HEAT KERNELS ON METRIC-MEASURE SPACES AND AN APPLICATION TO SEMI-LINEAR ELLIPTIC EQUATIONS

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Abstract. We consider a metric measure space (M, d, μ) such that there exists a *heat kernel* $p_t(x, y)$ on M satisfying upper and lower estimates (1.4) below, which depend on two parameters α and β. We show that these parameters are determined by the intrinsic properties of the space (M, d, μ). Namely, α is the *Hausdorff dimension* of this space whereas β, termed the *walk dimension*, is determined via the properties of the family of *Besov spaces* $W^{\sigma,2}$ on M.

We prove the embedding theorems for the space $W^{\beta/2,2}$ and use them to obtain the existence results for weak solutions to semi-linear elliptic equations on M of the form

$$-\Delta u + f(x, u) = g(x),$$

where Δ is the generator of the semigroup associated with p_t.

The framework in this paper is applicable for a large class of fractal domains, including the generalized Sierpinski carpet in $\mathbb{R}^n$ introduced in [3].

1. Introduction

Let (M, d, μ) be a metric-measure space, that is (M, d) be a metric space, and μ be a Borel measure on M. A family $\{p_t\}_{t>0}$ of non-negative measurable functions $p_t(x, y)$ on $M \times M$ is called a *heat kernel* or *a transition density* if the following conditions are satisfied, for all $x, y \in M$ and $s, t > 0$:

(1) Symmetry: $p_t(x, y) = p_t(y, x)$.

(2) Normalization:

$$\int_M p_t(x, y) d\mu(y) = 1.$$

(3) Semigroup property:

$$p_{s+t}(x, y) = \int_M p_s(x, z) p_t(z, y) d\mu(z).$$

(4) Identity approximation: for any $f \in L^2(M, \mu)$,

$$\int_M p_t(x, y) f(y) d\mu(y) \xrightarrow{L^2} f(x) \quad \text{as } t \to 0+ .$$

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For example, the classical Gauss-Weierstrass function in $\mathbb{R}^n$

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \tag{1.1}$$

satisfies this definition.

Any heat kernel gives rise to the *heat semigroup* $\{T_t\}_{t>0}$ where T_t is the operator in $L^2(M,\mu)$ defined by

$$T_t u(x) = \int_M p_t(x,y)u(y)d\mu(y). \tag{1.2}$$

The above properties of p_t imply that T_t is a bounded self-adjoint operator, and $\{T_t\}$ is a strongly continuous, positivity preserving, contraction semigroup in $L^2(M,\mu)$. Another way of constructing such a semigroup is to set

$$T_t = \exp(t\Delta), \tag{1.3}$$

where Δ is a non-positive definite self-adjoint operator in $L^2(M,\mu)$ satisfying in addition the Markov property. Typically, such operators arise as generators of Dirichlet forms. It is not always the case that the semigroup $\{T_t\}$ defined by (1.3) possesses an integral kernel. If it does then the integral kernel will be a heat kernel in the above sense (although some additional restrictions are needed to ensure the normalization condition).

In this note we would like to adopt the axiomatic approach to heat kernels, which to some extent is opposite to the above scheme. Namely, we will assume that a heat kernel is defined on a metric-measure space, and show that this implies many interesting consequences for the analysis on such a space. Similar approach was used in [2] and [15], although there the notion of a heat kernel was linked to diffusion processes on M, and in [15] the underlying space M was a subset of $\mathbb{R}^n$.

Let p_t be a heat kernel on (M,d,μ). Assume in addition that the heat kernel satisfies the following two-sided estimate, for all $x,y\in M$ and $t\in(0,\infty)$,

$$\frac{1}{t^{\alpha/\beta}}\Phi_1\left(\frac{d(x,y)^\beta}{t}\right) \le p_t(x,y) \le \frac{1}{t^{\alpha/\beta}}\Phi_2\left(\frac{d(x,y)^\beta}{t}\right), \tag{1.4}$$

where α,β are positive constants, and Φ_1 and Φ_2 are monotone decreasing positive functions on $[0,+\infty)$; moreover, Φ_2 has to decay sufficiently fast at $+\infty$ (see Theorems 3.1, 4.1, 4.2 for the exact conditions).

The Gauss-Weierstrass heat kernel (1.1) satisfies (1.4) with $\alpha = n$, $\beta = 2$, and

$$\Phi_1(s) = \Phi_2(s) = \frac{1}{(4\pi)^{n/2}} \exp\left(-\frac{s}{4}\right).$$

The development of analysis on fractals has brought plenty of examples of heat kernels satisfying (1.4) with functions Φ_1 and Φ_2 of the form

$$\Phi(s) = c' \exp\left(-c'' s^\gamma\right),$$

where $\gamma > 0$, and the positive constants c' and c'' may be different for Φ_1 and Φ_2. In these examples the parameters α and β may vary within the following limits:

$$2 \le \beta \le \alpha + 1.$$

In particular, β is typically larger than 2.

The nature of the parameters α and β is of great interest. Although originally they are defined through the heat kernel, a posteriori they happen to be the invariants of the space (M,d,μ) itself. Indeed, we prove that the measure of any metric ball $B(x,r) = \{y : d(x,y) < r\}$ in M satisfies the estimates

$$C^{-1}r^{\alpha} \le \mu\left(B(x,r)\right) \le C\,r^{\alpha}, \quad r > 0, \tag{1.5}$$

where C is a positive constant (Theorem 3.1). In particular, this implies that α is the *Hausdorff dimension* of M (see also [6]).

The nature of the parameter β is more complicated. If the heat kernel p_t is the transition density of a diffusion process X_t on M, then β is called the *walk dimension* of X_t. This terminology comes from the following observation: if the heat kernel satisfies (1.4) then the time t needed for the diffusion to move away at the distance r from the origin is of the order r^{β} (see [2, Lemma 3.9]).

On the other hand, we prove the following analytic characterization of β. Following [12], we introduce on the space (M,d,μ) the family $W^{\sigma,2}$ of *Besov spaces*, which generalizes the Sobolev space $W^{1,2}$ in $\mathbb{R}^n$ (see Section 4). If a heat kernel p_t is defined on M then it induces the energy form $\mathcal{E}$ defined on a dense subspace of $L^2(M,\mu)$. If p_t satisfies (1.4) then $W^{\beta/2,2}$ coincides with the domain $\mathcal{D}\left(\mathcal{E}\right)$ of $\mathcal{E}$ (Theorem 4.1), whereas for any $\sigma > \beta/2$ the space $W^{\sigma,2}$ consists only of constants (Theorem 4.2).

As a consequence, we see that β is the maximal number such that the space $W^{\beta/2,2}$ is non-trivial. Hence, β does not depend on the particular choice of a heat kernel, and it can be referred to as the walk dimension of the space (M,d,μ) itself.

Apart from the aforementioned results, we prove also certain embedding theorems for Besov spaces. In particular, if $\alpha > \beta$ then

$$W^{\beta/2,2}(M,\mu) \hookrightarrow L^{2^*}(M,\mu)$$

where

$$2^* := \frac{2\alpha}{\alpha - \beta} \tag{1.6}$$

(see Theorem 4.3 where the case $\alpha \le \beta$ is also considered).

We apply the embedding results to treat the following semi-linear elliptic equation on M

$$-\Delta u + f(x,u) = g(x), \tag{1.7}$$

where Δ is the generator of the semigroup T_t (the equation (1.7) arises when investigating the potential u in porous or other irregular domains). We prove the existence and uniqueness results for weak solutions of (1.7), which in particular imply that for all $q \ge p \ge 2^*$, the equation

$$-\Delta u + |u|^{q-2}\,u = g$$

has a unique weak solution $u \in \mathcal{D}\left(\mathcal{E}\right) \cap L^p \cap L^q$, for any $g \in L^{p'}$, where $p' = \frac{p}{p-1}$ (Theorem 5.1).

Note that the classical existence results of the equation (1.7) in $\mathbb{R}^n$, $n > 2$, depend on the critical parameter $2^* = \frac{2n}{n-2}$ (see [13]) that matches (1.6) since $\alpha = n$ and $\beta = 2$.

Notation. The letters C, c are used to denote positive constants whose values are unimportant but depend only on the hypotheses. The values of C, c may be different on different occurrences.

For two non-negative functions $f(s)$ and $g(s)$ defined on a set S, we write

$$f(s) \simeq g(s),$$

if there is a constant c such that for all $s \in S$

$$c^{-1} g(s) \leq f(s) \leq c\, g(s).$$

2. Some examples

Let $l \geq 3$ be an integer and let $M_0 = [0,1]^n$ $(n \geq 2)$. We divide M_0 into l^n equal subcubes. Remove a symmetric pattern of subcubes from M_0, and denote by M_1 what remains. Repeat the same procedure for each subcube in M_1: divide each subcube into l^n equal parts and remove the same pattern from each subcube, and denote by M_2 what remains. Continuing this way infinitely, we obtain a sequence of sets $\{M_k\}$. Set

$$\widetilde{M} = \bigcap_{k=0}^{\infty} M_k$$

and define

$$M = \bigcup_{k=0}^{\infty} l^k \, \widetilde{M}\,,$$

where we write $a\,K = \{ax : x \in K\}$ for a real number a and a set K.

The set M is called an unbounded *generalized Sierpinski carpet* (cf. [3]); see Figure 1, which corresponds to the case $n = 2$ and $l = 3$.

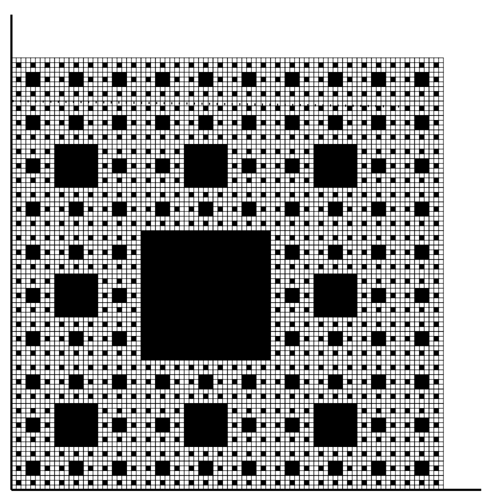

Figure 1.

The distance d on M is set to be the Euclidean distance, and the measure μ is the Hausdorff measure of the dimension α, where α is the Hausdorff dimension of M. For any generalized Sierpinski carpet, there exists a heat kernel satisfying the following estimate

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d(x,y)^\beta}{t}\right)^{\frac{1}{\beta-1}}\right), \tag{2.1}$$

which is a particular case of (1.4) (see [3]). There are also plenty of other fractals such that (2.1) or (1.4) holds, see for example [2], [7].

See also [8, 9], [16] for the heat kernel estimates in the setting of graphs or manifolds.

3. Volume of balls

For a metric space (M,d), denote by $B(x,r)$ the metric ball in M of the radius r centered at the point $x \in M$, that is

$$B(x,r) = \{y \in M : d(x,y) < r\}.$$

Theorem 3.1. *Let (M,d,μ) be a metric-measure space, and p_t be a heat kernel on M satisfying (1.4) with function Φ_2 such that*

$$\int^\infty s^{\alpha/\beta}\Phi_2(s)\frac{ds}{s} < \infty. \tag{3.1}$$

Then for any ball $B(x,r)$ in M we have

$$C^{-1}r^\alpha \le \mu(B(x,r)) \le Cr^\alpha. \tag{3.2}$$

Proof. Fix $x \in M$ and prove the upper bound for the volume function

$$V(r) := \mu(B(x,r)) \le Cr^\alpha, \tag{3.3}$$

for any $r > 0$. Indeed, for any $t > 0$, we have

$$\int_{B(x,r)} p_t(x,y)d\mu(y) \le \int_M p_t(x,y)d\mu(y) = 1 \tag{3.4}$$

whence

$$V(r) \le \left(\inf_{y\in B(x,r)} p_t(x,y)\right)^{-1}.$$

Taking $t = r^\beta$ and applying the lower bound in (1.4) we obtain

$$\inf_{y\in B(x,r)} p_t(x,y) \ge \frac{1}{t^{\alpha/\beta}}\Phi_1(1) = c\, r^{-\alpha},$$

whence (3.3) follows.

Let us prove the opposite inequality

$$V(r) \ge c\, r^\alpha. \tag{3.5}$$

We first show that the upper bound in (1.4) and (3.3) imply the following inequality

$$\int_{M\setminus B(x,r)} p_t(x,y)d\mu(y) \le \frac{1}{2}, \quad \forall t \le \varepsilon r^\beta \tag{3.6}$$

provided that $\varepsilon > 0$ is sufficiently small. Setting $r_k = 2^k r$ and using the monotonicity of Φ_2, we obtain

$$\begin{aligned}
\int_{M\setminus B(x,r)} p_t(x,y)d\mu(y) &\leq \int_{M\setminus B(x,r)} t^{-\alpha/\beta}\Phi_2\left(\frac{d(x,y)^\beta}{t}\right)d\mu(y)\\
&\leq \sum_{k=0}^{\infty}\int_{B(x,r_{k+1})\setminus B(x,r_k)} t^{-\alpha/\beta}\Phi_2\left(\frac{r_k^\beta}{t}\right)d\mu(y)\\
&\leq C\sum_{k=0}^{\infty} r_k^\alpha t^{-\alpha/\beta}\Phi_2\left(\frac{r_k^\beta}{t}\right)\\
&= C\sum_{k=0}^{\infty}\left(\frac{\left(2^k r\right)^\beta}{t}\right)^{\alpha/\beta}\Phi_2\left(\frac{\left(2^k r\right)^\beta}{t}\right)\\
&\leq C\int_{r^\beta/(2t)}^{\infty} s^{\alpha/\beta}\Phi_2(s)\frac{ds}{s},
\end{aligned}$$

where the last inequality is proved as follows: setting $x_k = (2^k r)^\beta/t$ and using the monotonicity of Φ_2, we obtain

$$\begin{aligned}
&\int_{r^\beta/(2t)}^{\infty} s^{\alpha/\beta}\Phi_2(s)\frac{ds}{s}\\
&= \int_{x_0/2}^{x_0} s^{\alpha/\beta-1}\Phi_2(s)ds + \sum_{k=0}^{\infty}\int_{x_k}^{x_{k+1}} s^{\alpha/\beta-1}\Phi_2(s)ds\\
&\geq \frac{\beta}{\alpha}\Phi_2(x_0)\left(x_0^{\alpha/\beta} - (x_0/2)^{\alpha/\beta}\right) + \frac{\beta}{\alpha}\sum_{k=0}^{\infty}\Phi_2(x_{k+1})\left(x_{k+1}^{\alpha/\beta} - x_k^{\alpha/\beta}\right)\\
&\geq c\,\Phi_2(x_0)x_0^{\alpha/\beta} + c\sum_{k=1}^{\infty}\Phi_2(x_k)x_k^{\alpha/\beta}.
\end{aligned}$$

Thus, by (3.1), the integral

$$\int_{M\setminus B(x,r)} p_t(x,y)d\mu(y)$$

can be made arbitrarily small, in particular smaller than $1/2$, provided r^β/t is large enough. From the normalization property and (3.6), we conclude that for such r and t

$$\int_{B(x,r)} p_t(x,y)d\mu(y) \geq \frac{1}{2}\,, \tag{3.7}$$

whence

$$V(r) \geq \frac{1}{2}\left(\sup_{y\in B(x,r)} p_t(x,y)\right)^{-1}.$$

Finally, choosing $t := \varepsilon r^\beta$ and using the upper bound

$$p_t(x,y) \leq t^{-\alpha/\beta}\Phi_2(0) = Cr^{-\alpha},$$

we obtain (3.5). ■

Note that the method we have used in the proof is close to the one used in [9].

4. Besov spaces

Let (M, d, μ) be a metric-measure space. For any $q \in [1, +\infty]$, set $L^q = L^q(M, \mu)$ and

$$\|u\|_q = \|u\|_{L^q(M,\mu)}.$$

Fix a positive number ρ_0, and for any $\sigma > 0$ define the non-negative functional $W_\sigma(u)$ on measurable functions on M by

$$W_\sigma(u)^2 := \sup_{0<r<\rho_0} r^{-2\sigma} \int_M \left[\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u(y) - u(x)|^2 d\mu(y) \right] d\mu(x). \tag{4.1}$$

In particular, if the condition (3.2) is satisfied then for any $\beta > 0$

$$W_{\beta/2}(u)^2 \simeq \sup_{0<r<\rho_0} r^{-\alpha-\beta} \int_M \left[\int_{B(x,r)} |u(y) - u(x)|^2 d\mu(y) \right] d\mu(x). \tag{4.2}$$

Define the space $W^{\sigma,2}$ as follows:

$$W^{\sigma,2} = W^{\sigma,2}(M, d, \mu) := \left\{ u \in L^2 : W_\sigma(u) < \infty \right\}.$$

It is easy to see that W_σ is a semi-norm in $W^{\sigma,2}$, and $W^{\sigma,2}$ is a Banach space with the norm

$$\|u\|_{\sigma,2} := \|u\|_2 + W_\sigma(u)$$

(see for example [12]). The space $W^{\sigma,2}$ is one of the family of *Besov spaces*; it is similar to the space that was denoted by $\operatorname{Lip}(\sigma, 2, \infty)$ in [12].

4.1. The Laplace operator.

Define by (1.2) the semigroup $\{T_t\}_{t>0}$ in L^2, and consider its *infinitesimal generator* Δ defined by

$$\Delta u := \lim_{t \to 0} \frac{T_t u - u}{t}, \tag{4.3}$$

where the limit is taken in the L^2-norm. It is natural to refer to Δ as the *Laplace operator* of the heat kernel p_t. The *domain* of the Laplacian Δ is the space of functions u such that the limit in (4.3) exists. Since $\{T_t\}$ is a strongly continuous contraction semigroup, the *domain* $\operatorname{dom}(\Delta)$ of Δ is dense in L^2 (see [20, Theorem 1, p.237]), and Δ is a self-adjoint, non-positive definite operator.

4.2. The Dirichlet form.

For any $t > 0$ define the quadratic form $\mathcal{E}_t$ on L^2 by

$$\mathcal{E}_t[u] := \left(\frac{u - T_t u}{t}, u \right), \tag{4.4}$$

where $(\ ,\ )$ is the inner product in L^2. An easy computation shows that $\mathcal{E}_t$ can be equivalently defined by

$$\mathcal{E}_t[u] = \frac{1}{2t} \int_M \int_M (u(x) - u(y))^2 p_t(x,y) d\mu(y) d\mu(x). \tag{4.5}$$

In terms of the spectral resolution $\{E_\lambda\}$ of the operator $-\Delta$, $\mathcal{E}_t$ can be expressed as follows

$$\mathcal{E}_t[u] = \int_0^\infty \frac{1-e^{-t\lambda}}{t} d\|E_\lambda u\|_2^2,$$

which implies that $\mathcal{E}_t[u]$ is decreasing in t (see also [4]). Let us define the Dirichlet form $\mathcal{E}$ by

$$\mathcal{E}[u] := \lim_{t\to 0+} \mathcal{E}_t[u] \tag{4.6}$$

(where the limit may be $+\infty$ since $\mathcal{E}[u] \geq \mathcal{E}_t[u]$) and its *domain* by

$$\mathcal{D}(\mathcal{E}) := \{u \in L^2 : \mathcal{E}[u] < \infty\}.$$

It is known that $\mathcal{D}(\mathcal{E})$ is dense in L^2.

By the standard procedure, the quadratic form $\mathcal{E}[u]$ extends to the bilinear form $\mathcal{E}(u,v)$. Then the space $\mathcal{D}(\mathcal{E})$ is a Hilbert space with the inner product

$$[u,v] := (u,v) + \mathcal{E}(u,v). \tag{4.7}$$

Clearly (4.3) and (4.4) imply that

$$\mathcal{E}(u,v) = (-\Delta u, v), \tag{4.8}$$

for all $u \in \mathrm{dom}(\Delta)$ and $v \in \mathcal{D}(\mathcal{E})$.

4.3. **Besov space as the domain of the Dirichlet form.** The domain $\mathcal{D}(\mathcal{E})$ of the Dirichlet form is identified by the following theorem.

Theorem 4.1. *Let p_t be a heat kernel on (M,d,μ) satisfying (1.4). Assume in addition that the function $s^{1+\alpha/\beta}\Phi_2(s)$ is monotone decreasing in $[s_0,+\infty)$ for some $s_0>0$, and that*

$$\int_0^\infty s^{1+\alpha/\beta}\Phi_2(s)\frac{ds}{s} < \infty. \tag{4.9}$$

Let $\mathcal{E}$ be the Dirichlet form defined by (4.5) and (4.6). Then

$$\mathcal{D}(\mathcal{E}) = W^{\beta/2,2}$$

and for any $u \in \mathcal{D}(\mathcal{E})$

$$\mathcal{E}[u] \simeq W_{\beta/2}(u)^2. \tag{4.10}$$

Proof. Since the expressions $\mathcal{E}[u]$ and $W_{\beta/2}(u)$ are defined for all $u \in L^2$, it suffices to show that (4.10) holds for all $u \in L^2$ (allowing the infinite values for both sides). Note that the hypotheses of Theorem 3.1 hold so that we have the estimate (3.2) of the volumes of balls.

We first prove that

$$\mathcal{E}[u] \geq c\,W_{\beta/2}(u)^2, \tag{4.11}$$

using the approach of [15]. By the lower bound in (1.4) and using the monotonicity of Φ_1, we obtain from (4.5) that for any $r>0$ and $t=r^\beta$,

$$\begin{aligned}\mathcal{E}[u] &\geq \frac{1}{2t}\int_M\int_M (u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x)\\ &\geq \frac{1}{2t}\int_M\int_{B(x,r)} (u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x)\\ &\geq \frac{1}{2}\left(\frac{1}{t}\right)^{1+\alpha/\beta}\Phi_1\left(\frac{r^\beta}{t}\right)\int_M\int_{B(x,r)} (u(x)-u(y))^2 d\mu(y)d\mu(x)\\ &= \frac{\Phi_1(1)}{2} r^{-(\alpha+\beta)}\int_M\int_{B(x,r)} (u(x)-u(y))^2 d\mu(y)d\mu(x).\end{aligned}$$

This combines with (4.2) to yield (4.11).

Let us now prove the opposite inequality, that is

$$\mathcal{E}\left[u\right] \leq C\, W_{\beta/2}\left(u\right)^2.$$

For any $t>0$ we have

$$\begin{aligned}\mathcal{E}_t\left[u\right] &= \frac{1}{2t}\int_M\int_M (u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x)\\ &= \frac{1}{2t}\left\{\int_M\int_{M\setminus B(x,1)} + \int_M\int_{B(x,1)}\right\}(u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x)\\ &=: A(t)+B(t).\end{aligned}\tag{4.12}$$

For $x\in M$ we obtain, using (3.2),

$$\begin{aligned}\int_{M\setminus B(x,1)}\frac{1}{d(x,y)^{\alpha+\beta}}d\mu(y) &= \sum_{k=0}^\infty \int_{B(x,2^{k+1})\setminus B(x,2^k)}\frac{1}{d(x,y)^{\alpha+\beta}}d\mu(y)\\ &\leq \sum_{k=0}^\infty \mu(B(x,2^{k+1}))2^{-k(\alpha+\beta)}\\ &\leq C\sum_{k=0}^\infty 2^{-k\beta}\\ &\leq C.\end{aligned}\tag{4.13}$$

By the hypothesis, the function

$$\varphi(s) := s^{1+\alpha/\beta}\Phi_2(s)$$

is monotone decreasing in $[s_0,+\infty)$ and

$$\int_0^\infty \varphi(s)\frac{ds}{s}<\infty,$$

which implies that

$$\lim_{s\to\infty}\varphi(s)=0.\tag{4.14}$$

Using the elementary inequality $(a-b)^2 \leq 2(a^2+b^2)$, the upper bound in (1.4), and (4.13), we obtain that, for small $t>0$,

$$
\begin{aligned}
A(t) &= \frac{1}{2t}\int_M \int_{M\setminus B(x,1)} (u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x) \\
&\leq \frac{1}{t}\int_M \int_{M\setminus B(x,1)} (u(x)^2+u(y)^2) p_t(x,y)d\mu(y)d\mu(x) \\
&\leq 2\int_M u(x)^2 \int_{M\setminus B(x,1)} \left(\frac{1}{t}\right)^{1+\alpha/\beta} \Phi_2\left(\frac{d(x,y)^\beta}{t}\right) d\mu(y)d\mu(x) \\
&= 2\int_M u(x)^2 \int_{M\setminus B(x,1)} \frac{1}{d(x,y)^{\alpha+\beta}}\varphi\left(\frac{d(x,y)^\beta}{t}\right) d\mu(y)d\mu(x) \\
&\leq 2\varphi\left(\frac{1}{t}\right)\int_M u(x)^2 \int_{M\setminus B(x,1)} \frac{1}{d(x,y)^{\alpha+\beta}} d\mu(y)d\mu(x) \\
&\leq C\varphi\left(\frac{1}{t}\right)\|u\|_2^2, \qquad (4.15)
\end{aligned}
$$

whence by (4.14)

$$\lim_{t\to 0+} A(t) = 0. \qquad (4.16)$$

The quantity $B(t)$ is estimated as follows: using (4.2) and (4.9), we have that

$$
\begin{aligned}
B(t) &= \frac{1}{2t}\int_M \int_{B(x,1)} (u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x) \\
&= \frac{1}{2t}\sum_{k=1}^{\infty}\int_M \int_{B(x,2^{-(k-1)})\setminus B(x,2^{-k})} (u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x) \\
&\leq \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{t}\right)^{1+\alpha/\beta}\Phi_2\left(\frac{2^{-k\beta}}{t}\right)\int_M \int_{B(x,2^{-(k-1)})} (u(x)-u(y))^2 d\mu(y)d\mu(x) \\
&\leq C\, W_{\beta/2}(u)^2 \sum_{k=1}^{\infty}\left(\frac{2^{-k\beta}}{t}\right)^{1+\alpha/\beta}\Phi_2\left(\frac{2^{-k\beta}}{t}\right) \\
&\leq C\, W_{\beta/2}(u)^2 \int_0^\infty s^{1+\alpha/\beta}\Phi_2(s)\frac{ds}{s} \\
&\leq C\, W_{\beta/2}(u)^2. \qquad (4.17)
\end{aligned}
$$

It follows from (4.12), (4.16) and (4.17) that

$$\mathcal{E}[u] = \lim_{t\to 0+} \mathcal{E}_t[u] \leq C\, W_{\beta/2}(u)^2, \qquad (4.18)$$

which finishes the proof. ■

As we see from the proof, the inclusion $\mathcal{D}(\mathcal{E}) \subset W^{\beta/2,2}$ was obtained using only the lower estimate in (1.4), whereas the opposite inclusion was obtained using only the upper estimate in (1.4).

4.4. **Characterization of β in terms of Besov spaces.** The next theorem explains the parameter β (the walk dimension) from the viewpoint of the scale of the Besov spaces (see also [11]).

Theorem 4.2. *Let p_t be a heat kernel on (M, d, μ) satisfying the upper bound in (1.4). Assume in addition that $s^{2+\alpha/\beta}\Phi_2(s)$ is bounded on $[0, +\infty)$ and*

$$\int_0^\infty s^{2+\alpha/\beta}\Phi_2(s)\frac{ds}{s} < \infty. \tag{4.19}$$

Then, for any $\sigma > \beta/2$, the space $W^{\sigma,2}$ contains only constants.

Remark. Observe that the hypotheses that are imposed on Φ_2 in Theorems 3.1, 4.1, 4.2 become stronger each time. Certainly, every function of the form

$$\Phi_2(s) = \exp\left(-cs^\gamma\right)$$

satisfies all the hypotheses for $c > 0$ and $\gamma > 0$.

Proof. The proof is similar to that of Theorem 4.1. Fix some function $u \in W^{\sigma,2}$. We will show that $\mathcal{E}[u] = 0$, which implies that $\mathcal{E}_t[u] = 0$ for all $t > 0$ and hence u is constant on M.

Choose some $\varepsilon \in (0, 1)$ so small that

$$2\sigma - (1+\varepsilon)\beta > 0, \tag{4.20}$$

and we prove that

$$\mathcal{E}_t[u] \le Ct^\varepsilon\left(\|u\|_2^2 + W_\sigma(u)^2\right),$$

which gives $\mathcal{E}[u] = 0$ by letting $t\to 0+$.

To see this, using the notation $A(t)$, $B(t)$, and $\varphi(t)$ introduced in the previous proof, rewrite the estimate (4.15) for $A(t)$ as follows

$$t^{-\varepsilon}A(t) \le C\frac{1}{t^\varepsilon}\varphi\left(\frac{1}{t}\right)\|u\|_2^2.$$

By the hypothesis, the function

$$s^\varepsilon\varphi(s) = s^{\varepsilon+1+\sigma/\beta}\Phi_2(s)$$

is bounded on $[0, +\infty)$, whence we obtain

$$t^{-\varepsilon}A(t) \le C\|u\|_2^2. \tag{4.21}$$

Let us estimate $B(t)$. Similar to (4.17), but using W_σ instead of $W_{\beta/2}$, it follows from (4.20) and (4.19) that

$$\begin{aligned}
t^{-\varepsilon}B(t) &= \frac{1}{2t^{1+\varepsilon}}\int_M\int_{B(x,1)}(u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x)\\
&= \frac{1}{2t^{1+\varepsilon}}\sum_{k=1}^{\infty}\int_M\int_{B(x,2^{-(k-1)})\setminus B(x,2^{-k})}(u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x)\\
&\le \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{t}\right)^{1+\varepsilon+\alpha/\beta}\Phi_2\left(\frac{2^{-k\beta}}{t}\right)\int_M\int_{B(x,2^{-(k-1)})}(u(x)-u(y))^2 d\mu(y)d\mu(x)\\
&\le C\,W_\sigma(u)^2\sum_{k=1}^{\infty}2^{-(2\sigma-(1+\varepsilon)\beta)k}\left(\frac{2^{-k\beta}}{t}\right)^{1+\varepsilon+\alpha/\beta}\Phi_2\left(\frac{2^{-k\beta}}{t}\right)\\
&\le C\,W_\sigma(u)^2\int_0^\infty s^{1+\varepsilon+\alpha/\beta}\Phi_2(s)\frac{ds}{s}\\
&\le C\,W_\sigma(u)^2. \qquad (4.22)
\end{aligned}$$

Therefore, we obtain from (4.12), (4.21) and (4.22) that

$$\mathcal{E}_t[u] = A(t)+B(t) = t^{\varepsilon}\left(t^{-\varepsilon}A(t)+t^{-\varepsilon}B(t)\right) \le Ct^{\varepsilon}\left(\|u\|_2^2+W_\sigma(u)^2\right).$$

■

Let us mention that Theorems 4.1 and 4.2 were obtained by Jonsson [12] for the case of the Sierpinski gasket in $\mathbb{R}^n$, but starting from the Dirichlet form rather than from the heat kernel.

The next Corollary states that both parameters α, β in (1.4) are uniquely determined by the space (M, d, μ) itself.

Corollary 4.1. *Let $p_t^{(i)}$ $(i = 1, 2)$ be two heat kernels on (M, d, μ) satisfying (1.4) with the parameters α_i, β_i and the functions $\Phi_1^{(i)}, \Phi_2^{(i)}$, respectively. Assume that $\Phi_2^{(i)}(i = 1, 2)$ satisfy the same hypotheses as in Theorem 4.2. Then $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.*

Proof. By Theorem 3.1, $\mu(B(x, r))$ satisfies (3.2) for both α_1 and α_2, for all $r > 0$, whence we obtain $\alpha_1 = \alpha_2$. If $\beta_1 > \beta_2$ then set $\sigma = \beta_1/2$ so that by Theorem 4.2 applied to the heat kernel $p_t^{(2)}$, the space $W^{\beta_1/2,2} = W^{\sigma,2}$ contains only constant functions. However, this contradicts the fact that the heat kernel $p_t^{(1)}(., y)$ is in $W^{\beta_1/2,2}$ for all $t > 0$ and $y \in M$. Similarly, $\beta_2 > \beta_1$ is impossible. ■

4.5. **Embedding theorems.** In addition to the spaces L^p and $W^{\sigma,2}$ defined above, we introduce the Hölder space $C^\lambda = C^\lambda(M, d, \mu)$ as follows: we say $u \in C^\lambda$ if

$$\|u\|_{C^\lambda} := \mu\text{-}\operatorname*{ess\,sup}_{\substack{x,y\in M\\ 0<d(x,y)<\rho_0}}\frac{|u(x)-u(y)|}{d(x,y)^\lambda} < \infty.$$

Theorem 4.3. *Let (M, d, μ) be a metric-measure space with a heat kernel satisfying (1.4), and assume that the function Φ_2 satisfies the same hypotheses as in Theorem 4.1. Then the following is true.*

(*i*) *If* $\alpha > \beta$ *then for any* $2 \le q \le 2^*$

$$W^{\beta/2,2} \hookrightarrow L^q, \tag{4.23}$$

where

$$2^* := \frac{2\alpha}{\alpha - \beta}.$$

That is, $u \in W^{\beta/2,2}$ *implies* $u \in L^q$ *and*

$$\|u\|_q \le C\|u\|_{\beta/2,2}. \tag{4.24}$$

(*ii*) *If* $\alpha = \beta$ *then the embedding (4.23) holds for any* $2 \le q < \infty$.

(*iii*) *if* $\alpha < \beta$ *then*

$$W^{\beta/2,2} \hookrightarrow C^\lambda$$

where

$$\lambda = \frac{\beta - \alpha}{2}.$$

That is, $u \in W^{\beta/2,2}$ *implies* $u \in C^\lambda$ *and*

$$\|u\|_{C^\lambda} \le C\|u\|_{\beta/2,2}. \tag{4.25}$$

Remark. Observe that the definitions of the function spaces $W^{\beta/2,2}$,L^q,C^λ involved in the embedding theorems do not depend on a heat kernel. However, the proof of the parts (i) and (ii) uses the existence of a heat kernel satisfying the estimate (1.4).

Proof. By Theorem 4.1, we have $W^{\beta/2,2} = \mathcal{D}(\mathcal{E})$ and

$$\|u\|_{\beta/2,2} \simeq \|u\|_2 + \mathcal{E}[u]^{1/2},$$

for any $u \in W^{\beta/2,2}$. Hence, in parts (i), and (ii), it suffices to prove that

$$\mathcal{D}(\mathcal{E}) \hookrightarrow L^q$$

(where $2 \le q \le 2^*$ in the case (i) and $2 \le q < \infty$ in the case (ii)), and

$$\|u\|_q \le C\left(\|u\|_2 + \mathcal{E}[u]^{1/2}\right) \tag{4.26}$$

for any $u \in \mathcal{D}(\mathcal{E})$.

Proof for the case (i), $\alpha > \beta$. The upper bound in (1.4) implies that

$$\sup_{x,y\in M} p_t(x,y) \le Ct^{-\alpha/\beta}. \tag{4.27}$$

Using the definition (1.2) of the semigroup T_t, the Cauchy-Schwartz inequality, and the normalization property of the heat kernel, we obtain that for any $u \in L^2$, $x \in M$, and $t > 0$

$$\begin{aligned} |T_t u(x)| &\le \int_M p_t(x,y)|u(y)|\,d\mu(y) \\ &\le \left\{\int_M p_t(x,y)u(y)^2 d\mu(y)\right\}^{1/2}\left\{\int_M p_t(x,y)d\mu(y)\right\}^{1/2} \\ &\le C\,t^{-\frac{\alpha}{2\beta}}\|u\|_2. \end{aligned} \tag{4.28}$$

Therefore, for any $\nu \geq \frac{2\alpha}{\beta}$, we have that

$$\|T_t u\|_\infty \leq C\, t^{-\frac{\nu}{4}} \|u\|_2 \quad \text{for all } 0 < t < 1. \tag{4.29}$$

Hence, the heat semigroup $\{T_t\}$ is $L^2 \to L^\infty$ ultracontractive for $0 < t < 1$. Since $\alpha > \beta$, we see that $\nu \geq 2\alpha/\beta > 2$, and so by [5, Theorem 2.4.2, p.75] (or [4], [19]), $\mathcal{D}(\mathcal{E}) \hookrightarrow L^{\frac{2\nu}{\nu-2}}$, that is, for any $u \in \mathcal{D}(\mathcal{E})$

$$\|u\|^2_{\frac{2\nu}{\nu-2}} \leq C\mathcal{E}[u] + C_0 \|u\|_2^2. \tag{4.30}$$

When ν varies in $[\frac{2\alpha}{\beta}, +\infty)$, the exponent $q = \frac{2\nu}{\nu-2}$ varies in $(2, 2^*]$. Therefore, (4.30) implies (4.26) for all $2 < q \leq 2^*$. For the remaining case $q = 2$, (4.26) is trivial.

Note that if $\nu = \frac{2\alpha}{\beta}$, then (4.29) holds for all $t > 0$, which implies (4.30) with $C_0 = 0$ (see [5, Corollary 2.4.3]), that is

$$\|u\|^2_{2^*} \leq C\mathcal{E}[u]. \tag{4.31}$$

Proof for the case (ii), $\alpha = \beta$. The proof is the same as above, with the following modification. Since (4.29) holds for all $\nu \geq 2\alpha/\beta = 2$, we see that (4.30) holds for all $\nu > 2$. Therefore, $q = \frac{2\nu}{\nu-2}$ takes all the values in $(2, +\infty)$, whence the claim follows.

Proof for the case (iii), $\alpha < \beta$. For any $x \in M$ and $r > 0$, set

$$u_r(x) := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u(\xi) d\mu(\xi). \tag{4.32}$$

We claim that for any $u \in W^{\beta/2,2}$ and for all $x \in M$, $0 < r < \rho_0$,

$$|u_{2r}(x) - u_r(x)| \leq Cr^\lambda\, W_{\beta/2}(u). \tag{4.33}$$

To see this, denoting by $B_1 = B(x,r), B_2 = B(x,2r)$, we have that

$$u_r(x) = \frac{1}{\mu(B_1)} \int_{B_1} u(\xi) d\mu(\xi) = \frac{1}{\mu(B_1)\mu(B_2)} \int_{B_1} \int_{B_2} u(\xi) d\mu(\xi) d\mu(\eta),$$

and a similar expression for $u_{2r}(x)$. Applying the Cauchy-Schwartz inequality, (3.2) and (4.1), we obtain

$$\begin{aligned}
|u_{2r}(x) - u_r(x)|^2 &= \left\{ \frac{1}{\mu(B_1)\mu(B_2)} \int_{B_1} \int_{B_2} (u(\xi) - u(\eta)) d\mu(\xi) d\mu(\eta) \right\}^2 \\
&\leq \frac{1}{\mu(B_1)\mu(B_2)} \int_{B_1} \int_{B_2} (u(\xi) - u(\eta))^2\, d\mu(\xi) d\mu(\eta) \\
&\leq C\, r^{-\alpha} \int_M \left[\frac{1}{\mu(B(\eta, 2r))} \int_{B(\eta,2r)} (u(\xi) - u(\eta))^2\, d\mu(\xi) \right] d\mu(\eta) \\
&\leq C\, r^{-\alpha+\beta}\, W_{\beta/2}(u)^2,
\end{aligned}$$

proving (4.33).

In the same way, we have

$$|u_{2r}(x) - u_r(y)| \leq C\, r^\lambda\, W_{\beta/2}(u)$$

for all $x, y \in M$ such that $r := d(x,y) < \rho_0$. Therefore, for such x, y we obtain

$$|u_r(x) - u_r(y)| \le |u_r(x) - u_{2r}(x)| + |u_{2r}(x) - u_r(y)| \le C\, r^\lambda\, W_{\beta/2}(u). \tag{4.34}$$

Let x, y be fixed Lebesgue points in M with $r := d(x,y) < \rho_0$. Set $r_k = 2^{-k} r$ for all $k = 0, 1, 2, ...$ Then (4.33) implies

$$\begin{aligned}
|u(x) - u_r(x)| &= \lim_{k\to\infty} |u_{r_k}(x) - u_{r_0}(x)| \\
&\le \sum_{k=0}^{\infty} |u_{r_k}(x) - u_{r_{k+1}}(x)| \\
&\le C\, W_{\beta/2}(u) \sum_{k=0}^{\infty} \left(\frac{r}{2^{k+1}}\right)^\lambda \\
&\le C\, r^\lambda\, W_{\beta/2}(u).
\end{aligned}$$

Similarly, we have

$$|u(y) - u_r(y)| \le C\, r^\lambda W_{\beta/2}(u),$$

which, together with (4.34), yields (4.25). ■

The method we used in the proof of part (iii) is similar to the one used in [10, 14]. Note that in this part we have not used explicitly the heat kernel, although the proof does use the relation $\mu(B(x,r)) \simeq r^\alpha$ that holds by Theorem 3.1.

Theorem 4.4. (Compact embedding theorem) *Let (M, d, μ) be a metric-measure space with a heat kernel satisfying (1.4). Then for any bounded sequence $\{u_k\}$ in $\mathcal{D}(\mathcal{E})$ in the norm (4.7), there exists a subsequence $\{u_{k_i}\}$ that converges to a function $u \in L^2(M,\mu)$ in the following sense:*

$$\|u_{k_i} - u\|_{L^2(B,\mu)} \to 0,$$

for any bounded set $B \subset M$.

Proof. Let $\{u_k\}$ be a bounded sequence in $\mathcal{D}(\mathcal{E})$. Since $\{u_k\}$ is also bounded in L^2, there exists a subsequence, still denoted by $\{u_k\}$, such that $\{u_k\}$ weakly converges to some function $u \in L^2$. Let us show that in fact $\{u_k\}$ converges to u in $L^2(B) = L^2(B,\mu)$ for any bounded set $B \subset M$.

For any $t > 0$, we have that, using the triangle inequality,

$$\|u_k - u\|_{L^2(B)} \le \|u_k - T_t u_k\|_{L^2(M)} + \|T_t u_k - T_t u\|_{L^2(B)} + \|T_t u - u\|_{L^2(M)}.$$

For any function $v \in L^2$ we have

$$\begin{aligned}
\|v - T_t v\|_2^2 &= \int_M \left(\int_M (v(x) - v(y)) p_t(x,y) d\mu(y)\right)^2 d\mu(x) \\
&\le \int_M \left\{\int_M p_t(x,y) d\mu(y) \int_M (v(x) - v(y))^2 p_t(x,y) d\mu(y)\right\} d\mu(x) \\
&= 2t\, \mathcal{E}_t[v] \\
&\le 2t\, \mathcal{E}[v].
\end{aligned}$$

Since $\mathcal{E}[u_k]$ is uniformly bounded by the hypothesis, we obtain that for all k and $t>0$

$$\|u_k - T_t u_k\|_2 \le C\sqrt{t}.$$

Since $\{u_k\}$ converges to u weakly in L^2 and $p_t(x,\cdot)\in L^2$, we see that for any $x\in M$

$$T_t u_k(x) = \int_M p_t(x,y)u_k(y)d\mu(y) \overset{k\to\infty}{\longrightarrow} \int_M p_t(x,y)u(y)d\mu(y) = T_t u(x).$$

Also, we have by (4.28)

$$\|T_t u_k\|_\infty \le Ct^{-\frac{\alpha}{2\beta}}\|u_k\|_2$$

so that the sequence $\{T_t u_k\}$ is uniformly bounded in k for any $t>0$. Since $\{T_t u_k\}$ converges to $T_t u$ pointwise, the dominated converegnce theorem gives that

$$T_t u_k \to T_t u \quad \text{in} \quad L^2(B),$$

as $k\to\infty$. Hence, we obtain that, for any $t>0$,

$$\limsup_{k\to\infty} \|u_k - u\|_{L^2(B)} \le C\sqrt{t} + \|T_t u - u\|_2.$$

Since $T_t u \to u$ in $L^2(M)$ as $t\to 0$, we finish the proof by letting $t\to 0$. ■

Corollary 4.2. *Under the hypotheses of Theorem 4.4, there exists a subsequence $\{u_{k_i}\}$ that converges to a function $u\in L^2(M,\mu)$ almost everywhere.*

Proof. Fix a point $x\in M$ and consider the sequence of balls $B_N = B(x,N)$, where $N = 1,2,\ldots.$ By Theorem 4.4 we can assume that the sequence $\{u_k\}$ converges to u in $L^2(B_N)$ for any N. Therefore, there exists a subsequence that converges almost everywhere in B_1. From this seqeunce, let us select a subsequence that converges to u almost everywhere in B_2, and so on. Using the diagonal principle, we obtain a subsequence that converges to u almost everywhere in M. ■

5. Semi-linear elliptic equations

As above, let (M,d,μ) be a metric measure space which possesses a heat kernel satisfying (1.4). In this section we show the existence of generalized solutions of the equation

$$-\Delta u + f(x,u) = g(x), \tag{5.1}$$

where Δ is the Laplace operator in M defined by (4.3) or (4.8). More precisely, we say that $u\in\mathcal{D}(\mathcal{E})$ is a generalized solution of (5.1) if the following identity holds

$$\mathcal{E}(u,v) + \int_M f(x,u(x))v(x)d\mu(x) - \int_M g(x)v(x)d\mu(x) = 0, \tag{5.2}$$

for any test function v from a certain class to be defined below.

Fix a couple $p,q\in(1,\infty)$, set

$$E^{p,q} := \mathcal{D}(\mathcal{E})\cap L^p\cap L^q,$$

and define the norm in $E^{p,q}$ by

$$\|u\| := \|u\|_p + \|u\|_q + \mathcal{E}[u]^{1/2}.$$

Clearly $E^{p,q}$ is a Banach space, and its dual is

$$(E^{p,q})^* = E^{p',q'},$$

where p' and q' are the Hölder conjugates to p and q, respectively.

We assume throughout this section that

$$g \in L^{p'} \tag{5.3}$$

and

$$|f(x,u)| \leq C|u|^{q-1} + f_0(x), \quad \text{for all } x \in M \text{ and } u \in \mathbb{R}, \tag{5.4}$$

where f_0 is a non-negative function in $L^{q'}$.

Let us show that all the terms in (5.2) make sense if $u \in \mathcal{D}(\mathcal{E}) \cap L^q$ and $v \in E^{p,q}$. Indeed, $\mathcal{E}(u,v)$ is defined as $u, v \in \mathcal{D}(\mathcal{E})$, and the other two terms are finite by the Hölder inequality:

$$\left| \int_M gv \, d\mu \right| \leq \|g\|_{p'} \|v\|_p < \infty \tag{5.5}$$

and

$$\left| \int_M f(\cdot, u) v \, d\mu \right| \leq \|f(\cdot,u)\|_{q'} \|v\|_q \leq \left(C\|u\|_q^{q/q'} + \|f_0\|_{q'} \right) \|v\|_q < \infty. \tag{5.6}$$

Now we can give a precise definition of a generalized solution of (5.1).

Definition 1. Assuming that f and g satisfy (5.3) and (5.4), we say that $u \in E^{p,q}$ is a generalized solution of (5.1) if the identity (5.2) holds for all $v \in E^{p,q}$.

Let E be a Banach space and $I : E \to \mathbb{R}$ be a functional on E. Recall that I is *Fréchet differentiable* at $u \in E$ if there exists an element in the dual space E^* of E, denoted by $I'(u)$, such that for all $v \in E$

$$I(u + tv) = I(u) + tI'(u)v + o(t) \quad \text{as } t \to 0.$$

The functional $I'(u)$ is termed the *Fréchet derivative* of I at point u. We say that I is *continuously Fréchet differentiable* if I is Fréchet differentiable at any $u \in E$, and the mapping $u \mapsto I'(u)$ is a continuous mapping from E to E^* (see for example [13, 17]). Finally, we say that u is a *critical point* of I if $I'(u) = 0$.

We will show that a generalized solution of (5.1) may be obtained as a *critical point* of a functional $I(u)$, where $I(u)$ is defined by

$$I(u) := \frac{1}{2}\mathcal{E}[u] + \int_M F(x, u(x)) d\mu(x) - \int_M g(x)u(x) d\mu(x), \tag{5.7}$$

and

$$F(x,u) := \int_0^u f(x,s) \, ds. \tag{5.8}$$

Let us show that the functional I defined by (5.7) is continuously Fréchet differentiable for suitable f and g.

Proposition 5.1. *Assume that $f(x,u)$ is continuous in $u \in \mathbb{R}$ for all $x \in M$ and satisfies (5.4), and g satisfies (5.3). Then I defined as in (5.7) is continuously Fréchet differentiable in $E^{p,q}$. Moreover, we have*

$$I'(u)v = \mathcal{E}(u,v) + \int_M f(x, u(x)) v(x) d\mu(x) - \int_M g(x) v(x) d\mu(x), \tag{5.9}$$

for all $u, v \in E^{p,q}$.

Thus, u *is a generalized solution of (5.1) if and only if* u *is a critical point of* I.

Proof. The proof follows the same line as in [13]. For completeness, we sketch the proof. It is easy to see directly from the definition that the functional

$$I_0(u) = \frac{1}{2}\mathcal{E}[u] - \int_M g(x)u(x)d\mu(x)$$

is continuously Fréchet differentiable at any point $u \in E^{p,q}$ and

$$I_0'(u)v = \mathcal{E}(u,v) - \int_M g(x)v(x)d\mu(x).$$

Let us show that the remaining part of I, that is the functional

$$J(u) := \int_M F(x, u(x))d\mu(x),$$

is also continuously Fréchet differentiable, and

$$J'(u)v = \int_M f(x, u(x))v(x)d\mu(x). \tag{5.10}$$

Indeed, taking (5.10) as the definition of J', we have for all $u, v \in E^{p,q}$ and $-1 < t < 1$

$$\begin{aligned} J(u+tv) - J(u) - tJ'(u)v &= \int_M \left[\int_u^{u+tv} f(x,s)ds - tf(x,u(x))v(x)\right] d\mu(x) \\ &= t\int_M \left(f(x, u+\theta v) - f(x,u)\right) v(x)d\mu(x), \end{aligned}$$

where

$$\theta = \theta(x,t) \in [0,t] \subset (-1,1).$$

By (5.6) we have $f(\cdot, u)v \in L^1$. By (5.4) and (5.6) we obtain similarly

$$|f(\cdot, u+\theta v)| \leq C\,|u+\theta v|^{q-1} + f_0 \leq C\,(|u| + |v|)^{q-1} + f_0 \in L^1.$$

Since $f(x, u+\theta v) \to f(x,u)$ as $t \to 0$, we conclude by the dominated convergence theorem that

$$J(u+tv) - J(u) - tJ'(u)v = o(t) \quad \text{as } t \to 0,$$

proving that J is Fréchet differentiable.

It remains to show that $J'(u)$ is continuous. For any $u_1, u_2, v \in E^{p,q}$, we have

$$\begin{aligned} |J'(u_1)v - J'(u_2)v| &= \left|\int_M (f(x,u_1(x)) - f(x,u_2(x)))v(x)d\mu(x)\right| \\ &\leq \|f(\cdot,u_1) - f(\cdot,u_2)\|_{q'}\|v\|_q \\ &\leq \|f(\cdot,u_1) - f(\cdot,u_2)\|_{q'}\|v\|_{E^{p,q}}, \end{aligned}$$

whence

$$\|J'(u_1) - J'(u_2)\|_{(E^{p,q})^*} \leq \|f(\cdot,u_1) - f(\cdot,u_2)\|_{q'}.$$

Note that the *Nemytsky operator* $\mathcal{F}u := f(x, u(x))$ is continuous from L^q to $L^{q'}$, provided that f satisfies (5.4) (see [18, Theorem 19.1]; for a bounded domain, see [1, Theorem 2.2]). Indeed, if $\mathcal{F}u$ is not continuous, then there exists a sequence $\{u_k\}$ such that $\|u_k - u\|_q \to 0$ but

$$\|\mathcal{F}u_k - \mathcal{F}u\|_{q'} \geq \varepsilon \tag{5.11}$$

for all k and some $\varepsilon > 0$. Since $\{u_k\}$ converges to u in L^q, there is a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that $\{u_{k_j}\}$ converges to u almost everywhere in M. Fix some $R > 0$ and set $B_R = B(x_0, R)$ for a fixed point $x_0 \in M$. By (5.4) and the dominated convergence theorem, we have

$$\begin{aligned}
\lim_{j\to\infty} \|\mathcal{F}u_{k_j} - \mathcal{F}u\|_{q'}^{q'} &= \lim_{j\to\infty} \left\{ \int_{B_R} |f(\cdot, u_{k_j}) - f(\cdot, u)|^{q'} d\mu + \int_{M\setminus B_R} |f(\cdot, u_{k_j}) - f(\cdot, u)|^{q'} d\mu \right\} \\
&\leq C \lim_{j\to\infty} \int_{M\setminus B_R} (|u_{k_j}|^q + |u|^q + |f_0|^{q'}) d\mu \\
&= C \lim_{j\to\infty} \int_{M\setminus B_R} (|u_{k_j} - u + u|^q + |u|^q + |f_0|^{q'}) d\mu \\
&\leq C \int_{M\setminus B_R} (|u|^q + |f_0|^{q'}) d\mu
\end{aligned}$$

Choosing R large enough we can make the right-hand side arbitrarily small, which contradicts (5.11). Hence, $\mathcal{F}u$ is continuous from L^q to $L^{q'}$. Therefore, $J'(u)$ is continuous. ■

By Proposition 5.1, in order to prove the existence of a generalized solution of (5.1), it is enough to show that the funcitonal I defined by (5.7) has a critical point in $E^{p,q}$; this in turn will follow if I has a minimum point in $E^{p,q}$. The following statement provides the conditions, which ensure that a functional on a Banach space has a minimum point.

Proposition 5.2. ([13, Theorem 2.5, p.14]) *Let I be a real-valued functional in a reflexive Banach space E satisfying the following conditions:*

(i) I is bounded below; that is

$$\inf_{u\in E} I(u) > -\infty.$$

(ii) I is coercive; that is for any real a there exists b such that $I(u) \leq a$ implies $\|u\| \leq b$.

(iii) Any sequence $\{u_k\}$ that converges to u weakly in E has a subsequence $\{u_{k_i}\}$ such that

$$\liminf_{i\to\infty} I(u_{k_i}) \geq I(u).$$

Then I has a minimum point in E.

Now we prove the main result of this section.

Theorem 5.1. *Assume that M admits a heat kernel satisfying (1.4) with $\alpha > \beta$, and let the function Φ_2 satisfy the same hypotheses as in Theorem 4.1 (or 4.3). Fix two numbers p, q such that*

$$q \geq p \geq 2^* := \frac{2\alpha}{\alpha - \beta}. \tag{5.12}$$

Let $g \in L^{p'}$, and $f(x,u)$ be a measurable function on $M \times \mathbb{R}$ that is continuous in u for any $x \in M$ and satisfies (5.4). Moreover, assume that for all $x \in M$ and $u \in \mathbb{R}$

$$F(x,u) := \int_0^u f(x,s)ds \geq c|u|^q + F_0(x), \tag{5.13}$$

where $F_0 \in L^1$. Then (5.1) has a generalized solution $u \in E^{p,q}$.

Remark. Here is an example of function f that satisfies all the hypotheses of Theorem 5.1:

$$f(x,u) = |u|^{q-2}\, u.$$

Hence, the equation

$$-\Delta u + |u|^{q-2}\, u = g \tag{5.14}$$

has a generalized solution $u \in E^{p,q}$ for any $g \in L^{p'}$ provided p and q satisfy (5.12). We will see below that this solution is unique.

Proof. It suffices to show that the functional I defined by (5.7) satisfies the conditions $(i) - (iii)$ of Proposition 5.2.

Condition (i) - I *is bounded below.* By (5.12) there exists $\theta \in [0,1]$ such that

$$\frac{1}{p} = \frac{\theta}{2^*} + \frac{1-\theta}{q},$$

whence, for any $u \in E^{p,q}$,

$$\|u\|_p \leq \|u\|_{2^*}^{\theta}\ \|u\|_q^{1-\theta}$$

by using the Hölder inequality. By (4.31), we see that

$$\|u\|_{2^*} \leq C\,\mathcal{E}\,[u]^{1/2}\,,$$

which implies

$$\begin{aligned} \|u\|_p &\leq C\,\mathcal{E}\,[u]^{\theta/2}\ \|u\|_q^{1-\theta} \\ &\leq C\left(\|u\|_q + \mathcal{E}\,[u]^{1/2}\right). \end{aligned} \tag{5.15}$$

From (5.7), (5.13), (5.5) and (5.15), we obtain that, for any $u \in E^{p,q}$,

$$\begin{aligned} I(u) &\geq \frac{1}{2}\mathcal{E}\,[u] + \left(c\|u\|_q^q - \|F_0\|_1\right) - \|g\|_{p'}\|u\|_p \\ &\geq \frac{1}{2}\mathcal{E}\,[u] + c\|u\|_q^q - \|F_0\|_1 - C\|g\|_{p'}(\|u\|_q + \mathcal{E}\,[u]^{1/2}) \\ &\geq \left[\frac{1}{2}\mathcal{E}\,[u] - C\mathcal{E}\,[u]^{1/2}\right] + \left[c\|u\|_q^q - C\|u\|_q\right] - C \\ &= \left[\frac{1}{2}s^2 - Cs\right] + [ct^q - Ct] - C, \end{aligned} \tag{5.16}$$

where $s := \mathcal{E}(u,u)^{1/2}$ and $t := \|u\|_q$. By $q > 1$ the value of each square bracket is bounded below, whence we conclude that I is bounded below.

Condition (ii) - I is coercive. If $I(u) \leq a$ for some a, then by (5.16)

$$a \geq \left[\frac{1}{2}s^2 - Cs\right] + [ct^q - Ct] - C,$$

which implies that s and t must be bounded. Together with (5.15) this implies that u is bounded in $E^{p,q}$.

Condition (iii). Let a sequence $\{u_k\}$ converge to u weakly in $E^{p,q}$. Since $g \in L^{p'} \subset (E^{p,q})^*$, we have

$$\lim_{k\to\infty} \int_M g u_k \, d\mu = \int_M g u \, d\mu. \tag{5.17}$$

Since $u \in \mathcal{D}(\mathcal{E}) \subset (E^{p,q})^*$, we have

$$\lim_{k\to\infty} \mathcal{E}(u_k, u) = \mathcal{E}[u].$$

Applying the inequality

$$\mathcal{E}[u_k] \geq 2\mathcal{E}(u_k, u) - \mathcal{E}[u],$$

we obtain

$$\liminf_{k\to\infty} \mathcal{E}[u_k] \geq \mathcal{E}[u]. \tag{5.18}$$

We are left to verify that there exists a subsequence $\{u_{k_i}\}$ such that

$$\liminf_{i\to\infty} \int_M F(\cdot, u_{k_i}) d\mu \geq \int_M F(\cdot, u) d\mu. \tag{5.19}$$

The sequence $\{u_k\}$ is bounded in $\mathcal{D}(\mathcal{E})$. Therefore, by Corollary 4.2 there exists a subsequence $\{u_{k_i}\}$ that converges to u almost everywhere in M. Therefore, we have also $F(\cdot, u_{k_i}) \to F(\cdot, u)$ almost everywhere in M. By (5.13), we have $F(\cdot, u_{k_i}) \geq F_0 \in L^1$, and (5.19) follows by Fatou's lemma. Combining (5.17)-(5.19), we complete the proof. ∎

Finally, we complement Theorem 5.1 by the uniqueness result.

Proposition 5.3. *Let the function $f(x, u)$ be strictly monotone increasing in u for every $x \in M$. Then the equation (5.1) has at most one generalized solution $u \in E^{p,q}$.*

Hence, the equation (5.14) has exactly one generalized solution in $E^{p,q}$.

Proof. Let u_1 and u_2 be two generalized solutions of (5.1). Then for any $v \in E^{p,q}$ we have from (5.2)

$$\mathcal{E}(u_1 - u_2, v) + \int_M (f(\cdot, u_1) - f(\cdot, u_2)) \, v \, d\mu = 0.$$

Substituting $v = u_1 - u_2$, we obtain

$$\mathcal{E}[u_1 - u_2] + \int (f(\cdot, u_1) - f(\cdot, u_2)) (u_1 - u_2) \, d\mu = 0.$$

By the monotonicity of $f(x, u)$ in u, the both terms here are non-negative, whence each of them must vanish. In particular, we obtain

$$(f(\cdot, u_1) - f(\cdot, u_2)) (u_1 - u_2) = 0$$

almost everywhere, which by the strict monotonicity of f implies $u_1 = u_2$. ∎

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