A NOTE ON THE RIEMANN PROBLEM FOR GENERAL STRICTLY HYPERBOLIC SYSTEMS

STEFANO BIANCHINI

Abstract. We consider the construction and the properties of the Riemann solver for the hyperbolic system

$$u_t + f(u)_x = 0, \tag{0.1}$$

assuming only that Df is strictly hyperbolic. In the first part we prove a general regularity theorem on the admissible curves T_i of the i-family, depending on the number of inflection points of f: namely, if there is only one inflection point, T_i is $C^{1,1}$. Note that if the i-th eigenvalue of Df is genuinely nonlinear, by [7] T_i is $C^{2,1}$, and we give an example of a Lipschitz continuous admissible curve T_i if f has two inflection points.

In the second part, using the same analysis of [4], we show a general way for constructing the curves T_i, and we prove a stability result on the solution to the Riemann problem. In particular we prove the uniqueness of the admissible curves for (0.1).

Finally we apply the construction to various approximations to (0.1): vanishing viscosity, relaxation schemes and the semidiscrete upwind scheme. In particular, when the system is in conservation form, we obtain the existence of smooth travelling profiles for all small admissible jumps of (0.1).

1. Regularity of the admissible curves for general hyperbolic systems

Consider the $n \times n$ strictly hyperbolic system of conservation laws

$$u_t + f(u)_x = 0. \tag{1.1}$$

Let $\lambda_i(u)$ be the i-th eigenvector of $A(u) \doteq Df(u)$, and $r_i(u)$, $l_i(u)$ the corresponding right and left eigenvectors, normalized by

$$\big|r_i(u)\big| = 1, \qquad \big\langle l_j(u), r_i(u)\big\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Denote with $R_i(s,u)$, $S_i(s,u)$ the i-th rarefaction and shock curves starting in u, respectively. It is well known that these curves are defined for $s \in [-\delta_1, \delta_1]$, δ_1 small, and that can be parametrized by the i-th coordinates, i.e.

$$s = \big\langle l_i(u_0), R_i(s,u)\big\rangle, \qquad s = \big\langle l_i(u_0), S_i(s,u_0)\big\rangle.$$

See for example [5], [6].

In [8] it is shown how to construct the entropic self-similar solution a Riemann problem for (1.1), i.e. with the initial data

$$u(0,\cdot) = \begin{cases} u^- & x \leq 0 \\ u^+ & x > 0 \end{cases} \tag{1.2}$$

The fundamental step is the definition of the admissible i-curve $T_i(s,u)$ passing through u: each point $T_i(s,u)$ can be connected to u by a finite union of rarefactions and admissible shocks of the i-th family with increasing speed. Following [8], we say that the a shock $[u, S_i(\bar{s},u)]$ is admissible if satisfies the Rankine-Hugoniot conditions,

$$f\big(S_i(\bar{s},u)\big) - f(u) = \sigma(s,u)\big(S_i(\bar{s},u) - u\big), \tag{1.3}$$

Date: February 21, 2002.

1991 *Mathematics Subject Classification.* 35L65.

Key words and phrases. Hyperbolic systems, conservation laws, Riemann problem.

This work is supported by The Institute of Mathematical Sciences, The Chinese University of Hong Kong.

and for all $0 \le s \le \bar{s}$ we have that

$$\sigma\big(S_i(\bar{s}, u), u\big) \le \sigma\big(S_i(s, u), u\big). \tag{1.4}$$

In [8] it is shown that the above condition is equivalent to

$$\sigma\big(S_i(\bar{s}, u), u\big) \ge \sigma\big(S_i(s - \bar{s}, S_i(\bar{s}, u)), S_i(\bar{s}, u)\big), \tag{1.5}$$

and that $T_i(s, u)$ exists and it is unique in a neighborhood of u, under the assumption that the flux function f has a finite number of inflection points. The last condition means that for all $i = 1, \ldots, N$, the directional derivative of λ_i along $r_i(u)$, $D\lambda_i r_i(u)$, vanishes only on a finite number of hypersurfaces $\mathcal{F}_m$, $m = 1, \ldots, M_i$, and each $\mathcal{F}_i$ is transversal to the vector field $r_i(u)$.

As it is shown in [8], for fixed s, u^-, the point $T_i(s, u)$ can be constructed patching together a finite number of curves R_i and S_i. Moreover using the same proof of [8] or the results of Sections 2 and [4], one can prove that T_i is Lipschitz continuous. The following example shows that this is the best regularity we can expect in general.

Example 1.1. Consider the following triangular system:

$$\left\{ \begin{array}{rcl} u_t + \big(u(u-\alpha)^2(3\alpha - u)\big)_x & = & 0 \\ v_t + \lambda v_x - u^2/2 & = & 0 \end{array} \right. \tag{1.6}$$

with $\alpha \in (0, 1]$. Since we will consider solution with $u \in [0, 4\alpha]$, the above system is certainly strictly hyperbolic for all $0 < \alpha \le 1$ if $\lambda > 4$. Denote with $f(u)$ the flux function of u, namely

$$f(u) = u(u-\alpha)^2(3\alpha - u).$$

It is easy to see that the shock 1-curve for this system passing in (u, v) is given by

$$v(s) = v + \frac{s^2 - u^2}{2(\lambda - \sigma(s))}, \qquad \sigma(s) = \frac{f(s) - f(u)}{s - u}. \tag{1.7}$$

For this system, we can explicitly construct the mixed curve T_i starting in $(0, 0)$: in fact, for $s \in [0, \alpha]$, $T_i(s; (0, 0))$ coincides with the shock curve $S_i(s, (0, 0))$:

$$v(s) = \frac{s^2}{2(\lambda - \sigma(s))}, \qquad \sigma(s) = (s - \alpha)^2(3\alpha - s). \tag{1.8}$$

For $s \in [\alpha, 3\alpha)$, let $x(s)$ be the point in $[\alpha, s)$ determined by

$$f'(x(s))(s - x(s)) = f(s) - f(x(s)). \tag{1.9}$$

Then the curve $T_i(s; 0)$ is given by

$$v(s) = \frac{\alpha^2}{2\lambda} + \int_\alpha^{x(s)} \frac{s}{\lambda - \lambda_1(s)} ds + \frac{s^2 - x^2(s)}{2(\lambda - \sigma'(s))}, \qquad \sigma'(s) = \frac{f(s) - f(x(s))}{s - x(s)}, \tag{1.10}$$

where $\lambda_1(s) = f'(s)$. In fact, the point $(s, v(s))$ is connected to $(0, 0)$ by a shock, a rarefaction and a shock: the first shock start at $P_0 \doteq (0, 0)$ and ends in $P_1 \doteq S_1(\alpha; (0, 0)) = (\alpha, \alpha/2\lambda)$, and has speed 0. The rarefaction starts in P_1 and ends in $P_2 \doteq R_1(x(s) - \alpha, P_1)$, with speed increasing from 0 to $f'(x(s))$. The last shock is $S_1(s - x(s); P_2)$, and has speed equal to $f'(x(s))$.

Finally for $s \ge 3\alpha$, the curve $T_1(s; (0, 0))$ coincides with the shock curve $S_1(s; (0, 0))$, given by (1.8).

Similarly the mixed curve T_1 starting in P_1 is given by (1.10) for $\alpha < s < 3\alpha$ and by the shock curve $S_1(s; P_1)$ for $s > 3$, which is the given by

$$v(s) = \frac{\alpha^2}{2\lambda} + \frac{s^2 - \alpha^2}{2(\lambda - \sigma''(s))}, \qquad \sigma''(s) = s(s - \alpha)(3\alpha - s). \tag{1.11}$$

For $s = 3\alpha$ we have that

$$T_1(3\alpha, P_0) = S_1(3\alpha; P_0) = T_1(3\alpha; P_1) = S_2(3\alpha; P_1) = \frac{9\alpha^2}{2\lambda}.$$

We now compute the derivatives of these curves for $s = 3\alpha$. We have with elementary computations that the first and second derivatives of (1.10) are given by:

$$\left.\frac{dv}{ds}\right|_{s=3} = \frac{3\alpha}{\lambda} - \frac{24\alpha^4}{\lambda^2}, \quad \left.\frac{d^2 v}{ds^2}\right|_{s=3} = \frac{1}{\lambda} - \frac{67\alpha^3}{\lambda^2} + \frac{288\alpha^6}{\lambda^3}. \tag{1.12}$$

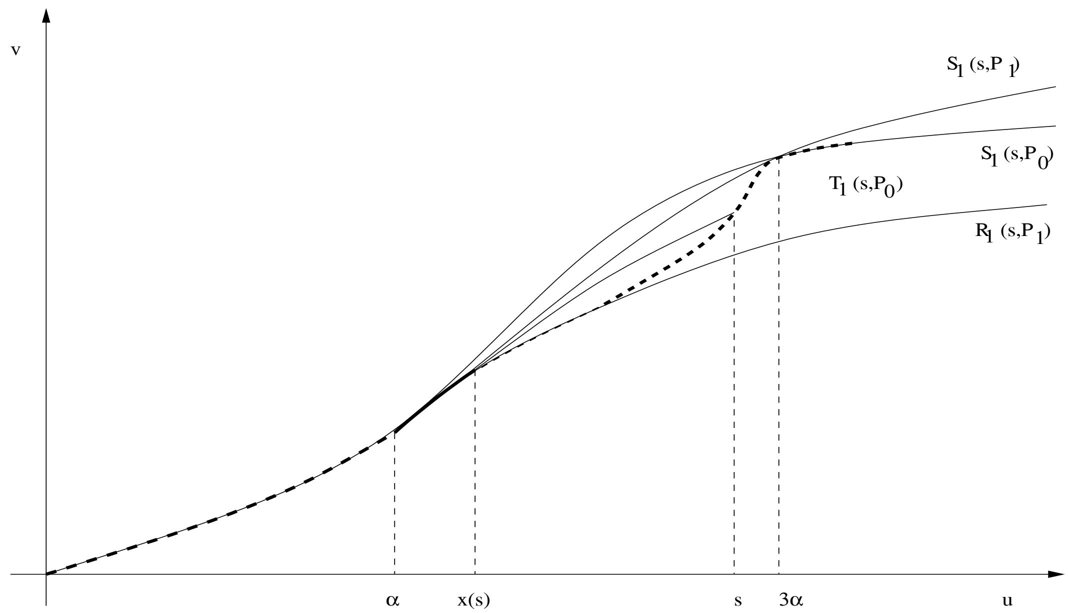

FIGURE 1. The curves $T_1(s,(0,0))$, $R_1(s,P_1)$ and $T_1(s,P_1)$ for the hyperbolic system (1.6).

On the other hand we have that for the Rankine-Hugoniot curve (1.8), starting in $(0,0)$,

$$\frac{dv}{ds}\bigg|_{s=3\alpha} = \frac{3\alpha}{\lambda} - \frac{18\alpha^4}{\lambda^2}. \tag{1.13}$$

Instead, the Rankine-Hugoniot curve (1.11) starting at P_1 has derivatives

$$\frac{dv}{ds}\bigg|_{s=3\alpha} = \frac{3\alpha}{\lambda} - \frac{24\alpha^4}{\lambda^2}, \qquad \frac{d^2v}{ds^2}\bigg|_{s=3\alpha} = \frac{1}{\lambda} - \frac{58\alpha^3}{\lambda^2} + \frac{288\alpha^6}{\lambda^3}. \tag{1.14}$$

Thus we obtain that the curve $T_1(s,P_1)$ is only $C^{1,1}$ in $s=3\alpha$, and the curve $T_1(s,(0,0))$ is only Lipschitz continuous in $s=3\alpha$.

Note that $T_1(s,P_1)$ is only $C^{1,1}$ because in the interval $[\alpha,3\alpha]$ there is an inflection point, and the jump in the second derivative is due to the fact that $x'=-3/2$ for $s\to 3\alpha^-$, but $x\equiv 1$ for $s\geq 3\alpha$: thus the function $x(s)$ is only Lipschitz continuous. On the other hand, there are two inflection points in $[0,3\alpha]$, and the Lipschitz continuity of $T_1(s,(0,0))$ is due to the fact that we switch from the shock curve $S_1(s-\alpha,P_1)$ to the shock curve $S_1(s,(0,0))$ as s crosses 3α.

The above example proves that if there are at least 2 inflection points, then the curve T_i is in general Lipschitz continuous. On the other hand, it is well known that if the field is genuinely nonlinear, then the curve T_i is $C^{2,1}$ [7], so that one expect an intermediate situation when there is only one inflection point: as example 1.1 suggests, T_i should be $C^{1,1}$.

Assume that f has only one inflection point in the i-th family, i.e. the i-th eigenvalue satisfies

$$D\lambda_i(u)r_i(u) = 0$$

in a hypersurface transversal $\mathcal{F}$ to the vector field $r_i(u)$. Consider a point u^-, and and let $T_i(s,u^-)$ be the mixed curve of the i-th family starting in u^- and parametrized by

$$\left\langle l_i(u^-), T_i(s,u^-) - u^- \right\rangle = s.$$

Assume for definiteness that $D\lambda_i(u^-)r_i(u^-) > 0$ and $D\lambda_i(u^- + sr_i(u^-))r_i(u^- + sr_i(u^-)) < 0$ for some $s>0$: this means that the rarefaction curve R_i will cross the hypersurface $\mathcal{F}$ for some $s_1>0$

In [8] it is shown that the curve T_i for $s>0$ is formed by a rarefaction until $s=s_1$, i.e. $T_i(s_1,u^-)\in F$. Then, for $s_1<s<s_2$, it is composed by a rarefaction $R_i(\tau,u^-)$, $\tau\in[0,x(s)]$, starting in u^- and ending in the point $P_1 = R_i(x(s),u^-)$, followed by a shock $S_i(\tau',P_1)$, $\tau'\in[0,s-x(s)]$, where $x(s)$ is determined

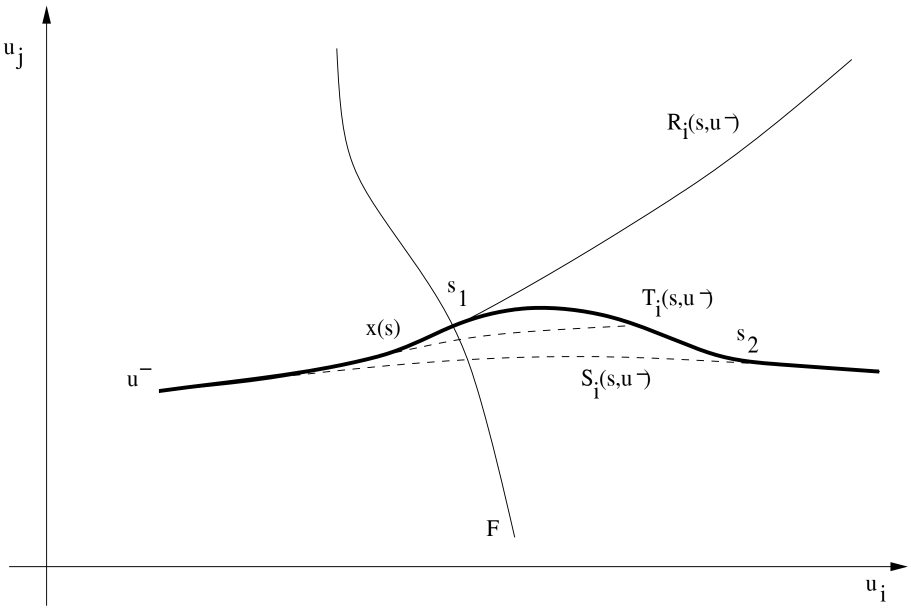

FIGURE 2. Single inflection point in the i-th family.

by the equation

$$f\big(S_i(s,P_1)\big) - f(P_1) = \lambda_i(P_1)\big(S_i(s,P_1) - P_1\big). \tag{1.15}$$

The value s_2 is determined by the relation

$$f\big(S_i(s,u^-)\big) - f(u^-) = \lambda_i(u^-)\big(S_i(s,u^-) - u^-\big). \tag{1.16}$$

Finally, for $s \geq s_2$, $T_i(s,u^-)$ coincides with the shock curve $S_i(s,u^-)$. Note that by letting $s \to \bar{s}$ the admissibility assumption (1.5) implies that $\lambda_i(T_i(\bar{s},u^-)) \leq \sigma(S_i(\bar{s},u^-),u^-)$, and by the genuinely nonlinearity for $s \geq s_1$ we obtain that

$$\lambda_i\big(T_i(s_2,u^-)\big) < \sigma\big(T_i(s_2,u^-)\big) = \lambda_i(u^-), \tag{1.17}$$

i.e. $\lambda_i(u^-)$ is not an eigenvalue of $A(T_i(s_2,u^-))$.

In [8] it is shown that the mixed curve $T_i(s,u^-)$ is C^2 for $s \neq s_2$, i.e. outside the point $P_2 \doteq T_i(s_2,u^-) = S_i(s_2,u^-)$. The proof is based on the fact that the point $x(s)$ depends smoothly on s.

We now prove that in that point the curve is C^1. In fact, differentiating (1.15) for $s = s_2^-$, we have

$$\begin{aligned}\big(A(P_2) - \lambda_i(u^-)I\big)\left(\frac{\partial S_i}{\partial s} + D_u S_i r_i(u^-)\frac{dx}{ds}\right) &= \big(A(u^-) - \lambda_i(u^-)I\big) r_i(u^-)\frac{dx}{ds} + D\lambda_i r_i(u^-)\frac{dx}{ds}\big(P_2 - u^-\big)\\ &= D\lambda_i r_i(u^-)\frac{dx}{ds}\big(P_2 - u^-\big).\end{aligned}$$

By definition we have

$$\left.\frac{\partial T_i}{\partial s}\right|_{s_2^-} = \left.\frac{\partial S_i}{\partial s} + D_u S_i r_i(u^-)\frac{dx}{ds}\right|_{s_2^-},$$

so that, using the fact that $\big\langle l_i(u^-), \partial T_i/\partial s\big\rangle = 1$ and (1.17), we obtain

$$\begin{aligned}\left.\frac{\partial T_i}{\partial s}\right|_{s_2^-} &= D\lambda_i r_i(u^-)\left.\frac{dx}{ds}\right|_{s_2^-}\big(A(P_2) - \lambda_i(u^-)I\big)^{-1}\big(P_2 - u^-\big)\\ &= \frac{\big(A(P_2) - \lambda_i(u^-)I\big)^{-1}\big(P_2 - u^-\big)}{\Big\langle l_i(u^-), \big(A(P_2) - \lambda_i(u^-)I\big)^{-1}\big(P_2 - u^-\big)\Big\rangle}.\end{aligned} \tag{1.18}$$

Repeating the above computation for $\partial T_i/\partial s\big|_{s_2^+}$ we obtain

$$\begin{aligned}\frac{\partial T_i}{\partial s}\bigg|_{s_2^+} &= \frac{d\sigma_i}{ds}\bigg|_{s_2^+}\big(A(P_2)-\lambda_i(u^-)I\big)^{-1}\big(P_2-u^-\big)\\ &= \frac{\big(A(P_2)-\lambda_i(u^-)I\big)^{-1}\big(P_2-u^-\big)}{\big\langle l_i(u^-),\big(A(P_2)-\lambda_i(u^-)I\big)^{-1}\big(P_2-u^-\big)\big\rangle} = \frac{\partial T_i}{\partial s}\bigg|_{s_2^-},\end{aligned}$$

and as a consequence

$$\frac{d\sigma_i}{ds}\bigg|_{s_2^+} = \frac{d\sigma_i}{ds}\bigg|_{s_2^-} = D\lambda_i r_i(u^-)\,\frac{dx}{ds}\bigg|_{s_2^-}.$$

This concludes the proof. One can also verify that if $u^-\in F$, then T_i is C^3.

2. Construction of the mixed curves

Consider the hyperbolic system (1.1) with diagonal viscosity,

$$u_t + f(u)_x - u_{xx} = 0 \tag{2.1}$$

It is well known that to identify a small travelling profile of the i-th family one needs $n+2$ parameters: the value u, the derivative of u in the i-th direction r_i and the speed σ_i of the profile [4]. In the case of (2.1), it is known that there is an invariant manifold, which contains all small i-th travelling profiles, invariant under the flow generated by the ODE

$$-\sigma u_x + f(u)_x - u_{xx} = 0. \tag{2.2}$$

In this manifold, the above ODE takes the form

$$\begin{cases} u_x &= v_i\tilde r_i(u,v_i,\sigma_i)\\ v_{i,x} &= v_i\phi_i(u,v_i,\sigma_i)\\ \sigma_{i,x} &= 0\end{cases} \tag{2.3}$$

The function $\tilde r_i$ gives the component of the derivative u_x when we know the i-th component $u_{i,x}=v_i$, while ϕ_i describes the internal dynamics of the travelling profile.

Aim of this section is to prove that it is possible to associate three curves to the system (2.3) under the assumptions that the functions $\tilde r_i$, ϕ_i are smooth and that

$$\frac{\partial\phi_i}{\partial\sigma} < 0. \tag{2.4}$$

These curves, which we will denote as $\mathcal{R}_i$, $\mathcal{S}_i$, $\mathcal{T}_i$, correspond to the rarefaction curves R_i, shock curves S_i and the mixed curves T_i for the hyperbolic system (1.1). Once fixed the functions $\tilde r_i$, ϕ_i, the curves $\mathcal{R}_i$, $\mathcal{S}_i$, $\mathcal{T}_i$ are unique, but of course they will depend on $\tilde r_i$, ϕ_i. However, we will prove that if the "rarefaction curves" $\mathcal{R}_i$ and the "shock curves" $\mathcal{S}_i$ of (2.3) coincide with their hyperbolic counterparts R_i, S_i, then also the "mixed curves" $\mathcal{T}_i$ coincide with the curves T_i. As a consequence the uniqueness of the admissible curves T_i follows.

In particular, using the functions $\tilde r_i$, ϕ_i obtained by the center manifold theorem applied to (2.1), we can construct the curves T_i without any assumption on the number of inflection points of f, see [4].

Consider a fixed base of vectors $\bar r_i$, $i=1,\dots,n$ in $\mathbb{R}^n$, and its dual base $\bar l_i$, normalized by

$$\big|\bar r_i\big| = 1, \qquad \big\langle \bar l_j,\bar r_i\big\rangle = \begin{cases}1 & j=i\\ 0 & j\neq i\end{cases}$$

We will use the following norm in $\mathbb{R}^n$:

$$\big|u\big| = \max\Big\{\big|\big\langle\bar l_i,u\big\rangle\big|; i=1,\dots,n\Big\}.$$

Let $\tilde r_i$ be a smooth vector valued function defined in a neighborhood of a the point $(\bar u,0,\bar\lambda_i)\in\mathbb{R}^{n+2}$,

$$\tilde r_i = \tilde r_i\big(u,v_i,\sigma_i\big), \qquad \text{with} \qquad \tilde r_i\big(\bar u,0,\bar\lambda_i\big) = \bar r_i, \tag{2.5}$$

normalized such that

$$\big\langle\bar l_i,\tilde r_i(u,v_i,\sigma_i)\big\rangle = 1. \tag{2.6}$$

The last condition is not a restriction because for any smooth function $\tilde{r}_i$ satisfying (2.5) we have

$$\Big|\tilde{r}_i\big(u, v_i, \sigma_i\big) - \tilde{r}_i\big(u', v_i', \sigma_i'\big)\Big| \leq C_0 \Big\{ \big|u - u'\big| + \big|v_i - v_i'\big| + \big|\sigma_i - \sigma_i'\big| \Big\}, \tag{2.7}$$

where C_0 is a sufficiently big constant and thus

$$\big\langle \bar{l}_i, \tilde{r}_i(u, v_i, \sigma_i) \big\rangle \geq \frac{1}{2},$$

if (u, v_i, σ_i) is sufficiently close to $(\bar{u}, 0, \bar{\lambda}_i)$. We will call $\tilde{r}_i$ the *i-th generalized eigenvector.*

Similarly, let ϕ_i be a smooth function satisfying

$$\phi_i = \phi_i(u, v_i, \sigma_i), \qquad \phi_i(\bar{u}, 0, \bar{\lambda}_i) = 0, \qquad \frac{\partial}{\partial \sigma_i} \phi(\bar{u}, 0, \bar{\lambda}) \leq -c < 0. \tag{2.8}$$

Since we have

$$\Big|\phi_i\big(u, v_i, \sigma_i\big) - \phi_i\big(u', v_i', \sigma_i'\big)\Big| \leq C_0 \Big\{ \big|u - u'\big| + \big|v_i - v_i'\big| + \big|\sigma_i - \sigma_i'\big| \Big\}, \tag{2.9}$$

the last conditions in (2.8) imply that

$$\Big|\phi_i\big(u, v_i, \sigma_i\big)\Big|, \bigg| \frac{1}{c} \frac{\partial \phi_i}{\partial \sigma} + 1 \bigg| \leq C_0 \big\{ |u| + |v_i| + |\sigma_i| \big\}, \tag{2.10}$$

for some constant C_0. For reasons which will be clear later, we define

$$\tilde{\lambda}_i(u, v_i, \sigma_i) \doteq \frac{1}{c} \phi_i(u, v_i, \sigma_i) + \sigma_i \tag{2.11}$$

as the *i-th generalized eigenvector*. By choosing $C_0 \geq 1$ sufficiently big, we can also assume that

$$\frac{1}{c} \Big\{ \big|D_u \phi_i\big| + \big|\phi_{i,v}\big| \Big\} \leq C_0. \tag{2.12}$$

Note that from (2.8) there is a unique smooth function $\tilde{\sigma}_i = \tilde{\sigma}_i(u, v_i)$ such that

$$\phi_i\big(u, v_i, \tilde{\sigma}_i(u, v_i)\big) = 0. \tag{2.13}$$

Fix a point $u^- \in \mathbb{R}^n$ sufficiently close to $\bar{u}$ and let δ_1 be a small constant. For any $s \leq \delta_1$ consider the family of Lipschitz continuous curves with values in $\mathbb{R}^{n+2}$

$$\Gamma_i(s, u^-) = \Big\{ \gamma : [0, s] \mapsto \mathbb{R}^{n+2}, \gamma(\tau) = \big(u(\tau), v_i(\tau), \sigma_i(\tau)\big) \Big\} \tag{2.14}$$

such that

$$u(0) = u^-, \ u_i(\tau) = u_i^- + \tau, \ \big|u(\tau) - u^-\big| = \tau, \qquad \big|v_i(0)\big| = 0, \ \big|v_i(\tau)\big| \leq \delta_1, \quad \big|\sigma_i(\tau) - \bar{\lambda}_i\big| \leq 2C_0\delta_1 \leq 1,$$

for some small $\delta_1 \leq 1/2C_0$. We define in Γ_i the norm

$$\big\|\gamma - \gamma'\big\| = \big\|u - u'\big\|_{L^\infty} + \big\|v_i - v_i'\big\|_{L^\infty} + \delta_1 \big\|\sigma_i - \sigma_i'\big\|_{L^\infty}. \tag{2.15}$$

For any $\gamma \in \Gamma_i(s, u^-)$, define the function $f_i(\tau; \gamma)$, $\tau \leq s$ as

$$f_i(\tau; \gamma) \doteq \int_0^\tau \tilde{\lambda}_i\big(\gamma_i(\varsigma)\big) d\varsigma = \int_0^\tau \bigg\{ \frac{1}{c} \phi_i\big(u(\varsigma), v_i(\varsigma), \sigma_i(\varsigma)\big) + \sigma_i(\varsigma) \bigg\} d\varsigma. \tag{2.16}$$

It is easy to verify that we have the estimates

$$\Big|f_i\big(\tau, \gamma\big) - f_i\big(\tau, \gamma'\big)\Big| \leq C_0 \tau \Big\{ \big\|u - u'\big\|_{L^\infty} + \big\|v_i - v_i'\big\|_{L^\infty} + 4C_0^2 \delta_1 \big\|\sigma_i - \sigma_i'\big\|_{L^1} = 4C_0^2 \tau \big\|\gamma - \gamma'\big\|, \tag{2.17}$$

where we used (2.10). For any function f defined in $[0, s]$, denote with $\text{conv} f$ its convex envelope, i.e. the set

$$\text{conv} f(x) = \inf \Big\{ \theta f(y) + (1 - \theta) f(z), \ x = \theta y + (1 - \theta) z; \ x, y, z \in [0, s], \ \theta \in [0, 1] \Big\}.$$

We now define the *i-th rarefaction curve* $\mathcal{R}_i(s, u^-)$ as the solution of the ODE

$$\dot{u} = \tilde{r}_i\big(u, 0, \tilde{\sigma}_i(u, 0)\big). \tag{2.18}$$

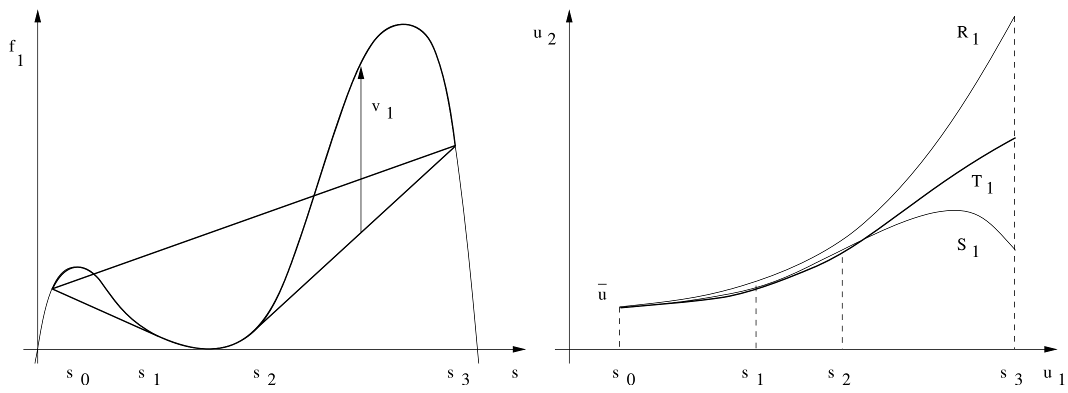

FIGURE 3. The lines $\mathcal{R}_1$, $\mathcal{S}_1$ and $\mathcal{T}_1$ in the triangular case.

The *i-th shock curve* $\mathcal{S}_i(s,u^-)$ is the value u at $\tau = s$ of the solution of the system

$$(2.19)\qquad \left\{\begin{array}{rcl} u(\tau) &=& u^- + \int_0^\tau \tilde{r}_i\big(u(\varsigma), v_i(\varsigma), \sigma_i(\varsigma)\big) d\varsigma \\ v_i(\tau) &=& c\Big(f_i\big(\tau; u, v_i, \sigma_i\big) - \tau\sigma_i\Big) \\ \sigma_i &=& f_i\big(s; u, v_i, \sigma_i\big)/s \end{array}\right.$$

for $\tau \in [0,s]$. Similarly, the *i admissible curve* $\mathcal{T}_i(s,u^-) = u(s)$, where, for any fixed $s > 0$, u is the solution of the system

$$(2.20)\qquad \left\{\begin{array}{rcl} u(\tau) &=& u^- + \int_0^\tau \tilde{r}_i\big(u(\varsigma), v_i(\varsigma), \sigma_i(\varsigma)\big) d\varsigma \\ v_i(\tau) &=& c\Big(f_i\big(\tau; u, v_i, \sigma_i\big) - \mathrm{conv} f_i\big(\tau; u, v_i, \sigma_i\big)\Big) \\ \sigma_i(\tau) &=& \frac{d}{d\tau}\mathrm{conv} f_i\big(\tau; u, v_i, \sigma_i\big) \end{array}\right.$$

with $\tau \in [0,s]$. For $s < 0$, we consider the concave envelope of f_i in the second and third equation of system (2.20).

Remark 2.1. Consider the triangular system of example 1.1 with diagonal viscosity

$$(2.21)\qquad \left\{\begin{array}{rcl} u_t + \big(u(u-\alpha)^2(3\alpha - u)\big)_x &=& u_{xx} \\ v_t + \lambda v_x - u^2/2 &=& v_{xx} \end{array}\right.$$

In [3] it is shown that, using the center manifold theorem, there is a function $\tilde{r}_1$ satisfying (2.5),(2.6). Moreover it is shown that the equations on the manifold are

$$(2.22)\qquad \left\{\begin{array}{rcl} u_\tau &=& \tilde{r}_i\big(u(\tau), v_1(\tau), \sigma_1(\tau)\big) \\ v_{1,\tau} &=& \lambda_1\big(u(\tau)\big) - \sigma_1 \\ \sigma_{1,\tau} &=& 0 \end{array}\right.$$

so that the function $\phi_i = \lambda_1(u) - \sigma$ satisfies (2.8). It is easy to check that in this special case $f_1(s) \equiv s(s-\alpha)^2(3\alpha - s)$, and then we have the identities $\mathcal{R}_1 \equiv R_1$, $\mathcal{S}_1 \equiv S_1$, $\mathcal{T}_1 \equiv T_1$.

We consider only the construction of $\mathcal{T}_i(s,u^-)$ for $s > 0$, since (2.18) is a standard ODE and the construction of $\mathcal{S}_i$ and of $\mathcal{T}_i$ for $s < 0$ are similar. We basically repeat the computations of [4].

On the set $\Gamma_i(s,u^-)$ consider the transformation $\Omega_{i,s} : \gamma = (u, v_i, \sigma_i) \mapsto \hat{\gamma} = (\hat{u}, \hat{v}_i, \hat{\sigma}_i)$ defined by (2.20), i.e.

$$(2.23)\qquad \left\{\begin{array}{rcl} \hat{u}(\tau) &=& u^- + \int_0^\tau \tilde{r}_i\big(u(\varsigma), v_i(\varsigma), \sigma_i(\varsigma)\big) d\varsigma \\ \hat{v}_i(\tau) &=& c\Big(f_i\big(\tau; u, v_i, \sigma_i\big) - \mathrm{conv} f_i\big(\tau; u, v_i, \sigma_i\big)\Big) \\ \hat{\sigma}_i(\tau) &=& \frac{d}{d\tau}\mathrm{conv} f_i\big(\tau; u, v_i, \sigma_i\big) \end{array}\right.$$

First of all it is easy to prove that the new line $\hat{\gamma} = (\hat{u}, \hat{v}_i, \hat{\sigma}_i)$ belongs to Γ_i: in fact, using (2.7) we have that

$$\begin{aligned}
\Big|u(\tau) - u^-\Big| &= \max_j \left|\left\langle \bar{l}_j, \int_0^\tau \tilde{r}_i(\varsigma) d\varsigma \right\rangle\right| \leq \max_j \Big\{\tau, 4C_0^2\tau\delta_1 + C_0\big|u^- - \bar{u}\big|\Big\} = \tau \leq \delta_1, \\
\big|v_i(\tau)\big| &\leq c\int_0^\tau \big|f'(\varsigma;\gamma) - f'(0;\gamma)\big| d\varsigma = c\Big\|\tilde{\lambda}_i(\gamma) - \tilde{\lambda}_i(u^-, 0, \sigma_i, (0))\Big\|_{L^1} \leq 8c\tau C_0^2\delta_1\tau \leq \delta_1, \\
\big|\sigma_i(\tau) - \bar{\lambda}_i\big| &\leq \frac{1}{c}\big\|\phi_i + c\sigma_i\big\|_{L^\infty} + C_0\big|u^- - \bar{u}\big| \leq C_0\delta_1 + 4C_0^3\delta_1^2 + C_0\big|u^- - \bar{u}\big| \leq 2C_0\delta_1.
\end{aligned}$$

for s, δ_1 sufficiently small. Moreover f_i is a $C^{1,1}$ function, which implies that $\hat{\sigma}_i$ is at least Lipschitz continuous, while $u(\tau)$ and $v_i(\tau)$ are $C^{1,1}$.

Next we show that the map $\Omega_{i,s}$ is a contraction in $\Gamma_i(s, u^-)$ if s is sufficiently small: in fact we have

$$\begin{aligned}
\big|u(\tau) - u'(\tau)\big| = &\left|\int_0^\tau \Big(\tilde{r}_i\big(u, v_i, \sigma_i\big) - \tilde{r}_i\big(u', (v_i)', \sigma_i'\big)\Big) d\tau\right| \\
&\leq C_0\tau\bigg\{\Big\|u - u'\Big\|_{L^\infty} + \Big\|v_i - v_i'\Big\|_{L^\infty} + \Big\|\sigma_i - \sigma_i'\Big\|_{L^\infty}\bigg\},
\end{aligned}$$

$$\begin{aligned}
\big|v_i(\tau) - v_i'(\tau)\big| \leq &\left|\int_0^\tau \Big\{\phi_i\big(u, v_i, \sigma_i\big) - \phi_i\big(u', v_i', \sigma_i'\big) + c\big(\sigma_i - \sigma_i'\big)\Big\} d\varsigma\right| \\
&\leq C_0\tau\bigg\{\Big\|u - u'\Big\|_{L^\infty} + \Big\|v_i - v_i'\Big\|_{L^\infty} + 4cC_0\delta_1\Big\|\sigma_i - \sigma_i'\Big\|_{L^\infty}\bigg\},
\end{aligned}$$

$$\begin{aligned}
\Big|\sigma_i(\tau) - \sigma_i'(\tau)\Big| \leq &\frac{1}{c}\Big\|\phi_i\big(u, v_i, \sigma_i\big) + c\sigma_i - \phi_i\big(u', v_i', \sigma_i'\big) + c\sigma_i'\Big\|_{L^\infty} \\
&\leq C_0\bigg\{\Big\|u - u'\Big\|_{L^\infty} + \Big\|v_i - v_i'\Big\|_{L^\infty} + 4C_0\delta_1\Big\|\sigma_i - \sigma_i'\Big\|_{L^\infty}\bigg\}.
\end{aligned}$$

Thus we conclude that

(2.24)

$$\begin{aligned}
\Big\|\hat{\gamma} - \hat{\gamma}'\Big\| &\leq C_0(2s + \delta_1)\big\|u - u'\big\|_{L^\infty} + C_0(2s + \delta_1)\big\|v_i - v_i'\big\|_{L^\infty} + C_0\big(s + 4cC_0\delta_1 s + 4C_0\delta_1^2\big)\big\|\sigma_i - \sigma_i'\big\|_{L^\infty} \\
&\leq 10C_0(1 + c)\delta_1\Big(\big\|u - u'\big\|_{L^\infty} + \big\|v_i - v_i'\big\|_{L^\infty} + \delta_1\big\|\sigma_i - \sigma_i'\big\|_{L^\infty}\Big) \leq \frac{1}{2}\Big\|\gamma - \gamma'\Big\|,
\end{aligned}$$

if $s = \mathcal{O}(1)\delta_1^2$ and δ_1 is sufficiently small.

Now we define $\mathcal{T}_i(s, u^-)$ by

$$\mathcal{T}_i\big(s, u^-\big) \doteq u(s), \tag{2.25}$$

i.e. the end point of the solution $\gamma(\tau) \in \Gamma_i(s, u^-)$ to system (2.20).

Remark 2.2. Note that to find the point $\mathcal{T}_i(s, u^-)$ we have to solve the system (2.20) for $\tau \in [0, s]$. This is similar to the hyperbolic case, where to construct a line $T_i(s, u^-)$ we have to find the point $u(s) = T_i(s, u^-)$ which can be connected to u^- using only admissible shocks and rarefactions of the i-th family.

We prove that the line $\mathcal{T}_i(s, u^-)$ is Lipschitz, and its derivative is close to $\bar{r}_i$. In fact, if $\gamma \in \Gamma_i(s, u^-)$, $\gamma' \in \Gamma_i(s + h, u^-)$ are the fixed points of the transformation $\Omega_{i,s}$, by the contraction property (2.24) we have

$$\Big\|\gamma - \gamma'\big|_{[0,s]}\Big\| \leq 2\Big\|\Omega_{i,s}\Big(\gamma'\big|_{[0,s]}\Big) - \gamma'\big|_{[0,s]}\Big\| \leq \mathcal{O}(1)sh.$$

Thus from the first equation of system (2.20) one obtains that

$$\mathcal{T}_i(s, u^-) - \mathcal{T}_i(s + h, u^-) = \mathcal{O}(1)sh. \tag{2.26}$$

In particular $\mathcal{T}_i(s, u^-)$ is differentiable in 0 and has derivative

$$\frac{\partial \mathcal{T}_i}{\partial s}\bigg|_{s=0} = \tilde{r}_i\big(u^-, 0, \tilde{\sigma}_i(u^-, 0)\big).$$

We now prove a stability result for the curves $\mathcal{T}_i$, analogous to the stability for shock of 1-dimensional scalar conservation laws.

Lemma 2.3. *Fix u^-, and let $0 < s < s'$. Denote with*

$$\gamma_i(\tau) = \big(u(\tau), v_i(\tau), \sigma_i(\tau)\big), \qquad \gamma_i'(\tau) = \big(u'(\tau), v_i'(\tau), \sigma_i'(\tau)\big),$$

the solutions to (2.23) *in* $\Gamma_i(s, u^-)$, $\Gamma_i(s', u^-)$. *Then*

$$\sigma_i(\tau) \geq \sigma_i'(\tau) \qquad \tau \in [0, s]. \tag{2.27}$$

Proof. Consider $f_i'(\tau; \gamma')$ and denote with $\mathrm{conv}_s f_i'$ is its convex envelope in $[0, s]$. Define the quantities

$$w_i(\tau) \doteq f_i'(\tau; \gamma') - \mathrm{conv}_s f_i'(\tau; \gamma'), \qquad \xi_i(\tau) \doteq \frac{d}{d\tau} \mathrm{conv}_s f_i'(\tau; \gamma'). \tag{2.28}$$

Note that by construction $w_i(\tau) \leq v_i'(\tau)$, and that $v_i' - w_i$, $\xi - \sigma_i'$ are increasing and positive.

We will now use the following norm on $\Gamma_i(s, u^-)$:

$$\|\gamma\|_X = \delta_1 \|u\|_{L^\infty} + \delta_1 \|v_i\|_{L^\infty} + \|\sigma\|_{L^1}. \tag{2.29}$$

It is easy to prove that the map (2.23) is contraction w.r.t. the norm $\|\cdot\|_X$, i.e.

$$\big\|\Omega_{i,s}(\gamma) - \Omega_{i,s}(\gamma')\big\|_X \leq \frac{1}{2}\|\gamma - \gamma'\|_X.$$

We can estimate $\Omega_{i,s}(u'|_{[0,s]}, w_i, \xi_i)$ as

$$\begin{aligned}
\Big\|\Omega_{i,s}\big(u'|_{[0,s]}, w_i, \xi_i\big) - \big(u'|_{[0,s]}, w_i, \xi_i\big)\Big\|_X \leq& \int_0^s \Big|\tilde r_i\big(u(\varsigma), w_i(\varsigma), \xi_i(\varsigma)\big) - \tilde r_i\big(u(\varsigma), v_i'(\varsigma), \sigma_i'(\varsigma)\big)\Big| d\varsigma \\
&+ c\int_0^s \Big|\tilde\lambda_i\big(u(\varsigma), w_i(\varsigma), \xi_i(\varsigma)\big) - \tilde\lambda_i\big(u(\varsigma), v_i'(\varsigma), \sigma_i'(\varsigma)\big)\Big| d\varsigma \\
&+ \int_0^s \Big|\tilde\lambda_i\big(u(\varsigma), w_i(\varsigma), \xi_i(\varsigma)\big) - \tilde\lambda_i\big(u(\varsigma), v_i'(\varsigma), \sigma_i'(\varsigma)\big)\Big| d\varsigma \\
\leq 5C_0(1+c)& \int_0^s \Big\{\big(v_i'(\varsigma) - w_i(\varsigma)\big) + \big(\xi(\varsigma) - \sigma_i'(\varsigma)\big)\Big\} d\varsigma \leq 10 C_0 (1+c) v_i(s).
\end{aligned}$$

Thus by the strict contraction property

$$\Big\|f_i - f_i'|_{[0,s]}\Big\| \leq C_0 s \|\gamma - \gamma'\| \leq 2C_0 s \Big\|\Omega_{i,s}\big(u, w_i, \xi_i\big) - \big(u, w_i, \xi_i\big)\Big\| \leq 10 C_0^2 \delta_1 v_i'(s) \leq \frac{1}{2}\big|v_i'(s)\big|. \tag{2.30}$$

This implies immediately that $f_i(s) \geq f_i'(s) + |v_i'(s)|/2$.

Assume now that $\sigma_i(\tau) < \sigma_i'(\tau)$ for some $\tau \in [0, s]$. Since $f_i(s) \geq f_i'(s)$, there is a point $\bar s \in [0, s]$ such that $f_i(\bar s) < f_i'(\bar s)$ and

$$\mathrm{conv} f_i(\bar s) = f_i(\bar s).$$

The last equality implies $v_i(\bar s) = 0$. It is easy to check that the curve γ restricted to $[0, \bar s]$ is the solution to (2.20) in $\Gamma_i(\bar s, u^-)$. But this is in contradiction with (2.30). □

For any u^- we define the jump $[u^-, \mathcal{S}_i(s', u^-)]$ *admissible* if for all $s \in [0, s']$ one has

$$\sigma_i(\tau) \geq \sigma_i' \qquad \tau \in [0, s], \tag{2.31}$$

where σ_i' is the speed of the shock and σ_i is obtained as the solution to (2.20) in $\Gamma_i(s, u^-)$. Using the same proof of Lemma 2.3, it is easy to prove that this is equivalent to the condition of admissibility given in [8],

$$\sigma_i \geq \sigma_i', \tag{2.32}$$

where σ_i is the speed of the jump $[u^-, \mathcal{S}_i(s, u^-)]$.

We conclude then with the following theorem:

Theorem 2.4. *For all u^- close to $\bar u$, and for any s sufficiently small, the curves $\mathcal{T}_i(s, u^-)$ solution to* (2.20) *are Lipschitz continuous and admit derivative for $s = 0$. Moreover these curves are the unique curves such that each point $u(s) = \mathcal{T}_i(s, u^-)$ can be connected to u^- by patching a countable number of rarefaction $\mathcal{R}_i$ and admissible shock $\mathcal{S}_i$, in such a way that the corresponding speed σ_i is increasing.*

Proof. By construction the line $\gamma \in \Gamma_i(s, u^-)$ solution to (2.20) is the union of generalized rarefaction or shocks. In fact, if $f_i(\tau) = \mathrm{conv} f_i(\tau)$ in some close interval $[s_m, s_{m+1}] \subseteq [0, s]$, then $\gamma(\tau)$ clearly coincides with the rarefaction $\mathcal{R}_i(\tau - s_m, \gamma_i(s_i))$ for $\tau \in [s_m, s_{m+1}]$. On the other hand, if $f_i(\tau) > \mathrm{conv} f_i(\tau)$ in some open interval $(s_n, s_{n+s}) \subseteq [0, s]$, and $f_i(s_n) = \mathrm{conv} f_i(s_n)$, $f_i(s_{n+1}) = \mathrm{conv} f_i(s_{n+1})$, then it is clear that $\gamma(s_{n+1}) = \mathcal{S}_i(s_{n+1} - s_n, \gamma(s_n))$. By Lemma 2.3 these shocks are admissible.

Suppose now that $\tilde{\gamma}$ is another curve obtained by patching rarefactions and admissible shocks such that σ_i is increasing. Then it is clearly a solution to (2.20). By the uniqueness of the solution the result follows. □

As a corollary we have that

Corollary 2.5. *Assume that the rarefactions and shock lines are obtained using the hyperbolic function f. Then for every u^- there is a unique admissible curve $T_i(s, u^-)$ for s sufficiently small.*

Proof. In [4] it is proved the existence of the admissible curves $T_i(s, u^-)$ obtained by patching admissible shocks and rarefactions by means of the center manifold for (2.1). The above theorem gives the uniqueness of the line $T_i \equiv \mathcal{T}_i$. □

Remark 2.6. Assume that we have the function $\tilde{r}_i$, ϕ_i for $i = 1, \dots, n$ and that

$$\mathrm{span}\Big\{\bar{r}_1, \dots, \bar{r}_n\Big\} = \mathbb{R}^n, \qquad \bar{\lambda}_1 < \dots < \bar{\lambda}_2. \tag{2.33}$$

We can construct the curves $\mathcal{T}_i(s_i, u)$, $i = 1, \dots, n$ for $|s_i| \leq \delta_1$, $|u - \bar{u}| \leq \delta_1$, with δ_1 sufficiently small, and moreover we have that the composed map

$$(s_1, \dots, s_n) \mapsto \mathcal{T}_n\bigg(s_n, \mathcal{T}_{n-1}\Big(s_{n-1}, \mathcal{T}_{n-2}\big(s_{n-2}, \dots \mathcal{T}_1(s_1, u)\big)\Big)\bigg) \tag{2.34}$$

has an invertible derivative in $\{s_i = 0\}$ because of (2.33). Thus, by the implicit function theorem, given u^-, u^+, we can connect u^- to u^+ by a sequence of rarefactions $\mathcal{R}_i$ and admissible shocks $\mathcal{S}_i$ with increasing speed.

The inverse of (2.34) defines a Riemann solver, which in the conservative case is unique by Corollary 2.5.

Remark 2.7. If instead of the last inequality in (2.8) we assume that

$$\frac{\partial}{\partial \sigma_i}\phi\big(\bar{u}, 0, bar\lambda_i\big) \geq c > 0,$$

then we can repeat the computations of this section by considering the system

$$\begin{cases} \hat{u}(\tau) &= & u^- + \displaystyle\int_0^\tau \tilde{r}_i\big(u(\varsigma), v_i(\varsigma), \sigma_i(\varsigma)\big) d\varsigma \\ \hat{v}_i(\tau) &= & c\Big(f_i\big(\tau; u, v_i, \sigma_i\big) - \mathrm{conc} f_i\big(\tau; u, v_i, \sigma_i\big)\Big) \\ \hat{\sigma}_i(\tau) &= & \dfrac{d}{d\tau}\mathrm{conc} f_i\big(\tau; u, v_i, \sigma_i\big) \end{cases}$$

where $\mathrm{conc} f_i$ is the concave envelope of f. In the hyperbolic setting, it means that we are going from $u+$ to u^-, or equivalently that t is reversed.

3. Applications

We now consider some applications of the construction of the curves $\mathcal{T}_i$. Our aim is to prove that we can obtain the functions $\tilde{r}_i$, ϕ_i, and thus the curves $\mathcal{R}_i$, $\mathcal{S}_i$, $\mathcal{T}_i$ using the center manifold theorem applied to many approximations of the hyperbolic system (1.1): vanishing viscosity, relaxation schemes and semidiscrete schemes. By Remark 2.6, we can then specify a Riemann solver "compatible" with the approximation.

In particular we can identify all the small travelling profiles of these approximations. If the system is in conservation form, i.e. the shock curve satisfy the Rankine-Hugoniot condition, Corollary 2.5 implies that all the small admissible jumps $[u^-, u^+]$ of the system (1.1) have a smooth travelling profile $\varphi(\xi)$ such that $\varphi(-\infty) = u^-$, $\varphi(+\infty) = u^+$ (see [1], [9], [10]).

3.1. **Vanishing viscosity.** Consider the parabolic system

$$u_t + A(u,u_x)u_x - B(u)u_{xx} = 0. \tag{3.1}$$

Note that particular case of the above system is the system in conservation form

$$u_t + f(u)_x - \big(B(u)u_x\big) = 0.$$

The matrix $A(u,u_x)$ is assumed to be strictly hyperbolic and $B(u)$ a positive definite matrix. Denote with $\lambda_i(u,u_x)$ the i-th eigenvalue of $A(u,u_x)$ and let $r_i(u,u_x)$, $l_i(u,u_x)$ be the corresponding right and left eigenvectors.

We assume that, by means of a change of coordinates $y = J(u)x$, $B(u)$ can be written as

$$B(u) = J(u)\begin{bmatrix} 0 & 0 \\ 0 & C(u) \end{bmatrix} J^{-1}(u), \tag{3.2}$$

where $C(u)$ is a $k \times k$ uniformly positive matrix. We assume moreover Kawashima's dissipative condition, i.e.

$$\big\langle l_i(u,u_x), B(u)r_i(u,u_x)\big\rangle > 0. \tag{3.3}$$

The change of coordinates $y = J(u)x$ transforms the matrix $A(u,u_x)$ in

$$J^{-1}(u)A(u,u_x)J(u) = \begin{bmatrix} A_{11}(u,u_x) & A_{12}(u,u_x) \\ A_{21}(u,u_x) & A_{22}(u,u_x) \end{bmatrix}, \tag{3.4}$$

where A_{11} is a $n-k$-dimensional matrix, and A_{22} is k-dimensional. Note that by (3.3), we have that

$$\mathrm{rank}\Big\{\Big[\ \big(A_{11}(\bar u,0) - \lambda(\bar u)I\big) \quad A_{12}(\bar u,0)\ \Big]\Big\} = n-k. \tag{3.5}$$

The equation for travelling profiles is the ODE

$$\big(A(u,u_x)u_x - \sigma I\big)u_x = B(u)u_{xx},$$

which can be rewritten as the first order system by setting $u_x = Jp$,

$$\left\{\begin{array}{ccc} u_x & = & J(u)p \\ B(u)J(u)p_x & = & \Big(A\big(u,J(u)p\big) - \sigma_i I - B(u)\big(DJ(u)J(u)\big)p\Big)J(u)p \\ \sigma_i & = & 0 \end{array}\right. \tag{3.6}$$

Due to the assumptions (3.2), and its consequence (3.5), the equation for $p = (p_1,p_2)$, with $p_1 \in \mathbb{R}^{n-k}$, $p_2 \in \mathbb{R}^k$, can be divided into two parts: $n-k$ algebraic relations and a system of k ODE for p_2.

Using (3.5), we can write

$$v = Q(\bar u)\alpha,$$

where $\alpha \in \mathbb{R}^k$ and Q is a $k \times (n-k)$-dimensional matrix. For simplicity we assume the condition

$$\det\big(A_{11}(\bar u,0) - \lambda_i(\bar u)I\big) \neq 0, \tag{3.7}$$

so that Q takes the form

$$Q(\bar u) = \begin{bmatrix} -\Big(A_{11}(\bar u,0) - \lambda_i(\bar u)I\Big)^{-1}A_{12}(\bar u,0) \\ I \end{bmatrix}, \tag{3.8}$$

but similar computations can be done under the assumption that

$$\mathrm{rank}\left\{\begin{bmatrix} 0 & 0 \\ 0 & S(\bar u) \end{bmatrix} Q\right\} = k.$$

Note that the above condition is certainly satisfied by (3.8), but it is not implied by (3.3).

Let $v = (v_1,v_2)$, where v_2 is k-dimensional. The assumption (3.7) implies that we can obtain v_1 as a function of v_2 by

$$v_1 = -\Big(A_{11} - \lambda_i I - \big(JB(DJJ)v\big)_{11}\Big)^{-1}\Big(A_{12} + \big(JB(DJJ)v\big)_{12}\Big)v_2, \tag{3.9}$$

if v is sufficiently small, so that the system (3.6) becomes

$$\left\{\begin{array}{rcl} u_x & = & J(u)v \\ C(u)v_{2,x} & = & \Big(A_{22} - A_{21}\big(A_{11}-\lambda_i I\big)^{-1}A_{12} - \sigma_i I - d(u,v)v\Big)v_2 \\ \sigma_i & = & 0 \end{array}\right. \tag{3.10}$$

for some smooth function $d(u,v)$.

The linearization of the system (3.10) around the equilibrium $(\bar{u},0,\lambda_i(\bar{u}))$ gives the linear system

$$\left\{\begin{array}{rcl} u_x & = & J(\bar{u})v \\ C(\bar{u},0)v_{2,x} & = & \Big(A_{22}(\bar{u},0) - A_{21}\big(A_{11}-\lambda_i I\big)^{-1}A_{12}(\bar{u},0) - \lambda_i(\bar{u}) I\Big)v_2 \\ \sigma_i & = & 0 \end{array}\right. \tag{3.11}$$

where $v=(v_1,v_2)$ can be obtained by

$$v_1 = -\big(A_{11}(\bar{u}) - \lambda_i(\bar{u})I\big)^{-1}A_{12}(\bar{u})v_2.$$

We can write this system as

$$\dot{X} = PX,$$

where the matrix P is the $n+k+1$ matrix

$$P = \left[\begin{array}{ccc} 0 & I & 0 \\ 0 & A_{22} - A_{21}\Big(A_{11}-\lambda_i I\Big)^{-1}A_{12} - \lambda_i I & 0 \\ 0 & 0 & 0 \end{array}\right]. \tag{3.12}$$

It is clear that P has a null space of dimension $n+2$ because $\lambda_i(\bar{u})$ is an eigenvalue of $A(\bar{u},0)$, so that there is a center manifold $\mathcal{C}_i$ of dimension $n+2$ for the original system (3.6).

In the space $(u,v,\sigma_i)\in\mathbb{R}^{2n+1}$, the invariant manifold is tangent to the eigenspace

$$M_i = \Big\{u, v_i r_i(\bar{u}), \sigma_i\Big\}, \tag{3.13}$$

so that we can write

$$v_j = \mathcal{C}_{ji}\big(u, v_i, \sigma_i\big). \tag{3.14}$$

Since for $(u, v_i = 0, \sigma)$ we have that the solution to (3.6) is constant, this implies that $\mathcal{C}_{ji}(u,0,\sigma_i)=0$, i.e.

$$v = v_i\tilde{r}_i\big(u, v_i, \sigma_i\big), \tag{3.15}$$

for some smooth vector function $\tilde{r}_i$, normalized by $\langle l_i(\bar{u}), \tilde{r}_i\rangle = 1$. Moreover $\mathcal{C}_i$ is tangent to the eigenspace M_i, so that

$$\tilde{r}_i\big(\bar{u},0,\lambda_i(\bar{u})\big) = r_i(\bar{u}).$$

The equations on this invariant manifold are

$$\left\{\begin{array}{rcl} u_x & = & v_i\tilde{r}_i(u,v_i,\sigma_i) \\ c_i(u,v_i,\sigma_i)v_{i,x} & = & \big(a_i(u,v_i,\sigma_i) - \sigma_i I\big)v_i \\ \sigma_i & = & 0 \end{array}\right. \tag{3.16}$$

where we defined the function

$$c_i(u,v_i,\sigma_i) \doteq \Big\langle l_i(\bar{u}), B(u)\big(\tilde{r}_i(u,v_i,\sigma_i) + v_i\tilde{r}_{i,v}(u,v_i,\sigma_i)\big)\Big\rangle, \tag{3.17}$$

$$a_i(u,v_i,\sigma_i) \doteq \Big\langle l_i(\bar{u}), A(u,v_i\tilde{r}_i)\tilde{r}_i(u,v_i,\sigma_i)\Big\rangle - v_i\Big\langle l_i(\bar{u}), B(u)D\tilde{r}_i\tilde{r}_i(u,v_i,\sigma_i)\Big\rangle. \tag{3.18}$$

Note that by the assumption (3.3) we obtain that in a neighborhood of $(\bar{u},0,\lambda_i(\bar{u}))$, c_i is strictly bigger than 0. Defining

$$\phi_i(u,v_i,\sigma_i) \doteq \frac{a_i(u,v_i,\sigma_i) - \sigma_i}{c_i(u,v_i,\sigma_i)}, \tag{3.19}$$

we can apply the results of Section 2: in fact,

$$\frac{\partial}{\partial \sigma_i}\phi_i\big(\bar u, 0, \lambda_i(\bar u)\big) = \frac{1}{c_i}\Big(\big\langle l_i(\bar u), A(\bar u, 0)\tilde r_{i,\sigma}\big\rangle - 1\Big) - \frac{1}{c_i^2}\Big(\big\langle l_i(\bar u), A(\bar u, 0) r_i(\bar u)\big\rangle - \lambda_i(\bar u)\Big) c_{i,\sigma}$$
$$= -\frac{1}{c_i\big(\bar u, 0, \lambda_i(\bar u)\big)},$$

because $\langle l_i(\bar u), \tilde r_{i,\sigma}\rangle = 0$.

3.2. **Relaxation schemes.** Consider the relaxation problem

$$\left\{\begin{array}{rcl} u_t + A_{11}(u,v)u_x + A_{12}(u,v)v_x &=& 0 \\ v_t + A_{21}(u,v)u_x + A_{22}(u,v)v_x &=& Q(u,v). \end{array}\right. \tag{3.20}$$

where u, v are n-dimensional and k-dimensional vectors, respectively.

The equation for travelling profiles is the ordinary differential equation

$$\left\{\begin{array}{rcl} \big(A_{11}(u,v) - \sigma I\big)u_x + A_{12}(u,v)v_x &=& 0 \\ A_{21}(u,v)u_x + \big(A_{22}(u,v) - \sigma I\big)v_x &=& Q(u,v). \end{array}\right. \tag{3.21}$$

We assume that the condition $Q(u,v) = 0$ uniquely determines v as a function of u, i.e. a manifold of equilibria $v = h(u)$.

The linearization in the equilibrium $(\bar u, \bar v \doteq h(\bar u))$ gives the linear system

$$\left\{\begin{array}{rcl} \big(A_{11}(\bar u,\bar v) - \sigma I\big)u_x + A_{12}(\bar u,\bar v)v_x &=& 0 \\ A_{21}(\bar u,\bar v)u_x + \big(A_{22}(\bar u,\bar v) - \sigma I\big)v_x &=& Q_u(\bar u,\bar v)u + Q_v(\bar u,\bar v)v. \end{array}\right. \tag{3.22}$$

As in [10], we assume that there is an invertible $(n+k)\times(n+k)$ invertible matrix $P(u,v)$ such that

$$P(u,v)\left[\begin{array}{cc} 0 & 0 \\ Q_u(u,v) & Q_v(u,v) \end{array}\right] P^{-1}(u,v) = \left[\begin{array}{cc} 0 & 0 \\ 0 & S(u,v) \end{array}\right], \tag{3.23}$$

where S is strictly definite negative. With a linear change of coordinates $v \mapsto Lu + v$ for some $n \times n$ matrix L, we can set $P(\bar u, \bar v) = I$. We can thus rewrite (3.22) as

$$\left\{\begin{array}{rcl} \big(A_{11}(\bar u,\bar v) - \sigma I\big)u_x + A_{12}(\bar u,\bar v)v_x &=& 0 \\ A_{21}(\bar u,\bar v)u_x + \big(A_{22}(\bar u,\bar v) - \sigma I\big)v_x &=& S(\bar u,\bar v)v. \end{array}\right. \tag{3.24}$$

We assume that $\tilde A_{11}(\bar u, \bar v)$ is strictly hyperbolic and denote with $\lambda_i(u)$ its i-th eigenvalue, and let $\bar r_i$, $\bar l_i$ be its left and right eigenvectors, respectively.

The non characteristic condition says that $A - \lambda_i(u)I$ is invertible, so that, for σ_i close to $\lambda_i(u)$, the system (3.21) can be written as

$$\left\{\begin{array}{rcl} \left(\begin{array}{c} u_x \\ v_x \end{array}\right) &=& \big(A(u,v) - \sigma_i I\big)^{-1}\left(\begin{array}{c} 0 \\ Q(u,v) \end{array}\right) \\ \sigma_{i,x} &=& 0 \end{array}\right. \tag{3.25}$$

whose linearization around $(\bar u, 0, \lambda_i(\bar u))$ is

$$\left\{\begin{array}{rcl} \left(\begin{array}{c} \tilde u_x \\ \tilde v_x \end{array}\right) &=& \big(A(\bar u,\bar v) - \sigma I\big)^{-1}\left(\begin{array}{c} 0 \\ S(\bar u,\bar v)v \end{array}\right) \\ \sigma_{i,x} &=& 0 \end{array}\right. \tag{3.26}$$

In [10] it is shown that, under the assumptions that

$$\big(A(\bar u,\bar v) - \bar\lambda_i I\big)^{-1}\left[\begin{array}{cc} 0 & 0 \\ 0 & S(\bar u,\bar v) \end{array}\right]$$

has no nonzero purely imaginary eigenvalues, and that the following stability condition holds

$$\big\langle \bar l_i, A_{12}(\bar u,\bar v)S^{-1}(\bar u,\bar v)\tilde A_{21}\bar r_i\big\rangle < 0, \tag{3.27}$$

then there exists an invariant $n+2$-dimensional space M_i of (3.26),

$$M_i = \operatorname{span}\Big\{\bar r_i, S^{-1}(\bar u,\bar v)A_{21}(\bar u,\bar v)\bar r_i\Big\}, \tag{3.28}$$

and by the center manifold theorem there is an invariant manifold $\mathcal{C}_i$ tangent to $M_i(u)$ in $(\bar u, \bar v)$, which can be parametrized by u, a scalar component α_i and the speed σ_i. Since all the equilibria $v = h(u)$ belong to $\mathcal{C}_i$, we can write

$$v = h(u) + \alpha_i g_i(u, \alpha_i, \sigma_i), \tag{3.29}$$

with $g_i(\bar u, 0, \lambda_i(\bar u)) = S^{-1}(\bar u, \bar v) A_{21}(\bar u, \bar v) \bar r_i$ and $h(\bar u) = \bar v$, $Dh(\bar u) = 0$. The last conditions follow from the tangency of $\mathcal{C}_i$ with M_i.

Using the non characteristic condition, one see immediately that the equations on $\mathcal{C}_i$ can be written as

$$\left\{ \begin{array}{ccc} u_x & = & \alpha_i \tilde r_i(u, \alpha_i, \sigma_i) \\ \alpha_{i,x} & = & \alpha_i \phi_i(u, \alpha_i, \sigma_i) \\ \sigma_i & = & 0 \end{array} \right. \tag{3.30}$$

for some functions $\tilde r_i$ and ϕ_i. In fact for $\alpha_i = 0$ we are on the equilibrium manifold $v = h(u)$, and then $u_x = \alpha_{i,x} = 0$. We can also assume that $\langle \bar l_i, \tilde r_i \rangle = 1$. Because $\mathcal{C}_i$ is tangent to M_i, we obtain the relations

$$\tilde r_i(\bar u, 0, \bar\lambda_i) = \bar r_i, \qquad \phi_i(\bar u, 0, \bar\lambda_i) = 0. \tag{3.31}$$

Moreover a simple computation shows that

$$\frac{\partial}{\partial \sigma_i} \phi_i(\bar u, 0, \bar\lambda_i) = \frac{1}{\Big\langle \bar l_i, A_{12}(\bar u, \bar v) S^{-1}(\bar u, \bar v) A_{21}(\bar u, \bar v) \bar r_i \Big\rangle} < 0, \tag{3.32}$$

by (3.27).

3.3. **Semidiscrete schemes.** Consider the semidiscrete scheme

$$u^m_t + f(u^m) - f(u^{m-1}) = 0, \tag{3.33}$$

where for linear stability we assume that $\lambda_i(u) > 0$.

The equation for travelling profiles is the Retarded Functional Differential Equation (RFDE)

$$-\sigma u'(\xi) + f\big(u(\xi)\big) - f\big(u(\xi - 1)\big) = 0. \tag{3.34}$$

In [1] it is shown the existence of a center manifold $\mathcal{C}_i$ of dimension $n+2$ in $C^1([-1,0];\mathbb{R}^n)$, which can be parametrized by u, $v_i = u_{i,x} = \langle l_i(\bar u), u_x \rangle$, σ_i (see [2]):

$$(u, v_i, \sigma_i) \mapsto \phi\big(\cdot; u, v_i, \sigma_i\big) \in C^1\big((-1,0], \mathbb{R}^n\big), \quad \phi(0) = u, \phi_i'(0) = v_i.$$

In particular, since for $(u_0, v_i = 0, \sigma_i)$ we obtain the equilibria $u \equiv u_0$, from the map $(u, v_i, \sigma_i) \mapsto \phi(\cdot, u, v_i, \sigma_i)$ one can deduce the two functions

$$u_x = \frac{d}{dx}\phi(0; u, v_i, \sigma_i) \doteq v_i \tilde r_i(u, v_i, \sigma_i), \qquad v_i(-1) = \left\langle l_i(\bar u), \frac{d}{dx}\phi(-1; u, v_i, \sigma_i) \right\rangle \doteq v_i \tilde p_i\big(u, v_i, \sigma_i\big). \tag{3.35}$$

The function $\tilde r_i$ gives direction of the derivative u_x once we know the i-th component $v_i = u_{i,x}$, while $v_i \tilde p_i$ gives the value of the i-th component of the derivative at $\xi = -1$, i.e. $u_{i,x}(-1)$.

The equation for v_i can be obtained from (3.34): in fact, differentiating w.r.t. x and taking the scalar product with $l_i(\bar u)$, it follows

$$-\sigma_i v_{i,x} + \tilde\lambda_i\big(u, v_i, \sigma_i\big) v_i - \tilde\lambda_i\big(u(-1), v_i p_i, \sigma_i\big) v_i p_i(u, v_i, \sigma_i) = 0,$$

where $u(-1)$ can be computed from

$$-\sigma_i v_i + f(u) - f(u(-1)) = 0,$$

and where $\tilde\lambda_i$ is given by

$$\tilde\lambda_i(u, v_i, \sigma_i) = \big\langle l_i(\bar u), A(u) \tilde r_i(u, v_i, \sigma_i) \big\rangle. \tag{3.36}$$

Thus we obtain that on the manifold $\mathcal{C}_i$ the RFDE (3.34) takes the form of the system of ODE

$$\left\{ \begin{array}{ccc} u_x & = & v_i \tilde r_i(u, v_i, \sigma_i) \\ v_{i,x} & = & v_i \Big(\tilde\lambda_i(u, v_i, \sigma_i) - \tilde\lambda_i\big(u(-1), v_i p_i, \sigma_i\big) p_i \Big) / \sigma_i \\ \sigma_{i,x} & = & 0 \end{array} \right. \tag{3.37}$$

Since $\mathcal{C}_i$ is tangent in $u(x) \equiv \bar{u}$ to the manifold (see [2])

$$M_i = \left\{ u + v_i e^{\beta_i} r_i(\bar{u})\xi, \frac{\sigma_i}{\lambda_i(\bar{u})} = \frac{1 - e^{-\beta_i}}{\beta_i}; \xi \in (-1, 0] \right\} \in C^1\big((-1, 0], \mathbb{R}^2\big),$$

we deduce that

$$\tilde{r}_i\big(\bar{u}, 0, \lambda_i(\bar{u})\big) = r_i(\bar{u}), \qquad \tilde{\lambda}_i\big(\bar{u}, 0, \lambda_i(\bar{u})\big) = \lambda_i(\bar{u}). \tag{3.38}$$

Using the fact that in all points $u(x) \equiv u$ sufficiently close to $\bar{u}$ the center manifold $\mathcal{C}_i$ is also tangent to the set

$$M_i = \left\{ u + v_i e^{\beta_i} r_i(u)\xi, \frac{\sigma_i}{\lambda_i(u)} = \frac{1 - e^{-\beta_i}}{\beta_i}; \xi \in (-1, 0] \right\} \in C^1\big((-1, 0], \mathbb{R}^2\big),$$

in [2] it is shown that

$$p_i\big(u, 0, \sigma_i\big) = e^{-\beta_i}, \tag{3.39}$$

where β_i is given by the dispersion relation

$$\frac{\sigma_i}{\lambda_i(u)} = \frac{1 - e^{-\beta_i}}{\beta_i}.$$

Let ϕ_i be the function

$$\phi_i(u, v_i, \sigma_i) \doteq \frac{1}{\sigma_i}\Big(\tilde{\lambda}_i(u, v_i, \sigma_i) - \tilde{\lambda}_i\big(u(-1), v_i p_i, \sigma_i\big) p_i(u, v_i, \sigma_i)\Big). \tag{3.40}$$

Using (3.38) and $\langle l_i(\bar{u}), \tilde{r}_{i,\sigma} \rangle = 0$, we obtain that

$$\phi_i\big(\bar{u}, 0, \lambda_i(\bar{u})\big) = 0, \tag{3.41}$$

and

$$\frac{\partial}{\partial \sigma_i}\phi_i\big(\bar{u}, 0, \lambda_i(\bar{u})\big) = -\frac{\partial p_i}{\partial \sigma_i} = \frac{1}{\lambda_i(\bar{u})} \left.\frac{\beta_i^2 e^{-\beta_i}}{(1 + \beta_i)e^{-\beta_i} - 1}\right|_{\beta_i = 0} = -\frac{2}{\lambda_i(\bar{u})}. \tag{3.42}$$

References

[1] S. Benzoni-Gavage. Semidiscrete shock profiles for hyperbolic systems of conservation laws. *Physica D*, 115:109–123, 1998.

[2] S. Bianchini. Bv solutions to semidiscrete schemes. in preparation.

[3] S. Bianchini and A. Bressan. A center manifold technique for tracing viscous travelling waves. preprint.

[4] S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. preprint.

[5] A. Bressan. *Hyperbolic systems of conservation laws.* Oxford Univ. Press, 2000.

[6] C. Dafermos. *Hyperbolic conservation laws in continuous physics.* Springer, 2000.

[7] P.D. Lax. Hyperbolic systems of conservation laws ii. *Comm. Pure Appl. Math.*, 10:537–566, 1957.

[8] T.-P. Liu. *Admissible solutions of hyperbolic conservation laws*, volume 240. Memoir A.M.S., 1981.

[9] A. Majda and R. L. Pego. Stable viscosity matrices for systems of conservation laws. *J. Diff. Eq.*, 56:229–262, 1985.

[10] W.-A. Yong and K. Zumbrum. Existence of relaxation shock profiles for hyperbolic conservation laws. *SIAM J. Appl. Math.*, 60(5):1565–1575, 2000.

Istituto per le Applicazioni del Calcolo "M. Picone" - CNR, Viale del Policlinico 137, 00161 Roma (ITALY)

and

The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong (CHINA)

E-mail address: bianchin@iac.rm.cnr.it

URL: http://www.iac.rm.cnr.it/~bianchin/