left(\frac{(\gamma-1)b_0^2}{2A\gamma(1+b_0^2)}\right)^{\frac{1}{\gamma-1}} q_0^{\frac{\gamma+1}{\gamma-1}}(1+O(q_0^{-\frac{2}{\gamma-1}})),$$

$$B_2 = \frac{1-b_0^2}{1+b_0^2}\left(\frac{(\gamma-1)b_0^2}{2A\gamma(1+b_0^2)}\right)^{\frac{1}{\gamma-1}} q_0^{\frac{\gamma+1}{\gamma-1}}(1+O(q_0^{-\frac{2}{\gamma-1}})),$$

$$B_3 = -\frac{b_0}{(1+b_0^2)^2}\left(\frac{(\gamma-1)b_0^2}{2A\gamma(1+b_0^2)}\right)^{\frac{1}{\gamma-1}} q_0^{\frac{2\gamma}{\gamma-1}}(1+O(q_0^{-\frac{2}{\gamma-1}})).$$

Dividing (3.4) by $B_1$ we can write (3.3) as

$$\partial_r\dot{\varphi} + \mu_1\partial_z\dot{\varphi} + \mu_2\xi = \kappa_1(\xi, \nabla_{r,z}\dot{\varphi}) \qquad on \qquad r = \chi(z) \tag{3.5}$$

where $\mu_1 = \frac{1-b_0^2}{2b_0}(1+O(q_0^{-\frac{2}{\gamma-1}}))$, $\mu_2 = -\frac{q_0}{2(1+b_0^2)}(1+O(q_0^{-\frac{2}{\gamma-1}}))$.

Besides, (1.14) implies $\chi'(z) = -\frac{\partial_z\varphi}{\partial_r\varphi}$ on $r = \chi(z)$, it follows from Taylor's expansion and the fact $s_0\hat{U}'(s_0) + \hat{W}'(s_0) = 0$ that

$$\partial_z(z\xi) + \frac{1}{U_+}((\partial_z\dot{\varphi})(\chi(z), z) + s_0(\partial_r\dot{\varphi})(\chi(z), z)) = O_2(\xi, \nabla\dot{\varphi}).$$

Since $\partial_z(\dot{\varphi}(\chi(z), z)) = (\partial_z\dot{\varphi})(\chi(z), z) + \chi'(z)(\partial_r\dot{\varphi})(\chi(z), z) = (\partial_z\dot{\varphi})(\chi(z), z) + s_0(\partial_r\dot{\varphi})(\chi(z), z) + O_2(\xi, \nabla\dot{\varphi})$, then by substituting it into the above equation we have

$$\partial_z(z\xi + \frac{1}{U_+}\dot{\varphi}(\chi(z), z)) = \kappa_2(\xi, \nabla_{r,z}\dot{\varphi}) \tag{3.6}$$

(3.5) and (3.6) are the new forms of the Rankine-Hugoniot condition (1.13) and the continuity condition (1.14) on the shock front.

After such a reformulation of the problem (1.11)-(1.14), to prove the main theorem we only need to solve the problem (3.1), (3.2), (3.5) and (3.6) with small initial data $\dot{\varphi}(r,z)|_{z=1}, \xi(z)|_{z=1}$ in the domain $\{(r,z) : z \ge 1, b_0 z \le r \le \chi(z)\}$. The smallness means

$$\sum_{l\le k_2}|\nabla_{r,z}^l\dot{\varphi}|, \sum_{l\le k_2}|\partial_z^l\xi| \le C\varepsilon_0 \qquad on \qquad z = 1 \tag{3.7}$$

where $k_2$ and $\varepsilon_0$ are given in Theorem 1.1. We notice that (3.7) can be derived from the result on the local existence and stability in [1].

## §4. Uniform estimate on the linearized operator

Now we give an energy estimate for the linear part of (3.1). The following conclusion plays the key role in our analysis.

**Theorem 4.1.** Set $D_T = \{1 \le z \le T, b_0 z \le r \le \chi(z)\}$ for any $T > 1$. $\Gamma_T = \{1 \le z \le T, r = \chi(z)\}$ and $B_T = \{1 \le z \le T, r = b_0 z\}$ are the lateral boundaries of $D_T$. If $\dot\varphi \in C^\infty(D_T)$ satisfies the boundary condition (3.2), $|\xi(z)| + |z\xi'(z)| \le C\varepsilon_0$ is sufficiently small for $z \in [1, T]$. Then there exists a multiplier $\mathcal{M}\dot\varphi = ra(\frac{r}{z})\partial_z\dot\varphi + zb(\frac{r}{z})\partial_r\dot\varphi$, such that

$$\begin{aligned}
&\frac{C_1}{\sqrt{T}}\int_{b_0T}^{\chi(T)} |\nabla_{r,z}\dot\varphi(r,T)|^2 dr + C_2\iint_{D_T} z^{-\frac{3}{2}}|\nabla_{r,z}\dot\varphi|^2 drdz + C_3\int_{\Gamma_T} z^{-\frac{1}{2}}|\partial_z\dot\varphi|^2 dl + C_4\int_{B_T} z^{-\frac{1}{2}}|\partial_z\dot\varphi|^2 dl \\
&\le \iint_{D_T} z^{-\frac{3}{2}}\mathcal{L}\dot\varphi\mathcal{M}\dot\varphi drdz + C_5\int_{\Gamma_T} z^{-\frac{1}{2}}(\mathcal{B}_0\dot\varphi)^2 dl + C_6\int_{b_0}^{\chi(1)} (|\dot\varphi(r,1)|^2 + |\partial_z\dot\varphi(r,1)|^2) dr \qquad (4.1)
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{L}\dot\varphi = \partial_z^2\dot\varphi + 2P_1(\frac{r}{z})\partial_{zr}^2\dot\varphi + P_2(\frac{r}{z})\partial_r^2\dot\varphi + P_3(r,z)\partial_z\dot\varphi + P_4(r,z)\partial_r\dot\varphi, \\
&\mathcal{B}_0\dot\varphi = (\partial_r + \mu_1\partial_z)\dot\varphi.
\end{aligned}$$

and $C_i (1 \le i \le 6)$ are positive constants independent of $q_0$ and $\varepsilon_0$, in particular,

$$\begin{aligned}
C_4 &= \frac{\sqrt{2(\gamma-1)}}{4}b_0^2 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \\
C_5 &= \frac{\gamma-1}{2}b_0^4 + O(b_0^5) + O(\varepsilon_0) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})
\end{aligned}$$

**Remark 4.1.** The values of constants $C_4$ and $C_5$ will play an important role in the energy estimates for the nonlinear problem (3.1),(3.2), (3.5)-(3.7) in §5. Because our nonlinear problem is with the Neumann boundary condition (3.2), the usual Poincare inequality doesn't hold for the solution $\dot\varphi$. Hence from the equation (3.5) and (3.6) we know that the term $\int_{\Gamma_T} z^{-\frac{1}{2}}(\mathcal{B}_0\dot\varphi)^2 dl$ which appears in the right hand side of (4.1) will bring the much more troubles for us. Thanks to the choice of $C_4$ and $C_5$, we can show that the term $C_5\int_{\Gamma_T} z^{-\frac{1}{2}}(\mathcal{B}_0\dot\varphi)^2 dl$ can be asorbed by the left hand side of (4.1). The details see below §5.

**Remark 4.2.** The integral on $\Gamma_T$ in the left side of (4.1) only contains the term $\int_{\Gamma_T} z^{-\frac{1}{2}}|\partial_z\dot\varphi|^2 dl$ and doesn't contain $\int_{\Gamma_T} z^{-\frac{1}{2}}|\partial_r\dot\varphi|^2 dl$, but we can get the estimate on $\int_{\Gamma_T} z^{-\frac{1}{2}}|\partial_r\dot\varphi|^2 dl$ if $\int_{\Gamma_T} z^{-\frac{1}{2}}|\partial_z\dot\varphi|^2 dl$ and $\int_{\Gamma_T} z^{-\frac{1}{2}}|\mathcal{B}_0\dot\varphi|^2 dl$ are known. In fact, we have

$$\int_{\Gamma_T} z^{-\frac{1}{2}}|\partial_r\dot\varphi|^2 dl \le 2(\mu_1^2\int_{\Gamma_T} z^{-\frac{1}{2}}|\partial_z\dot\varphi|^2 dl + \int_{\Gamma_T} z^{-\frac{1}{2}}|\mathcal{B}_0\dot\varphi|^2 dl)$$

**Proof.** Let $A = A(r,z)$ and $B = B(r,z)$ be determined. Denoting $M\dot\varphi = A(r,z)\partial_z\dot\varphi + B(r,z)\partial_r\dot\varphi$, we have by the integration by parts that

$$\begin{aligned}
\iint_{D_T} z^{-\frac{3}{2}}\mathcal{L}\dot\varphi\mathcal{M}\dot\varphi drdz &= \iint_{D_T} z^{-\frac{3}{2}}\big(K_0(\partial_z\dot\varphi)^2 + K_1(\partial_r\dot\varphi)^2 + K_2\partial_z\dot\varphi\partial_r\dot\varphi)\big)drdz \\
&+ T^{-\frac{3}{2}}\int_{b_0T}^{\chi(T)} K_3(r,T)dr - \int_{b_0}^{\chi(1)} K_3(r,1)dr + \int_{B_T} z^{-\frac{3}{2}}(b_0K_3 - K_4)dl \\
&+ \int_{\Gamma_T} z^{-\frac{3}{2}}(K_4 - \chi' K_3)dl \qquad (4.2)
\end{aligned}$$

where

$$K_0 = -\frac{\partial_z A}{2} + \frac{\partial_r B}{2} - \partial_r(P_1 A) + P_3 A + \frac{3A}{4z}$$
$$K_1 = -\partial_z(P_1 B) + \frac{1}{2}\partial_z(P_2 A) - \frac{1}{2}\partial_r(P_2 B) + P_4 B + \frac{3}{4z}(2P_1 B - P_2 A)$$
$$K_2 = -\partial_z B - \partial_r(P_2 A) + P_3 B + P_4 A + \frac{3B}{2z}$$
$$K_3 = \frac{A}{2}(\partial_z\dot\varphi)^2 + B\partial_z\dot\varphi\partial_r\dot\varphi + (P_1 B - \frac{P_2 A}{2})(\partial_r\dot\varphi)^2$$
$$K_4 = (P_1 A - \frac{B}{2})(\partial_z\dot\varphi)^2 + P_2 A\partial_z\dot\varphi\partial_r\dot\varphi + \frac{P_2 B}{2}(\partial_r\dot\varphi)^2$$

Our purpose is to choose suitable coefficients $A(r,z)$ and $B(r,z)$ so that all integrals on $D_T, B_T$ and $t = T$ in the right hand side of (4.2) are definitely positive and the integral on $\Gamma_T$ gives the control on $\dot\varphi$ "in some sense". We will derive some sufficient conditions for $A(r,z)$ and $B(r,z)$ in the process of investigating each integral. Assume $A(r,z) = ra(\frac{r}{z})$ and $B(r,z) = zb(\frac{r}{z})$ with $a(s) > 0$ and $b(s) > 0$. Then $a(s)$ and $b(s)$ will be determined by the following five steps. In what follows, we will denote by $C$ a generic positive constant independent of $q_0$ and $\varepsilon_0$, it may take different value in different expressions.

**Step 1.** Positivity of $\int_{B_T} z^{-\frac{3}{2}}(b_0K_3 - K_4)dl$.

Since $b_0K_3 - K_4 = (\frac{b_0}{2}A - (P_1A - \frac{B}{2}))(\partial_z\dot\varphi)^2 + (b_0B - P_2A)\partial_r\dot\varphi\partial_z\dot\varphi + (b_0(P_1B - \frac{P_2A}{2}) - \frac{P_2B}{2})(\partial_r\dot\varphi)^2$, using the boundary condition (3.2) we have on $r = b_0z$:

$$\begin{aligned} b_0K_3 - K_4 =& z(\partial_z\dot\varphi)^2\{b(b_0)(\frac{1}{2} + b_0^2 + b_0^3P_1(b_0) - \frac{P_2(b_0)}{2}b_0^2) + a(b_0)b_0(-P_1(b_0) \\ &+ \frac{b_0}{2} - P_2(b_0)b_0 - \frac{P_2(b_0)}{2}b_0^3)\} \end{aligned}$$

In view of $\frac{1}{2} + b_0^2 + b_0^3P_1(b_0) - \frac{P_2(b_0)}{2}b_0^2 = \frac{1}{2} + \frac{(q_0+W(b_0))^2b_0^2}{(q_0+W(b_0))^2 - c^2(b_0)} > \frac{1}{2}$, the inequality

$$b(b_0) > b_0^2a(b_0) > 0, \tag{4.3}$$

implies

$$b_0K_3 - K_4 > z(\partial_z\dot\varphi)^2b_0a(b_0)\{b_0 + b_0^3 + b_0^4P_1(b_0) - P_2(b_0)b_0^3 - P_1(b_0) - P_2(b_0)b_0\}$$

By the boundary condition $U(b_0) = b_0(q_0 + W(b_0))$ and the expressions of $P_1(b_0)$ and $P_2(b_0)$, we have

$$\begin{aligned} & b_0 + b_0^3 + b_0^4P_1(b_0) - P_2(b_0)b_0^3 - P_1(b_0) - P_2(b_0)b_0 \\ &= b_0 + b_0^3 + \frac{c^2 - U^2}{(q_0+W)^2 - c^2}b_0^3 - \frac{b_0(q_0+W)^2}{(q_0+W)^2 - c^2} + \frac{c^2 - U^2}{(q_0+W)^2 - c^2} + b_0^4P_1(b_0) = 0 \end{aligned}$$

Hence $b_0K_3 - K_4 = z(\partial_z\dot\varphi)^2(b(b_0) - b_0^2a(b_0))(\frac{1}{2} + b_0^2 + b_0^3P_1(b_0) - \frac{P_2(b_0)}{2}b_0^2) > 0$ under the condition (4.3). This leads to the first constraint (4.3) for $a(s)$ and $b(s)$. In addition, $b_0K_3 - K_4$ will be computed explicitly in the Lemma A.8 of Appendix.

**Step 2.** Positivity of $\int_{b_0T}^{\chi(T)} K_3(r,T)dr$.

On $z = T$, we have

$$\begin{aligned} K_3(r,T) &= \frac{ra}{2}(\partial_z\dot\varphi)^2 + zb\partial_z\dot\varphi\partial_r\dot\varphi + (zP_1b - \frac{P_2ra}{2})(\partial_r\dot\varphi)^2 \\ &= z(\frac{sa(s)}{2}(\partial_z\dot\varphi)^2 + b(s)\partial_z\dot\varphi\partial_r\dot\varphi + (P_1b(s) - \frac{P_2s}{2}a(s))(\partial_r\dot\varphi)^2) \end{aligned}$$

To ensure the positivity of $K_3$ one requires that the discriminant of the quadratic form should be negative

$$\Delta = b^2(s)(1 - 2P_1 \frac{sa(s)}{b(s)} + P_2(\frac{sa(s)}{b(s)})^2) < 0$$

Denote $P_1^2 - P_2$ by $D_1$. Then the above inequality leads to

$$\frac{sa(s)}{b(s)} > \frac{P_1 - \sqrt{D_1}}{P_2} = \frac{1}{P_1 + \sqrt{D_1}} = \frac{1}{\lambda_2(s)}.$$

Therefore, $\int_{b_0 T}^{\chi(T)} K_3(r,T)dr \geq CT \int_{b_0 T}^{\chi(T)} |\nabla_{r,z}\dot{\varphi}(r,T)|^2 dr$ as long as $a(s)$ and $b(s)$ are appropriately selected to satisfy

$$0 < \frac{b(s)}{sa(s)} < \lambda_2(s) \tag{4.4}$$

**Step 3.** Positivity of the integral on $D_T$.
We look for the requirements for $a(s)$ and $b(s)$, so that

$$K_0(\partial_z\dot{\varphi})^2 + K_1(\partial_r\dot{\varphi})^2 + K_2\partial_z\dot{\varphi}\partial_r\dot{\varphi} \geq C((\partial_z\dot{\varphi})^2 + (\partial_r\dot{\varphi})^2)$$

The above estimate holds if the coefficients $K_0, K_1$ and $K_2$ satisfy

$$K_0 > 0, K_2^2 - 4K_0K_1 < 0 \tag{4.5}$$

This is a system of nonlinear ordinary differential inequalities. Indeed, substituting $a(s)$ and $b(s)$ into the expressions of $K_0, K_1$ and $K_2$ yields

$$\begin{aligned}
K_0 &= (\frac{s^2}{2} - P_1 s)a'(s) + \frac{b'(s)}{2} - P_1' sa(s) - P_1 a(s) + \tilde{P}_3 a(s) + \frac{3}{4} sa(s)\\
K_2^2 - 4K_0K_1 &= \{-P_2 sa'(s) + sb'(s) - P_2 a(s) + \frac{1}{2}b(s) - P_2' sa(s) + \frac{\tilde{P}_3}{s} b(s) + \tilde{P}_4 a(s)\}^2\\
&- 4\{(\frac{s^2}{2} - P_1 s)a'(s) + \frac{b'(s)}{2} - P_1' sa(s) - P_1 a(s) + \tilde{P}_3 a(s) + \frac{3}{4} sa(s)\}\{-\frac{P_2}{2} s^2 a'(s)\\
&+ (P_1 s - \frac{P_2}{2})b'(s) + P_1' sb(s) - \frac{1}{2}P_2' s^2 a(s) - \frac{1}{2}P_2' b(s) + \frac{P_1}{2} b(s) + \frac{\tilde{P}_4}{s} b(s) - \frac{3}{4} P_2 sa(s)\}
\end{aligned}$$

Denote by $Q_0, Q_1$ and $Q_2$ the terms which only involve $a(s)$ and $b(s)$, but not their derivatives in $K_0, K_1$ and $K_2$ , namely,

$$\begin{cases}
Q_0 = (\frac{3}{4}s - P_1' s - P_1 + \tilde{P}_3)a(s)\\
Q_1 = P_1' sb(s) - \frac{1}{2}P_2' s^2 a(s) - \frac{1}{2}P_2' b(s) + \frac{1}{2}P_1 b(s) - \frac{3}{4}P_2 sa(s) + \frac{\tilde{P}_4}{s} b(s)\\
Q_2 = -P_2 a(s) + \frac{1}{2}b(s) - P_2' sa(s) + \frac{\tilde{P}_3}{s} b(s) + \tilde{P}_4 a(s)
\end{cases}$$

Then

$$\begin{aligned}
K_2^2 - 4K_0K_1 &= (-P_2 sa'(s) + sb'(s))^2 - 4((\frac{s^2}{2} - P_1 s)a'(s) + \frac{b'(s)}{2})(-\frac{P_2}{2}s^2 a'(s) + (P_1 s - \frac{P_2}{2})b'(s))\\
&+ 2Q_2(-P_2 sa'(s) + sb'(s)) + 4Q_0(\frac{P_2}{2}s^2 a'(s) - (P_1 s - \frac{P_2}{2})b'(s)) - 4Q_1((\frac{s^2}{2} - P_1 s)a'(s) + \frac{b'(s)}{2})\\
&+ Q_2^2 - 4Q_0Q_1.
\end{aligned}$$

The right hand side is a quadratic form of $a'(s)$ and $b'(s)$. Denoting the coefficients of the linear terms by $a_1(s)$ and $a_2(s)$, namely

$$\begin{cases} a_1 = -P_2Q_2s + P_2Q_0s^2 - Q_1(s^2 - 2P_1s) \\ a_2 = Q_2s - Q_0(2P_1s - P_2) - Q_1 \end{cases} \tag{4.6}$$

then we have

$$K_2^2 - 4K_0K_1 = (P_2 + s^2 - 2P_1s)(P_2s^2a'(s)^2 - 2P_1sa'(s)b'(s) + b'(s)^2) + 2a_1a'(s) + 2a_2b'(s) + Q_2^2 - 4Q_0Q_1. \tag{4.7}$$

The coefficient $P_2 + s^2 - 2P_1s$, which will be denoted by $-\tilde{A}$, is equal to $-(\lambda_2(s) - s)(s - \lambda_1(s)) < 0$ in $[b_0, s_0 + \eta_0]$ due to Lemma 2.1 (ii) and Lemma 2.3 (i).

To transform (4.7) to a standard quadratic form, we introduce

$$\begin{cases} Y_1 = a'(s) + \frac{a_1 + a_2P_1s}{\tilde{A}s^2D_1}, \\ Y_2 = -P_1sa'(s) + b'(s) - \frac{a_2}{\tilde{A}}. \end{cases}$$

Substituting them into the expressions of $K_0$ and $K_2^2 - 4K_0K_1$ one gets

$$\begin{cases} K_0 = \frac{s^2 - P_1s}{2}Y_1 + \frac{Y_2}{2} + Q_0 + \frac{a_2}{2\tilde{A}} - \frac{(s-P_1)(a_1+a_2P_1s)}{2\tilde{A}sD_1}, \\ K_2^2 - 4K_0K_1 = \tilde{A}s^2D_1Y_1^2 - \tilde{A}Y_2^2 + Q_2^2 - 4Q_0Q_1 + \frac{a_2^2}{\tilde{A}} - \frac{(a_1+a_2P_1s)^2}{\tilde{A}s^2D_1}. \end{cases}$$

A key observation is the fact that the sum of the last four terms in the right hand side of the second equality above is nonnegative. Indeed, setting $Y_3 = -(Q_0 + \frac{a_2}{2\tilde{A}} - \frac{(s-P_1)(a_1+a_2P_1s)}{2\tilde{A}sD_1})$, one has the following identity

$$Q_2^2 - 4Q_0Q_1 + \frac{a_2^2}{\tilde{A}} - \frac{(a_1 + a_2P_1s)^2}{\tilde{A}s^2D_1} = 4D_1Y_3^2$$

Hence $K_0 > 0$, $K_2^2 - 4K_0K_1 < 0$ are equivalent to

$$\begin{cases} (s^2 - P_1s)Y_1 + Y_2 - 2Y_3 > 0 \\ \tilde{A}s^2D_1Y_1^2 - \tilde{A}Y_2^2 + 4D_1Y_3^2 < 0 \end{cases} \tag{4.8}$$

**Step 4.** Construction of $a(s)$ and $b(s)$.

By studying the solvability condition for (4.8) carefully, one can show that (4.8) is equivalent to the following differential system

$$\begin{cases} (s^2 - P_1s)Y_1 + Y_2 - 2Y_3 = \sqrt{\tilde{\delta}_0(s) + k^2(s)s^2a^2(s)D_1 + 4Y_3^2} - 2Y_3 \\ \tilde{A}s^2D_1Y_1^2 - \tilde{A}Y_2^2 + 4D_1Y_3^2 = -\tilde{\delta}_0(s)D_1, \end{cases} \tag{4.9}$$

where the new functions $\tilde{\delta}_0(s) > 0$ and $k(s) \geq 0$ are to be determined together with $a(s)$ and $b(s)$.

By solving $Y_1$ and $Y_2$ in (4.9) and rewriting it in terms of $a(s)$ and $b(s)$, one gets that

$$\begin{cases} a'(s) + \frac{a_1+a_2P_1s}{\tilde{A}s^2D_1} = -\frac{s-P_1}{sD_1}(b'(s) - P_1sa'(s) - \frac{a_2}{\tilde{A}}) + k(s)a(s) \\ b'(s) - P_1sa'(s) - \frac{a_2}{\tilde{A}} = \frac{D_1}{\tilde{A}}\left(\sqrt{\tilde{\delta}_0(s) + k^2(s)s^2a^2(s)D_1 + 4Y_3^2} - k(s)sa(s)(s - P_1)\right) \end{cases} \tag{4.10}$$

We now show that there exist $a(s), b(s), \tilde{\delta}_0(s)$ and $k(s)$ satisfying (4.3),(4.4) and (4.10), provided $q_0$ is large. Set $b(s) = s\tilde{\lambda}(s)a(s)$. Then (4.3) and (4.4) are the consequence of the following inequality

$$s < \tilde{\lambda}(s) < \lambda_2(s) \tag{4.11}$$

and the first equation in (4.10) becomes

$$\begin{aligned}\left(1+\frac{(s-P_1)(\tilde{\lambda}(s)-P_1)}{D_1}\right)a'(s) = -\frac{s-P_1}{sD_1}\left((\tilde{\lambda}(s)+s\tilde{\lambda}'(s))a(s)-\frac{a_2}{\tilde{A}}\right)+k(s)a(s)\\ -\frac{a_1+a_2P_1s}{\tilde{A}s^2D_1}\end{aligned} \tag{4.12}$$

It is shown in Lemma A.1 of Appendix that the coefficient of $a'(s)$ is positive for large $q_0$. Then (4.12) can be written as:

$$a'(s) = \left(\tilde{Q}_0(s) + \frac{D_1}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)}k(s)\right)a(s) \tag{4.13}$$

where $\tilde{Q}_0(s) = \big(-\frac{s-P_1}{s}(\tilde{\lambda}(s)+s\tilde{\lambda}'(s)-\frac{\tilde{a}_2(s)}{\tilde{A}})-\frac{\tilde{a}_1(s)+\tilde{a}_2(s)P_1s}{\tilde{A}s^2}\big)/(D_1+(s-P_1)(\tilde{\lambda}(s)-P_1))$ with $\tilde{a}_i(s) = a_i(s)/a(s)(i=1,2)$. Clearly, for $a(b_0)=1$ the linear ordinary differential equation (4.13) has a unique positive solution $a(s)$ in $[b_0, s_0+\eta_0]$.

It remains to determine $\tilde{\lambda}(s), \tilde{\delta}_0(s)$ and $k(s)$. It follows from the second equation of (4.10) that we have the following algebraic equation for $k(s)$

$$A_0(s)k^2(s) + A_1(s)k(s) = A_2(s), \tag{4.14}$$

where

$$\begin{aligned}
A_0(s) &= \left(\frac{s(\tilde{\lambda}(s)-P_1)D_1}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)}+\frac{D_1}{\tilde{A}}s(s-P_1)\right)^2-\frac{D_1^3s^2}{\tilde{A}^2}\\
A_1(s) &= 2\left(\frac{s(\tilde{\lambda}(s)-P_1)D_1}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)}+\frac{D_1}{\tilde{A}}s(s-P_1)\right)\Bigg(\tilde{\lambda}(s)+s\tilde{\lambda}'(s)-\frac{\tilde{a}_2(s)}{\tilde{A}}\\
&\quad+s(\tilde{\lambda}(s)-P_1)\tilde{Q}_0(s)\Bigg)\\
A_2(s) &= \frac{D_1^2}{\tilde{A}^2}\left(\frac{\tilde{\delta}_0(s)}{a^2(s)}+4\frac{Y_3^2(s)}{a^2(s)}\right)-\big(\tilde{\lambda}(s)+s\tilde{\lambda}'(s)-\frac{\tilde{a}_2(s)}{\tilde{A}}+s(\tilde{\lambda}(s)-P_1)\tilde{Q}_0(s)\big)^2
\end{aligned}$$

Thus, there exists a positive solution $k(s)$ in (4.14), provided

$$A_0(s) < 0 \qquad and \qquad A_2(s) < 0, \tag{4.15}$$

The fact $A_0(s) < 0$ can be checked easily, see Lemma A.2 of Appendix. To prove the negativity of $A_2(s)$ we first choose $\tilde{\lambda}(s)$ so that

$$\frac{4D_1^2Y_3^2(s)}{\tilde{A}^2a^2(s)} < \left(\tilde{\lambda}(s)+s\tilde{\lambda}'(s)+s(\tilde{\lambda}(s)-P_1)\tilde{Q}_0(s)-\frac{\tilde{a}_2(s)}{\tilde{A}}\right)^2 \tag{4.16}$$

To this end we set

$$\tilde{\lambda}(s) = \frac{s_0+\eta_0-s}{s_0+\eta_0-b_0}(1-\theta_0b_0^2)\lambda_2(s_0)+\frac{s-b_0}{s_0+\eta_0-b_0}(\theta_0b_0^2\lambda_2(s_0)+(1-\theta_0b_0^2)s_0) \tag{4.17}$$

Below we choose $\theta_0 = 2\sqrt{2(\gamma-1)}$.

The choice of the special form of $\tilde{\lambda}(s)$ comes from the following considerations. The first one is to let its value is near $\lambda_2(s_0)$ on the boundary $r = b_0 z$ so that the coefficient of $z(\partial_z\dot{\varphi})^2$ in $b_0K_3 - K_4$ of step 1 is "large" as soon as possible when $\tilde{\lambda}(s)$ is between $s$ and $\lambda_2(s)$. The second one is to let its value is near $s_0$ on the shock $r = \chi(z)$ so that the coefficient of $\int_{\Gamma_T} z^{-\frac{1}{2}}(\mathcal{B}_0\dot{\varphi})^2 dl$ in the right hand side of (4.1) is "small" as soon as possible. Certainly, in order to guarantee the "smallness" of coefficient of $\int_{\Gamma_T} z^{-\frac{1}{2}}(\mathcal{B}_0\dot{\varphi})^2 dl$, from the below expression (4.20) we also require that $a(s_0)$ is very near $a(b_0) = 1$ for small $b_0$. This property can be obtained when $A_1(s) < 0$ in (4.14) (the details see Lemma A.6 of Appendix). In fact, the choice of $\theta_0 = 2\sqrt{2(\gamma-1)}$ just only leads to $A_1(s) < 0$ (see Lemma A.7 of Appendix). The third one is that we hope that $\tilde{\lambda}(s)$ isn't too "close" to $\lambda_2(s_0)$ so that $\int_{b_0T}^{\chi(T)} K_3(r,T)dr \geq CT\int_{b_0T}^{\chi(T)} |\nabla_{r,z}\dot{\varphi}(r,T)|^2 dr$ in Step 2, where $C > 0$ only depends on $b_0$ and $\gamma$. This is another reason that we choose a factor $\theta_0 b_0^2$ in the expression of $\tilde{\lambda}(s)$. The fourth one is to let the derivative of $\tilde{\lambda}(s)$ be large, so that (4.16) holds.

Because of the special choice of $\tilde{\lambda}(s)$, it is obvious from the entropy condition (2.4) and lemma 2.1 (ii) that for small $b_0$

$$s < \tilde{\lambda}(s) < \lambda_2(s) \tag{4.18}$$

In addtion, we can show that the inequality (4.16) holds true for $1 < \gamma < 3$ and large $q_0$. The proof of this fact is given in Lemma A.3 of Appendix. Then with a suitable choice of $\tilde{\delta}_0(s)$ ( see below ) one gets $A_2(s) < 0$ from (4.16).

Summarizing the above analysis, we can define the multipliers $a(s), b(s)$ and corresponding $\tilde{\delta}_0(s), k(s)$ as follows.

$$\begin{aligned}
&\tilde{\lambda}(s) = \frac{s_0+\eta_0-s}{s_0+\eta_0-b_0}(1-\theta_0b_0^2)\lambda_2(s_0) + \frac{s-b_0}{s_0+\eta_0-b_0}(\theta_0b_0^2\lambda_2(s_0) + (1-\theta_0b_0^2)s_0),\\
&\tilde{\delta}_0(s) = \frac{a^2(s)\tilde{A}^2}{D_1^2}\left\{\left(\tilde{\lambda}(s) + s\tilde{\lambda}'(s) + s(\tilde{\lambda}(s)-P_1)\tilde{Q}_0(s) - \frac{\tilde{a}_2(s)}{\tilde{A}}\right)^2 - \frac{4D_1^2Y_3^2(s)}{\tilde{A}^2a^2(s)} - 1\right\} > 0,\\
&k(s) = \frac{-A_1(s) - \sqrt{A_1^2(s) + 4A_0(s)A_2(s)}}{2A_0(s)} > 0,\\
&a(s) = exp\{\int_{b_0}^{s}\left(\tilde{Q}_0(s) + \frac{D_1}{D_1 + (s-P_1)(\tilde{\lambda}(s)-P_1)}k(s)\right)ds\},\\
&b(s) = s\tilde{\lambda}(s)a(s).
\end{aligned}$$

In this case, $A_2(s) = -1$ in (4.14). This will give us the convenience to analyze the property of $k(s)$ in Lemma A.6 of Appendix.

**Step 5.** The estimate on $\int_{\Gamma_T} z^{-\frac{3}{2}}(K_4 - \chi' K_3)dl$.

With the choice of the multipliers given in the previous steps, we can show on $r = \chi(z)$

$$\begin{aligned}
K_4 - \chi'K_3 \geq z\{(&\frac{\gamma-1}{8}b_0^2 + O(b_0^3) + O(\varepsilon_0) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}))(\partial_z\dot{\varphi})^2\\
&-(\frac{\gamma-1}{2} + O(b_0) + O(\varepsilon_0) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}))b_0^4(\mathcal{B}_0\dot{\varphi})^2\}
\end{aligned} \tag{4.19}$$

In fact, by the assumption on $\xi(z)$ in Theorem 4.1 and $\eta_0 \leq q_0^{-\frac{2}{\gamma-1}}(s_0-b_0)$, it follows that from the expressions of $K_3$ and $K_4$,

$$\begin{aligned}
K_4 - \chi'K_3 &= ra(\frac{\chi(z)}{z})\{(\beta_0(\partial_z\dot{\varphi})^2 + \beta_1\partial_z\dot{\varphi}\partial_r\dot{\varphi} + \beta_2(\partial_r\dot{\varphi})^2) + (O(q_0^{-\frac{2}{\gamma-1}}) + O(\varepsilon_0))(|\nabla_{r,z}\varphi|^2)\}\\
&= zb_0a(s_0)\{(\beta_0(\partial_z\dot{\varphi})^2 + \beta_1\partial_z\dot{\varphi}\partial_r\dot{\varphi} + \beta_2(\partial_r\dot{\varphi})^2) + (O(q_0^{-\frac{2}{\gamma-1}}) + O(\varepsilon_0))(|\nabla_{r,z}\varphi|^2)\}\\
&\equiv zb_0a(s_0)(I + II)
\end{aligned} \tag{4.20}$$

where $\beta_0 = P_1(s_0) - s_0 - \frac{\theta_0 b_0^2(\lambda_2(s_0)-s_0)}{2}, \beta_1 = P_2(s_0) - s_0^2 - \theta_0 b_0^2 s_0(\lambda_2(s_0) - s_0)$ and $\beta_2 = P_2(s_0)s_0 - P_1(s_0)s_0^2 + \theta_0 b_0^2(\lambda_2(s_0) - s_0)(\frac{P_2(s_0)}{2} - P_1(s_0))$. Noting $\partial_r\dot{\varphi} = B_0\dot{\varphi} - \mu_1\partial_z\dot{\varphi}$, one has

$$I = \{\beta_0 - \mu_1\beta_1 + \mu_1^2\beta_2\}(\partial_z\dot{\varphi})^2 + \{\beta_1 - 2\mu_1\beta_2\}\partial_z\dot{\varphi}B_0\dot{\varphi} + \beta_2(B_0\dot{\varphi})^2 \tag{4.21}$$

From the Lemma A.4 in Appendix, we have

$$\beta_0 - \mu_1\beta_1 + {\mu_1}^2\beta_2 = \frac{\gamma-1}{8}b_0 + O(b_0^2) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \tag{4.22}$$

Additionally, by the Lemma A.5 in Appendix, we have

$$\begin{aligned} \beta_2 &= -\frac{\gamma-1}{2}b_0^3 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \\ \beta_1 - 2\mu_1\beta_2 &= \sqrt{\frac{\gamma-1}{2}}\theta_0 b_0^3 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \end{aligned} \tag{4.23}$$

Using $\partial_z\dot{\varphi}\mathcal{B}_0\dot{\varphi} \geq -\frac{1}{2}(b_0(\mathcal{B}_0\dot{\varphi})^2 + \frac{1}{b_0}(\partial_z\dot{\varphi})^2)$, then substituting (4.22) and (4.23) into (4.21) we get

$$\begin{aligned} I \geq &(\frac{\gamma-1}{8}b_0 + O(b_0^2) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}))(\partial_z\dot{\varphi})^2 \\ &- (\frac{\gamma-1}{2}b_0^3 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}))(\mathcal{B}_0\dot{\varphi})^2 \end{aligned} \tag{4.24}$$

Finally, from Lemma A.6 in Appendix, we have

$$a(s_0) = e^{O(b_0^2)+O(q_0^{-\frac{2}{\gamma-1}})+O(\frac{1}{q_0^2})} \tag{4.25}$$

Hence substituting (4.24) and (4.25) into (4.20) yields the inequality (4.19).

Summing up the estimates in Step 1 to Step 5 and noting that $a(s)$ is bounded and $|a'(s)| \geq Cq_0^{\frac{2}{\gamma-1}}$ in $[b_0, s_0 + \eta_0]$ and using Lemma A.8 in Appendix we have from (4.2)

$$\begin{aligned} &\frac{C_1}{\sqrt{T}}\int_{b_0T}^{\chi(T)} |\nabla_{r,z}\dot{\varphi}(r,T)|^2 dr + C_2\iint_{D_T} z^{-\frac{3}{2}}|\nabla_{r,z}\dot{\varphi}|^2 drdz + (\frac{\gamma-1}{8}b_0^2 + O(b_0^3) + O(\varepsilon_0) + O(q_0^{-\frac{2}{\gamma-1}}) \\ &+ O(\frac{1}{q_0^2}))\int_{\Gamma_T} z^{-\frac{1}{2}}|\partial_z\dot{\varphi}|^2 dl + (\frac{\sqrt{2(\gamma-1)}}{4}b_0^2 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}))\int_{B_T} z^{-\frac{1}{2}}|\partial_z\dot{\varphi}|^2 dl \\ &\leq \iint_{D_T} z^{-\frac{3}{2}} L\dot{\varphi}M\dot{\varphi}drdz + (\frac{\gamma-1}{2}b_0^4 + O(b_0^5) + O(\varepsilon_0) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}))\int_{\Gamma_T} z^{-\frac{1}{2}}(B_0\dot{\varphi})^2 dl \\ &+ C_6\int_{b_0}^{\chi(1)} (|\dot{\varphi}(r,1)|^2 + |\partial_z\dot{\varphi}(r,1)|^2)dr \end{aligned} \tag{4.26}$$

where the constants $C_1, C_2$ and $C_6$ are all independent of $q_0$ and $\varepsilon_0$ thanks to the choice of $a(s)$ and $b(s)$. Therefore Theorem 4.1 is proved.

## §5. The proof of Theorem 1.1 for the case $\sigma(z) = b_0 z$

In order to prove Theorem 1.1 with the boundary $r = b_0 z$, we first derive the following higher order energy estimates.

**Theorem 5.1.** Assume that $\dot\varphi \in C^{k_0}(D_T)$ and $\xi(z) \in C^{k_0}[1,T]$ with $k_0 \geq 5$ is a solution of (3.1), (3.2), (3.5)-(3.7). In addition, $|\xi(z)| + |z\xi'(z)| \leq C\varepsilon_0$ in $[1,T]$, $\sum\limits_{0\leq l\leq [\frac{k_0}{2}]+1} z^l|\nabla_{r,z}^{l+1}\dot\varphi(r,z)| \leq C\varepsilon_0$, and $\varepsilon_0 > 0$ is sufficiently small, then

$$\begin{aligned}
&\int_{b_0 T}^{\chi(T)} \sum_{0\leq l\leq k_0-1} T^{2l-\frac{1}{2}}|\nabla_{r,z}^{l+1}\dot\varphi(r,T)|^2 dr + \iint_{D_T} \sum_{0\leq l\leq k_0-1} z^{2l-\frac{3}{2}}|\nabla_{r,z}^{l+1}\dot\varphi|^2 drdz \\
&+ \int_{\Gamma_T} \sum_{0\leq l\leq k_0-1} z^{2l-\frac{1}{2}}|\nabla_{r,z}^{l+1}\dot\varphi|^2 dl + \int_{B_T} \sum_{0\leq l\leq k_0-1} z^{2l-\frac{1}{2}}|\nabla_{r,z}^{l+1}\dot\varphi|^2 dl \\
&\leq C(q_0,b_0,\gamma)\Big(\int_{b_0}^{\chi(1)} \sum_{0\leq l\leq k_0} |\nabla_{r,z}^{l}\dot\varphi(r,1)|^2 dr + \dot\varphi^2(\chi(1),1) + \dot\varphi^2(b_0,1) + \xi^2(1)\Big),
\end{aligned} \tag{5.1}$$

here and below $C(q_0,b_0,\gamma) > 0$ denotes a generic constant depending on $q_0, b_0$ and $\gamma$.

Next, we turn to the main arguments for the proof of Theorem 5.1. As in [4] or [6], we will use the vector fields which are tangent to the surface of the cone and nearly tangential to the shock front. Then we use the standard commutation argument to raise the order of the energy estimate. The difference from the usual commutation argument is that the radial vector field is only nearly tangential to the shock front boundary, and thus there will appear some error terms caused by the perturbation of the shock front to be estimated. Furthermore, we cannot adapt the analysis in [4], since we have to deal with Neumann type boundary condition on the fixed boundary, while [4] treats Dirichlet type boundary condition so that Poincare type inequality (see [4]Lemma 1.), which is one of the key elememts in the analysis in [4], is available. However, by making use of the delicate energy estimate in §4, we will be able to drive the desired estimates. To prove this theorem, we first need an elementary estimate.

**Lemma 5.1** Assume that $\dot\varphi$ is a $C^{k_0}$ solution, then there is a constant independent of $\dot\varphi$ and $T$, so that

$$\sum_{0\leq l\leq k_0-1} z^l|\nabla_{r,z}^{l+1}\dot\varphi| \leq C(q_0,b_0,\gamma)\sum_{0\leq l\leq k_0-1} |\nabla_{r,z}S^l\dot\varphi| \quad in \quad D_T, \tag{5.2}$$

where $S = z\partial_z + r\partial_r$.

**Proof.** This lemma can be proved as in [4] or [6]. But for the convenience to readers, we verify the case for $k_0 = 2$, the general case can be completed by the inductive method. Since the differential operators $\partial_r^2, \partial_{zr}^2$ can be expressed as follows

$$\begin{aligned}
\partial_r^2 &= \frac{\partial_r S}{r} - \frac{z}{r^2}\partial_z S + \frac{z^2}{r^2}\partial_z^2 + \frac{z}{r^2}\partial_z - \frac{\partial_r}{r} \\
\partial_{zr}^2 &= \frac{\partial_z S}{r} - \frac{z}{r}\partial_z^2 - \frac{\partial_z}{r}
\end{aligned}$$

meanwhile both the boundary of the surface and the shock front are not characteristic, by using the equation (3.1) we can solve $\partial_z^2\dot\varphi$. Indeed, from the equation (3.1) we have

$$\begin{aligned}
&(1 - f_{11} - (2P_1 - f_{12})\frac{z}{r} + (P_2 - f_{22})\frac{z^2}{r^2})\partial_z^2\dot\varphi = \frac{f_{12} - 2P_1}{r}(\partial_z S\dot\varphi - \partial_z\dot\varphi) + \frac{f_{22} - P_2}{r}(\partial_r S\dot\varphi \\
&- \frac{z}{r}\partial_z S\dot\varphi + \frac{z}{r}\partial_z\dot\varphi - \partial_r\dot\varphi) - P_3\partial_z\dot\varphi - P_4\partial_r\dot\varphi
\end{aligned}$$

Since $P_1^2 - P_2 > 0$ and $\varepsilon_0 > 0$ is very small, then

$$z|\partial_z^2\dot\varphi| \leq C(|\nabla_{r,z}S\dot\varphi| + |\nabla_{r,z}\dot\varphi|)$$

Hence from the above expressions we know (5.2) holds for $k_0 = 2$.

The general case can be derived similarly, so we omit the details.

Return to the proof of Theorem 5.1. Since the vector field $S$ is tangent to the boudary $r = b_0 z$, then $\partial_r S^m \dot{\varphi} = b_0 \partial_z S^m \dot{\varphi}$ on $r = b_0 z$ in view of the boundary condition (3.2), so we can apply Theorem 4.1 and the Remark 4.2 to $S^m \dot{\varphi} (0 \le m \le k_0 - 1)$ (at this time, we can contemporarily neglect the concrete expressions of the constants in (4.1)), we have

$$\begin{aligned}
&\frac{1}{\sqrt{T}} \int_{b_0 T}^{\chi(T)} \sum_{0\le m\le k_0-1} |\nabla_{r,z} S^m \dot{\varphi}(r,T)|^2 dr + \iint_{D_T} z^{-\frac{3}{2}} \sum_{0\le m\le k_0-1} |\nabla_{r,z} S^m \dot{\varphi}|^2 drdz \\
&+ \int_{\Gamma_T} z^{-\frac{1}{2}} \sum_{0\le m\le k_0-1} |\nabla_{r,z} S^m \dot{\varphi}|^2 dl + \int_{B_T} z^{-\frac{1}{2}} \sum_{0\le m\le k_0-1} |\nabla_{r,z} S^m \dot{\varphi}|^2 dl \\
&\le C(q_0, b_0, \gamma) ( \iint_{D_T} z^{-\frac{3}{2}} \sum_{0\le m\le k_0-1} \mathcal{L} S^m \dot{\varphi} \mathcal{M} S^m \dot{\varphi} drdz + \int_{\Gamma_T} z^{-\frac{1}{2}} \sum_{0\le m\le k_0-1} (\mathcal{B}_0 S^m \dot{\varphi})^2 dl \\
&+ \int_{b_0}^{\chi(1)} \sum_{0\le m\le k_0} |\nabla_{r,z}^m \dot{\varphi}(r,1)|^2 dr)
\end{aligned} \tag{5.3}$$

To estimate the first term in the right hand side of (5.3), we need an explicit representation of $\mathcal{L} S^m \dot{\varphi}$. Thanks to $SP_1(\frac{r}{z}) = SP_2(\frac{r}{z}) = 0$ and $S(\frac{1}{r}) = -\frac{1}{r}$, we have $\mathcal{L} S \dot{\varphi} = S \mathcal{L} \dot{\varphi} - 2\mathcal{L} \dot{\varphi}$. It follows from the equation (3.1) that

$$\begin{aligned}
\mathcal{L} S^m \dot{\varphi} = \sum_{0\le l\le m} C_l \Big\{ \sum_{l_1+l_2\le l} C_{l_1 l_2} \Big( & S^{l_1}(f_{11}) \partial_z^2 S^{l_2} \dot{\varphi} + S^{l_1}(f_{12}) \partial_{zr}^2 S^{l_2} \dot{\varphi} + S^{l_1}(f_{22}) \partial_r^2 S^{l_2} \dot{\varphi} \\
& + \frac{(-1)^{l_1}}{r} S^{l_2}(f_0) \Big) \Big\},
\end{aligned} \tag{5.4}$$

where $f_{i,j}, f_0$ are the functions appeared in (3.1). By the properties of $f_{ij}$, $f_0$ and the assumptions in Theorem 5.1, one can show that for $m \le k_0 - 1$

$$|S^{l_1} f_{ij}| \le C \sum_{m\le k_0-1} |\nabla_{r,z} S^m \dot{\varphi}|, \qquad |S^{l_1}(f_0)| \le C \sum_{m\le k_0-1} |\nabla_{r,z} S^m \dot{\varphi}|^2 \tag{5.5}$$

We will treat $\iint_{D_T} z^{-\frac{3}{2}} S^{l_1}(f_{11}) \partial_z^2 S^{l_2} \dot{\varphi} \mathcal{M} S^m \dot{\varphi} drdz$ only, because the other terms can be disposed similarly. There are two cases:

if $l_2 \le m - 1$, from Lemma 5.1 and assumptions in Theorem 5.1 one can get

$$|S^{l_1}(f_{11}) \partial_z^2 S^{l_2} \dot{\varphi} \mathcal{M} S^m \dot{\varphi}| \le C\varepsilon_0 \sum_{m\le k_0-1} |\nabla_{r,z} S^m \dot{\varphi}|^2 \tag{5.6}$$

if $l_1 = 0, l_2 = m$, then

$$\begin{aligned}
S^{l_1}(f_{11}) \partial_z^2 S^{l_2} \dot{\varphi} \mathcal{M} S^m \dot{\varphi} &= \partial_z (f_{11} B \partial_z S^m \dot{\varphi} \partial_r S^m \dot{\varphi} - \frac{1}{2} A f_{11} (\partial_z S^m \dot{\varphi})^2) - \frac{1}{2} \partial_r (f_{11} B (\partial_z S^m \dot{\varphi})^2) \\
&+ \frac{1}{2} (\partial_r (f_{11} B) - \partial_z (f_{11} A)) (\partial_z S^m \dot{\varphi})^2 - \partial_z (f_{11} B) \partial_z S^m \dot{\varphi} \partial_r S^m \dot{\varphi}.
\end{aligned} \tag{5.7}$$

Hence by the integration by parts we get

$$\iint_{D_T} z^{-\frac{3}{2}} \sum_{0\le m\le k_0-1} \mathcal{L}S^m\dot\varphi \mathcal{M}S^m\dot\varphi drdz \le C\varepsilon_0\Big(\frac{1}{\sqrt{T}}\int_{b_0T}^{\chi(T)} \sum_{m\le k_0-1} |\nabla_{r,z}S^m\dot\varphi(r,T)|^2dr$$
$$+\iint_{D_T} z^{-\frac{3}{2}} \sum_{0\le m\le k_0-1} |\nabla_{r,z}S^m\dot\varphi|^2drdz + \int_{\Gamma_T} z^{-\frac{1}{2}} \sum_{0\le m\le k_0-1} |\nabla_{r,z}S^m\dot\varphi|^2dl$$
$$+\int_{B_T} z^{-\frac{1}{2}} \sum_{0\le m\le k_0-1} |\nabla_{r,z}S^m\dot\varphi|^2dl + \int_{b_0}^{\chi(1)} \sum_{0\le m\le k_0} |\nabla^m_{r,z}\dot\varphi(r,1)|^2dr\Big) \tag{5.8}$$

Next, we estimate the second term on the right hand side of (5.3), that is $\int_{\Gamma_T} z^{-\frac{1}{2}}\sum_m(\mathcal{B}_0S^m\dot\varphi)^2dl$, which is a major term, because it involves the boundary of shock front. Write $\mathcal{B}_0S^m\dot\varphi = [\mathcal{B}_0, S^m]\dot\varphi + (S^m - S^m_\Gamma)\mathcal{B}_0\dot\varphi + S^m_\Gamma\mathcal{B}_0\dot\varphi$ , we estimate each term separately. The first term has the form

$$[\mathcal{B}_0, S^m]\dot\varphi = \sum_{0\le l\le m-1} C_l S^l\mathcal{B}_0\dot\varphi \tag{5.9}$$

To estimate other two terms, we notice that from the equation (3.6)

$$\sum_{0\le m\le k_0-1} z^m|\partial_z^m\xi| \le C(\sum_{0\le m\le k_0-2} z^m|\nabla^{m+1}_{r,z}\dot\varphi| + |\xi|) \qquad on \qquad r=\chi(z) \tag{5.10}$$

Hence by the assumptions in Theorem 5.1, we have

$$\sum_{0\le m\le [\frac{k_0}{2}]+1} z^m|\partial_z^m\xi| \le C\varepsilon_0 \tag{5.11}$$

In addition, the equation (3.5) yields

$$S^m_\Gamma\mathcal{B}_0\dot\varphi + \mu_2S^m_\Gamma\xi = S^m_\Gamma\kappa_0(\xi,\nabla_{r,z}\dot\varphi) \qquad on \qquad r=\chi(z) \tag{5.12}$$

where $S_\Gamma = z\partial_z + z\chi'(z)\partial_r$ is tangent to the shock surface $r=\chi(z)$. It should be noted that $|\mu_2|$ is a large constant with the same order as $q_0$. Using (5.11) and (5.12), for $m\le k_0-1$ we have the following estimate:

$$|S^m_\Gamma\mathcal{B}_0\dot\varphi| \le C(q_0\sum_{0\le l\le m} z^l|\partial_z^l\xi| + \varepsilon_0\sum_{0\le l\le m} z^l|\nabla^{l+1}_{r,z}\dot\varphi|). \tag{5.13}$$

As in the Lemma 10 of [4], one can prove that

$$|(S^m - S^m_\Gamma)\mathcal{B}_0\dot\varphi| \le C\varepsilon_0(\sum_{0\le l\le m} z^l|\nabla^{l+1}_{r,z}\dot\varphi| + |\xi|). \tag{5.14}$$

Now collecting (5.9), (5.13) and (5.14) and using (5.10) and Lemma 5.1 one can get that

$$\int_{b_0T}^{\chi(T)} \sum_{0\le l\le k_0-1} T^{2l-\frac{1}{2}}|\nabla^{l+1}_{r,z}\dot\varphi(r,T)|^2dr + \iint_{D_T}\sum_{0\le l\le k_0-1} z^{2l-\frac{3}{2}}|\nabla^{l+1}_{r,z}\dot\varphi|^2drdz$$
$$+\int_{\Gamma_T}\sum_{0\le l\le k_0-1} z^{2l-\frac{1}{2}}|\nabla^{l+1}_{r,z}\dot\varphi|^2dl + \int_{B_T}\sum_{0\le l\le k_0-1} z^{2l-\frac{1}{2}}|\nabla^{l+1}_{r,z}\dot\varphi|^2dl$$
$$\le C(q_0,b_0,\gamma)\Big(\int_{\Gamma_T}\sum_{0\le l\le k_0-2} z^{2l-\frac{1}{2}}|\nabla^{l+1}_{r,z}\dot\varphi|^2dl + \int_{\Gamma_T} z^{-\frac{1}{2}}|\xi|^2dl + \int_{b_0}^{\chi(1)}\sum_{0\le l\le k_0}|\nabla^l_{r,z}\dot\varphi(r,1)|^2dr\Big) \tag{5.15}$$

In the special case $k_0 = 1$, by the estimate (4.1) and the equation (3.5) and the inequality (5.8),(5.15) actually is

$$
\begin{aligned}
&\frac{C_1}{\sqrt{T}}\int_{b_0T}^{\chi(T)}|\nabla_{r,z}\dot\varphi(r,T)|^2dr + C_2\iint_{D_T} z^{-\frac{3}{2}}|\nabla_{r,z}\dot\varphi|^2drdz + C_3\int_{\Gamma_T} z^{-\frac{1}{2}}|\partial_z\dot\varphi|^2dl\\
&+(\frac{\sqrt{2(\gamma-1)}}{4}b_0^2 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}))\int_1^T z^{-\frac{1}{2}}|\partial_z\dot\varphi(b_0z,z)|^2dz\\
&\le q_0^2(\frac{\gamma-1}{8}b_0^4 + O(b_0^5) + O(\varepsilon_0) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}))\int_{\Gamma_T} z^{-\frac{1}{2}}|\xi|^2dl\\
&+C_6\int_{b_0}^{\chi(1)}(|\dot\varphi(r,1)|^2 + |\partial_z\dot\varphi(r,1)|^2)dr
\end{aligned}
\tag{5.16}
$$

Here $C_1, C_2, C_3$ and $C_6$ are generic constants independent of $q_0$ and $\varepsilon_0$.

It follows from (5.15) and the inductive argument that the crucial point to prove (5.1) is to estimate the first term in the right hand side of (5.16). Note that the first term in the right side of (5.16) has a large factor $q_0^2$. We will absorb this term into the left hand side of (5.16).

Indeed, by the assumption on $\xi(z)$, we have

$$
\int_{\Gamma_T} z^{-\frac{1}{2}}|\xi(z)|^2dl = (1 + O(b_0^2) + O(\varepsilon_0^2) + O(q_0^{-\frac{2}{\gamma-1}}))\int_1^T z^{-\frac{1}{2}}|\xi(z)|^2dz \tag{5.17}
$$

In order to estimate $\int_1^T z^{-\frac{1}{2}}|\xi(z)|^2dz$, we treat it as follows

$$
\begin{aligned}
&\int_1^T z^{-\frac{1}{2}}|\xi(z)|^2dz = \int_1^T z^{-\frac{5}{2}}|z\xi(z)|^2dz\\
&\le (1+\frac{1}{b_0^2})\int_1^T z^{-\frac{1}{2}}|z\xi(z) + \frac{1}{U_+}\dot\varphi(\chi(z),z)|^2dz + (1+b_0^2)\int_1^T z^{-\frac{5}{2}}|\frac{1}{U_+}\dot\varphi(\chi(z),z)|^2dz\\
&\equiv I + II
\end{aligned}
\tag{5.18}
$$

Here and below we often use the inequality $(x+y)^2 \le (1+\frac{1}{b_0^2})x^2 + (1+b_0^2)y^2$.

By use of the Hardy type inequality in Lemma A.9 of Appendix, the equation (3.6) and the assumptions in Theorem 5.1, we have

$$
\begin{aligned}
|I| &\le \frac{16}{9}(1+\frac{1}{b_0^2})(1+b_0^2)\int_1^T z^{-\frac{1}{2}}|\partial_z(z\xi(z) + \frac{1}{U_+}\dot\varphi(\chi(z),z))|^2dz + C(b_0,\gamma)(\xi^2(1) + \dot\varphi^2(\chi(1),1))\\
&\le C(b_0,\gamma)(\varepsilon_0^2\int_1^T z^{-\frac{1}{2}}(|\xi(z)|^2 + |\nabla_{r,z}\dot\varphi(\chi(z),z)|^2)dz + \xi^2(1) + \dot\varphi^2(\chi(1),1))
\end{aligned}
\tag{5.19}
$$

Now we decompose $II \le II_1 + II_2$ so that $II$ can associate with the integral on $r = b_0z$ and the interior of $D_T$, where

$$
\begin{aligned}
II_1 &= (1+\frac{1}{b_0^2})\frac{(1+b_0^2)}{U_+^2}\int_1^T z^{-\frac{5}{2}}|\dot\varphi(\chi(z),z) - \dot\varphi(b_0z,z)|^2dz\\
II_2 &= \frac{(1+b_0^2)^2}{U_+^2}\int_1^T z^{-\frac{5}{2}}|\dot\varphi(b_0z,z)|^2dz
\end{aligned}
$$

$II_1$ can be treated as follows

$$\begin{aligned}|II_1| &= \frac{(1+b_0^2)^4}{b_0^4 q_0^2}(1+O(q_0^{-\frac{2}{\gamma-1}}))\int_1^T z^{-\frac{5}{2}}(\int_{b_0 z}^{\chi(z)} \partial_r\dot\varphi(r,z))^2 dz \\ &\le \frac{(1+b_0^2)^4}{b_0^4 q_0^2}(1+O(q_0^{-\frac{2}{\gamma-1}}))\int_1^T z^{-\frac{3}{2}}(\int_{b_0 z}^{\chi(z)} |\partial_r\dot\varphi(r,z)|^2 dr)\frac{\chi(z)-b_0 z}{z}dz \\ &\le \frac{(1+b_0^2)^4}{b_0^4 q_0^2}(1+O(q_0^{-\frac{2}{\gamma-1}}))(O(\varepsilon_0)+O(q_0^{-\frac{2}{\gamma-1}}))\iint_{D_T} z^{-\frac{3}{2}}|\partial_r\dot\varphi(r,z)|^2 drdz \qquad (5.20)\end{aligned}$$

Using again the Hardy type inequality in Lemma A.9 and the boundary condition (3.2), we have

$$\begin{aligned}|II_2| &\le \frac{16(1+b_0^2)^5}{9b_0^2 q_0^2}(1+O(q_0^{-\frac{2}{\gamma-1}}))\int_1^T z^{-\frac{1}{2}}|b_0\partial_r\dot\varphi(b_0 z,z)+\partial_z\dot\varphi(b_0 z,z)|^2 dz + C(b_0,\gamma)\dot\varphi^2(b_0,1) \\ &= \frac{16(1+b_0^2)^7}{9b_0^2 q_0^2}(1+O(q_0^{-\frac{2}{\gamma-1}}))\int_1^T z^{-\frac{1}{2}}|\partial_z\dot\varphi(b_0 z,z)|^2 dz + C(b_0,\gamma)\dot\varphi^2(b_0,1) \qquad (5.21)\end{aligned}$$

Substituting (5.21), (5.20), (5.19) into (5.18), (5.17) and (5.16), for the fixed $b_0$ and $\frac{1}{q_0}$ which are very small but $\frac{1}{q_0}$ is much smaller than $b_0$ and the arbitrary smallness of $\varepsilon_0$, we have

$$\begin{aligned}&\frac{C_1}{\sqrt{T}}\int_{b_0 T}^{\chi(T)} |\nabla_{r,z}\dot\varphi(r,T)|^2 dr + C_2\iint_{D_T} z^{-\frac{3}{2}}|\nabla_{r,z}\dot\varphi|^2 drdz + C_3\int_{\Gamma_T} z^{-\frac{1}{2}}|\nabla_{r,z}\dot\varphi|^2 dl \\ &+(\frac{\sqrt{2(\gamma-1)}}{4}b_0^2+O(b_0^4)+O(q_0^{-\frac{2}{\gamma-1}})+O(\frac{1}{q_0^2}))\int_1^T z^{-\frac{1}{2}}|\partial_z\dot\varphi(b_0 z,z)|^2 dz \\ &\le \frac{16}{9}(\frac{\gamma-1}{8}b_0^2+O(b_0^3)+O(\varepsilon_0)+O(q_0^{-\frac{2}{\gamma-1}})+O(\frac{1}{q_0^2}))\int_1^T z^{-\frac{1}{2}}|\partial_z\dot\varphi(b_0 z,z)|^2 dz \\ &+C(q_0,b_0,\gamma)(\int_{b_0}^{\chi(1)} |\nabla_{r,z}\dot\varphi(r,1)|^2 dr+\xi^2(1)+\dot\varphi^2(\chi(1),1)+\dot\varphi^2(b_0,1)) \qquad (5.22)\end{aligned}$$

Where $C_i(1\le i\le 3)$ only depends on $b_0$ and $\gamma$.

Now we compare the coefficients of $\int_1^T z^{-\frac{1}{2}}|\partial_z\dot\varphi(b_0 z,z)|^2 dz$ in two sides of (5.22). Obviously, the main parts of the coefficients in the left hand side and right hand side are $\frac{\sqrt{2(\gamma-1)}}{4}b_0^2$ and $\frac{2(\gamma-1)}{9}b_0^2$ respectively.

Since $1<\gamma<3$, then we have

$$\frac{\sqrt{2(\gamma-1)}}{4} > \frac{2(\gamma-1)}{9} \qquad (5.23)$$

Therefore, for the small $b_0$, $\frac{1}{q_0}$ and $\varepsilon_0$, the right side term $\int_{\Gamma_T} z^{-\frac{1}{2}}|\partial_z\dot\varphi(b_0 z,z)|^2 dz$ in (5.22) can be absorbed by the corresponding left side. Hence we can get the estimates on $\int_{B_T} z^{-\frac{1}{2}}|\partial_z\dot\varphi|^2 dl$ and $\int_{\Gamma_T} z^{-\frac{1}{2}}|\partial_z\dot\varphi|^2 dl$ from (5.22). This also leads to the estimate on $\int_{\Gamma_T} z^{-\frac{1}{2}}\xi^2(z)dl$ through the insertion of (5.21),(5.20) and (5.19) into (5.18) and (5.17), that is, $\int_{\Gamma_T} z^{-\frac{1}{2}}|\mathcal{B}_0\dot\varphi|^2 dl$ is known. Noting the Remark 4.2, then we obtain (5.1) for $k_0=1$.

(5.15) shows that the higher order derivatives of $\dot\varphi$ can be dominated by its lower order derivatives, then by inductive argument we obtain (5.1). This completes the proof of Theorem 5.1.

**The proof of Theorem 1.1**

Based on the the energy estimate of higher order we can easily prove the global existence of shock by using the local existence theorem and the standard continuity extension method. The local existense of the solution of (1.8),(1.9) and (1.10) near the tip is achieved in [1] and [2], while for any given $z_1$, the solution of (1.8) with the initial data given on $z = z_1$ and the boundary conditions (1.9),(1.10) in $[z_1, z_1+\zeta]$ for some $\zeta > 0$ can be obtained by using the characteristics method ( see [13] ), provided that the initial data is smooth and satisfies the compatibility conditions. Moreover, if the perturbation of the initial data given on $z = z_0$ is small as $O(\varepsilon_0)$, the lifespan of the solution is at least as large as $C\varepsilon_0^{-1}$ with $C > 0$. Therefore, as long as we can establish that the maximun norm of $\dot{\varphi}, \xi$ and their derivatives decays with a rate in $z$, then the solution can be extended continuously to the whole domain. That is, by using the local existence theorem and the property of decay of the solution we can obtain the uniform boundedness of $\dot{\varphi}, \xi$ and their derivatives, and then extend the solution continuously from $z_0 < z < z_1$ to $z_0 < z < z_1 + \zeta$ with $\zeta$ being independent of $z_1$. Hence the key element of the proof of Theorem 1.1 is to show the decay rate of the maximum norm of $\dot{\varphi}, \xi$ and their derivatives.

It follows from the Sobolev's imbedding theorem (or see [4] Lemma 14) and the assumptions of Theorem 5.1 that for $b_0 z \le r \le \chi(z)$ and $1 \le z \le T$, one has

$$\sum_{0\le l\le k_0-2} |z^l \nabla_{r,z}^{l+1}\dot{\varphi}|^2 \le Cz^{-1}\int_{b_0 z}^{\chi(z)} \sum_{0\le l\le k_0-1} |z^l\nabla_{r,z}^{l+1}\dot{\varphi}(r,z)|^2 dr \tag{5.22}$$

On the other hand, (5.1) shows that

$$\int_{b_0 z}^{\chi(z)} \sum_{0\le l\le k_0-1} |z^l\nabla_{r,z}^{l+1}\dot{\varphi}(r,z)|^2 dr \le C\varepsilon_0^2 z^{\frac{1}{2}} \tag{5.23}$$

Hence $\sum_{0\le l\le k_0-2} |z^l \nabla_{r,z}^{l+1}\dot{\varphi}|^2 \le C\varepsilon_0^2 z^{-1/2}$ for $b_0 z \le r \le \chi(z)$ and $1 \le z \le T$. For $k_0 \ge 5$, one has $\sum_{l\le[\frac{k_0}{2}]+1} |z^l\nabla_{r,z}^{l+1}\dot{\varphi}| \le C\varepsilon_0 z^{-\frac{1}{4}}$. In addition, due to $k_0 - 2 \ge [\frac{k_0}{2}] + 1$, the equations (3.4) and (3.5) yield $|\xi(z)| + |z\xi'(z)| \le C\varepsilon_0 z^{-\frac{1}{4}}$. Noting that the constant $C$ is independent of $T$, we complete the proof of Theorem 1.1 under the addtional assumption that $\sigma(z) = b_0 z$ as mentioned above.

## §6. The treatment for the general boundary

In this section, we discuss the general case when the surface of the symmetric obstacle is curved. We change the boundary of curved cone into a straight one by the following coordinates transform.

$$\begin{cases} \tilde{z} = \frac{\sigma(z)}{b_0} \\ \tilde{r} = r \end{cases} \tag{6.1}$$

Under the transformation (6.1), we use the notations $\tilde{\varphi}$,$\dot{\tilde{\varphi}}$ and $\tilde{\chi}$ instead of $\varphi$, $\dot{\varphi}$ and $\chi$. Similar to the computations in §3, the equation (1.11) can be rewritten as:

$$\begin{aligned}
&\partial_{\tilde{z}\tilde{z}}^2\dot{\tilde{\varphi}} + 2P_1(\frac{\tilde{r}}{\tilde{z}})\partial_{\tilde{z}\tilde{r}}^2\dot{\tilde{\varphi}} + P_2(\frac{\tilde{r}}{\tilde{z}})\partial_{\tilde{r}\tilde{r}}^2\dot{\tilde{\varphi}} + P_3(\tilde{r},\tilde{z})\partial_{\tilde{z}}\dot{\tilde{\varphi}} + P_4(\tilde{r},\tilde{z})\partial_{\tilde{r}}\dot{\tilde{\varphi}} = f_{11}(\frac{\tilde{r}}{\tilde{z}}, \nabla_{\tilde{r},\tilde{z}}\dot{\tilde{\varphi}})\partial_{\tilde{z}\tilde{z}}^2\dot{\tilde{\varphi}} \\
&\quad + f_{12}(\frac{\tilde{r}}{\tilde{z}}, \nabla_{\tilde{r},\tilde{z}}\dot{\tilde{\varphi}})\partial_{\tilde{r}\tilde{z}}^2\dot{\tilde{\varphi}} + f_{22}(\frac{\tilde{r}}{\tilde{z}}, \nabla_{\tilde{r},\tilde{z}}\dot{\tilde{\varphi}})\partial_{\tilde{r}\tilde{r}}^2\dot{\tilde{\varphi}} + \frac{1}{\tilde{r}}f_0(\frac{\tilde{r}}{\tilde{z}}, \nabla_{\tilde{r},\tilde{z}}\dot{\tilde{\varphi}}) + (1-\frac{\sigma'(z)}{b_0})\{\tilde{f}_{11}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0})\partial_{\tilde{z}\tilde{z}}^2\tilde{\varphi} \\
&\quad + \tilde{f}_{12}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0})\partial_{\tilde{r}\tilde{z}}^2\tilde{\varphi} + \tilde{f}_{22}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0})\partial_{\tilde{r}\tilde{r}}^2\tilde{\varphi}\} + \tilde{f}_{01}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0})\sigma''(z)\partial_{\tilde{z}}\tilde{\varphi} \\
&\quad + \frac{1}{\tilde{r}}\tilde{f}_{02}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0})(1-\frac{\sigma'(z)}{b_0})\partial_{\tilde{r}}\tilde{\varphi}
\end{aligned} \tag{6.2}$$

where $P_1, P_2, P_3, P_4, f_{11}, f_{12}, f_{22}$ and $f_0$ are the same as in (3.1), $\tilde{f}_{ij}$ is smooth on its arguments. We emphasize here that $\frac{\sigma'(z)}{b_0}$ appears in the expression of $\tilde{f}_{i,j}$ as a whole argument. The fact that all derivatives of this quantity is small as shown in (1.16) will play the essential role in estimating the influence of the perturbation of the boundary later.

Set $\tilde{\xi}(\tilde{z}) = \frac{\tilde{\chi}(\tilde{z})-s_0\tilde{z}}{\tilde{z}}$. Then the boundary conditions (1.12),(1.13) and the continuity condition (1.14) take the forms

$$-b_0\dot{\tilde{\varphi}}_{\tilde{z}} + \dot{\tilde{\varphi}}_{\tilde{r}} + b_0(1-(\frac{\sigma'(z)}{b_0})^2)\dot{\tilde{\varphi}}_{\tilde{z}} + b_0(1-(\frac{\sigma'(z)}{b_0})^2)\hat{\varphi}_{\tilde{z}} + q_0(b_0-\sigma'(z)) = 0 \qquad on \qquad \tilde{r} = b_0\tilde{z} \quad (6.3)$$

$$\partial_{\tilde{r}}\dot{\tilde{\varphi}} + \mu_1\partial_{\tilde{z}}\dot{\tilde{\varphi}} + \mu_2\tilde{\xi} = \kappa_3(\tilde{\xi}, \nabla_{\tilde{r},\tilde{z}}\dot{\tilde{\varphi}}) + \tilde{f}_0(\tilde{\xi}, \nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0})(1-\frac{\sigma'(z)}{b_0}) \qquad on \quad \tilde{r} = \tilde{\chi}(\tilde{z}) \qquad (6.4)$$

$$(\tilde{z}\tilde{\xi})' + \frac{1}{U_+}(\partial_{\tilde{z}}\dot{\tilde{\varphi}}(\tilde{\chi}(\tilde{z}),\tilde{z}) + s_0\partial_{\tilde{r}}\dot{\tilde{\varphi}}(\tilde{\chi}(\tilde{z}),\tilde{z})) = \kappa_4(\tilde{\xi}, \nabla_{\tilde{r},\tilde{z}}\dot{\tilde{\varphi}}(\tilde{\chi}(\tilde{z}),\tilde{z})) \qquad (6.5)$$

where the meaning of the function $\kappa_3, \kappa_4$ is shown in Section 3, $\tilde{f}_0$ is a function similar to $\tilde{f}_{i,j}$.

In order to prove Theorem 1.1 for general $\sigma(z)$, we have to analyze the contribution due to the perturbation of the boundary. It turns out that we can modify the arguments in the proof of Theorem 5.1 slightly to deal this general case. As in (5.3), we first estimate the term $\iint_{D_T}\tilde{z}^{-\frac{3}{2}}\sum\limits_{0\le m\le k_0-1}\mathcal{L}\tilde{S}^m\dot{\tilde{\varphi}}\mathcal{M}\tilde{S}^m\dot{\tilde{\varphi}}d\tilde{r}d\tilde{z}$, where $\tilde{S} = \tilde{r}\partial_{\tilde{r}} + \tilde{z}\partial_{\tilde{z}}$. Note that the first four terms on the right side of (6.2) have been estimated in §5. Without loss of generality, we only estimate $\iint_{D_T}\tilde{z}^{-\frac{3}{2}}\tilde{S}^m((1-\frac{\sigma'(z)}{b_0})\tilde{f}_{11}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0})\partial^2_{\tilde{z}\tilde{z}}\tilde{\varphi})\mathcal{M}\tilde{S}^m\dot{\tilde{\varphi}}d\tilde{r}d\tilde{z}$ and $\iint_{D_T}\tilde{z}^{-\frac{3}{2}}\tilde{S}^m(\tilde{f}_{01}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0})\sigma''(z)\partial_{\tilde{z}}\tilde{\varphi})\ \mathcal{M}\tilde{S}^m\dot{\tilde{\varphi}}d\tilde{r}d\tilde{z}$, the other terms can be analyzed similarly.

To estimate the integrals, we use the following decomposition:

$$(1-\frac{\sigma'(z)}{b_0})\tilde{f}_{11}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0})\partial^2_{\tilde{z}\tilde{z}}\tilde{\varphi} = I_1 + II_1 + III_1$$
$$\tilde{f}_{01}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0})\sigma''(z)\partial_{\tilde{z}}\tilde{\varphi} = I_2 + II_2 + III_2$$

where

$$I_1 = (1-\frac{\sigma'(z)}{b_0})\left(\tilde{f}_{11}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0}) - \tilde{f}_{11}(\nabla_{\tilde{r},\tilde{z}}\hat{\varphi}, \frac{\sigma'(z)}{b_0})\right)\partial^2_{\tilde{z}\tilde{z}}\hat{\varphi}$$
$$II_1 = (1-\frac{\sigma'(z)}{b_0})\tilde{f}_{11}(\nabla_{\tilde{r},\tilde{z}}\hat{\varphi}, \frac{\sigma'(z)}{b_0})\partial^2_{\tilde{z}\tilde{z}}\hat{\varphi}$$
$$III_1 = (1-\frac{\sigma'(z)}{b_0})\tilde{f}_{11}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0})\partial^2_{\tilde{z}\tilde{z}}\dot{\tilde{\varphi}}$$
$$I_2 = \sigma''(z)(\tilde{f}_{01}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0}) - \tilde{f}_{01}(\nabla_{\tilde{r},\tilde{z}}\hat{\varphi}, \frac{\sigma'(z)}{b_0}))\partial_{\tilde{z}}\hat{\varphi}$$
$$II_2 = \sigma''(z)\tilde{f}_{01}(\nabla_{\tilde{r},\tilde{z}}\hat{\varphi}, \frac{\sigma'(z)}{b_0})\partial_{\tilde{z}}\hat{\varphi}$$
$$III_2 = \sigma''(z)\tilde{f}_{01}(\nabla_{\tilde{r},\tilde{z}}\tilde{\varphi}, \frac{\sigma'(z)}{b_0})\partial_{\tilde{z}}\dot{\tilde{\varphi}}$$

Note also that $|\sigma(z) - b_0 z| \le \varepsilon_0$ and $|z(z\partial_z)^k(\sigma'(z) - b_0)| \le \varepsilon_0$ for $0 \le k \le k_2 - 1$ with $k_2 \ge k_0 + 1$ due to (1.16), here $k_0$ is the number appeared in Theorem 5.1. Additionally, $\partial_{\tilde{r}}\hat{\varphi} = \hat{U}(\frac{\tilde{r}}{\tilde{z}})$ and $\partial_{\tilde{z}}\hat{\varphi} = \hat{W}(\frac{\tilde{r}}{\tilde{z}})$

are positively homogeneous of degree 0. Hence we have the following estimates for $m \le k_0 - 1$

$$|\tilde{S}^m I_1 \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}}| \le \frac{C\varepsilon_0}{\tilde{z}} \sum_{l \le m} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^l \dot{\tilde{\varphi}}|^2$$
$$|\tilde{S}^m II_1 \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}}| \le \frac{C\varepsilon_0}{\tilde{z}} \sum_{l \le m} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^l \dot{\tilde{\varphi}}|$$
$$|\tilde{S}^m I_2 \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}}| \le \frac{C\varepsilon_0}{\tilde{z}} \sum_{l \le m} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^l \dot{\tilde{\varphi}}|^2$$
$$|\tilde{S}^m II_2 \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}}| \le \frac{C\varepsilon_0}{\tilde{z}} \sum_{l \le m} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^l \dot{\tilde{\varphi}}|$$
$$|\tilde{S}^m III_2 \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}}| \le C\frac{\varepsilon_0}{\tilde{z}} \sum_{l \le m} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^l \dot{\tilde{\varphi}}|^2$$

In addition, the term $\tilde{S}^m III_1 \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}}$ can be treated similarly as in (5.5), (5.6) and (5.7) of §5. Using the inequality $\frac{\varepsilon_0}{\tilde{z}}|g| \le \eta |g|^2 + C(\eta)\frac{\varepsilon_0^2}{\tilde{z}^2}$, here $\eta > 0$ is an appropriate small constant, then these estimates and the integration by parts leads to

$$\begin{aligned}
\iint_{D_T} \tilde{z}^{-\frac{3}{2}} \sum_{0 \le m \le k_0 - 1} \mathcal{L}\tilde{S}^m \dot{\tilde{\varphi}} \mathcal{M} \tilde{S}^m \dot{\tilde{\varphi}} d\tilde{r} d\tilde{z} &\le O(\varepsilon_0)\Big(\frac{1}{\sqrt{T}} \int_{b_0 T}^{\tilde{\chi}(T)} \sum_{m \le k_0 - 1} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^m \dot{\tilde{\varphi}}(r,T)|^2 d\tilde{r} \\
&+ \iint_{D_T} \tilde{z}^{-\frac{3}{2}} \sum_{0 \le m \le k_0 - 1} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^m \dot{\tilde{\varphi}}|^2 d\tilde{r} d\tilde{z} + \int_{\Gamma_T} \tilde{z}^{-\frac{1}{2}} \sum_{0 \le m \le k_0 - 1} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^m \dot{\tilde{\varphi}}|^2 dl \\
&+ \int_{B_T} \tilde{z}^{-\frac{1}{2}} \sum_{0 \le m \le k_0 - 1} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^m \dot{\tilde{\varphi}}|^2 dl + \int_{b_0}^{\tilde{\chi}(1)} \sum_{0 \le m \le k_0} |\nabla^m_{\tilde{r},\tilde{z}} \dot{\tilde{\varphi}}(\tilde{r},1)|^2 d\tilde{r} + \varepsilon_0\Big) \\
&+ \frac{1}{2C(q_0,b_0,\gamma)} \iint_{D_T} \tilde{z}^{-\frac{3}{2}} \sum_{0 \le m \le k_0 - 1} |\nabla_{\tilde{r},\tilde{z}} \tilde{S}^m \dot{\tilde{\varphi}}|^2 d\tilde{r} d\tilde{z}
\end{aligned}$$

here $C(q_0, b_0, \gamma)$ is the constant in (5.3).

Secondly, as in §5 we need to estimate the term $\int_{\Gamma_T} \tilde{z}^{-\frac{1}{2}} \sum_{0 \le m \le k_0 - 1} |\mathcal{B}_0 \tilde{S}^m \dot{\tilde{\varphi}}|^2 dl$. Since the equations (6.4) and (6.5) are very similar to (3.5) and (3.6) respectively, then this term can be estimated by the same method in §5.

Finally, we treat the integral on the boundary $\tilde{r} = b_0 \tilde{z}$. Since the strict inequality (4.3) holds and the coefficient of the first perturbed term $b_0(1 - (\frac{\sigma'(z)}{b_0})^2)\dot{\tilde{\varphi}}_{\tilde{z}}$ in (6.3) is sufficiently small due to the assumption (1.16), additionally, the second perturbed term in (6.3) satisfies $\int_{B_T} \tilde{z}^{-\frac{1}{2}} |\tilde{S}^m (b_0(1 - (\frac{\sigma'(z)}{b_0})^2)\hat{\varphi}_{\tilde{z}}) + q_0(b_0 - \sigma'(z))|^2 dl \le O(\varepsilon_0^2) \int_1^T \tilde{z}^{-\frac{5}{2}} d\tilde{z} = O(\varepsilon_0^2)$, hence the change of the form (6.2) will not influence the validity of the estimate (5.1). So Theorem 1.1 is proved in general case.

## Appendix

**Lemma A.1** The coefficient of $a'(s)$ in (4.12) is positive.

**Proof.** Since for large $q_0$

$$D_1 = \frac{(\gamma-1)b_0^2(1+b_0^2)^2(1-\frac{1}{2}(\gamma-1)b_0^2)}{2(1-\frac{1}{2}(\gamma-1)b_0^2(1+b_0^2))^2} + O(\frac{1}{q_0^2}) + O(q_0^{-\frac{2}{\gamma-1}}) > 0$$
$$s - P_1(s) = -\frac{(\gamma-1)b_0^3(1+b_0^2)}{2-(\gamma-1)b_0^2(1+b_0^2)} + O(\frac{1}{q_0^2}) + O(q_0^{-\frac{2}{\gamma-1}}) < 0$$

then by use of $\tilde{\lambda}(s) < \lambda_2(s)$ we have

$$D_1 + (s - P_1)(\tilde{\lambda}(s) - P_1) \geq D_1 + (s - P_1)(\lambda_2(s) - P_1) = \sqrt{D_1}(s - \lambda_1(s)) > 0$$

Hence $1 + (s - P_1)(\tilde{\lambda}(s) - P_1)/D_1 > 0$.

**Lemma A.2.** The coefficient $A_0(s)$ in (4.14) is negative.

**Proof** Factorize $A_0(s) = A_0^1(s)A_0^2(s)$, where

$$A_0^1(s) = \frac{s(\tilde{\lambda}(s) - P_1)D_1}{D_1 + (s - P_1)(\tilde{\lambda}(s) - P_1)} + \frac{D_1}{\tilde{A}}s(s - P_1) + \frac{D_1}{\tilde{A}}s\sqrt{D_1}$$
$$A_0^2(s) = \frac{s(\tilde{\lambda}(s) - P_1)D_1}{D_1 + (s - P_1)(\tilde{\lambda}(s) - P_1)} + \frac{D_1}{\tilde{A}}s(s - P_1) - \frac{D_1}{\tilde{A}}s\sqrt{D_1}.$$

Since

$$\begin{aligned} A_0^1(s) &= sD_1\left(\frac{\tilde{\lambda}(s) - P_1}{D_1 + (s - P_1)(\tilde{\lambda}(s) - P_1)} + \frac{1}{\lambda_2(s) - s}\right) \\ &= \frac{sD_1^{\frac{3}{2}}(\tilde{\lambda}(s) - \lambda_1(s))}{(\lambda_2(s) - s)(D_1 + (s - P_1)(\tilde{\lambda}(s) - P_1))} \\ &\geq \frac{sD_1^{\frac{3}{2}}(s - \lambda_1(s))}{(\lambda_2(s) - s)(D_1 + (s - P_1)(\tilde{\lambda}(s) - P_1))} > 0 \\ A_0^2(s) &= \frac{sD_1^{\frac{3}{2}}(\tilde{\lambda}(s) - \lambda_2(s))}{(\lambda_2(s) - s)(D_1 + (s - P_1)(\tilde{\lambda}(s) - P_1))} < 0 \end{aligned}$$

then $A_0(s) < 0$.

**Lemma A.3.** (4.16) holds for $1 < \gamma < 3$ and large $q_0$.

**Proof** In fact, (4.16) is equivelent to the following inequality

$$\begin{aligned} &2D_1(P_1^2 - P_2 + (s - P_1)(\tilde{\lambda}(s) - P_1))\left|\frac{Y_3(s)}{a(s)}\right| < \Big|\tilde{A}(P_1^2 - P_2)(\tilde{\lambda}(s) + s\tilde{\lambda}'(s)) - \frac{Q_0}{a}\Big((\tilde{\lambda}(s) - P_1)P_2 s \\ &\quad - (2P_1 s - P_2)(\tilde{\lambda}(s)P_1 - P_2)\Big) + \frac{Q_1}{a}\Big((\tilde{\lambda}(s) - P_1)(s - 2P_1) + (\tilde{\lambda}(s)P_1 - P_2)\Big) \\ &\quad - \frac{Q_2}{a}\Big(s(\tilde{\lambda}(s)P_1 - P_2) - (\tilde{\lambda}(s) - P_1)P_2\Big)\Big| \end{aligned} \tag{A.1}$$

When $q_0$ is large, one gets from Lemma 2.2 and 2.3

$$\begin{aligned} P_1 &= \frac{b_0}{1 - \frac{1}{2}(\gamma - 1)b_0^2(1 + b_0^2)} + O(\frac{1}{q_0^2}) + O(q_0^{-\frac{2}{\gamma-1}}) \\ P_2 &= \frac{b_0^2(1 - \frac{1}{2}(\gamma - 1)(1 + b_0^2))}{1 - \frac{1}{2}(\gamma - 1)b_0^2(1 + b_0^2)} + O(\frac{1}{q_0^2}) + O(q_0^{-\frac{2}{\gamma-1}}) \\ P_1^2 - P_2 &= \frac{(\gamma - 1)b_0^2(1 + b_0^2)^2(1 - \frac{1}{2}(\gamma - 1)b_0^2)}{2(1 - \frac{1}{2}(\gamma - 1)b_0^2(1 + b_0^2)} + O(\frac{1}{q_0^2}) + O(q_0^{-\frac{2}{\gamma-1}}) \\ \lambda_2(s) - s &= \frac{\sqrt{\gamma - 1}b_0(1 + b_0^2)(\sqrt{2 - (\gamma - 1)b_0^2} + \sqrt{\gamma - 1}b_0^2)}{2 - (\gamma - 1)b_0^2(1 + b_0^2)} + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \\ \tilde{A} &= \frac{(\gamma - 1)b_0^2(1 + b_0^2)}{2 - (\gamma - 1)b_0^2(1 + b_0^2)} + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \end{aligned}$$

$$|P_1'| = |\frac{(q_0+\hat{W})\hat{U}'}{(q_0+\hat{W})^2-\hat{c}^2(s)} + \frac{2\hat{c}\hat{c}'(q_0+\hat{W})\hat{U}}{((q_0+\hat{W})^2-\hat{c}^2(s))^2} - \frac{((q_0+\hat{W})^2+\hat{c}^2)\hat{U}\hat{W}'}{((q_0+\hat{W})^2-\hat{c}^2(s))^2}| \le C$$

$$|P_2'| = |\frac{2\hat{U}\hat{U}'-2\hat{c}\hat{c}'}{(q_0+\hat{W})^2-\hat{c}^2(s)} + \frac{2(\hat{c}^2-\hat{U}^2)((q_0+\hat{W})\hat{W}'-\hat{c}\hat{c}')}{((q_0+\hat{W})^2-\hat{c}^2(s))^2}| \le C$$

$$|\tilde{P}_3| \le Cq_0^{\frac{2(\gamma-2)}{\gamma-1}}$$

$$|\tilde{P}_4| \le Cq_0^{\frac{2(\gamma-2)}{\gamma-1}}$$

$$|\frac{Q_0(s)}{a}|, |\frac{Q_1(s)}{a}|, |\frac{Q_2(s)}{a}|, |\frac{Y_3(s)}{a}| \le Cq_0^{\frac{2(\gamma-2)}{\gamma-1}}$$

$$|\tilde{\lambda}'(s)| \ge Cq_0^{\frac{2}{\gamma-1}}$$

Substituting the above expressions into (A.1), we find that the left hand side of (A.1) is less than $C(1+q_0^{\frac{2(\gamma-2)}{\gamma-1}})$, the right hand side of (A.1) is larger than $Cq_0^{\frac{2}{\gamma-1}}$. Therefore, (A.1) holds, if $q_0$ is large enough and $1<\gamma<3$. So (4.16) is proved.

**Lemma A.4** $\beta_0 - \mu_1\beta_1 + \mu_1^2\beta_2 = \frac{\gamma-1}{8}b_0 + O(b_0^2) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})$.

**Proof.** We denote $\beta_0 - \mu_1\beta_1 + \mu_1^2\beta_2 = I_1(s_0) + I_2(s_0)$, where

$$I_1(s_0) = P_1(s_0) - s_0 - \mu_1(P_2(s_0) - s_0^2) + \mu_1^2(P_2(s_0)s_0 - P_1(s_0)s_0^2)$$

$$I_2(s_0) = \theta_0 b_0^2(\lambda_2(s_0) - s_0)(-\frac{1}{2} + \mu_1 s_0 + \mu_1^2(\frac{P_2(s_0)}{2} - P_1(s_0)))$$

Since

$$P_1(s_0) - s_0 = \frac{(\gamma-1)b_0^3(1+b_0^2)}{2-(\gamma-1)b_0^2(1+b_0^2)} + O(\frac{1}{q_0^2}) + O(q_0^{-\frac{2}{\gamma-1}}) = \frac{\gamma-1}{2}b_0^3 + O(b_0^5) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})$$

$$P_2(s_0) - s_0^2 = -\frac{(\gamma-1)b_0^2(1-b_0^2)(1+b_0^2)}{2-(\gamma-1)b_0^2(1+b_0^2)} + O(\frac{1}{q_0^2}) + O(q_0^{-\frac{2}{\gamma-1}}) = -\frac{\gamma-1}{2}b_0^2 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})$$

$$P_2(s_0)s_0 - P_1(s_0)s_0^2 = -\frac{(\gamma-1)b_0^3(1+b_0^2)}{2-(\gamma-1)b_0^2(1+b_0^2)} + O(\frac{1}{q_0^2}) + O(q_0^{-\frac{2}{\gamma-1}}) = -\frac{\gamma-1}{2}b_0^3 + O(b_0^5) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})$$

then for much large $q_0$ and small $b_0$,

$$\begin{aligned} I_1(s_0) &= \frac{(\gamma-1)b_0(1+b_0^2)^3}{4(2-(\gamma-1)b_0^2(1+b_0^2))} + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \\ &= \frac{\gamma-1}{8}b_0 + O(b_0^3) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \end{aligned} \tag{A.2}$$

Additionally, from the expressions in the proof of Lemma A.3, we have

$$\lambda_2(s_0) - s_0 = \frac{\sqrt{2(\gamma-1)}}{2}b_0 + O(b_0^3) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})$$

$$P_1(s_0) = b_0 + O(b_0^3) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})$$

$$P_2(s_0) = \frac{3-\gamma}{2}b_0^2 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})$$

Noting $\mu_1 = \frac{1-b_0^2}{2b_0}(1+O(q_0^{-\frac{2}{\gamma-1}}))$, hence we can obtain

$$I_2(s_0) = O(b_0^2) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \tag{A.3}$$

Combining (A.2) and (A.3), we know that Lemma A.4 holds.

**Lemma A.5**

$$\beta_2 = -\frac{\gamma-1}{2}b_0^3 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})$$

$$\beta_1 - 2\mu_1\beta_2 = \sqrt{\frac{\gamma-1}{2}}\theta_0 b_0^3 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})$$

**Proof.** By the computation in the proof of Lemma A.4, we have

$$\begin{aligned}
\beta_2 &= P_2(s_0)s_0 - P_1(s_0)s_0^2 + \theta_0 b_0^2(\lambda_2(s_0)-s_0)(\frac{P_2(s_0)}{2} - P_1(s_0)) \\
&= -\frac{\gamma-1}{2}b_0^3 + \theta_0 b_0^2(\sqrt{\frac{\gamma-1}{2}}b_0 + O(b_0^3))(-b_0 + O(b_0^2)) + O(b_0^5) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \\
&= -\frac{\gamma-1}{2}b_0^3 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})
\end{aligned}$$

Similarly,

$$\begin{aligned}
\beta_1 - 2\mu_1\beta_2 &= P_2(s_0) - s_0^2 - \theta_0 b_0^2 s_0(\lambda_2(s_0)-s_0) - \frac{1-b_0^2}{b_0}\beta_2 + O(q_0^{-\frac{2}{\gamma-1}}) \\
&= -\frac{\gamma-1}{2}b_0^2 - \frac{1-b_0^2}{b_0}(-\frac{\gamma-1}{2}b_0^3 - \sqrt{\frac{\gamma-1}{2}}\theta_0 b_0^4) + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \\
&= \sqrt{\frac{\gamma-1}{2}}\theta_0 b_0^3 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})
\end{aligned}$$

Hence Lemma A.5 is proved.

**Lemma A.6** $a(s_0) = e^{O(b_0^2)+O(q_0^{-\frac{2}{\gamma-1}})+O(\frac{1}{q_0^2})}$.

**Proof.** We know that

$$a(s_0) = exp\{\int_{b_0}^{s_0}\left(\tilde{Q}_0(s) + \frac{D_1}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)}k(s)\right)ds\} \tag{A.4}$$

Since

$$\tilde{Q}_0(s) = \frac{1}{D_1+(s-P_1)(\tilde{\lambda}(s)-P_1)}\left((P_1-s)\tilde{\lambda}'(s) + \frac{P_1-s}{s}\tilde{\lambda}(s) + \frac{(s-P_1)\tilde{a}_2(s)}{\tilde{A}s}\right) - \frac{\tilde{a}_1(s)+\tilde{a}_2(s)P_1 s}{\tilde{A}s^2})$$

and

$$\tilde{\lambda}'(s) = -\frac{\lambda_2(s_0)-s_0+O(b_0^3)}{s_0+\eta_0-b_0}$$

then by use of the computation in Lemma A.3, we have

$$\begin{aligned}
\tilde{Q}_0(s) &= \frac{1}{\frac{(\gamma-1)b_0^2}{2}+O(b_0^4)}\frac{1}{s_0+\eta_0-b_0}(-(P_1-s)(\lambda_2(s_0)-s_0) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})) \\
&= \frac{1}{s_0+\eta_0-b_0}(-\sqrt{\frac{\gamma-1}{2}}b_0^2 + O(b_0^3) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}))
\end{aligned}$$

Hence

$$\int_{b_0}^{s_0} \tilde{Q}_0(s)ds = O(b_0^2) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \tag{A.5}$$

Additionally, it is easy to get

$$\frac{D_1}{D_1 + (s-P_1)(\tilde{\lambda} - P_1)} = \frac{\frac{\gamma-1}{2}b_0^2 + O(b_0^4)}{\frac{\gamma-1}{2}b_0^2 + O(b_0^4)} + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) = 1 + O(b_0^2) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \tag{A.6}$$

By the expression of $k(s)$, and noting $A_2(s) = -1$, then we have

$$k(s) = \frac{2}{\sqrt{A_1^2(s) - 4A_0(s)} - A_1(s)}$$

From the below Lemma A.7, we know $A_1(s) \le -Cq_0^{\frac{2}{\gamma-1}}$, here $C > 0$ is a constant depending only on $b_0$ and $\gamma$. Hence $k(s)$ is bounded for large $q_0$. Combining this with (A.6), we have

$$\int_{b_0}^{s_0} \frac{D_1}{D_1 + (s-P_1)(\tilde{\lambda} - P_1)} k(s)ds = O(q_0^{-\frac{2}{\gamma-1}}) \tag{A.7}$$

Substituting (A.5) and (A.7) into (A.4), we heve

$$a(s_0) = e^{O(b_0^2) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})}$$

**Lemma A.7** $A_1(s) \le -Cq_0^{\frac{2}{\gamma-1}}$, here $C > 0$ is a constant depending only on $b_0$ and $\gamma$.
**Proof.** We denote $A_1(s) = 2sD_1A_1^1(s)A_1^2(s)$, where

$$A_1^1(s) = \frac{\tilde{\lambda} - P_1}{D_1 + (s-P_1)(\tilde{\lambda} - P_1)} + \frac{s - P_1}{\tilde{A}}$$
$$A_1^2(s) = \tilde{\lambda}(s) + s\tilde{\lambda}'(s) - \frac{\tilde{a}_2(s)}{\tilde{A}} + s(\tilde{\lambda}(s) - P_1)\tilde{Q}_0(s)$$

Since for small $b_0$, we have

$$\begin{aligned}\tilde{\lambda}(s) - s &= \frac{s_0 + \eta_0 - s}{s_0 + \eta_0 - b_0}((1 - \theta_0 b_0^2)\lambda_2(s_0) - s_0) + \frac{s - b_0}{s_0 + \eta_0 - b_0}\theta_0 b_0^2(\lambda_2(s_0) - s_0) \\ &\ge \sqrt{\frac{\gamma-1}{2}}\theta_0 b_0^3 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})\end{aligned}$$

hence

$$\begin{aligned}A_1^1(s) &= \frac{(\tilde{\lambda} - s) + (s - P_1)}{\frac{\gamma-1}{2}b_0^2 + O(b_0^4)} - \frac{\frac{\gamma-1}{2}b_0^3 + O(b_0^5)}{\frac{\gamma-1}{2}b_0^2 + O(b_0^4)} + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \\ &\ge \theta_0 b_0 \sqrt{\frac{2}{\gamma-1}} - 2b_0 + O(b_0^2) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) \\ &= 2b_0 + O(b_0^2) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}) > 0 \end{aligned} \tag{A.8}$$

Thanks to the choice of $\theta_0 = 2\sqrt{2(\gamma-1)}$.

For the term $A_1^2(s)$, we substitute the expression of $\tilde{Q}_0(s)$ into it, we have

$$A_1^2(s) = \tilde{\lambda}'(s)\Big(\frac{sD_1}{D_1+(s-P_1)(\tilde{\lambda}-P_1)} + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2})\Big)$$
$$= -\frac{\lambda_2(s_0)-s_0+O(b_0^3)}{s_0+\eta_0-b_0}(b_0+O(b_0^2)+O(q_0^{-\frac{2}{\gamma-1}})+O(\frac{1}{q_0^2})) < 0 \qquad (A.9)$$

Noting $0 < s_0+\eta_0-b_0 \le Cq_0^{-\frac{2}{\gamma-1}}$, then combining (A.8) and (A.9), we know the lemma holds.

**Lemma A.8** $b_0K_3 - K_4 = z(\partial_z\dot{\varphi})^2(\frac{\sqrt{2(\gamma-1)}}{4}b_0^2 + O(b_0^4) + O(q_0^{-\frac{2}{\gamma-1}}) + O(\frac{1}{q_0^2}))$ on the boundary $r = b_0z$.

**Proof.** By the expression of $b_0K_3 - K_4$ in Step 1 of §4, we have on $r = b_0z$:

$$b_0K_3 - K_4 = z(\partial_z\dot{\varphi})^2 b_0(\tilde{\lambda}(b_0)-b_0^2)(\frac{1}{2}+b_0^2+b_0^3P_1(b_0)-\frac{P_2(b_0)}{2}b_0^3)$$
$$= z(\partial_z\dot{\varphi})^2 b_0(\lambda_2(b_0)-b_0+O(b_0^3))(\frac{1}{2}+O(b_0^2)+O(q_0^{-\frac{2}{\gamma-1}})+O(\frac{1}{q_0^2}))$$
$$= z(\partial_z\dot{\varphi})^2(\frac{\sqrt{2(\gamma-1)}}{4}b_0^2+O(b_0^4)+O(q_0^{-\frac{2}{\gamma-1}})+O(\frac{1}{q_0^2}))$$

**Lemma A.9**(Hardy type inequality) If $u(z) \in C^1[1,T]$, then

$$\int_1^T z^{-\frac{5}{2}}u^2(z)dz \le \frac{16}{9}(1+b_0^2)\int_1^T z^{-\frac{1}{2}}|u'(z)|^2dz + \frac{2}{3}(1+\frac{1}{b_0^2})u^2(1)$$

**Proof.** From [14] Theorem 330, the Hardy inequality is

$$\int_0^\infty z^{-r}F^p(z)dz \le (\frac{p}{r-1})^p\int_0^\infty z^{-r}(zf(z))^pdz \qquad (A.10)$$

where $p>1, r>1, f(z)\ge 0$ and $F(z) = \int_0^z f(s)ds$ for $z\in(0,+\infty)$.

As in [4]Lemma 12, we put $G(z) = \int_1^z |u'(s)|ds$, then $u^2(z) \le (G(z)+|u(1)|)^2 \le (1+b_0^2)G^2(z) + (1+\frac{1}{b_0^2})u^2(1)$, that is,

$$\int_1^T z^{-\frac{5}{2}}u^2(z)dz \le (1+b_0^2)\int_1^T z^{-\frac{5}{2}}G^2(z)dz + \frac{2}{3}(1+\frac{1}{b_0^2})u^2(1) \qquad (A.11)$$

Set $f(z) = u'(z)$ if $z\in[1,T]$, $f(z)=0$ if $z\in[0,1)\cup(T,\infty)$; $F(z) = \int_0^z |f(s)|ds$ for $z\in[0,+\infty)$. Let $r=\frac{5}{2}, p=2$ in (A.10), then we have

$$\int_0^\infty z^{-\frac{5}{2}}F^2(z)dz \le \frac{16}{9}\int_0^\infty z^{-\frac{1}{2}}f^2(z)dz \qquad (A.12)$$

In terms of the definitions of $f(z)$ and $F(z)$, (A.12) gives

$$\int_1^T z^{-\frac{5}{2}}G^2(z)dz \le \frac{16}{9}\int_1^T z^{-\frac{1}{2}}|u'(z)|^2dz \qquad (A.13)$$

Substituting (A.13) into (A.11), we know that Lemma A.9 holds.

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