On the Global Existence of Solutions to the Prandtl's System

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Abstract

In this paper we establish a global existence of weak solutions to the 2-dimensional Prandtl's system for unsteady boundary layers in the class considered by O. A. Oleinik in [5] provided that the pressure is favourable. This generalizes the local well-posdness results due to Oleinik [5,6]. For the proof, we introduce a viscous splitting method so that the asymptotic behavior of the solution near the boundary can be estimated more accurately by methods applicable to the degenerate parabolic equations.

keywords: Boundary layers, Prandtl equations, BV space

1 Introduction

In this paper, we consider the following initial-boundary value problem for the 2-dimensional unsteady Prandtl's system

$$(1.1)\qquad \begin{cases} \partial_t u + u\,\partial_x u + v\,\partial_y u + \partial_x p = \nu\,\partial_y^2 u, \quad 0 < x < L, \quad y > 0 \\ \partial_x u + \partial_y v = 0, \\ u|_{t=0} = u_0(x,y), \quad u|_{y=0} = 0, \\ v|_{y=0} = v_0(x,t), \quad u|_{x=0} = u_1(t,y), \\ u(x,y,t) \to U(x,t), \quad y \to +\infty. \end{cases}$$

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where ν is a fixed positive constant, and the pressure p is determined by the so called Bernoulli's law:

$$\partial_t U + U\,\partial_x U + \partial_x p = 0, \tag{1.2}$$

which corresponds to a plane unsteady flow of viscous incompressible fluid in the presence of an arbitrary injection and removal of the fluid across the boundaries. It follows from the physical ground that one may assume that

$$U(x,t) > 0, \quad u_0(x,t) > 0, \quad u_1(y,t) > 0, \quad \text{and} \quad v_0(x,t) \le 0. \tag{1.3}$$

It is well-known that small forces of viscous friction may perceptibly affect the motion of a fluid, so that the solution to ideal inviscid fluid equations cannot approximate those to the Navier-Stokes system uniformly up to the boundaries. This gave rise to the theory of boundary layers which was first proposed by Prandtl in [7]. The Prandtl's system, which governs the first order approximation of the flow velocity in the boundary, serves as a basis for the development of the boundary layer theory which now is one of the fundamental parts of the fluid dynamics [8]. There are a lot of literature on theoretical, numerical and experimental studies on the Prandtl's system, see [6,8]. In particular, Oleinik and Samokhin give a systematic exposition of the main rigorous mathematical results as well as some open problems in [6]. A relatively deeper understanding has been achieved for the steady flows, see [6] and references cited there. However, in the case that the flow is unsteady, very little has been known except the local well-posedness theory for analytical data [9], the finite time blow-up result in [3], and a series of important works of Oleinik in [6] dealing with the well-posedness theory of classical solutions to (1.1) in Hõlder spaces for the data which are in the monotonic class in the sense that

$$\partial_y\, u_0(x,y) > 0, \quad \partial_y\, u_1(x,y) > 0. \tag{1.4}$$

The main results of Oleinik and her co-workers can be summarized as that there exists a unique (for short-time if L is given and fixed, and for arbitrary time if L is small) classical smooth solution to the initial-boundary value problem (1.1) provided that the initial data satisfy conditions (1.3) and (1.4). One of the open problems listed at the end of [6] by Oleinik and Samokhin is: what are the conditions ensuring the global in time existence and uniqueness of solution to (1.1) for arbitrary given L?

The main purpose of this paper is to establish the global (in time) existence of weak solution to the problem (1.1) for arbitrary finite L and data satisfying (1.3) and (1.4) provided that the pressure is favourable, i.e.,

$$\partial_x p(x,t) \leq 0 \quad \text{for} \quad t>0, \quad 0<x<L. \tag{1.5}$$

In fluid dynamics, one talks of a boundary layer in a favourable pressure gradient in the case that (1.5) is satisfied [8]. It is expected that boundary layers in favourable pressure gradients are relatively thin and thus stable since the effect of the pressure gradient counteracts the viscous spreading process. Thus we would expect a long time existence of regular solution to the Prandtl's system. On the contrary, for a boundary layer in an adverse pressure gradient, i.e., $\partial_x p > 0$, it is prone to the phenomenon of separation. Thus, it seems difficult to obtain global in time existence of a regular solution.

To give a precise statement of our results, we use the following Crocco transformation

$$\tau = t, \quad \xi = x, \quad \eta = \frac{u(x,y,t)}{U(x,t)}, \quad w(\tau,\xi,\eta) = \frac{\partial_y u(x,y,t)}{U(x,t)} \tag{1.6}$$

Then the original initial-boundary value problem (1.1) is transformed into the following initial-boundary value problem

$$\begin{cases} \partial_\tau w^{-1} + \eta U \partial_\xi w^{-1} + A\,\partial_\eta w^{-1} - B\,w^{-1} = -\nu\,\partial_\eta^2 w \\ \qquad \text{on} \quad Q = \{(\xi,\eta,\tau) | 0<\tau<\infty, \quad 0<\xi<L, \quad 0<\eta<1\} \\ w|_{\tau=0} = \frac{\partial_y u_0}{U} \equiv w_0, \quad w|_{\eta=1} = 0 \\ w|_{\xi=0} = w_1, \quad \text{and} \quad (\nu w\,\partial_\eta w - v_0\,w)|_{\eta=0} = \frac{\partial_x p}{U}, \end{cases} \tag{1.7}$$

where $A = (1-\eta^2)\,\partial_x U + (1-\eta)\,\frac{\partial_t U}{U}$, $B = \eta\,\partial_x U + \frac{\partial_t U}{U}$, and $w_1(\tau,\eta) = \frac{\partial_y U_1(0,y,t)}{U(0,t)}$.

We now can define the weak solution to the initial boundary value problem (1.7) as follows.

Definition 1.1 A function $w \,\epsilon\, BV(Q_T) \cap L^\infty(Q_T)$ (with $Q_T = \{(\xi, \eta, \tau) |\, 0 < \tau < T, \quad 0 < x < L, \quad 0 < \eta < 1\}$) is said to be a weak solution to problem (1.7) if the following conditions are satisfied:

i) There exists a positive constant C such that

$$C^{-1}(1-\eta) \leq w(\xi, \eta, \tau) \leq C(1-\eta) \quad \forall\, (\xi, \eta, \tau) \,\epsilon\, Q_T;$$

ii) w satisfies the partial differential equation in (1.7) in the sense of distribution;

iii) u_{yy} is a locally bounded measure in Q_T;

iv) The initial and boundary conditions are satisfied in the sense of trace.

Then our main result in this paper can be stated as

Theorem 1.1 Assume that the data satisfy the conditions (1.3) and (1.4). Then there exists a weak solution $w \,\epsilon\, BV\,(Q_T) \cap L^\infty\,(Q_T)$ to the initial-value problem (1.7) provided that the pressure is favourable, i.e., (1.5) holds for $0 < x < L$ and, $t > 0$.

As an immediate corollary, we obtain the global (in time) existence of a weak solution for the initial-boundary value problem (1.1). In fact, this weak solution is unique, however, this will be given in a forthcoming paper [11].

Remark: The requirement of monotonic data, (1.4), is crucial for the validity of Theorem 1.1. Indeed, for a class of data for which (1.5) holds true but (1.4) fails, then the corresponding classical solutions to the Prandtl's system blow-up in finite time in [3]. So the global in time existence of regular solution becomes impossible.

We now comment on the proof of theorem 1.1. The local well-posedness for the problem (1.7) has been established by Oleinik by an iteration method [6]. To obtain the global existence for the data satisfying (1.3) - (1.5), we need to obtain some global pointwise estimate of w for $\eta \to 1$ and uniform gradient estimates on the solutions. So it is crucial in our analysis that w stays always positive globally with precisely decay rate $(1-\eta)$ and admits a uniform total variation estimate. Our basic idea is using a viscous splitting method to solve the problem (1.7) in several time steps. In the first time step, we solve an initial-boundary value problem for a porous media type equation.

$$\begin{cases} \frac{1}{2}\,\partial_\tau\, w - \nu w^2\, \partial_\eta^2\, w = 0 \qquad (0, t_1) \times \Omega \\ w|_{\tau=0} = w_0 \qquad \textit{or given in the last step} \\ \textit{boundary conditions}, \end{cases} \tag{1.8}$$

and in the next time step, we solve a transport equation,

$$\begin{cases} \frac{1}{2}\,\partial_\tau\, w + \eta\, U\, \partial_\xi\, w + A\, w_\eta - Bw = 0 \qquad (t_1, t_2) \times \Omega \\ w|_{t=t_1} = \textit{given in the last step} \\ \textit{boundary conditions}, \end{cases} \tag{1.9}$$

where $\Omega = \{(\xi, \eta)|\; 0 < \xi < L, \quad 0 < \eta < 1\}$. Then we iterate these processes until $t_n = t$. Finally we let $n \to \infty$ to obtain a solution to the problem (1.7). The advantage of this splitting method is that the desired a prior estimates can be obtained more easily for each individual problem in (1.8) - (1.9).

We finish this introduction by outlining the rest of the paper. To simplify the presentation and avoid the technical details, we will concentrate on the simpler case that:

$$U(x,t) \equiv d = \textit{constant} \tag{1.10}$$

First, in Section 2, we prove the existence and uniqueness of smooth solutions to the problem (1.8) for the one dimensional porous medium type equation and derive some uniform estimates. Based on the estimates obtained in Section 2, we can establish the Theorem 1.1 in the special case that the boundary data w_1 satisfies some additional constraint in Section 3. This is done by using the viscous splitting method. To treat the general case, we need to study solutions for more general porous medium type equation than the one appearing in (1.8). This is done in Section 4. Then we prove the Theorem 1.1 in the general case in Section 5. Finally, we point out the modifications needed for the case that $U(x,t)$ is not a constant in Section 6.

2 A porous Medium Type Equation

In this section, we study an one dimensional porous medium type equation which is an analogy of (1.8) derived from the Prandtl's system by a splitting method as mentioned in the introduction. We shall establish the the existence and uniqueness of smooth solutions to such equation with appropriate initial and boundary data, as well as some uniform estimates on the solutions. The problem under consideration is

$$\begin{cases} u_t - u^2 u_{yy} = 0 \qquad 0 < y < d \\ u|_{t=0} = u_0 > 0 \\ \frac{\partial u}{\partial y}|_{y=0} = v_0, \quad u|_{y=d} = 0. \end{cases} \tag{2.1}$$

Here and in the rest of this section, we are using notations which are independent of the other sections of the paper. It should be clear that (2.1) is an analogy of the problem (1.8) in the introduction. It is always assumed that the initial and boundary data are as smooth and compatible as our following analysis requires. We assume that $v_0 \le 0$. One of the difficulties in the analysis is that the equation in (2.1) is degenerate near the boundary $y = d$. This problem can be approximated by the following problem where the equation is a uniform parabolic equation.

$$\begin{cases} u_t - (u^2+\varepsilon) u_{yy} = 0 \qquad 0 < y < d \\ u|_{t=0} = u_0 > 0 \\ \frac{\partial u}{\partial y}|_{y=0} = v_0, \quad u|_{y=d} = 0, \end{cases} \tag{2.2}$$

where $\varepsilon > 0$. It is well known that the problem (2.2) always has a unique smooth local in time solution which exists globally in time if it is bounded. Moreover, the solution u remains positive by the maximum principle.

Set

$$L^{\varepsilon} = \frac{\partial}{\partial t} - (u^2 + \varepsilon)\frac{\partial^2}{\partial y^2},$$
$$L = \frac{\partial}{\partial t} - u^2\frac{\partial^2}{\partial y^2}.$$

The following estimate is standard by the maximum principle.

Lemma 2.1 *The solution of problem (2.2) satisfies*

$$(2.3) \qquad -|v_0|_{L^\infty}(d-y) + u \le \max_y(-|v_0|(d-y) + u)|_{t=0}.$$

The same holds true when the plus sign is replaced by minus sign in front of u. In particular,

$$|u(t,y)| \le |u_0|_{L^\infty} + d|v_0|_{L^\infty}.$$

Proof. Set

$$w = u - (|v_0|_{L^\infty} + \varepsilon_0)(d-y),$$

where $\varepsilon_0 > 0$ is a constant. Then it is easy to check that $L^{\varepsilon} w = 0$. Applying the maximum principle to w we deduce that w can only achieve its positive maximum at $t = 0$ or $y = 0$.

If it achieves its maximum at $t = 0$, then (2.3) is obtained by letting $\varepsilon_0 \to 0$.

If it achieves its maximum at $y = 0$, then $\frac{\partial w}{\partial y}|_{y=0} \le 0$. But a direct calculation shows that $w_y|_{y=0}$ is positive, which is a contradiction. This proves Lemma 2.1.

The result of Lemma 2.1 holds true for the solution to the problem (2.1). In fact, we can obtain better estimates on the solution.

Lemma 2.2 *Let* $\phi = e^{\alpha y} \sin \alpha(d-y)$*, where* $\alpha = \frac{\pi}{2d}$*. Suppose* $v_0 \le 0$*. Then there exist constants* β *and* C_0 *which depend only on* d*,* $|u_0|_{L^\infty}$ *and* $|v_0|_{L^\infty}$*, such that the solution of (2.1) satisfies*

$$|u| \le C_0(d-y), \tag{2.4}$$

$$u \ge \theta_0 e^{-\beta t}\phi, \tag{2.5}$$

provided that (2.4) and (2.5) hold true initially, where $\theta_0 = \min \frac{u_0}{\phi}$*.*

Proof. Let u be a solution of (2.2), and set $w = \frac{u}{d+\varepsilon-y}$. Then

$$w_t - (u^2+\varepsilon)w_{yy} + \frac{2(u^2+\varepsilon)}{d+\varepsilon-y}w_y = 0.$$

It follows from the maximum principle that $|w|$ can only achieve its nonzero maximum at $y=0$ or $t=0$. That is, either

$$|w| \le \frac{1}{d+\varepsilon}|u|_{L^\infty} \le C_0,$$

or

$$|w| \le w|_{t=0} \le C_0.$$

Then (2.4) follows by letting $\varepsilon \to 0$.

To prove the second part of Lemma 2.2, we consider the function

$$w = u - e^{-\beta t}\theta_0\phi_\varepsilon,$$

where $\phi_\varepsilon = e^{\alpha y}\sin\alpha(d+\varepsilon-y)$. In the following proof, we omit the subindex ε for the sake of convenience. It can be checked easily that

$$L^\varepsilon(u - \theta_0 e^{-\beta t}\phi) = \theta_0 e^{-\beta t}(\beta\phi + (u^2+\varepsilon)\phi_{yy}).$$

Since

$$\phi_{yy} = -e^{\alpha y}2\alpha^2\cos\alpha(d+\varepsilon-y),$$

then by noticing (2.4), we have for $0 \le y \le d$,

$$\begin{aligned}&\beta\phi + (u^2+\varepsilon)\phi_{yy}\\ &= [\beta \sin\alpha(d+\varepsilon-y) - 2\alpha^2(u^2+\varepsilon)\cos\alpha(d+\varepsilon-y)]e^{\alpha y}\\ &\ge [\tfrac{\beta}{2d}(d+\varepsilon-y) - 2\alpha^2 C_0^2(d-y)^2 - 2\alpha^2\varepsilon]e^{\alpha y}.\end{aligned}$$

Then we can choose β sufficiently large so that

$$\beta\phi + (u^2+\varepsilon)\phi_{yy} \ge 0, \tag{2.6}$$

where β depends only on C_0 and d. By the maximum principle, $u - \theta_0 e^{-\beta t}$ can only achieve its minimum at $t = 0$ or $y = 0$ or $y = d$.

If the minimum is achieved at $y = d$, then (2.5) is proved by letting $\varepsilon \to 0$.

At $y = 0$,

$$\frac{\partial w}{\partial y}\Big|_{\varepsilon=0} = v_0 - \alpha\theta_0 e^{-\beta t} < 0.$$

by the assumption on v_0. Therefore $y = 0$ is not the minimum point.

Since $w|_{t=0} \ge 0$, then again we obtain (2.5) by letting $\varepsilon \to 0$. Thus the proof of Lemma 2.2 is complete.

Next, we consider the estimates of the derivatives of u. Note that u_y satisfies

$$(u_y)_t - (u^2+\varepsilon)(u_y)_{yy} - 2uu_y(u_y)_y = 0. \tag{2.7}$$

Thus

$$L^\varepsilon(u_y^2) - 2uu_y(u_y^2)_y = -2(u^2+\varepsilon)u_{yy}^2. \tag{2.8}$$

Set $a_0 = \max\{|v_0|_{L^\infty}, |u_{0,y}|_{L^\infty}\}$. Then we have

Lemma 2.3

$$|u_y^2|_{L^\infty} \le a_0^2 \tag{2.9}$$

Proof. Consider the function $u_y^2 + \delta u$ for positive δ. It follows from (2.2) and (2.8) that

$$L^\varepsilon(u_y^2 + \delta u) = -2(u^2 + \varepsilon)u_{yy}^2 + 2uu_y(u_y^2 + \delta u)_y - 2\delta u u_y^2.$$

Thus

$$L^\varepsilon(u_y^2 + \delta u) - 2uu_y(u_y^2 + \delta u)_y \leq 0. \tag{2.10}$$

By the maximum principle, $u_y^2 + \delta u$ can only achieve its maximum at $t = 0$ or $y = 0$ or $y = d$.

If the maximum is achieved at $y = d$, then

$$\frac{\partial}{\partial y}(u_y^2 + \delta u)|_{y=d} = (2u_y u_{yy} + \delta u_y)|_{y=d} \geq 0.$$

But from (2.2) and $u|_{y=d} = 0$, we deduce $u_{yy}|_{y=d} = 0$, then

$$\frac{\partial}{\partial y}(u_y^2 + \delta u)|_{y=d} = \delta u_y|_{y=d} \leq 0,$$

which yields a contradiction unless $\partial_y\, u|_{y=d} = 0$, that is a trivial case.

Therefore $u_y^2 + \delta u$ can only achieve its maximum at $y = 0$ or $t = 0$. Then (2.9) follows by letting $\delta \to 0$.

Now, we consider the case that the initial and boundary values as well as the solution of problem (2.1) depend on the parameter x. More precisely, the initial and boundary values depend smoothly on x, that is, for $0 \leq x \leq L$

$$u_0 = u_0(x, y), \qquad v_0 = v_0(t, x),$$

where $u_0(x, y)$ and $v_0(t, x) \leq 0$ are smooth in x as well as in t and y. Let

$$\Omega = \{(x, y)| \quad 0 \leq x < L, 0 \leq y < d\}.$$

First, we estimate u_x which solves the following problem.

$$\begin{cases} u_{xt} - (u^2 + \varepsilon)u_{xyy} - 2uu_{yy}u_x = 0 \\ u_x|_{t=0} = u_{0,x} \\ \frac{\partial u_x}{\partial y}|_{y=0} = v_{0,x}, \quad u_x|_{y=d} = 0. \end{cases} \tag{2.11}$$

Set $w = u_x$, then (2.11) becomes

$$\left\{ \begin{array}{l} w_t - (u^2+\varepsilon) w_{yy} - 2uu_{yy}w = 0 \\ w|_{t=0} = u_{0,x} \\ \frac{\partial w}{\partial y}|_{y=0} = v_{0,x}, \quad w|_{y=d} = 0. \end{array} \right. \tag{2.12}$$

Then we have the following weighted L^1 estimate on $w = u_x$.

Lemma 2.4 *There exists a constant C_2 which only depends on $|v_{0,x}|$, such that*

$$\int_0^d \frac{|w(t,\cdot)|}{u^2(t,\cdot)+\varepsilon} dy \le \int_0^d \frac{|w(t_0,\cdot)|}{u^2(t_0,\cdot)+\varepsilon} dy + C_2(t-t_0), t \ge t_0. \tag{2.13}$$

Proof. It follows from (2.2) and (2.12) that

$$\left(\frac{w}{u^2+\varepsilon}\right)_t = w_{yy}. \tag{2.14}$$

Then

$$\frac{d}{dt}\int_0^d \frac{|w|}{u^2+\varepsilon} dy \le -|w|_y|_{y=0} \le |v_{0,x}|. \tag{2.15}$$

We obtain (2.13) from (2.15) with $C_2 = d|v_{0,x}|_{L^\infty}$.

Remark 2.1 *Let $\varepsilon \to 0$ in Lemma 2.1 and Lemma 2.3, then we obtain that the solution of problem (2.1) satisfies*

$$|v_0|_{L^\infty}(y-d) + u(t,\cdot) \le \max\{|v_0|_{L^\infty}(y-d) + u(t_0,\cdot)\}, \tag{2.16}$$

$$|u_y(t,\cdot)|^2_{L^\infty} \le \max\{|u_y(t_0,\cdot)|^2_{L^\infty}, |v_0|^2_{L^\infty}\}, \tag{2.17}$$

for $t > t_0$. Same estimate holds if the plus sign in front of u in (2.16) is replaced by minus sign. Unfortunately, the estimate (2.13) is not valid when $\varepsilon = 0$, which will be improved later (see lemma 2.6 below).

The L^∞ estimate of u_y in Lemma 2.2 depends on the same norm at $t = t_0$ when we consider it at the time period (t_0, t). This is however, too strong requirement on the solutions obtained from the transport step in our splitting algorithm as we will see later. But we can also obtain the L^1 estimate of u_y.

Lemma 2.5 *Suppose $v_0 \leq 0$. Then the solution to the problem (2.2) as well as (2.1) satisfies for all $t \geq t_0$,*

$$\int_0^d |u_y(t,x,y)|dy \leq \int_0^d |u_y(t_0,x,y)|dy + u(t,x,0) - u(t_0,x,0). \tag{2.18}$$

In particular,

$$\int_0^d |u_y(t,x,y)|dy \leq \int_0^d |u_y(t_0,x,y)|dy + 2|u|_{L^\infty}. \tag{2.19}$$

Proof. We may assume $v_0 < 0$. Otherwise, one may replace v_0 by $v_0 - \varepsilon$ and let $\varepsilon \to 0$. It follows from (2.7) that

$$\frac{d}{dt}\int_0^d |u_y|dy = \int_0^d [(u^2+\varepsilon)u_{yy}]_y \; sign\, u_y \, dy \leq -(u^2+\varepsilon)u_{yy} sign u_y|_{y=0}.$$

Hence,

$$\frac{d}{dt}\int_0^d |u_y|dy \leq -u_t sign u_y|_{y=0} \leq u_t|_{y=0}. \tag{2.20}$$

Then (2.18) follows easily from (2.20).

We can also obtain the L^1 estimate of u_x for the solution of the problem (2.1).

Lemma 2.6 *The solution to the problem (2.1) satisfies*

$$\begin{aligned} \int_0^d \frac{|u_x(t,\cdot)|}{u^2(t,\cdot)}(d-y)^2 dy \quad &\leq \quad \int_0^d \frac{|u_x(t_0,\cdot)|}{u^2(t_0,\cdot)}(d-y)^2 dy + d^2|v_{0,x}|_{L^\infty}(t-t_0) \\ &\qquad +2\int_{t_0}^t \int_0^d |u_x(s,\cdot)|dyds, \end{aligned} \tag{2.21}$$

for $t > t_0$. Furthermore, if $v_0 \leq 0$, then there exists C_3 which depends only on $|u_0|_{L^\infty}$ and $|v_0|_{L^\infty}$ such that

$$\int_0^d |u_x(t,\cdot)|dy \leq e^{2C_3(t-t_0)}(C_3^2 \int_0^d |u_x(t_0,\cdot)|dy + \frac{d^2}{2}|v_{0,x}|_{L^\infty}). \tag{2.22}$$

Proof. We first consider the solution to the problem (2.2). Setting $w = u_x$, we have from (2.14) that

(2.23)
$$\begin{aligned} \frac{d}{dt}\int_0^d (d-y)^2 \frac{|w|}{u^2+\varepsilon} dy &= \int_0^d (d-y)^2 (sign\, w) w_{yy} dy \\ &= (d-y)^2 |w|_y|_0^d + 2\int_0^d (d-y)|w|_y\, dy \\ &\le -d^2 |w|_y|_{y=0} - 2(d-y)|w||_{y=0} + 2\int_0^d |w| dy \\ &\le d^2 |v_{0x}| + 2\int_0^d |w| dy. \end{aligned}$$

It follows from the assumption and Lemma 2.2 that there exists C_3 such that

$$\frac{1}{C_3}(d-y)^2 \le u^2 \le C_3 (d-y)^2. \tag{2.24}$$

Therefore

$$C_3 \int_0^d (d-y)^2 \frac{|w|}{u^2} dy \ge \int_0^d |w| dy \ge \frac{1}{C_3} \int_0^d (d-y)^2 \frac{|w|}{u^2} dy.$$

Integrating (2.23) and letting $\varepsilon \to 0$, one gets

$$\begin{aligned} \int_0^d \frac{|w(t,\cdot)|}{u^2(t,\cdot)} (d-y)^2 dy &\le \int_0^d \frac{|w(t_0,\cdot)|}{u^2(t_0,\cdot)} (d-y)^2 dy + d^2 |v_{0,x}|_{L^\infty} (t-t_0) + \\ &\quad 2\int_{t_0}^t \int_0^d |w(s,\cdot)| dy ds. \end{aligned} \tag{2.25}$$

Then (2.21) follows from (2.25). And (2.24) together with (2.25) implies

$$\begin{aligned} \int_0^d |w(t,\cdot)| dy &\le C_3 [C_3 \int_0^d |w(t_0,\cdot)| dy + d^2 |v_{0,x}|_{L^\infty} (t-t_0) \\ &\quad + 2\int_{t_0}^t \int_0^d |w(s,\cdot)| dy ds], \end{aligned} \tag{2.26}$$

which yields (2.22) by the Gronwall's inequality.

Remark 2.2 *The estimates (2.21) and (2.22) are independent of $|u_y|_{L^\infty}$. They will be used to obtain the estimates for the solutions to the Prandtl's equation.*

Next, we estimate $|u_x|_{L^\infty}$ in terms of $|u_0|_{L^\infty}$, $|v_0|_{C^2}$ and $|u_y|_{L^\infty}$. To this end, we set

$$w = u_x - v_{0,x}(y-d)$$

Then the problem (2.11) becomes

$$\begin{cases} w_t - (u^2+\varepsilon)w_{yy} = 2uu_{yy}w + 2uu_{yy}v_{0,x}(y-d) - v_{0,tx}(y-d) \\ w|_{t=0} = u_{0,x} - v_{0,x}(y-d) \equiv w_0 \\ \frac{\partial w}{\partial y}|_{y=0} = 0, \quad w|_{y=d} = 0. \end{cases} \tag{2.27}$$

We start with an integral estimate.

Lemma 2.7 *Suppose that $v_0 \leq 0$. Then there exist constants C_4 and C_5, which depend only on $|u|_{L^\infty}$, $|v_0|_{C^2}$ and d, such that*

$$\int_0^d w^2(t,\cdot)dy + \int_0^t \int_0^d (u^2+\varepsilon)w_y^2 dyds \leq [C_5 + \int_0^d w^2(0,\cdot)dy]e^{C_4(1+a_0^2)t}, \tag{2.28}$$

$$\begin{aligned} &\int_0^d \frac{w^2(t,\cdot)}{(y-d)^2+\varepsilon}dy + \int_0^t \int_0^d \frac{(u^2+\varepsilon)}{(y-d)^2+\varepsilon} w_y^2 dyds \\ &\leq [C_5 + \int_0^d \frac{w^2(0,\cdot)}{(y-d)^2+\varepsilon}dy]e^{C_4(1+a_0^2)t}, \end{aligned} \tag{2.29}$$

where a_0 is given in Lemma 2.3.

Proof. Multiplying w and integrating by parts in (2.27), we obtain

(2.30)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^d w^2 dy + \int_0^d (u^2+\varepsilon)w_y^2 dy \\ &= -\int_0^d (2uu_y w w_y + 2u_y(uw^2)_y + v_{0,tx}(y-d)w)dy \\ &\quad -2\int_0^d u_y v_{0,x}[u(y-d)w]_y dy + 2\int_0^d (v_0(w^2u)_y + v_0 v_{0,x}[u(y-d)w]_y)dy. \end{aligned}$$

Since

$$6\int_0^d |uu_y w w_y| dy \leq \frac{1}{2}\int_0^d u^2 w_y^2 dy + 18\int_0^d u_y^2 w^2 dy,$$

then there exists a constant C_4 depending only on $|u|_{L^\infty}$, $|v_0|_{C_2}$ and d, such that

$$(2.31) \qquad \frac{d}{dt}\int_0^d w^2 dy + \int_0^d (u^2+\varepsilon) w_y^2 dy \le C_4 + C_4(1+|u_y|_{L^\infty}^2)\int_0^d w^2 dy,$$

which implies (2.28).

Similarly, one may multiply (2.27) by $\frac{w}{(y-d)^2+\varepsilon}$ and integrate by parts to get

$$\frac{1}{2}\frac{d}{dt}\int_0^d \frac{w^2}{(y-d)^2+\varepsilon}dy + \int_0^d \frac{(u^2+\varepsilon)}{(y-d)^2+\varepsilon} w_y^2\, dy$$

$$(2.32) \qquad = -\int_0^d [(\frac{u^2+\varepsilon}{(y-d)^2+\varepsilon})_y w w_y + 2u_y(\frac{uw^2}{(y-d)^2+\varepsilon})_y + v_{0,tx}\frac{(y-d)w}{(y-d)^2+\varepsilon}$$

$$+2u_y v_{0,x}(\frac{u(y-d)w}{(y-d)^2+\varepsilon})_y]dy + 2\int_0^d v_0[(\frac{w^2 u}{(y-d)^2+\varepsilon})_y + v_{0,x}(\frac{u(y-d)w}{(y-d)^2+\varepsilon})_y]dy.$$

Since

$$\int_0^d \frac{2(u^2+\varepsilon)(d-y)}{((y-d)^2+\varepsilon)^2}|ww_y|dy \le \frac{1}{4}\int_0^d \frac{(u^2+\varepsilon)w_y^2}{(y-d)^2+\varepsilon}dy + 4\int_0^d \frac{(u^2+\varepsilon)(d-y)^2}{((y-d)^2+\varepsilon)^3}w^2 dy,$$

$$6\int_0^d \frac{|uu_y|}{(y-d)^2+\varepsilon}|ww_y|dy \le \frac{1}{4}\int_0^d \frac{u^2 w_y^2}{(y-d)^2+\varepsilon}dy + 36\int_0^d \frac{u_y^2 w^2}{(y-d)^2+\varepsilon}dy,$$

together with (2.24) we deduce

$$\frac{d}{dt}\int_0^d \frac{w^2}{(y-d)^2+\varepsilon}dy + \int_0^d \frac{u^2+\varepsilon}{(y-d)^2+\varepsilon} w_y^2 dy$$

$$\le C_5 + C_4(1+|u_y|_{L^\infty}^2)\int_0^d \frac{w^2}{(y-d)^2+\varepsilon}dy,$$

where C_4 and C_5 are constants depending only on $|u|_{L^\infty}$, $|v_0|_{C^2}$ and d. Then (2.29) follows as before.

Remark 2.3 *The estimates (2.24) and (2.29) imply that for* $0 < t < T$

$$\int_0^t \int_0^d w_y^2 dy\, ds \le e^{C_4(1+a_0^2)t}(C_5 + \int_0^d \frac{w^2(0,\cdot)}{(y-d)^2+\varepsilon}dy).$$

Similarly, one can multiply (2.27) by $\frac{w^3}{(y-d)^2+\varepsilon}$ to obtain

$$\int_0^t \int_0^d w^2 w_y^2 dyds \le e^{C_4(1+a_0^2)t}(C_5 + \int_0^d \frac{w^4(0,\cdot)}{(y-d)^2+\varepsilon}dy). \tag{2.33}$$

The estimate (2.28), together with (2.24), implies that for $0 < t < T$

$$\int_0^t \int_0^d u^2 w_y^2 dy\, ds \le e^{C_4(1+a_0^2)t}(C_5 + \int_0^d w^2(0,\cdot)dy),$$

and

$$\int_0^d w^2(t,\cdot)dy \le e^{C_4(1+a_0^2)t}(C_5 + \int_0^d w^2(0,\cdot)dy).$$

Remark 2.4 *For fixed k_0, we can obtain in the similar way that*

$$\int_0^d |w(t,\cdot)|^{k_0+1}dy \le e^{C_4(1+a_0^2)t}(C_5 + \int_0^d |w(0,\cdot)|^{k_0+1}dy), \quad t \ge 0, \tag{2.34}$$

where C_5 and C_4 depend only on $|u|_{L^\infty}$, $|v_0|_{C^2}$, d and k_0.

To continue our analysis, we need the following result which can be found in [1] (Lemma 5.45).

Proposition 2.1 *There exists an absolute constant C, such that*

$$[\int_0^d w^6 y^2\, dy]^{\frac{1}{3}} \le C \int_0^d (\frac{w^2}{d} + w_y^2) y^2\, dy. \tag{2.35}$$

We are now ready to derive the super-norm estimate on u_x.

Lemma 2.8 *Suppose that $v_0 \le 0$. Then there exists a constant C_6 depending only on T, d, $|u_{0,x}|_{L^\infty}$, $|u_0|_{L^\infty}$, $|u_y|_{L^\infty}$, $|v_0|_{C^2}$ and $\int_0^d \frac{u_{0,x}^2}{(y-d)^2}dy$, such that the solution to the problem (2.27) satisfies*

$$|u_x(t,x,y)| \le C_6 \tag{2.36}$$

Proof. We make use of the Nash-Moser iteration method. For convenience, changing the independent variable as

$$y \to y - d,$$

we need only to consider the following problem

$$\begin{cases} w_t - (u^2+\varepsilon)w_{yy} = 2uu_{yy}w - 2uu_{yy}v_{0,x}y - v_{0,tx}y \\ w|_{t=0} = w_0 \\ \frac{\partial w}{\partial y}|_{y=-d} = 0, \quad w|_{y=0} = 0, \end{cases} \tag{2.37}$$

For $k \geq 1$, multiplying the equation in (2.37) by $|w|^{k-1}w$ and integrating by parts yield

$$\begin{aligned} &\tfrac{1}{k+1}\tfrac{d}{dt}\textstyle\int_{-d}^0 |w|^{k+1}dy + \tfrac{4k}{(k+1)^2}\int_{-d}^0(u^2+\varepsilon)[(|w|^{\frac{k+1}{2}})_y]^2dy \\ &= -\textstyle\int_{-d}^0(2uu_y|w|^{k-1}ww_y + v_{0,tx}y|w|^{k-1}w)dy \\ &\quad - \textstyle\int_{-d}^0 2(u_y - v_0)[u|w|^{k+1} + uv_{0,x}y|w|^{k-1}w]_y dy. \end{aligned} \tag{2.38}$$

Note that

$$2|\textstyle\int_{-d}^0 uu_y|w|^{k-1}ww_ydy| \leq \tfrac{1}{(k+1)^2}\int_{-d}^0 u^2[(|w|^{\frac{k+1}{2}})_y]^2dy + C\int_{-d}^0|w|^{k+1}dy,$$

$$|\textstyle\int_{-d}^0 v_{0,tx}y|w|^{k-1}wdy| \leq C\int_{-d}^0(1+y^2)|w|^kdy,$$

$$|\textstyle\int_{-d}^0 u_y^2|w|^{k+1}dy| \leq C\int_{-d}^0|w|^{k+1}dy,$$

and

$$\begin{aligned} &2|\textstyle\int_{-d}^0 uu_y(|w|^{k+1})_ydy| \\ &\leq \tfrac{1}{(k+1)^2}\textstyle\int_{-d}^0 u^2[(|w|^{\frac{k+1}{2}})_y]^2dy + C(k+1)^2\int_{-d}^0|w|^{k+1}dy; \end{aligned}$$

also

$$\begin{aligned} &2|\textstyle\int_{-d}^0 u_y(uv_{0,x}y|w|^{k-1}w)_ydy| \\ &\leq \tfrac{1}{(k+1)^2}\textstyle\int_{-d}^0 u^2[(|w|^{\frac{k+1}{2}})_y]^2dy + Ck^2\int_{-d}^0(|w|^k+1)dy, \end{aligned}$$

where C is a constant depending only on d, $|w_0|_{L^\infty}$, $|u_y|_{L^\infty}$, and $|v_0|_{C^2}$.

Substituting these estimates into (2.38) and integrating from 0 to t, we obtain

$$\begin{aligned} &\tfrac{1}{k+1}\int_{-d}^{0}|w(t,\cdot)|^{k+1}dy+\tfrac{1}{C_3(k+1)^2}\int_0^t\int_{-d}^{0}y^2[(|w|^{\frac{k+1}{2}})_y]^2dyds \\ &\le C(k+1)^2\int_0^t\int_{-d}^{0}|w|^{k+1}dyds+\tfrac{1}{k+1}\int_{-d}^{0}|w_0|^{k+1}dy+C, \end{aligned} \tag{2.39}$$

where C is a constant with dependence as before. Noting that

$$\int_0^t\int_{-d}^{0}|w|^{k+1}dyds\le(\int_0^t\int_{-d}^{0}|w|^{\frac{4}{3}k}y^2dyds)^{\frac{3}{4}}(\int_0^t\int_{-d}^{0}\frac{|w|^4}{y^2}dyds)^{\frac{1}{4}}, \tag{2.40}$$

we have by the Hardy's inequality that

$$\int_0^t\int_{-d}^{0}|w|^{k+1}dyds\le 4(\int_0^t\int_{-d}^{0}|w|^{\frac{4}{3}k}y^2dyds)^{\frac{3}{4}}(\int_0^t\int_{-d}^{0}|w|^2{w_y}^2dyds)^{\frac{1}{4}}. \tag{2.41}$$

It follows from (2.33), (2.39) and (2.41) that for $0<t<T$,

$$\begin{aligned} &C_3(k+1)\int_{-d}^{0}|w(t,\cdot)|^{k+1}dy+\int_0^t\int_{-d}^{0}y^2(|w|^{\frac{k+1}{2}})_y^2dyds \\ &\le C_7(k+1)^4[\int_0^t\int_{-d}^{0}y^2|w|^{\frac{4}{3}k}dyds]^{\frac{3}{4}}+C_7[(k+1)^4+|w_0|_{L^\infty}^{k+1}], \end{aligned} \tag{2.42}$$

where C_7 is a constant depending on T, d, $|w_0|_{L^\infty}$, $|u_0|_{L^\infty}$, $|u_y|_{L^\infty}$, $|v_0|_{C^2}$ and $\int_0^d\frac{w_0^2}{(y-d)^2}dy$. Note that

$$\int_{-d}^{0}y^2|w|^{\frac{5}{3}(k+1)}dy\le[\int_{-d}^{0}y^2|w|^{3(k+1)}dy]^{\frac{1}{3}}[\int_{-d}^{0}y^2w^{(k+1)}dy]^{\frac{2}{3}}.$$

By the Sobolev inequality and Proposition 2.1, we can obtain from (2.42) that

$$\begin{aligned} &\int_0^t\int_{-d}^{0}y^2|w|^{\frac{5}{3}(k+1)}dyds \\ &\le C_7(k+1)^4[(\int_0^t\int_{-d}^{0}y^2|w|^{\frac{4}{3}k}dyds)^{\frac{3}{4}}+\tfrac{1}{d^2}\int_0^t\int_{-d}^{0}y^2|w|^{k+1}dyds \\ &\quad+(1+|w_0|_{L^\infty}^{k+1})][\sup_{s\le t}\int_{-d}^{0}y^2|w(s,\cdot)|^{(k+1)}dy]^{\frac{2}{3}}, \end{aligned}$$

which, together with (2.42) again, shows that

$$
\begin{aligned}
&[\int_0^t \int_{-d}^0 y^2 |w|^{\frac{5}{3}(k+1)} dyds]^{\frac{3}{5(k+1)}} \\
&\le C_8^{\frac{1}{k+1}} (k+1)^{\frac{4}{k+1}} [(\int_0^t \int_{-d}^0 y^2 |w|^{\frac{4}{3}(k+1)} dyds)^{\frac{3}{4(k+1)}} + 1 + |w_0|_{L^\infty}],
\end{aligned}
\tag{2.43}
$$

where C_8 is a constant depending on T, d, $|w_0|_{L^\infty}$, $|u_0|_{L^\infty}$, $|u_y|_{L^\infty}$, $|v_0|_{C^2}$ and $\int_0^d \frac{w_0^2}{(y-d)^2} dy$.

For convenience, we may assume that

$$
1 + |w_0|_{L^\infty} \le (\int_0^t \int_{-d}^0 y^2 |w|^{\frac{4}{3}(k+1)} dyds)^{\frac{3}{4(k+1)}},
$$

otherwise, the proof is complete. Let

$$
||w||_p = (\int_0^t \int_{-d}^0 y^2 |w|^p dyds)^{\frac{1}{p}},
$$

then (2.43) can be written as

$$
||w||_{\frac{5}{3}(k+1)} \le (C_8 (k+1)^4)^{\frac{1}{k+1}} ||w||_{\frac{4}{3}(k+1)}. \tag{2.44}
$$

We choose k so that $\frac{4}{3}(k+1) = (\frac{5}{4})^j$, $j = 2, 3, \cdots$. Then (2.44) implies that

$$
\begin{aligned}
&||w||_{(\frac{5}{4})^{j+1}} \\
&\le C_8^{(\frac{4}{5})^j} e^{8(\frac{4}{5})^j \log((\frac{5}{4})^j)} ||w||_{(\frac{5}{4})^j} \\
&\le C_8^{\sum_{i=6}^j (\frac{4}{5})^i} e^{\sum_{i=6}^j 8(\frac{4}{5})^i \log((\frac{5}{4})^i)} ||w||_{(\frac{5}{4})^6}.
\end{aligned}
$$

Since $\sum_{i=6}^\infty (\frac{4}{5})^i < \infty$ and $\sum_{i=6}^\infty 8(\frac{4}{5})^i \log((\frac{5}{4})^i) < \infty$, then

$$
|w| \le C_9 ||w||_{(\frac{5}{4})^6}. \tag{2.45}
$$

For $k_0 + 1 = (\frac{5}{4})^6$, one can deduce (2.34) from (2.33). Then we finished the proof of Lemma 2.8.

Similarly we can estimate u_t. Indeed, note that u_t satisfies the same equation as u_x does. Thus $w = u_t$ solves

$$(2.46)\qquad \begin{cases} w_t - (u^2+\varepsilon)w_{yy} - 2uu_{yy}w = 0 \\ w|_{t=0} = (u_0^2+\varepsilon)u_{0,yy} \\ \frac{\partial w}{\partial y}|_{y=0} = v_{0t}, \quad w|_{y=d} = 0. \end{cases}$$

Therefore all the estimates for u_x are also true for u_t. In particular, we have

Corollary 2.1 *Under the same assumptions as in Lemma 2.6, the solution to the problem (2.1) satisfies*

$$(2.47)\qquad \begin{aligned} &\textstyle\int_0^d \frac{|u_t(t,\cdot)|}{u(t,\cdot)^2}(d-y)^2dy \\ &\le \textstyle\int_0^d \frac{|u_t(t_0,\cdot)|}{u(t_0,\cdot)^2}(d-y)^2dy + d^2|v_{0,t}|_{L^\infty}(t-t_0) + 2\int_{t_0}^t\int_0^d |u_t(s,\cdot)|dyds. \end{aligned}$$

Moreover, there exists a constant $C_{10} = C_{10}(|u_0|_{L^\infty}, |v_0|_{L^\infty})$, such that

$$(2.48)\qquad \int_0^d |u_t(t,\cdot)|dy \le e^{2C_{10}(t-t_0)}(C_{10}^2\int_0^d |u_t(t_0,\cdot)|dy + \frac{d^2}{2}|v_{0,t}|_{L^\infty}).$$

The proof is the same as that of Lemma 2.6.

Remark 2.5 *As in Lemma 2.8, there exists a constant C_{11} depending on T, d, $|u_0|_{C^2}$, $|u_y|_{L^\infty}$ and $|v_0|_{C^2}$, such that*

$$(2.49)\qquad |u_t(t,x,y)| \le C_{11}.$$

Collecting all the estimates we have obtained, we have arrived at the following conclusion:

Theorem 2.1 *Assume that $\nu_0 \le 0$ and the data are smooth and compatible. Then the problem (2.1) has a unique positive bounded smooth solution in $(0,T]\times\Omega$. Moreover*

$$u \in Lip([0,T]\times\overline{\Omega}),$$

and u satisfies the estimates (2.4), (2.5), (2.9), (2.36) and (2.49).

Proof. The existence and estimates on the solution follow from the above lemmas and the standard limiting argument. To prove the theorem, it suffices to show the uniqueness of the solution.

Suppose that there are two solutions u and $\overline{u}$. Then their difference

$$w = u - \overline{u}$$

solves

$$\begin{cases} w_t - \overline{u}^2 w_{yy} - (u+\overline{u})u_{yy}w = 0, \\ w|_{t=0} = 0, \\ \frac{\partial w}{\partial y}|_{y=0} = 0, \quad w|_{y=d} = 0. \end{cases} \tag{2.50}$$

Multiplying (2.50) by w and integrating by parts, one gets

$$\frac{1}{2}\frac{d}{dt}\int_0^d w^2 dy + \int_0^d u^2 w_y^2 dy = \int_0^d (v_0 - u_y)[(u+\overline{u})w^2]_y dy - 2\int_0^d u u_y w w_y dy.$$

Since

$$-2\int_0^d u u_y w w_y dy \le \tfrac{1}{2}\int_0^d u^2 w_y^2 dy + 2\int_0^d u_y^2 w^2 dy,$$

$$2\int_0^d (v_0 - u_y)(u+\overline{u}) w_y w dy \le \tfrac{1}{2}\int_0^d u^2 w_y^2 dy + 2\int_0^d (v_0 - u_y)^2 (1 + \tfrac{\overline{u}}{u})^2 w^2 dy.$$

It follows from (2.24) and (2.9)that there exists a constant C such that

$$\frac{d}{dt}\int_0^d w^2 dy \le C\int_0^d w^2 dy. \tag{2.51}$$

Then the uniqueness follows by (2.51) and $w|_{t=0} = 0$.

3 Global Existence I, A Special Case

In this section, we will prove the Theorem 1.1 in the simple case that (1.10) holds and under some additional assumption on the boundary data. For the convenience of notations, we take $d = 1$ and rewrite the initial-boundary value problem (1.7) in this case as

$$
(3.1) \qquad \begin{cases} u_t + yu_x - u^2 u_{yy} = 0 & (0,T)\times\Omega \\ u|_{t=0} = u_{0,y} \quad u|_{x=0} = u_{1,y} \\ \frac{\partial u}{\partial y}|_{y=0} = v_0, \quad u|_{y=1} = 0, \end{cases}
$$

where $\Omega = \{(x,y)|0 < x < L, \quad 0 < y < 1\}$, u_0 and u_1 are given in (1.1) (the correspondence between (1.7) and (3.1) is given by identifying w as u, τ as t, ξ as x, and η as y, moreover, we have taken $\nu = 1$ for simplicity of presentation). Oleinik proved the local existence of smooth solution to the problem (3.1) in [5]. We shall show that under our assumptions (1.3)-(1.5), the solution to problem (3.1) will remain globally in the class that Oleinik considered. One of the crucial point in our argument is to obtain uniform total variation estimates on the solutions. As outlined in the introduction, this will be achieved by a viscous splitting method (which is motivated by the numerical method for Navier-Stokes system in [2]). This idea works directly in some special case. More precisely, setting

$$
(3.2) \qquad b(t,x,y) = \frac{\xi(x)}{u_{1,y}}(u_{1,yt} - u_{1,y}^2 u_{1,yyy}),
$$

we assume throughout this section that

$$
b(t,x,y) = 0.
$$

Here $\xi(x)$ is the nonnegative truncated function which is smooth and monotone non-increasing such that $\xi(0) = 1$ and $\xi(\frac{L}{2}) = 0$. The general case will be treated in Section 4-6. Our assumption implies that

$$
u_{1,yt} - u_{1,y}^2 u_{1,yyy} = 0.
$$

In this case, we shall see that it is easy to match the boundary conditions.

Now let $t_i = \frac{i}{n}T$, $1 \le i \le n$, for any given T. We first consider the problem

$$
(3.3) \qquad \begin{cases} u_t - u^2 u_{yy} = 0 & (0,t_1]\times\Omega \\ u|_{t=0} = u_{0,y}, \quad u|_{y=1} = 0 \\ \frac{\partial u}{\partial y}|_{y=0} = v_0. \end{cases}
$$

To simplify the notations we set $\overline{u}_1 = u_{1,y}$ and $\overline{u}_0 = u_{0,y}$. It follows from Theorem 2.1 that there exists a unique solution u to (3.3). Moreover, $u|_{x=0} = \overline{u}_1$ by our assumption. Then in next time step, we solve the problem

$$\begin{cases} u_t + yu_x = 0, & (t_1, t_2] \times \Omega \\ u|_{t=t_1} = u(t_1, x, y), & u|_{x=0} = \overline{u}_1(t, y). \end{cases} \tag{3.4}$$

In fact the solution to problem (3.4) can be written down explicitly

$$u(t,x,y) = \begin{cases} u(t_1, x-(t-t_1)y, y) & x > (t-t_1)y, \\ \overline{u}_1(t-\frac{x}{y}, y) & x \leq (t-t_1)y. \end{cases}$$

Suppose that the solution is obtained for $0 < t \leq t_{i-1}$. When i is odd, for $t_{i-1} < t \leq t_i$, we solve

$$\begin{cases} u_t - u^2 u_{yy} = 0 & (t_{i-1}, t_i] \times \Omega \\ u|_{t=t_{i-1}} = u(t_{i-1}, x, y) & u|_{y=1} = 0 \\ \frac{\partial u}{\partial y}|_{y=0} = v_0(t,x,y). \end{cases} \tag{3.5}$$

When i is even, for $t_{i-1} < t \leq t_i$, we define

$$u(t,x,y) = \begin{cases} u(t_{i-1}, x-(t-t_{i-1})y, y) & x > (t-t_{i-1})y \\ \overline{u}_1(t-\frac{x}{y}, y) & x \leq (t-t_{i-1})y. \end{cases} \tag{3.6}$$

Remark 3.1 *The function u constructed above depends on n. We omitted the index n for the sake of convenience.*

We now estimate this approximate solution. We start with the simple uniform super-norm estimate.

Lemma 3.1 *For $0 \leq t \leq T$, $(x, y) \in \Omega$, it holds that*

$$|u(t,x,y)| \leq |\overline{u}_0|_{L^\infty} + 2|v_0|_{L^\infty} + |\overline{u}_1|_{L^\infty}. \tag{3.7}$$

Proof. (3.7) is satisfied for $0 \leq t \leq t_1$ due to Lemma 2.1. It is easy to see that (3.7) is true for $t_1 \leq t \leq t_2$ by the explicit construction. In both cases $|v_0|_{L^\infty}(1-y)\mp u$ is bounded from above by

$$||v_0|_{L^\infty}(1-y)\mp u_0|_{L^\infty} \quad \text{or} \quad |\overline{u}_1|_{L^\infty} + 1|v_0|_{L^\infty}.$$

Suppose for $0 \le t \le t_{i-1}$

$$(3.8) \qquad |v_0|_{L^\infty}(1-y) \mp u \le \max\{|[|v_0|_{L^\infty}(1-y) \mp u_0]|_{L^\infty}, |\overline{u}_1|_{L^\infty} + |v_0|_{L^\infty}\}.$$

Now we consider $t_{i-1} \le t \le t_i$. When i is odd, Lemma 2.1 shows that

$$|v_0|_{L^\infty}(1-y) \mp u \le \max[|v_0|_{L^\infty}(1-y) \mp u_(t_{i-1}, \cdot)].$$

Therefore (3.8) holds for this case. In the case that i is even, (3.8) still holds true due to (3.6). Then (3.7) follows easily from (3.8). This proves Lemma 3.1.

This bound can be improved to show that the function u is always positive.

Lemma 3.2 *Suppose $v_0 \le 0$. Then there exist constants C_0 and β depending only on $|v_0|_{L^\infty}$, $|\overline{u}_0|_{L^\infty}$ and $|\overline{u}_1|_{C^1}$, such that*

$$(3.9) \qquad u(t,x,y) \le C_0(1-y),$$

$$(3.10) \qquad u(t,x,y) \ge \theta_0 \varepsilon^{-\beta t} \phi,$$

where ϕ , and θ_0 are given in Lemma 2.2.

Proof. When i is odd and $t_{i-1} \le t \le t_i$, (3.9) follows from Lemma 2.2 with $C_0 = |u|_{L^\infty}$. For i being even and $t_{i-1} \le t \le t_i$, (3.9) holds true by (3.6) and Lemma 2.2 with $C_0 = C_0(|u|_{L^\infty}, |\overline{u}_1|_{C^1})$. This proves the first part of the lemma.

Next, (3.10) is true for $0 \le t \le t_1$ by Lemma 2.2. Suppose that (3.10) remains valid for $0 \le t \le t_{i-1}$. We consider $t_{i-1} \le t \le t_i$. If i is odd, it follows from Lemma 2.2 that there exists β depending on C_0, so that

$$(3.11) \qquad \frac{u(t,x,y)}{\phi(y)} \ge e^{-\beta(t-t_{i-1})} \min \frac{u(t_{i-1},x,y)}{\phi(y)}.$$

Then (3.10) follows by (3.11) and our induction assumptions. In the case that i is even, (3.10) is an immediate consequence of the assumption $\overline{u}_1 > 0$ for $0 \le y < 1$ and $\overline{u}_1|_{y=1} = 0$. The proof of Lemma 3.2 is complete.

Now we consider the estimates of the gradient of u. We can show that the function u is C^1 smooth except at the origin of the space. Moreover, we can prove that its $W^{1,1}$ norm is uniformly bounded.

Lemma 3.3 *Suppose that $v_0 \le 0$. For any fixed ε_0 and n, there exists a constant $C = C(\varepsilon_0, n)$, such that*

$$|u_x| + |u_y| \le C, \tag{3.12}$$

for $0 \le t \le T$, $(x, y) \in \Omega \setminus \{(x, y)|x + y < \varepsilon_0\}$, and,

$$|u_t| \le C, \tag{3.13}$$

for $0 \le t \le T$, $(x, y) \in \Omega \setminus \{(x, y)|x + y < \varepsilon_0 \quad y < 1 - \varepsilon_0\}$.

Proof. For $0 \le t \le t_1$, (3.12) follows from Lemma 2.3 and Lemma 2.8, while (3.13) is a consequence of Remark 2.5.

Suppose that (3.12) and (3.13) hold for $0 \le t \le t_{i-1}$. We consider $t_{i-1} \le t \le t_i$. For i odd, both estimates remain true again by Lemma 2.3, Lemma 2.8 and Remark 2.5. While for even i, the desired estimates follow by (3.6) through direct computations.

Remark 3.2 *The constant C in Lemma 3.3 may depend on n, so we cannot obtain a Lipschitz estimate for solutions of (3.1). But the estimates (3.7), (3.9) and (3.10) are independent of n.*

Corollary 3.1 *The function u is continuous in t, x and y in $\Omega \setminus (0, 0)$, and takes on the initial and boundary data in the sense that*

$$\lim_{t \to 0} u(t, x, y) = \overline{u}_0(x, y), \qquad (x, y) \in \Omega, \tag{3.14}$$

(3.15) $$\lim_{y\to 1} u(t,x,y) = 0, \qquad (t,x) \in (0,T] \times (0,L],$$

Moreover, by (3.6), Theorem 2.1 and our assumption $b(t,x,y)=0$ we have

(3.16) $$\lim_{x\to 0} u(t,x,y) = \overline{u}_1(t,y) \qquad (t,y) \in (0,T] \times (0,1].$$

And for $t_{i-1} < t < t_i$, i is odd, we have by Theorem 2.1 that

(3.17) $$\lim_{y\to 0} u_y(t,x,y) = v_0(t,x) \quad x \in (0,L].$$

Now we consider the L^1 estimates of $|u_x|$ and $|u_y|$, which are crucial in our analysis.

Lemma 3.4 *There exists a constant C_3 depending on $|\overline{u}_0|_{L^\infty}$, $|v_0|_{L^\infty}$, and $|\overline{u}_1|_{C^1}$, such that*

(3.18)
$$\int\int_\Omega |u_x(t,\cdot)|dx\,dy \le C_3 e^{C_3 t}[\int\int_\Omega (C_3^2|\overline{u}_{0,x}|)dx\,dy + \frac{L}{2}|v_{0,x}|_{L^\infty} + \frac{1}{2}\int_0^1 \frac{|\overline{u}_{1,t}|}{\overline{u}_1^2}(1-y)^2\,dy]$$

Proof. For $0 < t \le t_1$, (3.18) follows from (2.22). Moreover, in this case, one gets from (2.21) that

(3.19)
$$\begin{aligned} \int\int_\Omega \tfrac{|u_x(t,\cdot)|}{u^2(t,\cdot)}(1-y)^2\,dy\,dx \le\ & \int_0^t \int\int_\Omega [|v_{0,x}|_{L^\infty} + 2|u_x|]dx\,dy\,ds \\ & + \int\int_\Omega \tfrac{|u_x(0,\cdot)|}{u^2(0,\cdot)}(1-y)^2\,dx\,dy. \end{aligned}$$

For $t_1 < t \le t_2$, one calculates from (3.6) that

(3.20)
$$\int\int_\Omega \frac{|u_x(t,\cdot)|}{u^2(t,\cdot)}(1-y)^2\,dx\,dy \le \int_{t_1}^t \int_0^1 \frac{|\overline{u}_{1,t}|}{\overline{u}_1^2}(1-y)^2\,dy\,ds + \int\int_\Omega \frac{|u_x(t_1,\cdot)|}{u^2(t_1,\cdot)}(1-y)^2\,dx\,dy.$$

In general, for $t_{i-1} < t \le t_i$, if i is odd, then

(3.21)
$$\begin{aligned} \int\int_\Omega \tfrac{|u_x(t,\cdot)|}{u^2(t,\cdot)}(1-y)^2\,dx\,dy \le\ & \int_{t_{i-1}}^t \int\int_\Omega [|v_{0,x}|_{L^\infty} + 2|u_x|]dx\,dy\,dx \\ & + \int\int_\Omega \tfrac{|u_x(t_{i-1},\cdot)|}{u^2(t_{i-1},\cdot)}(1-y)^2\,dx\,dy; \end{aligned}$$

while for even i,

$$\int\int_\Omega \frac{|u_x(t,\cdot)|}{u^2(t,\cdot)}(1-y)^2\ dx\,dy \le \int_{t_{i-1}}^t\int_0^1 \frac{|\overline{u}_{1,t}|}{\overline{u}_1^2}(1-y)^2\ dy\,ds + \int\int_\Omega \frac{|u_x(t_{i-1},\cdot)|}{u^2(t_{i-1},\cdot)}(1-y)^2\ dx\,dy. \tag{3.22}$$

Therefore for $t_{i-1} < t \le t_i$, it holds that

$$\int\int_\Omega \frac{|u_x(t,\cdot)|}{u^2(t,\cdot)}(1-y)^2\ dx\,dy \le \int_0^t\int_0^1 \frac{|\overline{u}_{1,t}|}{\overline{u}_1^2}(1-y)^2\ dy\,ds + \int_0^t\int\int_\Omega[|v_{0x}|_{L^\infty} + 2|u_x|]dx\,dy\,ds + \int\int_\Omega \frac{|u_x(0,\cdot)|}{u^2(0,\cdot)}(1-y)^2\ dx\,dy. \tag{3.23}$$

By Lemma 3.2, there exists C_3 depending on $|v_0|_{L^\infty}$ and $|\overline{u}_0|_{C^1}$ such that

$$\frac{1}{C_3}(1-y)^2 \le u^2 \le C_3(1-y)^2. \tag{3.24}$$

Hence,

$$\frac{1}{C_3}\int\int_\Omega |u_x(t,\cdot)|dx\,dy \le \int_0^t\int_0^1 \frac{|\overline{u}_{1,t}|}{\overline{u}_1^2}(1-y)^2\ dy\,ds + \int_0^t\int\int_\Omega[|v_{0,x}|_{L^\infty} + 2|u_x|]dx\,dy\,ds + C_3\int\int_\Omega |u_x(0,\cdot)|dx\,dy. \tag{3.25}$$

Then (3.18) follows from (3.25) and the Gronwall inequality.

Corollary 3.2 *For any $0 < x' \le x'' \le L$, it holds that for sufficiently large n,*

$$\int_{x'}^{x''}\int_0^1 |u_x(t,\cdot)|dx\,dy \le e^{C_3 t}\int_{x'-t}^{x''}\int_0^1 (C_3^2|\overline{u}_{0,x}| + \frac{1}{2}|v_{0,x}|_{L^\infty})dx\,dy$$

Proof. In fact, in the proof of Lemma 3.4, if $x' > \frac{T}{n}$, then the term from boundary at $x = 0$ vanishes in (3.20) and (3.23). Thus the proof of Corollary 3.2 follows easily.

Lemma 3.5 *It holds that for $0 < t \le t_i$,*

$$\int\int_\Omega |u_y(t,\cdot)|dx\,dy \le \int_0^t\int_0^1[|\overline{u}_{1,y}| + \frac{T|\overline{u}_{1,t}|}{n}]dy\,ds + 2\int_0^L |u(\cdot,x,0)|_{L^\infty}dx + \int\int_\Omega[|\overline{u}_{0,y}| + \frac{T}{n}\sum_{k=1}^{k=i-1}|u_x(t_k,\cdot)|]dx\,dy. \tag{3.26}$$

Proof. (3.26) follows from (2.19) for $0 < t \le t_1$. For $t_1 < t \le t_2$, (3.6) shows

$$\begin{aligned} &\int\int_\Omega |u_y(t,\cdot)| dx\, dy \\ = &\int\int_{x>(t-t_1)y} |u_x(t_1, x-(t-t_1)y, y)(t_1-t) + u_y(t_1, x-(t-t_1)y, y)| dx\, dy \\ &+ \int\int_{x\le(t-t_1)y} |\overline{u}_{1,y}(t-\tfrac{x}{y}, y) + \overline{u}_{1,t}(t-\tfrac{x}{y}, y)(-\tfrac{x}{y^2})| dx\, dy. \end{aligned}$$

Then

$$\int\int_\Omega |u_y(t,\cdot)| dx\, dy \le \int_{t_1}^t \int_0^1 [|\overline{u}_{1,y}| + \tfrac{T|\overline{u}_{1,t}|}{n}] dy\, ds + \int\int_\Omega [|u_y(t_1,\cdot)| + \tfrac{T}{n}|u_x(t_1,\cdot)|] dx\, dy. \tag{3.27}$$

And also by (3.6),

$$u(t_2, x, 0) = u(t_1, x, 0).$$

In general, for $t_{i-1}, t \le t_i$, when i is odd, we have by Lemma 2.5 that

$$\int\int_\Omega |u_y(t,\cdot)| dx\, dy \le \int\int_\Omega |u_y(t_{i-1},\cdot)| dx\, dy + \int_0^L [u(t,x,0) - u(t_{i-1},x,0)] dx, \tag{3.28}$$

while in the case that i is even,

$$\int\int_\Omega |u_y(t,\cdot)| dx\, dy \le \int_{t_{i-1}}^t \int_0^1 [|\overline{u}_{1,y}| + \tfrac{T|\overline{u}_{1,t}|}{n}] dy\, ds + \int\int_\Omega [|u_y(t_{i-1},\cdot)| + \tfrac{T}{n}|u_x(t_{i-1},\cdot)|] dx\, dy; \tag{3.29}$$

$$u(t_i, x, 0) = u(t_{i-1}, x, 0). \tag{3.30}$$

Then (3.26) follows from (3.28), (3.29) and (3.30), which proves Lemma 3.5.

Corollary 3.3 *For any $0 < x' \le x'' \le L$, if n is sufficiently large, then*

$$\int_{x'}^{x''} \int_0^1 |u_y(t,\cdot)| dy\, dx \le \int_{x'}^{x''} \int_0^1 [|\overline{u}_{0,y}| + \frac{T}{n} \sum_{k=1}^{i-1} |u_x(t_i,\cdot)|] dy\, dx + 2\int_{x'}^{x''} |u|_{L^\infty}\, dx.$$

Proof. Note that in the proof of Lemma 3.5, if $x' > \frac{T}{n}$, then the term from the boundary at $x = 0$ vanishes in (3.27) and (3.29). Then the corollary follows easily from the proof of Lemma 3.5.

Now we can give the main results of this section. Let $u^n(t,x,y)$ denote the function obtained through (3.3) to (3.6). Let $BV(\Omega)$ denote the space of function of finite total variation on Ω.

Theorem 3.1 *Assume that (1.3), (1.4), and (1.10) hold and $b(t,x,y) = 0$. Then the problem (3.1) has a weak solution*

$$u \in L^\infty(0,T;BV(\Omega)) \cap BV((0,t)\times\Omega),$$

which satisfies the estimates (3.7), (3.9) and (3.10). Furthermore, its derivatives satisfy (3.18) and

(3.31)

$$\int\int_\Omega |u_y(t,\cdot)|dx\,dy \le \int_0^t\int_0^1 |\overline{u}_{1,y}|dy\,ds + 2L|u|_{L^\infty} + \int\int_\Omega |\overline{u}_{0,y}|dx\,dy + \int_0^t\int\int_\Omega |u_x(s,\cdot)|dx\,dy\,ds,$$

(3.32)

$$\begin{aligned}\int\int_\Omega |u_t(t,\cdot)|dx\,dy \le\ & \int_0^1 \Big[\frac{y(1-y)}{u(t,0,y)} + \frac{y(1-y)}{u(t,L,y)}\Big]dy \\ & + \int_0^L |v_0(t,\cdot)|dx + \int\int_\Omega |u_y(t,\cdot)|dx\,dy.\end{aligned}$$

Proof. Let Y be the dual space of $H_0^2(\Omega)$. We claim

$$||\frac{\partial}{\partial t}(\frac{1-y}{u^n})||_{L^2(0,T;Y)} \le C_4, \quad (3.33)$$

where C_4 is independent of n.

In fact, if i is odd, then

$$\begin{aligned}\int_{t_{i-1}}^{t_i} ||\frac{\partial}{\partial t}(\frac{(1-y)^2}{u^n})||_Y^2\,ds &= \int_{t_{i-1}}^{t_i} ||((1-y)^2){u^n}_{yy}||_Y^2\,ds \\ &= \int_{t_{i-1}}^{t_i} \sup_{||\phi||_{H_0^2(\Omega}\le 1}\Big[\int\int_\Omega (1-y)^2 {u^n}_{yy}\phi\;dx\,dy\Big]^2 ds \\ &\le C\int_{t_{i-1}}^{t_i} |u^n(t,\cdot)|_{L^\infty}^2\,dt,\end{aligned}$$

with an uniform constant C depending only on L.

When i is even,

$$\begin{aligned} \int_{t_{i-1}}^{t_i} ||\frac{\partial}{\partial t}(\frac{(1-y)^2}{u^n})||_Y^2 \, dt &= \int_{t_{i-1}}^{t_i} ||((1-y)^2)u^n{}_x||_Y^2 \\ &\leq \int_{t_{i-1}}^{t_i} L|u^n(t,\cdot)|_{L^\infty}^2 \, dt. \end{aligned}$$

Then (3.33) follows easily.

As a consequence of Lemma 3.2, Lemma 3.4 and Lemma 3.5, we know

$$\frac{(1-y)^2}{u^n} \in L^2(0,T;W^{1,1}(\Omega))$$

and

$$||\frac{(1-y)^2}{u^n}||_{L^2(0,T;W^{1,1}(\Omega))} \leq C_5, \tag{3.34}$$

where C_5 is independent of n. By the Theorem 2.1 of [10, Chapter III], we conclude that $\{\frac{(1-y)^2}{u^n}\}$ is in a compact set of $L^2((0,T)\times\Omega)$ (note $\frac{1-y}{u^n} \in L^\infty$). Therefore one may assume

$$\frac{(1-y)^2}{u^n} \longrightarrow \frac{(1-y)^2}{u} \qquad \text{strongly in} \quad L^2((0,T)\times\Omega,$$

$$u^n \longrightarrow u \qquad \text{a.e in} \quad (0,T)\times\Omega.$$

In particular,

$$u^n \longrightarrow u \qquad \text{strongly in} \quad L^2((0,T)\times\Omega).$$

For any $\psi \in C_0^\infty((0,T)\times\Omega)$,

(3.35)

$$\int_0^t \int\int_\Omega [(\frac{1}{u^n})_t + y(\frac{1}{u^n})_x]\psi \, dx\, dy\, ds = \sum_{\text{i, odd}} \int_{t_{i-1}}^{t_i} \int\int_\Omega [-u^n_{yy} + y(\frac{1}{u^n})_x]\psi \, dx\, dy\, ds.$$

This and Corollary 2.1 lead to

(3.36)

$$\int_0^t \int\int_\Omega \frac{1}{u^n}(\psi_t + y\psi_x) dx\, dy\, ds = \sum_{\text{i, odd}} \int_{t_{i-1}}^{t_i} \int\int_\Omega (u^n\psi_{yy} + (\frac{y}{u^n})\psi_x) dx\, dy\, ds.$$

Let $n \to \infty$, because $t_i - t_{i-1} = \frac{T}{n}$, we have

$$\int_0^t \int\int_\Omega \frac{1}{u}(\psi_t + y\psi_x) dx\, dy\, ds = \frac{1}{2}\int_0^t \int\int_\Omega (u\psi_{yy} + (\frac{y}{u})\psi_x) dx\, dy\, ds. \tag{3.37}$$

Then u satisfies the equation of problem (3.1) in the sense of distribution by a suitable rescaling of the t variable. The estimates (3.7), (3.9), (3.10), (3.18) and (3.31) follow easily by letting $n \to \infty$. Finally, (3.32) can be obtained from the equation of problem (3.1) directly by multiplying $\frac{(1-y)^2}{u^2}$, differentiating with respect to t, and applying the standard L^1 estimate techniques to the resulting equation.

Now we check the boundary and initial conditions. It follows from Lemma 3.2 that

$$\lim_{y\to 1} u(t,x,y) = 0. \quad \text{a.e in} \quad (0,T)\times(0,L)$$

Thanks to (3.7),(3.18), (3.31) and (3.32), we can deduce that

$$\lim_{t\to 0} ||u(t,\cdot) - u_{0y}(\cdot)||_{L^2(\Omega)} = 0.$$

The boundary condition of u at $x = 0$ equals u_{1y} in the sense of trace of u (see Theorem 4.11.5 and Theorem 6.5.4, in [4]). The boundary condition of u_y at $y = 0$ is in the sense that for any

$$\psi \in C_0^\infty((0,T)\times(0,L)\times(-1,1))$$

and $\psi_y|_{y=0} = 0$, it holds that

$$\int_0^T \int\int_\Omega \frac{1}{u}(\psi_t + y\psi_x) dx\, dy\, dt = \frac{1}{2}\int_0^T \int\int_\Omega (u\psi_{yy} + (\frac{y}{u})\psi_x) dx\, dy\, dt - \int_0^T\int_0^L v_0\psi|_{y=0} dx\, dt. \tag{3.38}$$

Then we finished the proof of Theorem 3.1.

Remark 3.3 *It can be proved that the solution is smooth in short time. In fact, we can obtain the L^∞ gradient estimate of u in space variables in short time. Then the initial condition is satisfied in the sense*

$$\lim_{t\to 0} u(t,x,y) = \overline{u}_0(x,y).$$

4 More on Porous Medium Type Equations

In order to treat the general case without the assumption that $b(t, x, y) = 0$, we will show in this section that most of the results in section 2 are still true for more general porous medium type equations. With these results, we are able to show the global existence of Prandtl's equation for more general boundary conditions. Thus we consider the following problem

$$\left\{ \begin{array}{ll} u_t = u^2 u_{yy} + b(t, x, y)u & (t, x, y) \in (0, T) \times \Omega \\ u|_{t=0} = u_{0,y} > 0 & \\ \frac{\partial u}{\partial y}|_{y=0} = v_0, \quad u|_{y=1} = 0, & \end{array} \right. \tag{4.1}$$

where Ω, u_0, and $v_0 \le 0$ are the same as in section 3, and $b(t, x, y)$ is defined by (3.2) in section 3. We shall assume that there exists a constant B so that

$$|b(t, x, y)| \le B. \tag{4.2}$$

The problem (4.1) is approximated by

$$\left\{ \begin{array}{ll} u_t = (u^2 + \varepsilon)u_{yy} + bu & (t, x, y) \in (0, T) \times \Omega \\ u|_{t=0} = u_{0,y} > 0 & \\ \frac{\partial u}{\partial y}|_{y=0} = v_0, \quad u|_{y=1} = 0, & \end{array} \right. \tag{4.3}$$

It is well known that the problem (4.3) has a smooth solution when it is bounded. Then the solution of the problem (4.1) can be obtained by letting $\varepsilon \to 0$. For convenience, we will omit the subindex ε when we mention the solution to the problem (4.3) in the following discussion. The first estimate is obtained by the maximum principle, which is parallel to that of Lemma 2.1.

Lemma 4.1 *The solution of problem (4.1) satisfies*

$$|v_0|_{L^\infty}(1 - y) \mp u e^{-Bt} \le \max_{x,y}(|v_0|(1 - y) \mp u e^{-Bt_0})|_{t=t_0}, \tag{4.4}$$

where $t_0 < t \le T$. In particular,

$$|u(t, y)| \le e^{Bt}[|u(0, \cdot)|_{L^\infty} + 2|v_0|_{L^\infty}].$$

Proof. We first consider the solution of problem (4.3). Let

$$w_{\mp} = (|v_0|_{L^\infty} + \varepsilon_0)(y-1) \mp u e^{-Bt}.$$

As in the proof of Lemma 2.1, one can check by (4.2) that

$$L^\varepsilon w_{\mp} + (B-b)w_{\mp} = (b-B)(|v_0|_{L^\infty} + \varepsilon_0)(1-y) \leq 0,$$

here L^ε is defined in section 2. By the maximum principle and similar reasoning as in the proof of Lemma 2.1, one may deduce (4.4) by letting $\varepsilon \to 0$. This proves Lemma 4.1. The following result is parallel to Lemma 2.2.

Lemma 4.2 *There exist constants β and C_0 depending on B, $|u_0|_{L^\infty}$ and $|v_0|_{L^\infty}$, such that the solution to the problem (4.1) satisfies*

$$|u| \leq C_0(1-y)e^{Bt}, \tag{4.5}$$

$$u \geq \theta_0 e^{-\beta t}\phi, \tag{4.6}$$

where $\theta_0 = \min_{x,y} \frac{u_{0,y}}{\phi}$ and ϕ is given in Lemma 2.2 with $d = 1$.

Proof. The proof of (4.5) is similar to that of Lemma 2.2. To prove (4.6), we first show that the solution to (4.3) is positive, i.e.,

$$u \geq 0 \qquad \text{on} \quad (0,T)\times\Omega. \tag{4.7}$$

In fact, it follows from $L^\varepsilon(e^{-Bt}u) + (B-b)e^{-Bt}u = 0$ that u can only achieve its negative minimum at $y = 0$. Then at this points, $\frac{\partial u}{\partial y} \geq 0$. We may assume that $v_0 < 0$, otherwise we replace it by $v_0 - \varepsilon_0$. Then $\frac{\partial u}{\partial y}|_{y=0} = v_0 < 0$, which gives a contradiction. Therefore (4.7) is true by letting $\varepsilon_0 \to 0$.

Now, direct calculation shows that

$$\begin{aligned} L^\varepsilon(ue^{Bt} - \theta_0 e^{-\beta t}\phi) &= (B+b)e^{Bt}u + \theta_0 e^{-\beta t}(\beta\phi + (u^2+\varepsilon)\phi_{yy}) \\ &\geq \theta_0 e^{-\beta t}(\beta\phi + (u^2+\varepsilon)\phi_{yy}). \end{aligned}$$

Then as in the proof of Lemma 2.2, we can prove that (4.6) is true by letting $\varepsilon \to 0$. The details are omitted.

Now we consider the estimates of the derivatives of u. For the estimate of u_y, unlike the case that $b = 0$ where u_y can be estimated by using the maximum principle, we need to use the Nash-Moser iteration method. We first derive an L^2 estimate. Note that u_y solves

$$\begin{cases} (u_y)_t - [(u^2+\varepsilon)u_{yy}]_y - (bu)_y = 0 \\ u_y|_{t=0} = u_{0,yy} \\ u_y|_{y=0} = v_0, \quad u_{yy}|_{y=1} = 0. \end{cases} \tag{4.8}$$

Set

$$w = u_y - (1-y)^2\, v_0.$$

Then (4.8) becomes

$$\begin{cases} w_t = [(u^2+\varepsilon)(w_y + 2(y-1)\, v_0)]_y + (bu)_y - (1-y)^2\, v_{0,t} \qquad 0 < y < 1 \\ w|_{t=0} = u_{0,y} - (1-y)^2\, v_0 \equiv w_0 \\ w|_{y=0} = 0, \quad u_y|_{y=1} = 0. \end{cases} \tag{4.9}$$

Lemma 4.3 *There exist positive constants C_1 and C_2 depending only on B, $|v_0|_{C^1}$ and $|u|_{L^\infty}$, such that*

$$\int_0^d u_y^2(t,\cdot)dy + \int_0^t\int_0^d (u^2+\varepsilon)u_{yy}^2\ dy\,ds \le [C_2 + \int_0^d u_y^2(0,\cdot)dy]e^{C_1 t}, \tag{4.10}$$

$$\int_0^1 \frac{u_y^2(t,\cdot)}{(y-1)^2+\varepsilon}dy + \int_0^t\int_0^1 \frac{(u^2+\varepsilon)}{(y-1)^2+\varepsilon}u_{yy}^2 dy\,ds \le [C_2 + \int_0^1 \frac{u_y^2(0,\cdot)}{(y-1)^2+\varepsilon}dy]e^{C_1 t}. \tag{4.11}$$

Furthermore, for fixed $k_0 \ge 1$

$$\int_0^1 |u_y(t,\cdot)|^{k_0+1}\ dy \le [C_2 + \int_0^1 |u_y(0,\cdot)|^{k_0+1}\ dy]e^{C_1 t}, \tag{4.12}$$

The proof of Lemma 4.3 is similar to that of Lemma 2.7, so we omit the details.

As in the proof of Lemma 2.8, we can make use of the Nash-Moser iteration method to obtain the L^∞ estimates of u_y, which is given below without proof.

Lemma 4.4 *There exists a constant C_4 independent of ε, such that the solution to (4.3) as well as (4.1) satisfies*

$$|u_y(t,x,y)| \le C_4, \tag{4.13}$$

for $(t,x,y) \in (0,T)\times\Omega$.

Next, we turn to the estimate on u_x. Set $w = u_x$ as before. Then

$$\begin{cases} w_t - (u^2+\varepsilon)w_{yy} - 2uu_{yy}w - bw = b_x u \\ w|_{t=0} = u_{0,xy} \\ \frac{\partial w}{\partial y}|_{y=0} = v_{0,x}, \quad w|_{y=1} = 0, \end{cases} \tag{4.14}$$

This is a slighter modification of the problem (2.12). The same estimates in Lemma 2.7 and Lemma 2.8 hold. similar analysis goes to u_t. Here we give the L^∞ estimates without the proofs.

Lemma 4.5 *There exists a constant C_5 independent of ε, such that the solution to (4.3) as well as (4.1) satisfies*

$$|u_x(t,x,y)| + |u_t(t,x,y)| \le C_5, \tag{4.15}$$

for $(t,x,y) \in (0,T)\times\Omega$.

Summing up the estimates we have got thus far, we arrive at

Theorem 4.1 *Suppose that $v_0 \le 0$ and the data are smooth and compatible. Then the problem (4.1) has a unique positive bounded smooth solution and*

$$u \in Lip([0,T]\times\overline{\Omega}).$$

Moreover the solution satisfies (4.4)-(4.6), (4.13) and (4.15).

The proof is the same as that of Theorem 2.1.

Remark 4.1 *As a corollary of the uniqueness result, u is continuous in x, in particular, it holds that*

$$\lim_{x\to 0} u(t,x,y) = u_{1,y}(t,y). \tag{4.16}$$

In fact, by the definition of $b(t,x,y)$ (see (3.2)), we know that $u_{1,y}$ is a solution to the problem (4.1) with the parameter $x=0$. Then (4.16) follows by the uniqueness of solution to (4.1).

The following two lemmas are parallel to Lemma 2.5 and Lemma 2.6. These estimates enable one to obtain the total variation estimates of the solutions to the Prandtl's equation.

Lemma 4.6 *There exists a constants C_6 depending on B, T, and $|u|_{L^\infty}$, such that the solution to the problem (4.1) satisfies*

$$\begin{aligned} \int_0^1 |u_y(t,\cdot)|dy \le \quad & e^{B(t-t_0)} \int_0^1 |u_y(t_0,\cdot)|dy + C_6(t-t_0) \\ & +u(t,x,0) - u(t_0,x,0), \end{aligned} \tag{4.17}$$

for $(t,x)\in(0,T)\times(0,L)$.

Proof. Note that u_y satisfies

$$u_{yt} = (u^2 u_{yy})_y + (bu)_y.$$

Multiplying the above equation by $sign\, u_y$ and integrating by parts yield

$$\begin{aligned} \frac{d}{dt}\int_0^1 |u_y|dy &= \int_0^1 [(u^2u_{yy})_y \ sign\, u_y + (bu_y + b_y u) sign\, u_y]dy \\ &\le -(u^2 u_{yy} \ sign\, u_y)|_{y=0} + \int_0^1 B(|u|+|u_y|)dy \end{aligned}$$

Replacing v_0 by $v_0-\varepsilon$ and letting $\varepsilon\to 0$ if necessary, one may assume that $u_y|_{y=0} = v_0 < 0$ as before. It follows from the equation in (4.1) that

$$\frac{d}{dt}\int_0^1 |u_y|dy \le (u_t - bu)|_{y=0} + \int_0^1 B(|u|+|u_y|)dy,$$

or

$$\text{(4.18)}\qquad \begin{aligned}\int_0^1 |u_y(t,\cdot)|dy \leq\ & \int_0^1 |u_y(t_0,\cdot)|dy + u(t,x,0) - u(t_0,x,0)\\ & + B\int_{t_0}^t \int_0^1 |u_y|dy\,ds + C(t-t_0),\end{aligned}$$

where C is a constant depending only on T and $|u|_{L^\infty}$. Hence,

$$\text{(4.19)}\qquad \begin{aligned}\int_0^1 |u_y(t,\cdot)|dy \ \leq\ & e^{B(t-t_0)}[\int_0^1 |u_y(t_0,\cdot)|dy + C(t-t_0)]\\ & + u(t,x,0) - u(t_0,x,0) + C(t-t_0).\end{aligned}$$

Then (4.17) follows from (4.19).

Lemma 4.7 *The solution to (4.1) satisfies*

(4.20)

$$\begin{aligned}\int_0^1 \frac{|u_x(t,\cdot)|}{u(t,\cdot)^2}(1-y)^2\,dy \ \leq\ & \int_0^1 \frac{|u_x(t_0,\cdot)|}{u(t_0,\cdot)^2}(1-y)^2\,dy + \int_{t_0}^t \int_0^1 (2 + \frac{(1-y)^2B}{u^2})|u_x|dy\,ds\\ & + \int_{t_0}^t [|v_{0,x}|_{L^\infty} + \int_0^1 \frac{B(1-y)^2}{u}dy]ds.\end{aligned}$$

Hence there exists a constant C_7 depending on C_0, θ_0, and T, such that

$$\text{(4.21)}\qquad \int_0^1 |u_x(t,\cdot)|dy \leq e^{C_7(t-t_0)}[C_7^2 \int_0^1 |u_x(t_0,\cdot)|dy + C_7(t-t_0)],$$

for $(t,x) \in (0,T) \times (0,L)$.

Proof. As in the proof of Lemma 2.6, u_x satisfies

$$\text{(4.22)}\qquad (\frac{u_x}{u^2})_t = u_{xyy} + \frac{b_x}{u} - \frac{bu_x}{u^2}.$$

Multiplying (4.22) by $(1-y)^2 sign u_x$ and integrating by parts, one gets

$$\text{(4.23)}\qquad \begin{aligned}& \frac{d}{dt}\int_0^1 (1-y)^2 \frac{|u_x|}{u^2}dy\\ & = (1-y)^2|u_x|_y|_{y=0} + \int_0^1 [2(1-y)|u_x|_y + (\frac{b}{u})_x(1-y)^2\ sign\, u_x]dy\\ & \leq |v_{0,x}| + \int_0^1 [2|u_x| + B\frac{(1-y)^2|u_x|}{u^2} + B\frac{(1-y)^2}{u}]dy.\end{aligned}$$

Then (4.20) follows from (4.23).

It follows from Lemma 4.2 that there exists a constant C_3 depending on T, C_0 and θ_0 such that

$$\frac{1}{C_3} \leq \frac{(1-y)^2}{u^2} \leq C_3. \tag{4.24}$$

Then (4.20) becomes

$$\begin{aligned}\textstyle\int_0^1 |u_x(t,\cdot)|dy \leq \quad & C_3^2 \textstyle\int_0^1 |u_x(t_0,\cdot)|dy + C_3(2+C_3B) \int_{t_0}^t \int_0^1 |u_x(s,\cdot)|dy\,ds \\ & + C_8(t-t_0),\end{aligned}$$

with C_8 being a constant depending on C_3, B, $|v_0|_{C^1}$ and $|u|_{L^\infty}$, which implies (4.21) immediately.

Remark 4.2 *Similar estimates hold for u_t, i.e., in (4.20) and (4.21), u_x can be replaced by u_t.*

5 Global Existence II, General Case

In this section, we modify the arguments in Section 3 to establish the global existence of solutions to the problem (3.1) without the assumption $b(t,x,y) = 0$.

For given $T > 0$ and $t_i = \frac{iT}{n}$, $i = 1, \cdots n$, we consider the following problem in the first time step

$$\begin{cases} u_t = u^2 u_{yy} + bu & (0, t_1] \times \Omega \\ u|_{t=0} = \overline{u}_0, \quad u|_{y=1} = 0 \\ \frac{\partial u}{\partial y}|_{y=0} = v_0. \end{cases} \tag{5.1}$$

The choice of $b(t,x,y)$ enables us to match the boundary condition for the solution to (5.1) at $x = 0$ (see Remark 4.1). In the next time step, we solve the problem

$$(5.2)\qquad \begin{cases} u_t + yu_x + bu = 0 & (t_1, t_2] \times \Omega \\ u|_{t=t_1} = u(t_1, x, y) & u|_{x=0} = \overline{u}_1(t, y). \end{cases}$$

Where in (5.1) and (5.2), $\overline{u} = u_0, y$ and $\overline{u}_1 = u_1, y$ as in section 3. Again the solution of problem (5.2) can be written down explicitly

$$u(t,x,y) = \begin{cases} u(t_1, x-(t-t_1)y, y)e^{-\int_{t_1}^t b(s,x-(t-s)y,y)ds} & x > (t-t_1)y \\ \overline{u}_1(t-\frac{x}{y}, y)e^{-\int_0^x \frac{1}{y}b(t-\frac{x}{y}+\frac{s}{y},s,y)ds} & x \le (t-t_1)y. \end{cases}$$

Suppose that we have obtained u for $0 < t \le t_i$. For odd i, we solve the following problem:

$$(5.3)\qquad \begin{cases} u_t = u^2 u_{yy} + bu & (t_i, t_{i+1}] \times \Omega \\ u|_{t=t_i} = u(t_i, x, y) & u|_{y=1} = 0 \\ \frac{\partial u}{\partial y}|_{y=0} = v_0; \end{cases}$$

while for even i, we set

(5.4)

$$u(t,x,y) = \begin{cases} u(t_i, x-(t-t_i)y, y)e^{-\int_{t_i}^t b(s,x-(t-s)y,y)ds} & x > (t-t_i)y \\ \overline{u}_1(t-\frac{x}{y}, y)e^{-\int_0^x \frac{1}{y}b(t-\frac{x}{y}+\frac{s}{y},s,y)ds} & x \le (t-t_i)y. \end{cases}$$

Remark 5.1 *The function u given by (5.4) is the solution to*

$$(5.5)\qquad \begin{cases} u_t + yu_x + bu = 0 & (t_i, t_{i+1}] \times \Omega, \\ u|_{t=t_i} = u(t_i, x, y), & u|_{x=0} = \overline{u}_1(t, y). \end{cases}$$

u may depend on n in general. But for convenience, we omit the index n.

To estimate this approximate solution, we start with the L^∞ estimate of u. The following two lemmas are parallel to Lemma 3.1 and Lemma 3.2.

Lemma 5.1 *The function u obtained in the above process satisfies*

$$(5.6)\qquad |u(t,x,y)| \le e^{Bt}(|\overline{u}_0|_{L^\infty} + 2|v_0|_{L^\infty} + |\overline{u}_1|_{L^\infty}), \text{on } (0,T) \times \Omega$$

where B is given in (4.2).

Proof. By lemma 4.1, (5.6) holds for $0 \le t \le t_1$. It is easy to check that (5.6) remains true for $t_1 \le t \le t_2$ by (5.4). In both cases, it holds that

$$\begin{aligned}&|v_0|_{L^\infty}(y-1)\mp ue^{-Bt}\\ &\le \max\{|\mp u_0 + |v_0|_{L^\infty}(y-1)|_{L^\infty}, |\overline{u}_1|_{L^\infty} + |v_0|_{L^\infty}\}.\end{aligned}$$

Suppose that for $0 \le t \le t_{i-1}$, we have

$$|v_0|_{L^\infty}(y-1)\mp ue^{-Bt} \le \max\{|\mp u_0 + |v_0|_{L^\infty}(y-1)|_{L^\infty}, |\overline{u}_1|_{L^\infty} + |v_0|_{L^\infty}\}. \tag{5.7}$$

We will show that (5.7) remains true for $t_{i-1} \le t \le t_i$. For odd i, Lemma 4.1 implies that

$$|v_0|_{L^\infty}(y-1)\mp ue^{-Bt} \le \max[\mp u(t_{i-1},\cdot)e^{-Bt_{i-1}} + |v_0|_{L^\infty}(y-1)], \tag{5.8}$$

therefore (5.7) is valid. When i is even, it follows from (5.4) that

$$\begin{aligned}&|v_0|_{L^\infty}(y-1)\mp ue^{-Bt}\\ &\le \max\{|\mp u(t_{i-1},\cdot)e^{-Bt_{i-1}} + |v_0|_{L^\infty}(y-1)|_{L^\infty}, |\overline{u}_1|_{L^\infty} + |v_0|_{L^\infty}\}.\end{aligned}$$

Then (5.7) is also true in this case. Hence (5.7) holds for all $t \le T$. Lemma 5.1 is proved.

Lemma 5.2 *There exist positive constants C_0 and β depending on B, $|\overline{u}_0|_{L^\infty}$, $|v_0|_{L^\infty}$ and $|\overline{u}_1|_{C^1}$, such that u satisfies*

$$u \le C_0 e^{Bt}(1-y), \tag{5.9}$$

$$u \ge \theta_0 e^{-\beta t}\phi, \tag{5.10}$$

where ϕ and θ_0 are given in Lemma 2.2.

By using (5.4) and Lemma 4.2, one can prove this lemma in a similar way as for Lemma 3.2.

The next lemma is a counterpart of Lemma 3.3 and can be proved by using Lemma 4.4 and Lemma 4.5. We give it here without proof.

Lemma 5.3 *For any $\varepsilon_0 > 0$ and fixed n, there exists a constant C such that*

$$|u_x(t,x,y)| + |u_y(t,x,y)| \le C, \tag{5.11}$$

for $(t,x,y) \in (0,T) \times \Omega$ and $x + y > \varepsilon_0$. And $|u_t(t,x,y)|$ admits the same bounds for $(t,x,y) \in (0,T) \times \Omega$, $y \le 1 - \varepsilon_0$ and $x + y > \varepsilon_0$.

Remark 5.2 *For fixed n, the function u is continuously differentiable. Therefore the initial and boundary conditions are satisfied in the sense of (3.14)-(3.17).*

Next we consider the L^1 estimates of u_x and u_y. These estimates are independent of n therefore give us some kind compactness as well as the L^1 estimates to its limit.

Lemma 5.4 *There exists a constant C_9 depending on T, Ω, B, $|v_0|_{L^\infty}$, $|\overline{u}_0|_{L^\infty}$ and $|\overline{u}_1|_{C^1}$, such that*

$$\int\int_\Omega |u_x(t,\cdot)| dx\, dy \le e^{C_9 t}[\int\int_\Omega |\overline{u}_{0x}| + C_9] dx\, dy, \tag{5.12}$$

for $t \le T$.

Proof. For $t_0 < t \le t_1$, (5.12) can be deduced from (4.21). Moreover, it follows from (4.20) that

(5.13)

$$\begin{aligned} \int\int_\Omega \tfrac{|u_x(t_1,\cdot)|}{u(t_1,\cdot)^2}(1-y)^2\, dx\, dy \ \ \le \ \ & \int\int_\Omega \tfrac{|u_x(0,\cdot)|}{u(0,\cdot)^2}(1-y)^2\, dx\, dy + \int_0^{t_1}\int\int_\Omega (2 + \tfrac{(1-y)^2 B}{u^2})|u_x| dx\, dy\, ds \\ & + \int_0^{t_1}[L|v_{0,x}|_{L^\infty} + \int\int_\Omega \tfrac{B(1-y)^2}{u} dx\, dy] ds. \end{aligned}$$

For $t_1 < t \le t_2$, (5.4) implies that

(5.14)

$$\begin{aligned} \int\int_\Omega \tfrac{|u_x(t,\cdot)|}{u(t,\cdot)^2}(1-y)^2\, dx\, dy \ \ \le \ \ & e^{B(t-t_1)}[\int\int_\Omega \tfrac{|u_x(t_1,\cdot)|}{u(t_1,\cdot)^2}(1-y)^2\, dx\, dy \\ & + \int_{t_1}^t \int_0^1 \tfrac{|\overline{u}_{1,t}(s,y)|}{\overline{u}_1^2(s,y)}(1-y)^2 ds dy \\ & +(t-t_1)(\int_{t_1}^t \int_0^1 \tfrac{(1-y)^2}{\overline{u}_1(s,y)} ds dy + \int\int_\Omega \tfrac{(1-y)^2}{u(t_1,\cdot)} dx\, dy)]. \end{aligned}$$

For $t_{i-1} < t \le t_i$, one again deduces from (4.20) that for odd i,

(5.15)

$$\begin{aligned}\int\int_\Omega \frac{|u_x(t,\cdot)|}{u(t,\cdot)^2}(1-y)^2 dx\,dy \quad &\le \quad \int\int_\Omega \frac{|u_x(t_{i-1},\cdot)|}{u(t_{i-1},\cdot)^2}(1-y)^2\ dx\,dy \\ &+ \int_{t_{i-1}}^t \int\int_\Omega (2+\frac{(1-y)^2 B}{u^2})|u_x| dx\,dy\,ds \\ &+ \int_{t_{i-1}}^t [L|v_{0,x}|_{L^\infty} + \int\int_\Omega \frac{B(1-y)^2}{u} dx\,dy] ds;\end{aligned}$$

while for even i, (5.4) shows that

(5.16)

$$\begin{aligned}\int\int_\Omega \frac{|u_x(t,\cdot)|}{u(t,\cdot)^2}(1-y)^2\ dx\,dy \quad &\le \quad e^{B(t-t_{i-1})}[\int\int_\Omega \frac{|u_x(t_{i-1},\cdot)|}{u(t_{i-1},\cdot)^2}(1-y)^2\ dx\,dy \\ &+ \int_{t_{i-1}}^t \int_0^1 \frac{|\overline{u}_{1,t}(s,y)|}{\overline{u}_1^2(s,y)}(1-y)^2\ ds\,dy \\ &+ (t-t_{i-1})(\int_{t_{i-1}}^t \int_0^1 \frac{(1-y)^2}{\overline{u}_1(s,y)} ds dy + \int\int_\Omega \frac{(1-y)^2}{u(t_{i-1},\cdot)} dx\,dy)].\end{aligned}$$

By Lemma 5.1 and Lemma 5.2,

$$\frac{1}{C_3} \le \frac{(1-y)^2}{u^2(t,\cdot)} \le C_3, \tag{5.17}$$

where C_3 is a constant depending on T, Ω, B, $|v_0|_{L^\infty}$, $|\overline{u}_0|_{L^\infty}$ and $|\overline{u}_1|_{C^1}$. We may also assume

$$\frac{1}{C_3} \le \frac{1-y}{\overline{u}_1(t,\cdot)} \le C_3.$$

We thus have shown that for $t_{i-1} < t \le t_{i-1}$,

(5.18)

$$\int\int_\Omega \frac{|u_x(t,\cdot)|}{u(t,\cdot)^2}(1-y)^2 dx\,dy \le e^{B(t-t_{i-1})}[\int\int_\Omega \frac{|u_x(t_{i-1},\cdot)|}{u(t_{i-1},\cdot)^2}(1-y)^2\ dx\,dy + F_i],$$

$$F_i \le \int_{t_{i-1}}^t \int\int_\Omega (2+BC_3)|u_x| dx\,dy\,ds + C_{10}(t-t_{i-1}), \tag{5.19}$$

where C_{10} is a constant depending on T, Ω, B, $|v_0|_{L^\infty}$, $|\overline{u}_0|_{L^\infty}$ and $|\overline{u}_1|_{C^1}$. It follows from (5.18) and induction that

(5.20)

$$\int\int_\Omega \frac{|u_x(t,\cdot)|}{u(t,\cdot)^2}(1-y)^2\ dx\,dy \le e^{Bt}\int\int_\Omega \frac{|u_x(0,\cdot)|}{u(0,\cdot)^2}(1-y)^2\ dx\,dy + \sum_{j=1}^i e^{\frac{BT}{n}(i-j+1)} F_j,$$

This, together with (5.19), shows

$$\int\int_\Omega \frac{|u_x(t,\cdot)|}{u(t,\cdot)^2}(1-y)^2 dx\,dy \leq e^{Bt}[\int\int_\Omega \frac{|u_x(0,\cdot)|}{u(0,\cdot)^2}(1-y)^2\ dx\,dy + \int_0^t\int\int_\Omega (2+BC_3)|u_x| dx\,dy\,ds + C_{10}t] \tag{5.21}$$

Now, (5.12) follows from (5.21) and (5.17). Then we have proved Lemma 5.4.

Lemma 5.5 *There exists a constant C_{11} depending only on T, Ω, B, $|v_0|_{L^\infty}$, $|\overline{u}_0|_{L^\infty}$ and $|\overline{u}_1|_{C^1}$, such that for n sufficiently large*

$$\int\int_\Omega |u_y(t,\cdot)| dx\,dy \leq e^{Bt}\int\int_\Omega [|\overline{u}_{0,y}| + \frac{T}{n}\sum_{j=1}^{i}|u_x(t_{j-1},\cdot)|]\ dx\,dy + C_{11}. \tag{5.22}$$

Proof. For $0 < t \leq t_1$, (5.22) follows from (4.21). For $t_1 < t \leq t_2$, using (5.4), one may deduce as in (3.27) that

$$\int\int_\Omega |u_y(t,\cdot)| dx\,dy \leq e^{B(t-t_1)}[\int_{t_1}^t\int_0^1 (|\overline{u}_{1,y} + \frac{T|\overline{u}_{1,t}|}{n}) dy\,ds + \int\int_\Omega (\frac{T}{n}|u_x(t_1,\cdot)| + |u_y(t_1,\cdot)|) dx\,dy + B(t-t_1)(\int\int_\Omega \frac{T}{n}|u_x(t_1,\cdot)| dx\,dy) + \int_{t_1}^t\int_0^1 |\overline{u}_1|(\frac{2}{n}+1) dy\,ds], \tag{5.23}$$

and

$$u(t_2,x,0) = u(t_1,x,0)e^{-\int_{t_1}^{t_2} b(s,x,0)ds}. \tag{5.24}$$

In general, for $t_{i-1} < t \leq t_i$, we have by Lemma 4.6 that

$$\int\int_\Omega |u_y(t,\cdot)| dx\,dy \leq e^{B(t-t_{i-1})}\int\int_\Omega [|u_y(t_{i-1},\cdot)| + C_6(t-t_{i-1})] dx\,dy + \int_0^L [u(t,x,0) - u(t_{i-1},x,0)] dx \tag{5.25}$$

for odd i, and when i is even, (5.4) implies that

$$\begin{aligned}\int\int_\Omega |u_y(t,\cdot)|dx\,dy \le\ & e^{B(t-t_{i-1})}[\int_{t_{i-1}}^t \int_0^1 (|\overline{u}_{1,y} + \frac{T|\overline{u}_{1,t}|}{n})dy\,ds \\ & + \int\int_\Omega (\frac{T}{n}|u_x(t_{i-1},\cdot)| + |u_y(t_{i-1},\cdot)|)dx\,dy] \\ & + e^{B(t-t_{i-1})}C_{12}(t-t_{i-1})^2,\end{aligned} \tag{5.26}$$

where C_{12} is a constant with dependence as C_{11}, and

$$u(t_i,x,0) = u(t_{i-1},x,0)e^{-\int_{t_{i-1}}^{t_i} b(s,x,0)ds}. \tag{5.27}$$

Therefore for $t_{i-1} \le t \le t_i$, it holds that

$$\int\int_\Omega |u_y(t,\cdot)|dx\,dy \le e^{B(t-t_{i-1})}[\int\int_\Omega |u_y(t_{i-1},\cdot)|dx\,dy + F_{i-1}] + G_{i-1},$$

where

$$F_{i-1} = \int\int_\Omega \frac{T}{n}|u_x(t_{i-1},\cdot)|dx\,dy + \int_{t_{i-1}}^t \int_0^d (|\overline{u}_{1,y}| + \frac{T|\overline{u}_{1,t}|}{n})dy\,ds + (C_6 + C_{12}(\frac{T}{n}))\frac{T}{n},$$

$$G_{i-1} = \int_0^L [u(t_i,x,0) - u(t_{i-2},x,0)e^{-\int_{t_{i-2}}^{t_{i-1}} b(s,x,0)ds}]dx$$

for odd i, and $G_{i-1} = 0$ when i is even.

Hence,

$$\begin{aligned}\int\int_\Omega |u_y(t,\cdot)|dx\,dy \le\ & e^{Bt}\int\int_\Omega |u_y(0,\cdot)|dx\,dy + \sum_{j=1}^i e^{\frac{BT}{n}(i-j-1)}F_{j-1} \\ & + \sum_{j\ even}^{j\le i-1} e^{\frac{BT}{n}(i-j+1)}G_j.\end{aligned} \tag{5.28}$$

Note that

$$\begin{aligned}\sum_{j\ even}^{j\le i-1} e^{\frac{BT}{n}(i-j)}G_j \le\ & \sum_{k=1}^{[\frac{i}{2}]}\int_0^L e^{\frac{BT}{n}(i-2k-2)}(e^{\frac{2BT}{n}} - e^{-\int_{t_{2k}}^{t_{3k+1}} b\,ds})|u(t_{2k},x,0)|ds \\ & + \int_0^L (e^{\frac{BTi}{n}}|u(0,x,0)| + |u(t_i,x,0)|)dx,\end{aligned}$$

where $[\frac{i}{2}]$ is the largest integer less than or equal to i. Then for n sufficiently large, we have

$$\begin{aligned}\sum_{j\ even}^{j\leq i-1} e^{\frac{BT}{n}(i-j)}G_j \leq & \sum_{k=1}^{[\frac{i}{2}]}\int_0^L e^{\frac{BT}{n}(i-2k)}|u(t_{2k},x,0)|\frac{4BT}{n}\,dx \\ & +\int_0^L(e^{Bt}|u(0,x,0)|+|u(t_i,x,0)|)dx.\end{aligned} \tag{5.29}$$

And

$$\sum_{k=1}^{[\frac{i}{2}]}\int_0^L e^{\frac{BT}{n}(i-2k)}|u(t_{2k},x,0)|\frac{4BT}{n}\,dx \leq 4Bt\,L\,e^{Bt}|u|_{L^\infty((0,t)\times\Omega)}. \tag{5.30}$$

We also have as before

$$\begin{aligned}\sum_{j=1}^{i} e^{\frac{BT}{n}(i-j-1)}F_{j-1} \leq & \int\int_\Omega \frac{T}{n}\sum_{j=1}^{i}|u_x(t_{i-1},\cdot)|e^{\frac{BT}{n}((i-j+1)}dx\,dy \\ & +e^{Bt}\int_0^t\int_0^1(|\overline{u}_{1,y}|+\frac{1}{n}|\overline{u}_{1,t}|)dy\,ds \\ & +C_6te^{Bt}+C_{12}te^{Bt}\frac{T}{n}.\end{aligned} \tag{5.31}$$

Then (5.22) follows from Lemma 5.1 and (5.28) - (5.31). We finished the proof of Lemma 5.5.

Remark 5.3 *The corresponding results in Corollary 3.2 and Corollary 3.3 are also valid here.*

Now we state the main theorems of this section.

Theorem 5.1 *Under the assumptions (1.3), (1.4) and (1.10), the problem (3.1) has a solution*

$$u \in L^\infty(0,T;BV(\Omega)) \cap BV((0,T)\times\Omega)$$

satisfying (4.4)-(4.6). Furthermore, its derivatives are estimated by (5.12), (3.32) and

$$\int\int_\Omega |u_y(t,\cdot)|dx\,dy \leq e^{Bt}\int\int_\Omega[|\overline{u}_{0y}|+\int_0^t|u_x|ds]dx\,dy + C_{11}, \tag{5.32}$$

where C_{11} is the constant given in Lemma 5.5.

The proof of Theorem 5.1 is similar to that of the Theorem 3.1 with the aid of lemmas in section 4. We omit the details of the proof.

This also completes the proof of Theorem 1.1 in the simpler case that (1.10) holds.

Remark 5.4 *The corresponding results of Corollary 3.4 also hold under the assumptions of Theorem 5.1.*

Remark 5.5 *The weak solution to the (3.1) is unique. This fact will be proved in a forthcoming paper [11].*

6 Global Existence of the Prandtl's System

In this section, we indicate that the weak solution w to the problem (1.7), which we constructed in the previous sections, can be transformed to a weak solution to the problem (1.1) in some sense. We then point out that the general case (1.5) can be treated by modifying slightly the analysis for the special case (1.10).

First, by abusing the notations, we let $w(t, x, \eta)$ denote the weak solution to (1.7) constructed in the previous section (under the simplifying assumption that (1.10) holds). We then define $u(t, x, y)$ by

$$y = \int_0^u \frac{d\eta}{w(t, x, \eta)}. \tag{6.1}$$

Since

$$\frac{d - \eta}{\sqrt{C}_3} \leq w(t, x, \eta) \leq \sqrt{C}_3(d - \eta),$$

then $u \longrightarrow d$ as $y \longrightarrow \infty$. It is easy to see that $u > 0$ when $0 \leq y < \infty$ and $u|_{y=0} = 0$.

The initial condition $u|_{t=0} = u_0$ is satisfied by Remark 3.3 and Remark 5.3. The boundary condition $u|_{x=0} = u_1$ is fulfilled by virtue of the condition $w|_{x=0} = u_{1,y}$ in the sense of L^1 and the estimate (3.39)

The function u has derivatives $u_y = w$, $u_{yy} = ww_\eta$ and

$$u_t = w\int_0^u \frac{w_t(t,x,\eta)}{w^2(t,x,\eta)}d\eta, \tag{6.2}$$

$$u_x = w\int_0^u \frac{w_x(t,x,\eta)}{w^2(t,x,\eta)}d\eta. \tag{6.3}$$

Now we set

$$v = \frac{-u_t - uu_x + u_{yy}}{u_y}. \tag{6.4}$$

Then the first equation in (1.1) is satisfied. Since $u|_{y=0} = 0$, then

$$v|_{y=0} = (\frac{u_{yy}}{u_y})|_{y=0} = w_\eta|_{\eta=0} = v_0,$$

in the sense of (3.38).

Finally we check that v satisfies the second equation in (1.1). In fact, by (6.1)-(6.4) and the equation in (1.7),

$$\begin{aligned} v &= -\int_0^u \tfrac{w_t(t,x,\eta)+u(t,x,y)w_x(t,x,\eta)}{w^2(t,x,\eta)}d\eta + w_\eta \\ &= -\int_0^u \tfrac{[u(t,x,y)-\eta]w_x(t,x,\eta)}{w^2(t,x,\eta)}d\eta + v_0 \end{aligned}$$

Integrating by parts

$$\begin{aligned} v &= -u\int_0^u \tfrac{w_x(t,x,\eta)}{w^2(t,x,\eta)} + \int_0^\eta \tfrac{w_x(t,x,s)}{w^2(t,x,s)}ds\eta|_0^u \\ &\quad -\int_0^u\int_0^\eta \tfrac{w_x(t,x,s)}{w^2(t,x,s)}dsd\eta + v_0 \\ &= v_0 - \int_0^u \tfrac{u_x(t,x,\eta)}{w(t,x,\eta)}d\eta \\ &= v_0 - \int_0^y u_x(t,x,y)dy. \end{aligned}$$

Or

$$u_x + v_y = 0.$$

Moreover u, u_x and $u_y \in L^\infty$ by (4.5) and (3.39). We thus have obtained a weak solution to the problem (1.1).

Next, we note that all our analysis has been carried out under the simple assumption (1.10). However, the essential ingredients for the general case , i.e. (1.5), are similar to this special case. Indeed, under the assumption (1.5), one can see the a priori estimates (5.9) and (5.10) in Lemma 5.2 still hold true for the solutions of (1.7). With these precise bounds, one can check that the other estimates can be obtained by modifying the analysis in Section 5 accordingly. We omit the details of the analysis. The proof of Theorem 1.1 is considered complete.

Remark 6.1 *We believe that the weak solution constructed here is in fact smooth. However, this is left for the future.*

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