Some Regularity Criteria on Suitable Weak Solutions of the 3-D Incompressible Axisymmetric Navier-Stokes Equations

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Dedicated to Professor Louis Nirenberg for his 75th Birthday

Abstract: This paper is concerned with the partial regularity for the 3-D incompressible axisymmetric Navier-Stokes equations. It is shown that the gradient of the velocity field is locally uniformly bounded in L^∞-norm provided that one of the following two conditions is satisfied: (1) the scaled L^2-norm of ω^θ (the angular component of the vorticity) is finite and scaled total energy is small; (2) the scaled L^2-norm of ω^θ and u^θ (the angular component of the velocity) are both small. Our results imply that, under one of the above two conditions, the smooth solutions to the 3-D axisymmetric Navier-Stokes equations cannot develop finite time singularity and suitable weak solutions is in fact regular.

Key Words: 3-D axisymmetric Navier-Stokes equations, suitable weak solutions, partial regularity

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1 Introduction

Consider the Cauchy problem for the three-dimensional (3-D) incompressible Navier-Stokes equations

$$\begin{aligned} &\partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0, \quad (x,t)\in R^3\times(0,T),\\ &\text{div } u = 0, \end{aligned} \tag{1.1}$$

with the initial conditions

$$u(x,t)\mid_{t=0} = u_0(x). \tag{1.2}$$

For simplicity in presentation, here we assume that the viscosity is unit and the external force is zero. And the unknown functions are the velocity vector

$u = (u_1(x,t), u_2(x,t), u_3(x,t))$ and the pressure $p(x,t)$. In (1.1), $T > 0$ is a constant, and div $u = 0$ means that the fluid is incompressible.

Since Leray and Hopf's pioneering works (see [9],[7]), the well-posedness theory for 3-D incompressible Navier-Stokes equations has been a challenging open problem, even for C^∞-smooth initial data. It is well-known that the Cauchy problem, (1.1)-(1.2), has global weak solution, but whose uniqueness and regularity remain to be proved, while the unique and regular solution exists only locally in time. This can be also shown by the following interior regularity criterion (see [14], [15], [5]). A weak solution $u(x,t)$ of (1.1) is regular provided that $u \in L^p([0,T); L^q(R^3))$, where $2/p + 3/q \le 1$, $p \ge 2$ and $q > 3$. When $p = +\infty, q = 3$, one only has uniqueness for weak solutions but no regularity results at present (see [6]).

The three-dimensional axisymmetric Navier-Stokes equations is an important and interesting case. In this case, when $u^\theta \equiv 0$ (here u^θ is the angular component of the velocity in the cylindrical coordinate, see Section 2), which is usually called *no swirl*, the existence and uniqueness of (1.1) have been established ([8], [18]). In the case with *swirl*, that is, $u^\theta \not\equiv 0$, there are still some basic open problems including the global existence and uniqueness of the axisymmetric solutions of (1.1). Recently, based on the vorticity equations and the regular criteria mentioned above, Chae and Lee [3] obtained a regular criterion for Leray-Hopf weak solutions to the 3-D axisymmetric Navier-Stokes equations by imposing regular conditions on single component ω^θ, the angular component of the vorticity in the cylindrical coordinate. That is, if $\omega^\theta \in L^p([0,T); L^q(R^3))$, where $3/2 < q < \infty, 1 < p \le \infty$ and $2/p + 3/q \le 2$, then the Leray-Hopf weak solution of axisymmetric Navier-Stokes equations is regular.

In [13], Scheffer introduced the notions of *suitable weak solutions* and began to study the partial regularity of such weak solutions. Deeper results were obtained by Caffarelli-Kohn-Nirenberg in [2], where they showed that, for any such weak solutions, the singular set has one-dimensional Hausdorff measure zero. Lin gave a simplified proof of the main results in [10]. By using more natural scaling quantities and through different approach from [2], Tian and Xin in [17] obtained some new criteria for the regularity of suitable weak solutions to Navier-Stokes equations and improved somewhat the results implied in [2]. The singular set was also studied by Cheo and Lewis in [4]. Following from these general results, one concludes that singular points of solutions to 3-D axisymmetric Navier-Stokes equations must lie on the symmetry axis if there exists one at all.

In this paper, our aim is to improve the known results for the partial regularity of axisymmetric solutions to the 3-D axisymmetric Navier-Stokes equations. Motivated by [17], we obtain two criteria for partial regularity of the suitable weak solutions to the 3-D axisymmetric Navier-Stokes equations. First, it is shown that if the scaled L^2-norm of ω^θ is finite and the scaled total energy is small, then the suitable weak solutions to the 3-D axisymmetric Navier-Stokes equations is regular. In comparison with results in [17], only the single angular component of the vorticity is needed here. Second, we prove that the same result holds if the scaled L^2-norm of u^θ and ω^θ are both small. In our approach, we obtain that the gradient of the velocity field is locally uniformly bounded in L^∞-norm, as what has been done in [17]. In particular, in our proofs, we establish the gradient estimate of u^θ by using the generalized energy inequality and an estimate on $\|ru^\theta\|_{L^\infty}$, which is actually the maximum principle on ru^θ.

Finally, we point out that some other regular criteria for the suitable weak solutions were presented in [11] and [12]. In [11], it was shown that the essential boundedness of the Cartesian velocity component u_3 implies the suitable weak solutions is regular. While in [12], it was proved that, for the 3-D axisymmetric Navier-Stokes equations, a higher regularity of one of the velocity component u^θ implies that the regularity of all components.

This paper is organized as follows. Section 2 contains some the preliminaries and main results, and the proof of the main result is given in Section 3.

2 Preliminaries and Main Results

By an axisymmetric solution of (1.1), we mean a solution (u, p) to (1.1), which, in the cylindrical coordinate, takes the form that $p(x,t) = p(r, x_3, t)$ and

$$u(x,t) = u^r(r, x_3, t)e_r + u^\theta(r, x_3, t)e_\theta + u_3(r, x_3, t)e_3,$$

where

$$e_r = (\cos\theta, \sin\theta, 0), \quad e_\theta = (-\sin\theta, \cos\theta, 0), \quad e_3 = (0, 0, 1).$$

Here $u^\theta(r, x_3, t)$ is the angular component of $u(x,t)$. For the axisymmetric velocity field u, the corresponding vorticity $\omega = \nabla \times u$ is given by

$$\omega = \omega^r e_r + \omega^\theta e_\theta + \omega_3 e_3,$$

where,

$$\omega^r = \partial_3 u^\theta,\ \omega^\theta = \partial_r u^3 - \partial_3 u^r,\ \omega_3 = -\frac{1}{r}\partial_r(ru^\theta).$$

The 3-D axisymmetric Navier-Stokes equations have the forms

$$\frac{\tilde{D}u^r}{Dt} - (\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r)u^r + \frac{1}{r^2}u^r - \frac{1}{r}(u^\theta)^2 + \partial_r p = 0, \tag{2.1}$$

$$\frac{\tilde{D}u^\theta}{Dt} - (\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r)u^\theta + \frac{1}{r^2}u^\theta + \frac{1}{r}u^\theta u^r = 0, \tag{2.2}$$

$$\frac{\tilde{D}u_3}{Dt} - (\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r)u_3 + \partial_3 p = 0, \tag{2.3}$$

$$\partial_r(ru^r) + \partial_3(ru_3) = 0, \tag{2.4}$$

where

$$\frac{\tilde{D}}{Dt} = \frac{\partial}{\partial_t} + u^r\partial r + u_3\partial_3, \quad r = (x_1^2 + x_2^2)^{1/2}.$$

In the following, we set

$$\tilde{\nabla} = (\partial_r, \partial_3).$$

Now we give the definition of suitable weak solutions (see[2]).

Definition 2.1 A pair (u, p) is called a suitable weak solution of (1.1) if

(1) $u \in L^2([0,T); H^1(R^3)) \cap L^\infty([0,T); L^2(R^3)), p \in L^{5/4}([0,T) \times R^3)$, such that for some constants M_0 and M_1, the following inequalities hold:

$$\sup_{0\le t\le T} \|u(\cdot,t)\|^2 + \int_0^T \int_{R^3} |\nabla u|^2(x,t)dxdt \le M_0, \tag{2.5}$$

$$\int_0^T \int_{R^3} |p(x,t)|^{5/4} dxdt \le M_1. \tag{2.6}$$

(2) (u, p) satisfies (1.1) in the sense of distribution on $[0,T) \times R^3$;

(3) For each $0 \le \phi(x,t) \in C_0^\infty(R^3 \times (0,T))$, (u,p) satisfies the generalized energy inequality

$$\begin{aligned} 2\int_0^T \int_{R^3} |\nabla u|^2 \phi dxdt \;&\le \int_0^T \int_{R^3} |u|^2(\phi_t + \Delta\phi)dxdt \\ &+ \int_0^T \int_{R^3} (|u|^2 + 2p)u \cdot \nabla\phi dxdt. \end{aligned} \tag{2.7}$$

From (1) and (2) above, it is known that $u(x,t)$ is weakly continuous from [0,T] to $L^2([0,T])$ (see[16]), that is, for any $\omega(x) \in L^2(R^3)$,

$$\int_{R^3} u(x,t) \cdot \omega(x)dx \to \int_{R^3} u(x,t_0) \cdot \omega(x)dx$$

as $t \to t_0 \in [0,T]$. Therefore it follows from the generalized energy inequality (2.7) that

$$\begin{aligned} &\int_{R^3\times\{t\}} |u|^2\phi dx + 2\int_0^t\int_{R^3} |\nabla u|^2 \phi dxdt \\ &\le \int_0^t\int_{R^3} |u|^2(\phi_t + \Delta\phi)dxdt + \int_0^t\int_{R^3}(|u|^2+2p)u\cdot\nabla\phi dxdt \end{aligned} \tag{2.8}$$

for any $t \in (0,T)$.

It is noted that the existence of the suitable weak solution was proved in [2]. For any smooth function $\phi(x,t) = \phi(r,x_3,t) \in C_0^\infty(R^3\times[0,T))$ satisfying $\phi(x,t)\mid_{t=0} = 0$, the generalized energy inequality for the axisymmetric case is as follows.

$$\begin{aligned} &\int (u^r)^2\phi dx + 2\int\int[(\partial_r u^r)^2 + (\partial_3 u^r)^2]\phi dxdt + 2\int\int \frac{(u^r)^2}{r^2}\phi dxdt \\ &= \int\int (u^r)^2(\phi_t + \partial_r^2\phi + \partial_3^2\phi + \frac{1}{r}\partial_r\phi)dxdt + \int\int (u^r)^2(u^r\partial_r\phi + u^3\partial_3\phi)dxdt \\ &+2\int\int\frac{(u^\theta)^2}{r}u^r\phi dx - 2\int\int \partial_r p u^r\phi dxdt, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\int (u^\theta)^2\phi dx + 2\int\int[(\partial_r u^\theta)^2 + (\partial_3 u^\theta)^2]\phi dxdt + 2\int\int \frac{(u^\theta)^2}{r^2}\phi dxdt \\ &= \int\int (u^\theta)^2(\phi_t + \partial_r^2\phi + \partial_3^2\phi + \frac{1}{r}\partial_r\phi)dxdt + \int\int (u^\theta)^2(u^r\partial_r\phi + u^3\partial_3\phi)dxdt \\ &-2\int\int\frac{(u^\theta)^2}{r}u^r\phi dxdt, \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\int (u^3)^2\phi dx + 2\int\int[(\partial_r u^3)^2 + (\partial_3 u^3)^2]\phi dxdt \\ &= \int\int (u^3)^2(\phi_t + \partial_r^2\phi + \partial_3^2\phi + \frac{1}{r}\partial_r\phi)dxdt + \int\int (u^3)^2(u^r\partial_r\phi + u^3\partial_3\phi)dxdt \\ &-2\int\int \partial_3 p u^3\phi dxdt. \end{aligned} \tag{2.11}$$

Here

$$\int \cdot dx = \int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} \cdot r dr dx_3 d\theta, \quad \int\int \cdot dxdt = \int_0^T\int_0^{2\pi} \cdot r dr dt$$

We point out that (2.9)-(2.11) can be deduced from (2.8) by applying cylindrical coordinate transformation.

As we mentioned previously, the singular points in axisymmetric case can only possibly appear on z-axis. So we concentrate on the regularity at the point $Q = (0, 0, x_3, t)$. We will use the following notation: the parabolic ball centered at Q with radius R will be denoted by $P_R(Q) = B_R(x) \times (t - R^2, t)$, which will be denoted by P_R if there is no danger of confusion. The scaled total energy, scaled L^2-norm of vorticity and other scaled quantities are to be defined to be the following dimensionless quantities

$$\begin{aligned}
E(R) &\equiv \frac{1}{R^3} \int\int_{P_R} |u(x,t)|^2 dxdt, \\
W(R) &\equiv \frac{1}{R} \int\int_{P_R} |\nabla \times u(x,t)|^2 dxdt, \\
E_1(R) &= \sup_{-R^2 \le t < 0} \frac{1}{R} \int_{B_R} |u(x,t)|^2 dxdt, \\
E_2(R) &= \frac{1}{R} \int\int_{P_R} |\nabla u(x,t)|^2 dxdt \\
E_3(R) &= \frac{1}{R^2} \int\int_{P_R} |u(x,t)|^3 dxdt.
\end{aligned} \tag{2.12}$$

It was shown in [17] that the local behavior of the solution to the Navier-Stokes equations can be dominated by the above sealed quantities in (2.12). Indeed, the main results in [17] (Theorem 3.1 in [17]) are

Proposition 2.1 There exists an absolute constant $\varepsilon > 0$ with the following property. Let (u, p) be a smooth solutions to (1.1)-(1.2) satisfying the bounds (2.5) and (2.6). Assume that there exists a $R_0 > 0$ such that one of the following three conditions hold

(1) Either $\sup_{0<R\le R_0} E_1(R) < +\infty$ or $\sup_{0<R\le R_0} E_2(R) < +\infty$ and

$$E(R) \equiv \frac{1}{R^3} \int\int_{P_R} |u(x,t)|^2 dxdt \le \varepsilon \quad \text{for all} \quad R \le R_0,$$

(2)

$$\sup_{0<R\le R_0} W(R) \le \varepsilon,$$

(3)

$$\sup_{0<R\le R_0} E_3(R) \le \varepsilon,$$

then

$$\sup_{(x,t)\in P_{R/2}} |\nabla u(x,t)| \le CR^{-2} \quad \text{for} \quad R \le R_1$$

for some $R_1 < R_0$ with C being an absolute constant.

Remark. In (1), $\sup_{0<R\le R_0} E_2(R) < +\infty$ can be replaced by $\sup_{0<R\le R_0} W(R) < +\infty$.

For axisymmetric flow, we introduce the following dimensionless quantities:

$$
\begin{aligned}
E^\theta(R) &\equiv \frac{1}{R^3}\int\int_{P_R} |u^\theta(x,t)|^2 dxdt, \\
E^r(R) &\equiv \frac{1}{R^3}\int\int_{P_R} |u^r(x,t)|^2 dxdt, \\
E^3(R) &\equiv \frac{1}{R^3}\int\int_{P_R} |u^3(x,t)|^2 dxdt, \\
\tilde{E}(R) &\equiv \frac{1}{R^3}\int\int_{P_R} |u^r(x,t)|^2 + |u^3(x,t)|^2 dxdt, \\
W^\theta(R) &\equiv \frac{1}{R}\int\int_{P_R} |(\omega^\theta)|^2 dxdt, \\
E_1^\theta(R) &\equiv \sup_{-R^2\le t<0} \frac{1}{R}\int_{B_R} |u^\theta(x,t)|^2 dxdt, \\
E_2^\theta(R) &\equiv \frac{1}{R}\int\int_{P_R} |\nabla u^\theta(x,t)|^2 dxdt \\
\tilde{E}_2(R) &\equiv \frac{1}{R}\int\int_{P_R} |\nabla u^r(x,t)|^2 + |\nabla u^3(x,t)|^2 dxdt \\
F^\theta(R) &\equiv \frac{1}{R}\int\int_{P_R} (\frac{u^\theta}{r})^2 dxdt, \\
F^r(R) &\equiv \frac{1}{R}\int\int_{P_R} (\frac{u^r}{r})^2 dxdt.
\end{aligned}
\tag{2.13}
$$

In the following, we will use C to denote an absolute constant which may be different from line to line unless otherwise stated.

Our main results are as follows

Theorem 2.1 Suppose that $u_0(x) \in L^2(R^3)$ and $ru_0^\theta(x) \in L^\infty(R^3) \cap L^p(R^3)$ $(p \ge 2)$. There exists an absolute constant $\varepsilon > 0$ with the following property. Let (u, p) be a smooth solution to (1.1)-(1.2) satisfying the bounds (2.5) and (2.6). Assume that there exists a $R_0 > 0$ such that one of the following two conditions holds

(I) $\sup_{0<R\le R_0} W^\theta(R) < +\infty$ and

$$E(R) \equiv \frac{1}{R^3}\int\int_{P_R} |u(x,t)|^2 dxdt \le \varepsilon \quad \text{for all} \quad R \le R_0,$$

(II)

$$W^\theta(R) \le \varepsilon, \quad E^\theta(R) \le \varepsilon \quad \text{for all} \quad R \le R_0,$$

then

$$\sup_{(x,t)\in P_{R/2}} |\nabla u(x,t)| \le CR^{-2} \quad \text{for} \quad R \le R_1, \tag{2.14}$$

where $R_1 \le R_0$ is a positive number.

Remarks. 1. The conditions of (I) and (II) can be written as $\limsup_{R\to 0} E(R) = 0$ and $\limsup_{R\to 0} W^\theta(R) = 0$, $\limsup_{R\to 0} E^\theta(R) = 0$ respectively.

2. Just as shown in [2] and [17], the estimate (2.14) implies that $(x,t) \in P_{R/2}$ is a regular point and the set of sigular points for a suitable weak solution to the Navier-Stokes equations is one-dimensional Hausdorff measure zero.

3 Proof of Theorem 2.1

First, we give the L^∞ estimate of ru^θ which is actually the maximal principle for the equation on ru^θ (see also [3]).

Lemma 3.1 Suppose that u is a smooth axisymmetric solution of the Navier-Stokes equations with initial data $u_0 \in L^2(R^3)$. If $ru_0^\theta \in L^\infty \cap L^p$ for any $p \ge 2$, then $ru^\theta \in L^\infty([0,T)\times R^3) \cap L^p$,and

$$\|ru^\theta(t)\|_{L^\infty(R^3)} \le \|ru_0^\theta\|_{L^\infty(R^3)}, \quad \|ru^\theta(t)\|_{L^p(R^3)} \le \|ru_0^\theta\|_{L^p(R^3)}$$

Proof. It follows from (2.2) that

$$\frac{\tilde{D}}{Dt}(ru^\theta) - (\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r)(ru^\theta) + \frac{2}{r}\partial_r(ru^\theta) = 0.$$

Let $\xi(x) = \xi(R) \in C_0^\infty(R^3), 0 \le \xi \le 1$, $\xi(R) = 1$ on $R \le 1$ and supp$\xi \subset \{R < 2\}$, where $R^2 = x_1^2 + x_2^2 + x_3^2 = r^2 + x_3^2$. Let $\lambda(R) = \lambda_s(R) = \xi(R/s)$.

Multiplying the both side of the above equation by $|ru^\theta|^{p-2}(ru^\theta)\lambda^{pk}$ with $k \ge 3$ an integer and integrating over R^3, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{R^3}|ru^\theta\lambda^k|^p dx + \frac{4(p-1)}{p^2}\int_{R^3}|\tilde{\nabla}|ru^\theta\lambda^k|^{\frac{p}{2}}|^2 dx\\
&= k\int_{R^3} u^r\partial_r\lambda|ru^\theta\lambda^k|^{p-2}r^2(u^\theta)^2\lambda^{2k-1}dx - 2\int_{R^3}\partial_r(ru^\theta)\partial_r\lambda^k|ru^\theta\lambda^k|^{p-2}ru^\theta\lambda^k dx\\
&- \int_{R^3}\partial_r^2(\lambda^k)|ru^\theta\lambda^k|^{p-2}(ru^\theta\lambda^k)ru^\theta dx + \int_{R^3}\frac{1}{r}\partial_r(\lambda^k)(ru^\theta)|ru^\theta\lambda^k|^{p-2}(ru^\theta\lambda^k)dx\\
&- \int_{R^3}\frac{2}{r}\partial_r(ru^\theta\lambda^k)|ru^\theta\lambda^k|^{p-2}(ru^\theta\lambda^k)dx \equiv I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{3.1}$$

Noting that $ru^\theta\lambda^k$ is an axisymmetric smooth function which vanishes at infinity and on the x_3-axis, we obtain

$$I_5 = \frac{-4\pi}{p}\int_{-\infty}^{\infty}\int_0^\infty \partial_r|ru^\theta\lambda^k|^p dr dx_3 = 0.$$

Then, letting $s \to \infty$ in (3.1), one gets

$$\frac{1}{p}\frac{d}{dt}\int_{R^3}|ru^\theta|^p dx + \frac{4(p-1)}{p^2}\int_{R^3}|\tilde{\nabla}|ru^\theta|^{\frac{p}{2}}|^2 dx = 0$$

Thus, one concludes that for any large p,

$$\int_{R^3} |ru^\theta|^p dx + \frac{4(p-1)}{p} \int_0^T \int_{R^3} |\tilde{\nabla} |ru^\theta|^{\frac{p}{2}}|^2 dx \le \int_{R^3} |ru_0^\theta|^p dx.$$

Therefore,

$$\|ru^\theta(t)\|_{L^p(R^3)} \le \|ru_0^\theta\|_{L^p(R^3)}.$$

After letting $p \to \infty$, we arrive at the desired estimate

$$\|ru^\theta(t)\|_{L^\infty(R^3)} \le \|ru_0^\theta\|_{L^\infty(R^3)}.$$

The proof of the lemma is finished.

Set $\tilde{u} = u^r e_r + u^3 e_3$. Then it is easy to get

$$\text{div } \tilde{u} = 0, \nabla \times \tilde{u} = \omega^\theta e_\theta. \tag{3.2}$$

We now prove Thoerem 2.1. We start with the estimate on $\tilde{E}_2(R)$.

Lemma 3.2 There exists an absolute constant C such that for any $\lambda \in (0, 1/2], R = \lambda\rho$, and $\rho \le R_0$, one has

$$\tilde{E}_2(R) \le 72\lambda^2 \tilde{E}_2(\rho) + C(8\lambda^2 + \frac{1}{\lambda}) W^\theta(\rho). \tag{3.3}$$

Proof. The proof is similar to what has been done for Lemma 3.3 (ii) in [17], so we just give a sketch of the proof here. By (3.2), we have the following representation

$$\nabla\tilde{u}(x,t) = \int_{B_\rho} \nabla_x^2 \Gamma(x-y) \times \omega^\theta(y,t) e_\theta dy + \omega^\theta(x,t) e_\theta + H_0(x,t), \tag{3.4}$$

for all $(x,t) \in P_\rho$, where $H_0(x,t)$ is a harmonic function in $x \in B_\rho$ for each fixed $t \in (-\rho^2, 0)$, and $\Gamma(x)$ is the standard normalized fundamental solution of Laplace's equation in R^3. The integral on the right hand side of (3.3) is in the sense of the Cauchy principle value. Then by Calderon-Zygmund singular integral estimate and mean value property for harmonic function, one can derive the desired estimate (3.3). For details, see Lemma 3.3 in [17].

As a consequence of (3.3), one has

Lemma 3.3 (i) For any $\delta > 0$, there exists a positive number ε_0 such that if $W^\theta(R) \le \varepsilon_0$ for all $R \le R_0$, then $\tilde{E}_2(R) \le \delta$ for all $R \le R_1$, where $R_1 \le R_0$ is a positive number;

(ii) If there exists a constant C such that $W^\theta(R) \le C$ for all $R \le R_0$, then $\tilde{E}_2(R) \le C$ for all $R \le R_0$.

Proof. (i) In (3.3), we set $\mu = 72\lambda^2$ and $C(\lambda) = C(8\lambda^2 + \frac{1}{\lambda})$. Fix $0 < \lambda \le \frac{1}{2}$ such that $\mu = 72\lambda^2 < 1$. Thanks to (3.3), by iteration, we get

$$\tilde{E}_2(\lambda^k R_0) \le \mu^k \tilde{E}_2(R_0) + \frac{1-\mu^k}{1-\mu} C(\lambda)\varepsilon_0, \tag{3.5}$$

Now for any $\delta > 0$, we choose ε_0 small so that

$$\frac{1}{1-\mu} C(\lambda)\varepsilon_0 < \frac{\lambda\delta}{2}.$$

And then we choose an integer K_0 to get

$$\mu^{K_0} \tilde{E}_2(R_0) < \frac{\lambda\delta}{2}.$$

Define $R_1 = \lambda^{K_0} R_0$. For any $0 < R \le R_1$, there exists a $k \ge K_0$ so that $\lambda^{k+1} R_0 \le R \le \lambda^k R_0$. Thus

$$\begin{aligned} \tilde{E}_2(R) &\le \frac{1}{(\lambda^{k+1}R_0)^3} \int\int_{P_{\lambda^k R_0}} |\nabla \tilde{u}|^2 dx dt \le \frac{1}{\lambda} \tilde{E}_2(\lambda^k R_0) \\ &\le \frac{1}{\lambda} (\mu^{K_0} \tilde{E}_2(R_0) + \frac{1}{1-\mu} C(\lambda)\varepsilon_0) \le \delta. \end{aligned}$$

(ii) When $W^\theta(\rho) \le C$ for $R \le R_0$, it follows from (3.3) that for any fixed λ, $0 < \lambda \le 1$,

$$\tilde{E}_2(\lambda^k R_0) \le \mu^k \tilde{E}_2(R_0) + \frac{1-\mu^k}{1-\mu} C(\lambda) \le C(\lambda).$$

For any $0 < R \le R_0$, there exists a integer $k \ge 0$ so that $\lambda^{k+1} R_0 \le R \le \lambda^k R_0$, and

$$\tilde{E}_2(R) \le \frac{1}{\lambda} \tilde{E}_2(\lambda^k R_0) \le C(\lambda).$$

The proof of the Lemma is finished.

In the following, for any fixed positive numbers R and ρ satisfying $0 < R \le \frac{1}{4}\rho$ and $\rho \le R_0$, we set $R_* = 2R \le \frac{1}{2}\rho$. We also denote by $\bar{f}_R$ the average of f on the ball B_R, i.e. $\bar{f}_R = \frac{1}{\omega_3 R^3} \int_{B_R} f dx$. Let $\phi(x,t) = \phi(r, x_3, t)$ be a smooth axial function such that $0 \le \phi \le 1$, $\phi \equiv 1$ on P_R and $\mathrm{supp}(\phi(x,t))$ is P_{R_*} and

$$|\partial_r \phi| + |\partial_3 \phi| \le \frac{C}{R_*}, \quad |\partial_t \phi| + |\partial_r^2 \phi| + |\partial_3^2 \phi| \le \frac{C}{R_*^2} \tag{3.6}$$

Then, (2.10) shows

$$
\begin{aligned}
&\int_{B_{R_*}}(u^\theta)^2\phi dx+2\int\int_{P_{R_*}}[(\partial_r u^\theta)^2+(\partial_3 u^\theta)^2]\phi dxdt+2\int\int_{P_{R_*}}(\frac{u^\theta}{r})^2\phi dxdt\\
&=\int\int_{P_{R_*}}(u^\theta)^2(\phi_t+\partial_r^2\phi+\partial_3^2\phi+\frac{\partial_r\phi}{r})dxdt+\int\int_{P_{R_*}}(u^\theta)^2(u^r\partial_r\phi+u^3\partial_3\phi)dxdt\\
&-2\int\int_{P_{R_*}}\frac{(u^\theta)^2}{r}u^r\phi dxdt,\\
&\le\frac{C}{R_*^2}\int\int_{P_{R_*}}|u^\theta|^2dxdt+\frac{C}{R_*}\int\int_{P_{R_*}}|(u^\theta)^2-\overline{(u^\theta)^2_{R_*}}||\tilde{u}|dxdt\\
&+C\int\int_{P_{R_*}}|\frac{(u^\theta)^2}{r}u^r|dxdt.
\end{aligned}
\tag{3.7}
$$

Consequently,

$$
\begin{aligned}
&E_1^\theta(R)+E_2^\theta(R)+F^\theta(R)\le CE^\theta(R_*)\\
&+\frac{C}{R_*^2}\int\int_{P_{R_*}}|(u^\theta)^2-\overline{(u^\theta)^2_{R_*}}||\tilde{u}|dxdt+\frac{C}{R_*}\int\int_{P_{R_*}}|\frac{(u^\theta)^2}{r}u^r|dxdt.
\end{aligned}
\tag{3.8}
$$

We need to estimate the last two terms on the right hand of (3.8). First, we have

Lemma 3.4 For $0<\mu\le\rho/2$, it holds that

$$
\begin{aligned}
&\frac{1}{\mu^2}\int\int_{P_\mu}|(u^\theta)^2-\overline{(u^\theta)^2_\mu}||\tilde{u}|dxdt\\
&\le C(\frac{\rho}{\mu})^2(E_2^\theta)^{\frac12}(\rho)(\tilde{E}_2)^{\frac14}(\rho)(E_1^\theta)^{\frac12}(\rho)(\tilde{E})^{\frac14}(\rho)\\
&+C(\frac{\rho}{\mu})^{\frac52}(E_2^\theta)^{\frac12}(\rho)(\tilde{E})^{\frac12}(\rho)(E_1^\theta)^{\frac12}(\rho).
\end{aligned}
\tag{3.9}
$$

Proof. Applying the Hölder, Poincare, and Sobolev-Poincare's inequalities, we obtain

$$
\begin{aligned}
&\int\int_{P_\mu}|(u^\theta)^2-\overline{(u^\theta)^2_\mu}||\tilde{u}|dxdt\\
&\le C\int_{-\mu^2}^0\|\tilde{u}\|_{L^3(B_\mu)}\|u^\theta\|_{L^2(B_\mu)}\|\nabla u^\theta\|_{L^2(B_\mu)}dt\\
&\le C\int_{-\mu^2}^0(\|\tilde{u}\|^{\frac12}_{L^2(B_\mu)}\|\nabla\tilde{u}\|^{\frac12}_{L^2(B_\mu)}+\frac{1}{\mu^{\frac12}}\|\tilde{u}\|_{L^2(B_\mu)})\|u^\theta\|_{L^2(B_\mu)}\|\nabla u^\theta\|_{L^2(B_\mu)}dt\\
&\le C\int_{-\mu^2}^0\|\tilde{u}\|^{\frac12}_{L^2(B_\mu)}\|\nabla\tilde{u}\|^{\frac12}_{L^2(B_\mu)}\|u^\theta\|_{L^2(B_\mu)}\|\nabla u^\theta\|_{L^2(B_\mu)}dt\\
&+\frac{C}{\mu^{\frac12}}\int_{-\mu^2}^0\|\tilde{u}\|_{L^2(B_\mu)})\|u^\theta\|_{L^2(B_\mu)}\|\nabla u^\theta\|_{L^2(B_\mu)}dt\\
&\equiv I_1+I_2.
\end{aligned}
\tag{3.10}
$$

Furthermore,

$$\begin{aligned}
I_1 &\le C(\int_{-\mu^2}^0 \|\tilde{u}\|_{L^2(B_\mu}^2 \|u^\theta\|_{L^2(B_\mu)}^4 dt)^{\frac{1}{4}} (\int_{-\mu^2}^0 \|\nabla u^\theta\|_{L^2(B_\mu)}^2)^{\frac{1}{2}} (\int_{-\mu^2}^0 \|\nabla \tilde{u}\|_{L^2(B_\mu)}^2)^{\frac{1}{4}} \\
&\le C(\int_{-\mu^2}^0 \|\nabla u^\theta\|_{L^2(B_\mu)}^2 dt)^{\frac{1}{2}} (\int_{-\mu^2}^0 \|\nabla \tilde{u}\|_{L^2(B_\mu)}^2 dt)^{\frac{1}{4}} (\int_{-\mu^2}^0 \|\tilde{u}\|_{L^2(B_\mu)}^2 dt)^{\frac{1}{4}} \\
&\cdot \sup_{-\mu^2 \le t < 0} (\int_{B_\mu} |u^\theta|^2 dx)^{\frac{1}{2}} \\
&\le C\mu^2 (E_2^\theta)^{\frac{1}{2}}(\mu)(\tilde{E}_2)^{\frac{1}{4}}(\mu)(E_1^\theta)^{\frac{1}{2}}(\mu)(\tilde{E})^{\frac{1}{4}}(\mu).
\end{aligned}$$

And similarly, it holds that

$$\begin{aligned}
I_2 &\le \frac{C}{\mu^{\frac{1}{2}}} (\int_{-\mu^2}^0 (\|\tilde{u}\|_{L^2(B_\mu)}) \|u^\theta\|_{L^2(B_\mu)})^2 dt)^{\frac{1}{2}} (\int_{-\mu^2}^0 \|\nabla u^\theta\|_{L^2(B_\mu)}^2 dt)^{\frac{1}{2}} \\
&\le C\mu^2 (E_2^\theta)^{\frac{1}{2}}(\mu)(\tilde{E})^{\frac{1}{2}}(\mu)(E_1^\theta)^{\frac{1}{2}}(\mu).
\end{aligned}$$

Consequently (3.9) is obtained by putting I_1 and I_2 into (3.10). The proof of the Lemma is finished.

Next, the last term on the right hand of (3.8) can be estimated as follows.

Lemma 3.5 Suppose that $\|ru^\theta\|_{L^\infty} \le C$. Then for $0 < \mu \le \frac{\rho}{2}$, we have

$$\frac{1}{\mu} \int\int_{P_\mu} |\frac{(u^\theta)^2}{r} u^r| dxdt \le C(\frac{\rho}{\mu})(F^\theta(\rho))^{\frac{1}{2}}(\tilde{E}_2(\rho))^{\frac{1}{2}}. \qquad (3.11)$$

Proof. It is clear that

$$\begin{aligned}
&\int\int_{P_\mu} |\frac{(u^\theta)^2}{r} u^r| dxdt \\
&\le \|ru^\theta(t)\|_{L^\infty} (\int\int_{P_\mu} |\frac{u^\theta}{r}|^2 dxdt)^{\frac{1}{2}} (\int\int_{P_\mu} |\frac{u^r}{r}|^2 dxdt)^{\frac{1}{2}} \\
&\le C\mu (F^\theta(\mu))^{\frac{1}{2}} (\tilde{E}_2(\mu))^{\frac{1}{2}},
\end{aligned}$$

where (2.4) has been used in the second inequality. The lemma is then proved.

Finally we estimate $E(\mu)$ as follows.

Lemma 3.6 For $0 < \mu \le \rho/2$, we have

$$E(\mu) \le (\frac{\mu}{\rho})(E_1)^{\frac{1}{2}}(\rho)(E)^{\frac{1}{2}}(\rho) + C(\frac{\rho}{\mu})^{\frac{5}{2}}(E_1)^{\frac{1}{4}}(\rho)(E)^{\frac{1}{4}}(\rho)(E_2)^{\frac{1}{2}}(\rho). \qquad (3.12)$$

Or

$$E(\mu) \le C(\frac{\rho}{\mu})^2 (E_1)^{\frac{1}{2}}(\rho)(E_2)^{\frac{1}{2}}(\rho) + C(\frac{\mu}{\rho})^2 E_1(\rho). \qquad (3.13)$$

Here the notations E, E_1 and E_2 are same as in (2.12).

Proof. The proof of (3.12) can be found in [17] (p:244). Now we give the proof of (3.13).

It follows from Poincare's inequality that for almost all time,

$$\begin{aligned}\int_{B_\mu}|u|^2dx &= \int_{B_\mu}(|u|^2-\overline{(u)^2_\rho})dx+\int_{B_\mu}\overline{(u)^2_\rho}dx\\ &\le \int_{B_\rho}|\ |u|^2-\overline{(u)^2_\rho}\ |\ dx+\int_{B_\mu}\overline{(u)^2_\rho dx}\\ &\le C\rho\int_{B_\rho}|u||\nabla u|dx+C(\frac{\mu}{\rho})^3\int_{B_\rho}|u|^2dx.\end{aligned}$$

Thus

$$\int_{B_\mu}|u|^2dx\le C\rho^{\frac{3}{2}}E_1^{\frac{1}{2}}(\rho)(\int_{B_\rho}|\nabla u|^2dx)^{\frac{1}{2}}+C(\frac{\mu}{\rho})^3\rho E_1(\rho).$$

Integrating from $-\mu^2$ to 0 with respect to the time variable, and applying the Hölder inequality, we get

$$\begin{aligned}&\int_{-\mu^2}^0\int_{B_\mu}|u|^2dxdt\\ &\le C\rho^{\frac{3}{2}}E_1^{\frac{1}{2}}(\rho)\int_{-\mu^2}^0(\int_{B_\rho}|\nabla u|^2dx)^{\frac{1}{2}}dt+C(\frac{\mu}{\rho})^3\rho\mu^2E_1(\rho)\\ &\le C\rho^{\frac{3}{2}}E_1^{\frac{1}{2}}(\rho)\mu(\int_{-\mu^2}^0\int_{B_\rho}|\nabla u|^2dxdt)^{\frac{1}{2}}+C(\frac{\mu}{\rho})^3\rho\mu^2E_1(\rho),\end{aligned}$$

which yields (3.13) of the lemma.

Proof of Theorem 2.1 (I) Under the conditions of the theorem, one has from Lemma 3.1 that

$$\|ru^\theta(t)\|_{L^\infty}\le C.$$

And (ii) of Lemma 3.3 shows that

$$\tilde{E}_2(R)\le C, \tag{3.14}$$

for all $R\le R_0$ and some constant C. According to Proposition 2.1 (and its remark), we only need to prove that

$$E_2^\theta(R)<\infty, \tag{3.15}$$

for all $R\le R_1$ with $R_1<R_0$ a constant.

For any $\eta=\lambda\rho, 0<\lambda\le 1/4$ and $\eta_*=2\eta\le\frac{\rho}{2}$, in view of (3.8), one has

$$\begin{aligned}&E_1^\theta(\eta)+E_2^\theta(\eta)+F^\theta(\eta)\le CE^\theta(\eta_*)\\ &+\frac{C}{\eta_*^2}\int\int_{P_{\eta_*}}|(u^\theta)^2-\overline{(u^\theta)^2_{\eta_*}}||\tilde{u}|dxdt+\frac{C}{\eta_*}\int\int_{P_{\eta_*}}|\frac{(u^\theta)^2}{r}u^r|dxdt.\end{aligned}$$

Due to Lemma 3.4–Lemma 3.6, in which μ is replaced by η, one concludes that

$$\begin{aligned}&E_1^\theta(\eta)+E_2^\theta(\eta)+F^\theta(\eta)\\&\le C\lambda(E_1^\theta)^{\frac12}(\rho)(E^\theta)^{\frac12}(\rho)+C\lambda^{-\frac52}(E_1^\theta)^{\frac14}(\rho)(E^\theta)^{\frac14}(\rho)(E_2^\theta)^{\frac12}(\rho)\\&+C\lambda^{-2}(E_2^\theta)^{\frac12}(\rho)(\tilde E_2)^{\frac14}(\rho)(E_1^\theta)^{\frac12}(\rho)(\tilde E)^{\frac14}(\rho)\\&+C\lambda^{-\frac52}(E_2^\theta)^{\frac12}(\rho)(\tilde E)^{\frac12}(\rho)(E_1^\theta)^{\frac12}(\rho)\\&+C\lambda^{-1}(F^\theta(\rho))^{\frac12}(\tilde E_2(\rho))^{\frac12}.\end{aligned}\tag{3.16}$$

Let

$$\psi(\eta)=E_1^\theta(\eta)+E_2^\theta(\eta)+F^\theta(\eta).$$

Noticing that $E(\rho)\le C$ and $\tilde E_2(\rho)\le C$ (by (3.14)) for all $\rho\le R_0$ with C an absolute constant, one can employ Young's inequality to get the following iteration form

$$\psi(\eta)\le C\lambda\psi(\rho)+C(\lambda),$$

where $C(\lambda)$ denotes a constant depending on λ. Using the same iteration method as in Lemma 3.3, we obtain that there exists a constant $R_1<R_0$ such that

$$\psi(R)\le C$$

for all $R\le R_1$, where C is a constant. Therefore (3.15) is proved and the proof of part (I) of the theorem is finished.

To prove the second part of the theorem, we first recall the Hardy inequality (see [16], p:176):

$$\int_0^{+\infty}|\frac{\gamma(s)}{s}|^2\,ds\le 2\int_0^{+\infty}|\gamma'(s)|^2ds,\text{for}\qquad \gamma(s)\in C_0^\infty(0,+\infty).\tag{3.17}$$

Consequently,

Lemma 3.7 Suppose that $u(x)=u(r,x_3)\in H^1(R^3)$ is an axial function. Then

$$\|\frac{u}{r}\|^2_{L^2(R^3)}\le 4\|\partial_r u\|^2_{L^2(R^3)},\tag{3.18}$$

and for every $\mu>0$,

$$\|\frac{u}{r}\|^2_{L^2(B_\mu)}\le\frac{16}{\mu^2}\|u\|^2_{L^2(B_{2\mu})}+8\|\partial_r u\|^2_{L^2(B_{2\mu})}.\tag{3.19}$$

Proof. For $u \in C_0^\infty(R^3)$, we have from (3.17) that

$$\begin{aligned}&\int_0^{+\infty} \mid \frac{u(r,x_3)}{r} \mid^2 rdr \leq 2\int_0^{+\infty} \mid \partial_r(r^{\frac{1}{2}}u(r,x_3)) \mid^2 dr \\ &= \frac{1}{2}\int_0^{+\infty} r^{-1}|u|^2 dr + 2\int_0^{+\infty} u\partial_r u dr + 2\int_0^{+\infty} |\partial_r u|^2 rdr.\end{aligned} \tag{3.20}$$

So

$$\int_0^{+\infty} \mid \frac{u(r,x_3)}{r} \mid^2 rdr \leq 4\int_0^{+\infty} |\partial_r u|^2 rdr, \tag{3.21}$$

which implies (3.18). Moreover, for every $\mu > 0$, let $\zeta = \zeta(x) \in C_0^\infty(R^3)$ be a smooth function satisfying

$$\zeta(x) = 1, x \in B_\mu, \zeta(x) = 0, x \in R^3 \backslash B_{2\mu}, |\nabla\zeta| \leq 2/\mu. \tag{3.22}$$

Then it follows from (3.21) and (3.22) that

$$\begin{aligned}\int_0^\mu |\frac{u}{r}|^2 rdr &\leq \int_0^{+\infty} |\frac{\zeta^2 u}{r}|^2 rdr \\ &\leq 4\int_0^{+\infty} |\partial_r(\zeta^2 u|^2 rdr \\ &\leq \frac{16}{\mu^2}\int_0^{2\mu} |u|^2 rdr + 8\int_0^{2\mu} |\partial_r u|^2 rdr,\end{aligned}$$

which yields (3.19) immediately. The proof of the lemma is finished.

Lemma 3.8 Under the conditions (II) in Theorem 2.1, if $E^3(R) \leq C < +\infty$ for all $R \leq R_0$, then (2.14) holds.

Proof. Assume that the conditions (II) in Theorem 2.1 are satisfied. Applying (i) of Lemma 3.3, we obtain that, for any $\delta > 0$, there exists a positive number R_1 such that

$$\tilde{E}_2(R) \leq \delta, \quad R \leq R_1.$$

Due to (2) of Proposition 2.1, it suffices to prove

$$E_2^\theta(R) \leq \delta \tag{3.23}$$

for some sufficiently small $\delta > 0$ and for all $R \leq R_2 \leq R_1$.

Instead of (3.8), one can get from (2.10) that

$$\begin{aligned}&E_1^\theta(R) + E_2^\theta(R) + F^\theta(R) \leq CE^\theta(R_*) \\ &+\frac{C}{R_*^2}\int\int_{P_{R_*}} |u^\theta|(|\frac{u^r}{r}| + |\frac{u^3}{r}|)dxdt + \frac{C}{R_*}\int\int_{P_{R_*}} |\frac{(u^\theta)^2}{r}u^r|dxdt \\ &\equiv J_1 + J_2 + J_3\end{aligned} \tag{3.24}$$

for $R_* = 2R > 0$. The estimates of J_1 and J_3 are as same as before. To estimate the term J_2, we first use Hölder inequality to obtain

$$J_2 \le (E^\theta)^{\frac{1}{2}}(R_*) \left[(\frac{1}{R_*} \int\int_{P_{R_*}} |\frac{u^r}{r}|^2 dxdt)^{\frac{1}{2}} + (\frac{1}{R_*} \int\int_{P_{R_*}} |\frac{u^3}{r}|^2 dxdt)^{\frac{1}{2}} \right]. \tag{3.25}$$

Then, (2.4) yields

$$\frac{1}{R_*} \int\int_{P_{R_*}} |\frac{u^r}{r}|^2 dxdt \le \tilde{E}_2(R_*). \tag{3.26}$$

Thanks to (3.19) of Lemma 3.7, one may conclude that

$$\|\frac{u^3}{r}\|^2_{L^2(B_{R_*})} \le \frac{16}{{R_*}^2}\|u^3\|^2_{L^2(B_{2R_*})} + 8\|\partial_r u^3\|^2_{L^2(B_{2R_*})}.$$

Therefore,

$$\frac{1}{R_*} \int\int_{P_{R_*}} |\frac{u^3}{r}|^2 dxdt \le C(E^3(2R_*) + \tilde{E}_2(2R_*)). \tag{3.27}$$

Combining (3.25)-(3.27) with (3.24), one has

$$\begin{aligned} & E_1^\theta(R) + E_2^\theta(R) + F^\theta(R) \le J_1 + J_3 \\ & + C(E^\theta)^{\frac{1}{2}}(R_*) \left[(\tilde{E}_2)^{\frac{1}{2}}(R_*) + (E^3)^{\frac{1}{2}}(2R_*) + (\tilde{E}_2)^{\frac{1}{2}}(2R_*) \right] \\ & \le J_1 + J_3 + C(E^\theta(R_*))^{\frac{1}{2}} \end{aligned} \tag{3.28}$$

for all $2R_* \le R_0$, where one has used Lemma 3.3 and the assumption $E^3(R) \le C$ for $R \le R_0$. Then for any $\eta = \lambda\rho, 0 < \lambda \le 1/4$, using Lemma 3.5 and Lemma 3.6 we get an iteration form as follows,

$$\begin{aligned} & E_1^\theta(\eta) + E_2^\theta(\eta) + F^\theta(\eta) \\ & \le C\lambda(E_1^\theta)^{\frac{1}{2}}(\rho)(E^\theta)^{\frac{1}{2}}(\rho) + C\lambda^{-\frac{5}{2}}(E_1^\theta)^{\frac{1}{4}}(\rho)(E^\theta)^{\frac{1}{4}}(\rho)(E_2^\theta)^{\frac{1}{2}}(\rho) \\ & + C\lambda^{-1}(F^\theta(\rho))^{\frac{1}{2}}(\tilde{E}_2(\rho))^{\frac{1}{2}} \\ & + C\left[\lambda(E_1^\theta)^{\frac{1}{2}}(\rho)(E^\theta)^{\frac{1}{2}}(\rho) + C\lambda^{-\frac{5}{2}}(E_1^\theta)^{\frac{1}{4}}(\rho)(E^\theta)^{\frac{1}{4}}(\rho)(E_2^\theta)^{\frac{1}{2}}(\rho)\right]^{\frac{1}{2}} \end{aligned} \tag{3.29}$$

Let

$$\psi(\eta) = E_1^\theta(\eta) + E_2^\theta(\eta) + F^\theta(\eta).$$

Noticing that $E^\theta(\rho) \le \varepsilon$ and $\tilde{E}_2(\rho) \le \varepsilon$ (by Lemma 3.3 (i)) for all $\rho \le R_0$ and employing Young's inequality, we deduce

$$\psi(\eta) \le C\lambda\psi(\rho) + C(\lambda)\varepsilon,$$

where $C(\lambda)$ denotes a constant depending only on λ. Using the same iteration method as in (i) of Lemma 3.3, we obtain that for any $\delta > 0$, there exists a constant $R_1 < R_0$ such that

$$\psi(R) \leq \delta$$

for all $R \leq R_1$. The proof of the Lemma is complete.

Lemma 3.9 If $W^\theta(R) \leq C < +\infty$ for all $0 < R \leq R_0$, then $E^\theta(R)$ is bounded if and only if $E^3(R)$ is bounded, where $0 < R \leq \tilde{R}$ and $\tilde{R} \leq R_0$ is some constant.

Proof. As was proved before, it follows from $W^\theta(R) \leq C < +\infty$ for all $R \leq R_0$ that

$$\tilde{E}_2(R) \leq C < +\infty$$

for all $R \leq R_1 \leq R_0$. If $E^3(R) \leq C < +\infty$ for all $R \leq R_1$, then in view of (3.24)-(3.27), a similar argument as the proof of Lemma 3.8 gives

$$E_1^\theta(R) + E_2^\theta(R) + F^\theta(R) \leq C$$

for all $R \leq R_2 \leq R_1$ and with a constant C. Using (3.12) (Lemma 3.6), we get

$$E^\theta(R) \leq C < +\infty$$

for all $R \leq \tilde{R}$, where $\tilde{R} \leq R_2$ is some constant.

Conversely, if $W^\theta(R) \leq C < +\infty$, and $E^\theta(R) \leq C < +\infty$ for all $R \leq R_1$, we will prove

$$E^3(R) \leq C < +\infty \tag{3.30}$$

for all $R \leq \tilde{R}$.

From (2.9)-(2.11) one has

$$\begin{aligned}
&\int_{B_{R_*}} \phi|u|^2dx + \int\int_{P_{R_*}} |\nabla u|^2\phi dxdt + 2\int\int_{P_{R_*}} (\frac{u^r}{r})^2dxdt + 2\int\int_{P_{R_*}} (\frac{u^\theta}{r})^2dxdt\\
&\leq \frac{C}{R_*^2}\int\int_{P_{R_*}} |u|^2dxdt + \int\int_{P_{R_*}} [(u^r)^2+(u^3)^2](u^r\partial_r\phi + u^3\partial_3\phi)dxdt\\
&+\int\int_{P_{R_*}} (u^\theta)^2(u^r\partial_r\phi + u^3\partial_3\phi)dxdt + 2\int\int_{P_{R_*}} p(u^r\partial_r\phi + u^3\partial_3\phi)dxdt,
\end{aligned} \tag{3.31}$$

which implies that

$$\begin{aligned}
&E_1(R) + E_2(R) + F^\theta(R) + F^r(R) \leq CE(R_*) + \frac{C}{R_*^2}\int\int_{P_{R_*}} |(\tilde{u})^2 - \overline{(\tilde{u})^2_{R_*}}||\tilde{u}|dxdt\\
&+\frac{C}{R_*^2}\int\int_{P_{R_*}} (u^\theta)^2(|u^r|+|u^3|)dxdt + \frac{C}{R_*^2}\int\int_{P_{R_*}} |p||\tilde{u}|dxdt\\
&\equiv J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.32}$$

Now we first consider the term J_4. For any $\mu > 0$, it follows easily from the Hölder's inequality that

$$J_4(\mu) = \frac{1}{\mu^2}\int\int_{P_\mu} |p||\tilde{u}|dxdt \le \frac{1}{\mu^2}\int_{-\mu^2}^{0}\left(\int_{B_\mu}|\tilde{u}|^6 dx\right)^{\frac{1}{6}}\left(\int_{B_\mu}|p|^{\frac{6}{5}}dx\right)^{\frac{5}{6}} dt.$$

Using the interpolation inequality, we have

$$(\int_{B_\mu}|\tilde{u}|^6 dx)^{\frac{1}{6}} \le C(\int_{B_\mu}|\nabla\tilde{u}|^2 dx)^{\frac{1}{2}} + \frac{C}{\mu}(\int_{B_\mu}|\tilde{u}|^2 dx)^{\frac{1}{2}}.$$

Consequently,

$$\begin{aligned}
J_4(\mu) \ &\le \frac{C}{\mu^2}\int_{-\mu^2}^{0}\left(\int_{B_\mu}|\nabla\tilde{u}|^2 dx\right)^{\frac{1}{2}}\left(\int_{B_\mu}|p|^{\frac{6}{5}}dx\right)^{\frac{5}{6}} dt \\
&+\frac{C}{\mu^3}\int_{-\mu^2}^{0}\left(\int_{B_\mu}|\tilde{u}|^2 dx\right)^{\frac{1}{2}}\left(\int_{B_\mu}|p|^{\frac{6}{5}}dx\right)^{\frac{5}{6}} dt \\
&\le \frac{C}{\mu^2}\left(\int_{-\mu^2}^{0}\int_{B_\mu}|\nabla\tilde{u}|^2 dxdt\right)^{\frac{1}{2}}\left(\int_{-\mu^2}^{0}(\int_{B_\mu}|p|^{\frac{6}{5}}dx)^{\frac{5}{3}}dt\right)^{\frac{1}{2}} \\
&+(\tilde{E}_1)^{\frac{1}{2}}(\mu)\cdot\frac{C}{\mu^{5/2}}\int_{-\mu^2}^{0}\left(\int_{B_\mu}|p|^{\frac{6}{5}}dx\right)^{\frac{5}{6}} dt \\
&= (\tilde{E}_2)^{\frac{1}{2}}(\mu)\cdot\frac{C}{\mu^{3/2}}\left(\int_{-\mu^2}^{0}(\int_{B_\mu}|p|^{\frac{6}{5}}dxt)^{\frac{5}{3}}dt\right)^{\frac{1}{2}} \\
&+(\tilde{E}_1)^{\frac{1}{2}}(\mu)\cdot\frac{C}{\mu^{5/2}}\int_{-\mu^2}^{0}\left(\int_{B_\mu}|p|^{\frac{6}{5}}dx\right)^{\frac{5}{6}} dt.
\end{aligned} \tag{3.33}$$

Next we estimate the pressure. It is known that

$$\Delta p = -\text{div }(u\cdot\nabla)u \qquad \text{in} \quad B_\mu,$$

from which one can obtain the representation

$$p(x,t) = \int_{B_\mu}\nabla_x\Gamma(x-y)(u\cdot\nabla)u(y)dy + p_0(x,t),$$

where $p_0(x,t)$ is a harmonic function in B_μ for a.e. t. Note that in the axisymmetric case,

$$\begin{aligned}
(u\cdot\nabla)u(y) \ &= (u^r\partial_r + \frac{1}{r}u^\theta\partial_\theta + u^3\partial_3)\cdot(u^r e_r + u^\theta e_\theta + u^3 e_3) \\
&= (u^r\partial_r u^r)e_r + (u^r\partial_r u^\theta)e_\theta + (u^r\partial_r u^3)e_3 - (\frac{1}{r}u^\theta u^r)e_\theta - (\frac{1}{r}u^\theta u^\theta)e_r \\
&+(u^3\partial_3 u^r)e_r + (u^3\partial_3 u^\theta)e_\theta + (u^3\partial_3 u^3)e_3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
p(x,t) &= \int_{B_\mu} \nabla_x\Gamma(x-y)[(u^r\partial_r u^r)e_r + (u^r\partial_r u^3)e_3 - (\frac{1}{r}u^\theta u^r)e_\theta - (\frac{1}{r}u^\theta u^\theta)e_r \\
&+(u^3\partial_3 u^r)e_r + (u^3\partial_3 u^3)e_3]dy + \int_{B_\mu} \nabla_x\Gamma(x-y)[(u^r\partial_r u^\theta)e_\theta \\
&+(u^3\partial_3 u^\theta)e_\theta]dy + p_0(x,t) \\
&\le |\int_{B_\mu} \nabla_x\Gamma(x-y)[(u^r\partial_r u^r)e_r + (u^r\partial_r u^3)e_3 - (\frac{1}{r}u^\theta u^r)e_\theta - (\frac{1}{r}u^\theta u^\theta)e_r \\
&+(u^3\partial_3 u^r)e_r + (u^3\partial_3 u^3)e_3]dy| + |\int_{B_\mu}[\partial_r\nabla_x\Gamma(x-y)(u^r u^\theta)e_\theta \\
&+\partial_3\nabla_x\Gamma(x-y)(u^3 u^\theta)e_\theta]dy| + |(u^r u^\theta + u^3 u^\theta)| + |H_0(x,t)| \\
&\equiv H_1(x,t) + H_2(x,t) + H_3(x,t) + |H_0(x,t)|,
\end{aligned} \tag{3.34}$$

where $H_0(x,t)$ is a harmonic function in B_μ for *a.e.* t and the integrals above are in the sense of the Cauchy principle. It follows from Young's inequality that

$$\begin{aligned}
&\|\int_{B_\mu} \nabla_x\Gamma(x-y)(u^r\partial_r u^r)e_r dx\|_{L^{\frac{6}{5}}(B_\mu)} \\
&\le C\mu^{\frac{1}{2}}\|u^r\partial_r u^r\|_{L^1(B_\mu)} \\
&\le C\mu^{\frac{1}{2}}\|u^r\|_{L^2(B_\mu)}\|\partial_r u^r\|_{L^2(B_\mu)}.
\end{aligned}$$

Other terms in $H_1(x,t)$ can be estimated similarly. Thus we get

$$\begin{aligned}
\|H_1(\cdot,t)\|_{L^{\frac{6}{5}}(B_\mu)} &\le C\mu^{\frac{1}{2}}(\|\tilde{u}\|_{L^2(B_\mu)}\|\tilde{\nabla}\tilde{u}\|_{L^2(B_\mu)} + \|\frac{u^\theta}{r}\|_{L^2(B_\mu)}\|u^r\|_{L^2(B_\mu)} \\
&+\|\frac{u^\theta}{r}\|_{L^2(B_\mu)}\|u^\theta\|_{L^2(B_\mu)}).
\end{aligned}$$

Consequently,

$$\int_{-\mu^2}^0 \|H_1(\cdot,t)\|_{L^{\frac{6}{5}}(B_\mu)}dt \le C\mu^{\frac{5}{2}}(\tilde{E})^{\frac{1}{2}}(\mu)(\tilde{E}_2)^{\frac{1}{2}}(\mu) + C\mu^{\frac{5}{2}}(F^\theta)^{\frac{1}{2}}(\mu)[(E^r)^{\frac{1}{2}}(\mu) + (E^\theta)^{\frac{1}{2}}(\mu)].$$

Furthermore, we have

$$\begin{aligned}
&\int_{-\mu^2}^0 \|H_1(\cdot,t)\|^2_{L^{\frac{6}{5}}(B_\mu)}dt \\
&\le C\mu\int_{-u^2}^0[\|\tilde{u}\|^2_{L^2(B_\mu)}\|\tilde{\nabla}\tilde{u}\|^2_{L^2(B_\mu)} + \|\frac{u^\theta}{r}\|^2_{L^2(B_\mu)}\|u^r\|^2_{L^2(B_\mu)} \\
&+\|\frac{u^\theta}{r}\|^2_{L^2(B_\mu)}\|u^\theta\|^2_{L^2(B_\mu)})]dt \\
&\le C\mu^2\tilde{E}_1(\mu)\int_{-\mu^2}^0\int_{B_\mu}|\tilde{\nabla}\tilde{u}|^2 dxdt + C\mu^2(E_1^r(\mu) + E_1^\theta(\mu))\int_{-\mu^2}^0\int_{B_\mu}|\frac{u^\theta}{r}|^2 dxdt,
\end{aligned}$$

and

$$(\int_{-\mu^2}^0 \|H_1(\cdot,t)\|^2_{L^{\frac{6}{5}}(B_\mu)}dt)^{\frac{1}{2}} \leq C\mu^{\frac{3}{2}}(\tilde{E}_1)^{\frac{1}{2}}(\mu)(\tilde{E}_2)^{\frac{1}{2}}(\mu)+C\mu^{\frac{3}{2}}[(E_1^r)^{\frac{1}{2}}(\mu)+(E_1^r)^{\frac{1}{2}}(\mu)](F^\theta)^{\frac{1}{2}}(\mu).$$

By using the classical Calderon-Zygmund singular integral estimate and the interpolation inequality, we get

$$\begin{aligned}
&\|H_2(\cdot,t)\|_{L^{\frac{6}{5}}(B_\mu)} \leq C(\int_{B_\mu}|\tilde{u}|^{\frac{6}{5}}|u^\theta|^{\frac{6}{5}}dx)^{\frac{5}{6}} = C\|\tilde{u}u^\theta\|_{L^{\frac{6}{5}}(B_\mu)} \leq C\|u^\theta\|_{L^2(B_\mu)}\|\tilde{u}\|_{L^3(B_\mu)}\\
&\leq C\|u^\theta\|_{L^2(B_\mu)}\left[(\int_{B_\mu}|\nabla\tilde{u}|^2dx)^{\frac{1}{4}}(\int_{B_\mu}|\tilde{u}|^2dx)^{\frac{1}{4}}+\frac{1}{\mu^{1/2}}(\int_{B_\mu}|\tilde{u}|^2dx)^{\frac{1}{2}}\right].
\end{aligned}$$

So,

$$\int_{-\mu^2}^0 \|H_2(\cdot,t)\|_{L^{\frac{6}{5}}(B_\mu)}dt \leq C\mu^{\frac{5}{2}}(E^\theta)^{\frac{1}{2}}(\mu)(\tilde{E}_2)^{\frac{1}{4}}(\mu)(\tilde{E})^{\frac{1}{4}}(\mu)+C\mu^{\frac{5}{2}}(E^\theta)^{\frac{1}{2}}(\mu)(\tilde{E})^{\frac{1}{2}}(\mu).$$

$$(\int_{-\mu^2}^0 \|H_2(\cdot,t)\|^2_{L^{\frac{6}{5}}(B_\mu)}dt)^{\frac{1}{2}} \leq C\mu^{\frac{3}{2}}(E_1^\theta)^{\frac{1}{2}}(\mu)[(\tilde{E}_2)^{\frac{1}{4}}(\mu)(\tilde{E})^{\frac{1}{4}}(\mu)+(\tilde{E})^{\frac{1}{2}}(\mu).$$

Moreover, one can estimate $H_3(x,t)$ similarly to get

$$\|H_3(\cdot,t)\|_{L^{\frac{6}{5}}(B_\mu)} \leq C\|u^\theta\|_{L^2(B_\mu)}[(\int_{B_\mu}|\nabla\tilde{u}|^2dx)^{\frac{1}{4}}(\int_{B_\mu}|\tilde{u}|^2dx)^{\frac{1}{4}}+\frac{1}{\mu^{1/2}}(\int_{B_\mu}|\tilde{u}|^2dx)^{\frac{1}{2}}],$$

$$\int_{-\mu^2}^0 \|H_3(\cdot,t)\|_{L^{\frac{6}{5}}(B_\mu)}dt \leq C\mu^{\frac{5}{2}}(E^\theta)^{\frac{1}{2}}(\mu)(\tilde{E}_2)^{\frac{1}{4}}(\mu)(\tilde{E})^{\frac{1}{4}}(\mu)+C\mu^{\frac{5}{2}}(E^\theta)^{\frac{1}{2}}(\mu)(\tilde{E})^{\frac{1}{2}}(\mu),$$

and

$$(\int_{-\mu^2}^0 \|H_3(\cdot,t)\|^2_{L^{\frac{6}{5}}(B_\mu)}dt)^{\frac{1}{2}} \leq C\mu^{\frac{3}{2}}(E_1^\theta)^{\frac{1}{2}}(\mu)[(\tilde{E}_2)^{\frac{1}{4}}(\mu)(\tilde{E})^{\frac{1}{4}}(\mu)+(\tilde{E})^{\frac{1}{2}}(\mu)].$$

Set

$$H(x,t) = H_1(x,t)+H_2(x,t)+H_3(x,t).$$

Then collecting all the estimates above, we arrive at

$$\begin{aligned}
\int_{-\mu^2}^0 \|H(\cdot,t)\|_{L^{\frac{6}{5}}(B_\mu)}dt &\leq C\mu^{\frac{5}{2}}[(\tilde{E})^{\frac{1}{2}}(\mu)(\tilde{E}_2)^{\frac{1}{2}}(\mu)+(F^\theta)^{\frac{1}{2}}(E^\theta)^{\frac{1}{2}}(\mu)\\
&+(F^\theta)^{\frac{1}{2}}(E^r)^{\frac{1}{2}}(\mu)+(E^\theta)^{\frac{1}{2}}(\mu)(\tilde{E})^{\frac{1}{4}}(\mu)(\tilde{E}_2)^{\frac{1}{4}}(\mu)\\
&+(E^\theta)^{\frac{1}{2}}(\mu)(\tilde{E})^{\frac{1}{2}}(\mu)],
\end{aligned}$$

and

$$\begin{aligned}
(\int_{-\mu^2}^0 \|H(\cdot,t)\|^2_{L^{\frac{6}{5}}(B_\mu)}dt)^{\frac{1}{2}} &\leq C\mu^{\frac{3}{2}}[(\tilde{E}_1)^{\frac{1}{2}}(\mu)(\tilde{E}_2)^{\frac{1}{2}}(\mu)+(F^\theta)^{\frac{1}{2}}(\mu)(E_1^\theta)^{\frac{1}{2}}(\mu)\\
&+(F^\theta)^{\frac{1}{2}}(\mu)(E_1^r)^{\frac{1}{2}}(\mu)+(E_1^\theta)^{\frac{1}{2}}(\mu)(\tilde{E})^{\frac{1}{4}}(\mu)(\tilde{E}_2)^{\frac{1}{4}}(\mu)\\
&+(E_1^\theta)^{\frac{1}{2}}(\mu)(\tilde{E})^{\frac{1}{2}}(\mu)].
\end{aligned}$$

Finally, since $H_0(x,t)$ is a harmonic function, the mean value property of a harmonic function gives

$$\begin{aligned}\|H_0(x,t)\|_{L^{\frac{6}{5}}(B_\mu)} &\le C(\frac{\mu}{\rho})^3\|H_0(x,t)\|_{L^{\frac{6}{5}}(B_\rho)}\\ &\le C(\frac{\mu}{\rho})^3\|p(x,t)\|_{L^{\frac{6}{5}}(B_\rho)}+C(\frac{\mu}{\rho})^3\|H(x,t)\|_{L^{\frac{6}{5}}(B_\rho)}\end{aligned}$$

for any $0<\mu\le\rho$. Consequently,

$$\begin{aligned}&\int_{-\mu^2}^0\|H_0(\cdot,t)\|_{L^{\frac{6}{5}}(B_\mu)}dt\\ &\le C(\frac{\mu}{\rho})^3\int_{-\mu^2}^0\|p(\cdot,t)\|_{L^{\frac{6}{5}}(B_\rho)}dt+C(\frac{\mu}{\rho})^3\int_{-\mu^2}^0\|H(\cdot,t)\|_{L^{\frac{6}{5}}(B_\rho)}dt\\ &\le C(\frac{\mu}{\rho})^3\int_{-\mu^2}^0\|p(\cdot,t)\|_{L^{\frac{6}{5}}(B_\rho)}dt+C(\frac{\mu}{\rho})^3\rho^{\frac{5}{2}}[(\tilde{E})^{\frac{1}{2}}(\rho)(\tilde{E}_2)^{\frac{1}{2}}(\rho)+(F^\theta)^{\frac{1}{2}}(\rho)(E^r)^{\frac{1}{2}}(\rho)\\ &+(F^\theta)^{\frac{1}{2}}(\rho)(E^\theta)^{\frac{1}{2}}(\rho)+(E^\theta)^{\frac{1}{2}}(\rho)(\tilde{E}_2)^{\frac{1}{4}}(\rho)(\tilde{E})^{\frac{1}{4}}(\rho)+(E^\theta)^{\frac{1}{2}}(\rho)(\tilde{E})^{\frac{1}{2}}(\rho)],\end{aligned}$$

and thus,

$$\begin{aligned}&(\int_{-\mu^2}^0\|H_0(\cdot,t)\|^2_{L^{\frac{6}{5}}(B_\nu)}dt)^{\frac{1}{2}}\\ &\le C(\frac{\mu}{\rho})^3(\int_{-\mu^2}^0\|p(\cdot,t)\|^2_{L^{\frac{6}{5}}(B_\rho)}dt)^{\frac{1}{2}}+C(\frac{\mu}{\rho})^3(\int_{-\mu^2}^0\|H(\cdot,t)\|^2_{L^{\frac{6}{5}}(B_\rho)}dt)^{\frac{1}{2}}\\ &\le C(\frac{\mu}{\rho})^3(\int_{-\mu^2}^0\|p(\cdot,t)\|^2_{L^{\frac{6}{5}}(B_\rho)}dt)^{\frac{1}{2}}+C(\frac{\mu}{\rho})^3\rho^{\frac{3}{2}}[(\tilde{E}_1)^{\frac{1}{2}}(\rho)(\tilde{E}_2{}^{\frac{1}{2}}(\rho)\\ &+(F^\theta)^{\frac{1}{2}}(\rho)(E_1^r)^{\frac{1}{2}}(\rho)+(F^\theta)^{\frac{1}{2}}(\rho)(E_1^\theta)^{\frac{1}{2}}(\rho)+(E_1^\theta)^{\frac{1}{2}}(\rho)(\tilde{E}_2)^{\frac{1}{4}}(\rho)(\tilde{E})^{\frac{1}{4}}(\rho)\\ &+(E_1^\theta)^{\frac{1}{2}}(\rho)(\tilde{E})^{\frac{1}{2}}(\rho)].\end{aligned}$$

Denote

$$\begin{aligned}P_1(\mu)&=\frac{1}{\mu^{5/2}}\int_{-\mu^2}^0(\int_{B_\mu}|p|^{\frac{6}{5}}dx)^{\frac{5}{6}}dt,\\ P_2(\mu)&=(\frac{1}{\mu^3}\int_{-\mu^2}^0(\int_{B_\mu}|p|^{\frac{6}{5}}dx)^{\frac{5}{3}}dt)^{\frac{1}{2}}.\end{aligned}$$

Then

$$\begin{aligned}\mu^{\frac{5}{2}}P_1(\mu) &\le \int_{-\mu^2}^0\|H(\cdot,t\|_{L^{\frac{6}{5}}(B_\mu)}dt+\int_{-\mu^2}^0\|H_0(\cdot,t\|_{L^{\frac{6}{5}}(B_\mu)}dt\\ &\le C(\frac{\mu}{\rho})^3\int_{-\mu^2}^0\|p(\cdot,t)\|_{L^{\frac{6}{5}}(B_\rho)}dt+C\mu^{\frac{5}{2}}[(\tilde{E})^{\frac{1}{2}}(\mu)(\tilde{E}_2)^{\frac{1}{2}}(\mu)\\ &+(F^\theta)^{\frac{1}{2}}(\mu)(E^\theta)^{\frac{1}{2}}(\mu)+(F^\theta)^{\frac{1}{2}}(\mu)(E^r)^{\frac{1}{2}}(\mu)+(E^\theta)^{\frac{1}{2}}(\mu)(\tilde{E})^{\frac{1}{4}}(\mu)(\tilde{E}_2)^{\frac{1}{4}}(\mu)\\ &+(E^\theta)^{\frac{1}{2}}(\mu)(\tilde{E})^{\frac{1}{2}}(\mu)]+C(\frac{\mu}{\rho})^3\rho^{\frac{5}{2}}[(\tilde{E})^{\frac{1}{2}}(\rho)(\tilde{E}_2)^{\frac{1}{2}}(\rho)+(F^\theta)^{\frac{1}{2}}(\rho)(E^r)^{\frac{1}{2}}(\rho)\\ &+(F^\theta)^{\frac{1}{2}}(\rho)(E^\theta)^{\frac{1}{2}}(\rho)+(E^\theta)^{\frac{1}{2}}(\rho)(\tilde{E}_2)^{\frac{1}{4}}(\rho)(\tilde{E})^{\frac{1}{4}}(\rho)+(E^\theta)^{\frac{1}{2}}(\rho)(\tilde{E})^{\frac{1}{2}}(\rho)].\end{aligned}$$

So

$$\begin{aligned}P_1(\mu) &\le C(\tfrac{\mu}{\rho})^{\frac12}P_1(\rho)+C[(\tilde E)^{\frac12}(\mu)(\tilde E_2)^{\frac12}(\mu)+(F^\theta)^{\frac12}(E^\theta)^{\frac12}(\mu)\\ &+(F^\theta)^{\frac12}(E^r)^{\frac12}(\mu)+(E^\theta)^{\frac12}(\mu)(\tilde E)^{\frac14}(\mu)(\tilde E_2)^{\frac14}(\mu)\\ &+(E^\theta)^{\frac12}(\mu)(\tilde E)^{\frac12}(\mu)]+C(\tfrac{\mu}{\rho})^{\frac12}[(\tilde E)^{\frac12}(\rho)(\tilde E_2)^{\frac12}(\rho)+(F^\theta)^{\frac12}(\rho)(E^r)^{\frac12}(\rho)\\ &+(F^\theta)^{\frac12}(\rho)(E^\theta)^{\frac12}(\rho)+(E^\theta)^{\frac12}(\rho)(\tilde E_2)^{\frac14}(\rho)(\tilde E)^{\frac14}(\rho)+(E^\theta)^{\frac12}(\rho)(\tilde E)^{\frac12}(\rho)].\end{aligned}$$

Similarly, one can derive that

$$\begin{aligned}\mu^{\frac32}P_2(\mu) &= (\int_{-\mu^2}^0(\int_{B_\mu}|p|^{\frac65}dx)^{\frac53}dt)^{\frac12}\\ &\le C(\textstyle\int_{-\mu^2}^0\|H(\cdot,t)\|^2_{L^{\frac65}(B_\mu)}dt)^{\frac12}+(\int_{-\mu^2}^0\|H_0(\cdot,t)\|^2_{L^{\frac65}(B_\nu)}dt)^{\frac12}\\ &\le C\mu^{\frac32}[(\tilde E_1)^{\frac12}(\mu)(\tilde E_2)^{\frac12}(\mu)+(F^\theta)^{\frac12}(\mu)(E_1^\theta)^{\frac12}(\mu)+(F^\theta)^{\frac12}(\mu)(E_1^r)^{\frac12}(\mu)\\ &+(E_1^\theta)^{\frac12}(\mu)(\tilde E)^{\frac14}(\mu)(\tilde E_2)^{\frac14}(\mu)+(E_1^\theta)^{\frac12}(\mu)(\tilde E)^{\frac12}(\mu)]\\ &\le C(\frac{\mu}{\rho})^3(\int_{-\mu^2}^0\|p(\cdot,t)\|^2_{L^{\frac65}(B_\rho)}dt)^{\frac12}+C(\frac{\mu}{\rho})^3\rho^{\frac32}[(\tilde E_1)^{\frac12}(\rho)(\tilde E_2{}^{\frac12}(\rho)\\ &+(F^\theta)^{\frac12}(\rho)(E_1^r)^{\frac12}(\rho)+(F^\theta)^{\frac12}(\rho)(E_1^\theta)^{\frac12}(\rho)+(E_1^\theta)^{\frac12}(\rho)(\tilde E_2)^{\frac14}(\rho)(\tilde E)^{\frac14}(\rho)\\ &+(E_1^\theta)^{\frac12}(\rho)(\tilde E)^{\frac12}(\rho)],\end{aligned}$$

and

$$\begin{aligned}P_2(\mu) &\le C(\frac{\mu}{\rho})^{\frac32}P_2(\rho)+C[(\tilde E_1)^{\frac12}(\mu)(\tilde E_2)^{\frac12}(\mu)+(F^\theta)^{\frac12}(\mu)(E_1^\theta)^{\frac12}(\mu)+(F^\theta)^{\frac12}(\mu)(E_1^r)^{\frac12}(\mu)\\ &+(E_1^\theta)^{\frac12}(\mu)(\tilde E)^{\frac14}(\mu)(\tilde E_2)^{\frac14}(\mu)+(E_1^\theta)^{\frac12}(\mu)(\tilde E)^{\frac12}(\mu)]\\ &+C(\frac{\mu}{\rho})^{\frac32}[(\tilde E_1)^{\frac12}(\rho)(\tilde E_2{}^{\frac12}(\rho)\\ &+(F^\theta)^{\frac12}(\rho)(E_1^r)^{\frac12}(\rho)+(F^\theta)^{\frac12}(\rho)(E_1^\theta)^{\frac12}(\rho)+(E_1^\theta)^{\frac12}(\rho)(\tilde E_2)^{\frac14}(\rho)(\tilde E)^{\frac14}(\rho)\\ &+(E_1^\theta)^{\frac12}(\rho)(\tilde E)^{\frac12}(\rho)].\end{aligned}$$

Note that (3.33) gives

$$J_4(\mu)\le C(\tilde E_2)^{\frac12}(\mu)P_2(\mu)+(\tilde E_1)^{\frac12}(\mu)P_1(\mu).$$

So the estimates on P_1 and P_2 above give an iteration on $J_4(\mu)$. Noticing that

$$E(\mu)=E^\theta(\mu)+E^r(\mu)+E^3(\mu),$$

and after employing (3.13) in Lemma 3.6, we get

$$\begin{aligned}J_1(\mu) &\le C(\tfrac{\rho}{\mu})^2(E_1^\theta)^{\frac12}(\rho)(E_2^\theta)^{\frac12}(\rho)+C(\tfrac{\mu}{\rho})^2E_1^\theta(\rho)\\ &+C(\tfrac{\rho}{\mu})^2(\tilde E_1)^{\frac12}(\rho)(\tilde E_2)^{\frac12}(\rho)+C(\tfrac{\mu}{\rho})^2\tilde E_1(\rho).\end{aligned}$$

Moreover, a similar argument as for Lemma 3.4 leads to the estimate of J_2 as follows

$$\begin{aligned} J_2(\mu) &\leq C(\tfrac{\rho}{\mu})^2(\tilde{E}_2)^{\frac{1}{2}}(\rho)(\tilde{E}_2)^{\frac{1}{4}}(\rho)(\tilde{E}_1)^{\frac{1}{2}}(\rho)(\tilde{E})^{\frac{1}{4}}(\rho) \\ &+C(\frac{\rho}{\mu})^{\frac{5}{2}}(\tilde{E}_2)^{\frac{1}{2}}(\rho)(\tilde{E})^{\frac{1}{2}}(\rho)(\tilde{E}_1)^{\frac{1}{2}}(\rho). \end{aligned}$$

It follows from (3.25)-(3.27) that

$$J_3(\mu) \leq C(E^\theta)^{\frac{1}{2}}(2\mu)\left[(\tilde{E}_2)^{\frac{1}{2}}(\mu) + (E^3)^{\frac{1}{2}}(2\mu) + (\tilde{E}_2)^{\frac{1}{2}}(2\mu)\right].$$

Applying the above estimates on J_1, J_2, J_3 and J_4, and using Young's inequality, we eventually obtain an iteration form for $\Phi(\mu) = E_1(\mu) + E_2(\mu) + F^\theta(\mu) + F^r(\mu)$ as

$$\Phi(\mu) \leq \lambda\Phi(\rho) + C(\lambda),$$

where $\mu = \lambda\rho, 0 < \lambda \leq 1/4$, and $C(\lambda)$ is a constant depending on λ. A similar proof as for Lemma 3.3 gives

$$\Phi(R) \leq C$$

for all $R \leq \tilde{R}$. Here $\tilde{R} \leq R_0$ and C are some constants. Thus, (3.30) is obtained. And the proof of the lemma is finished.

Proof of Theorem 2.1 (II) Now this becomes a clear consequence of Lemma 3.8 and Lemma 3.9, and so we finish the proof of the theorem.

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