Existence of Positive Solution of a Class of Semi-linear Sub-elliptic Equation in the Entire Space $I\!H^n$

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Abstract In this paper, we study the following problem

$$\begin{cases} \Delta_{H^n} u - u + u^p = 0 & in H^n \\ u > 0 & in H^n \\ u(x) \to 0 & \rho(x) \to \infty \end{cases}$$

where $1 < p \leq \frac{Q+2}{Q-2}$, Q is the homogeneous dimension of Heisenberg group H^n. Our main result is that this problem have at least one positive solution. For subcritical exponent case $1 < p < \frac{Q+2}{Q-2}$, we give two methods to prove this. For the critical exponent case $p = \frac{Q+2}{Q-2}$, we first give a Lion's type concentration-compact Lemma in the Heisenberg group, then as an application of this Lemma, we use it to prove our main theorem.

Key Words and Phrases: Semilinear subelliptic equation, Heisenberg group.
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1 Introduction

Let H^n be the Heisenberg group, where $\Delta_{H^n} = \sum_{i=1}^{n}(X_i^2 + Y_i^2)$ is its subelliptic Laplacian operator, $\rho(x)$ is the distance function from x to the point 0. Under the real coordinate $(x_1, \cdots, x_n, y_1, \cdots, y_n, t)$, the vector field X_i and Y_i are defined by

$$\begin{matrix} X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \\ Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t} \end{matrix} \quad i = 1, 2, \cdots, n.$$

and the distance function $\rho(x)$ is defined by

$$\rho(x) = (\sum_{i=1}^{n}(x_i^2 + y_i^2)^2 + t^2)^{\frac{1}{4}}.$$

It is well known that $\{X_i, Y_i\}$ generate the real Lie algebra of Lie group H^n and

$$[X_i, Y_i] = 4\delta_{ij}\frac{\partial}{\partial t}, \quad i, j = 1, \cdots, n.$$

In this Lie group, there is a group of natural dilations defined by

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0$$

where $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$. With this group of dilations, the Lie group H^n is a two step stratified nilpotent Lie group of homogeneous dimension $Q = 2n+2$, and Δ_{H^n} is homogeneous partial differential operator of degree 2. In this paper, we deal with the existence of the positive solution to the following semi-linear subelliptic equation

$$\begin{cases} \Delta_{H^n} u - u + u^p = 0 & inH^n \\ u > 0 & inH^n \\ u(x) \to 0 & \rho(x) \to \infty \end{cases} \tag{1}$$

where $1 < p \leq \frac{Q+2}{Q-2}$

Equation (1) comes from the CR-Yamabe problem(see [14]) and has been studied by several authors(see [4], [10],[12] and the references therein). In the paper [12], they studied the problem

$$\begin{cases} \Delta_{H^n} u + u^p = 0 & inH^n \\ u > 0 & inH^n \end{cases}$$

and showed that if the problem's solution is cylindrical, then it must be 0. In the works [2] and [4], they have gotten some results on the existence of the boundary value problem of equation (1) on the bounded domain and unbounded domain with thin condition, and $1 < p < \frac{Q+2}{Q-2}$. In these condition, the corresponding functional satisfies P.S condition, and the normal variational methods works. In the entire space $H^n, 1 < p \leq \frac{Q+2}{Q-2}$, we lost the compactness of Folland-Stein-Soblev embedding. The corresponding functional lost P.S condition. Their methods don't work. To our knowledge, in these situation, there exists no report of progress on this problem up to now.

On the Euclidean space, the similar problem was studied by many peoples(see [2], [3],[11], and the references therein). In [2], W-Y Ding and W-M Ni gave some beautiful results on the similar semilinear problem in Euclidean. But for our problem, as a consequence of [12], it may not have radical symmetry solution. So our problem is more subtle then them.

Our main result is the following theorem.

Theorem 1 For $1 < p \leq \frac{Q+2}{Q-2}$, the problem (1) has a solution $u \in E$.

To proof this theorem, we divided it by two case, that is the subcritical exponent case $1 < p < \frac{Q+2}{Q-2}$ and the critical exponent case $p = \frac{Q+2}{Q-2}$.

To begin proof our theorem, we first give some preliminary definition and Lemmas. For $u \in C_0^\infty(H^n)$ the C^∞ smooth funciotn with compact support, we define

$$\|u\|^2 = \int\limits_{R^n} |\nabla_H u|^2 + u^2 \tag{2}$$

where $\nabla_H = (\nabla_{X_1}, \cdots, \nabla_{X_2}, \nabla_{Y_1}, \cdots, \nabla_{Y_n})$. Then we define the Folland-Stein-Sobolev space by $E = \overline{C_0^\infty(H^n)}$, the compactness of $C_0^\infty(H^n)$ under the norm (2). This is a Hilbert space. For $\Omega \subset H^n$, the compactness of $C_0^\infty(\Omega)$ in E is denoted by $E(\Omega)$, it is a Hilbert space too. There space have embedding theorem like the Sobolev embedding.

Lemma 1.1 $\forall u \in E, 1 < q \leq \frac{2Q}{Q-2}$, we have

$$\|u\|_{L^q} \leq C\|u\| \tag{3}$$

where C is a constant independent of u.

Lemma 1.2 Let $\Omega \subset H^n$ be bounded smooth domain in H^n, the embedding

$$E(\Omega) \hookrightarrow L^p(\Omega), \quad 1 \leq p < \frac{2Q}{Q-2} \tag{4}$$

is compact.

For the subcritical exponent case, $1 < p < \frac{Q+2}{Q-2}$, we use two methods to solve the problem (1).

The first method:

In the Folland-Stein-Sobolev space E, we define the energy functional

$$J(u) = \frac{1}{2} \int\limits_{H^n} |\nabla_H u|^2 + u^2 - \frac{1}{p+1} \int u^{p+1}, \quad u \in E \tag{5}$$

Let B_k be the ball $B_k = \{x \in H^n \big| \rho(x) < k\}$. Denoted the completion $C_0^\infty(B_k)$ in E by E_k, then

$$E_k \subset E_{k+1} \subseteq E$$

$$E = \overline{\bigcup_{k=1}^{\infty} E_k}$$

Set $J_k = J\Big|_{E_k}$, find an element $u_0 \in E_1 \subseteq E_1 \subseteq \cdots \subseteq E_k \subseteq E$, such that

$$J(u_0) < 0, \quad J_k(u_0) < 0 \tag{6}$$

Let Γ, Γ_k defined by

$$\begin{aligned} \Gamma &= \{r : [0,1] \to E \big| r(0) = 0,\ r(1) = u_0,\ r \text{ is continuous}\} \\ \Gamma_k &= \{r : [0,1] \to E_k \big| r(0) = 0,\ r(1) = u_0,\ r \text{ is continuous}\} \end{aligned} \tag{7}$$

Define

$$\begin{aligned} c &= \min_{r\in\Gamma} \max_{0\le t\le 1} I(r(t)), \\ c_k &= \min_{r\in\Gamma_k} \max_{0\le t\le 1} I_k(r(t)). \end{aligned} \tag{8}$$

For $\Gamma_k \subseteq \Gamma_{k+1} \subset \Gamma$, we have

$$c_k \ge c_{k+1} \ge c > 0 \tag{9}$$

By mountain-path Lemma, we know c_k is a critical value of the functional I_k. Let u_k be a critical point of I_k corresponding the critical value, that is $I_k(u_k) = c_k$ and $I_k'(u_k) = 0$. By some complex estimates of u_k, we shall proof c is a critical value of I, and $u_k \to u$ in E, u is a critical point and $I(u) = c$. By the maximum principle we get a positive solution of (1).

the second method:

Define $M = \{u \in E \big| \int\limits_{H^n} |u|^{p+1} = 1\} \subset E$. On the manifold, we define

$$I(u) = \frac{1}{2} \int\limits_{H^n} |\nabla_H u|^2 + u^2 \tag{10}$$

The main idea is, for I have bounded from below, we define

$$c = \inf_{u\in M} I(u) \tag{11}$$

Then we prove that the critical c can be arrived by $u \in M$. Then by Lagrane multiplier method, we know the problem have a positive solution.

For the case $p = \frac{Q+2}{Q-2}$, it is more delicate then the subcritical exponent case. Since the Folland-Stein-Sobolev embedding $E(\Omega) \hookrightarrow L^{\frac{2Q}{Q-2}}$, even Ω is a bounded domain, lost compactness. This induce that the functional J_k lost P.S condition, the mountain path lemma does not work. So the first method to the case $1 < p < \frac{Q+2}{Q-2}$ does not work. But on the manifold $M = \{u \in E \big| \int\limits_{H^n} |u|^{p+1} = 1\}$, the functional $I(u) = \frac{1}{2} \int\limits_{H^n} |\nabla_H u|^2 + u^2$ is bounded below too, so we shall use the second method of the case $1 < p < \frac{Q+2}{Q-2}$ to solve the problem.

As we know, the embedding $E \hookrightarrow L^{\frac{2Q}{Q-2}}$ is the limit case of Folland-Stein-Sobolev embedding. So there are many interesting phenomena. And it is more complex than the case $1 < p < \frac{Q+2}{Q-2}$ to prove the minimum

$$c = \inf_{u \in M} I(u) \tag{12}$$

can be arrived in the manifold $u \in M$.

For the second method of subcritical case and critical case, to overcome the difficult that the functional I lost P.S condition, we give some Lion's version concentration-compactness Lemmas. This is one of bones in this work.

For the case $1 < p < \frac{Q+2}{Q-2}$, by our proof we know, for every smooth bounded domain Ω, the Dirichlet problem

$$\begin{cases} \varepsilon^2 \Delta_{H^n} u - u + u^p = 0 & in\ \Omega \\ u > 0 & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases} \tag{13}$$

have a least energy solution u_ε. Let $x_\varepsilon \in \Omega, u(x_\varepsilon) = \max_{x \in \Omega} u(x)$, we like J.Wei in the paper [13], we want to know what is the $\lim_{\varepsilon \to 0} \mathrm{dist}(x_\varepsilon, \partial\Omega)$. In one of our preparing works, we shall proof that

$$\mathrm{dist}(x_\varepsilon, \partial\Omega) \to \max_{x \in \Omega} d(x, \partial\Omega), \quad \varepsilon \to 0$$

and we shall publish this result elsewhere.

2 The subcritical exponent case, $1 < p < \frac{Q+2}{Q-2}$.

2.1 The first method

In this subsection, we shall use the mountain-path lemma and domain extension method to proof the Theorem in the subcritical exponent case $1 < p < \frac{Q+2}{Q-2}$. But more, we get that the problem have a least energy solution, and proof that

$$c = \inf_{r \in \Gamma} \max_{0 \le t \le 1} J(rt)$$

can be arrived by a path $r_0 \in \Gamma$. This is the foundation of our paper [1].

For the Folland-Stein-Sobolev embedding $E_k \hookrightarrow L^{p+1}, 1 < p < \frac{Q+2}{q-2}$ is compact [5], by the standard method we have the following lemma.

Lemma 2.1 For $k \in \boldsymbol{N}$, the functional J_k defined in the Hilbert E_k satisfies P.S condition.

For an element $e \in E_1 \subset E_k \subset E, \|e\| = 1, \forall k \in \boldsymbol{N}$, we have

$$J_k(te) = \frac{t^2}{2} - \frac{t^{p+1}}{p+1} \int |e|^{p+1} dx \tag{14}$$

For $p+1>2$, we have the following Lemma 2.2.

Lemma 2.2 There exists an element $u \in (\bigcap_{k=1}^{\infty} E_k) \cap E$, such that

$$I_k(u_0) < 0, \quad J(u_0) < 0 \tag{15}$$

For $\|u\| = 1$, we have

$$J(tu) = \frac{t^2}{2} - \frac{t^{p+1}}{p+1} \int_{H^n} |u|^{p+1} \tag{16}$$

By the Lemma 2.1, there is a positive constant $C > 0$ independent of u, such that

$$\int_{H^n} |u|^{p+1} \leq C \tag{17}$$

Combine the inequality (17) and the formula (16) we have

$$J(tu) \geq \frac{t^2}{2} - \frac{t^{p+1}}{p+1} C \tag{18}$$

Since $p+1>2$, we have the following Lemma 2.3.

Lemma 2.3 There is a neighborhood U_k of 0 respectively in E_k, and a neighborhood U of 0 in E, such that

$$J_k(u) \geq \alpha, \quad J(u) \geq \alpha \tag{19}$$

for all $u \in U_k$ or $u \in U$ respectively, where $\alpha > 0$ is a positive constant.

From mountain path lemma and the above Lemma, we have the following Lemma 2.4.

Lemma 2.4 The value c_k is a critical value of functional I_k, and more we have

$$c_k \geq c_{k+} \geq c > \alpha > 0 \tag{20}$$

Suppose u_k is a critical point of J_k corresponding the critical value c_k. Then we have

$$J'(u_k)u_k = \|u_k\|^2 - \int_{H^n} |u|^{p+1} \tag{21}$$

$$J(u_k) = \|u_k\|^2 - \frac{1}{p+1} \int_{H^n} |u_k|^{p+1} = c_k > \alpha \tag{22}$$

From (21) and (22), we have

$$c_1 \geq \frac{p}{p+1} \|u_k\|^2 = c_k > \alpha \tag{23}$$

That is to say $\{u_k\}$ is a bounded point set in E. So there is a subset of $\{u_k\}$, we still denote it by $\{u_k\}$, and a point $\overline{u} \in E$, such that

$$u_k \rightharpoonup \overline{u}, \tag{24}$$

and $\overline{u} \geq 0$ is a weak solution of

$$\Delta_{H^n} u - u + u^p = 0 \qquad in H^n.$$

By the method of Ding and Ni(see [2]), If we can prove $\overline{u} \not\equiv 0$, then $\overline{u}$ is a critical of functional J, and

$$J(\overline{u}) = c.$$

Then by maximum principal, we know $\overline{u}$ is a positive solution of problem (1), and it is a positive least energy solution of it. So if we can prove $\overline{u} \not\equiv 0$, our theorem is proved. Next we locus on this problem.

For $u_k \in E$ is a solution of

$$\Delta_{H^n} u - u + u^p = 0 \qquad in H^n,$$

we have

$$\int |\nabla_{H^n} u_k|^2 + u_k^2 - \int u_k^{p+1} = 0.$$

Then we have

$$\int u_k^2 (u_k^{p-1} - 1) = \int |\nabla_{H^n} u_k|^2 \geq 0.$$

Since $u_k \not\equiv 0$, there must be exists $\xi_k \in H^n$, such that

$$u_k(\xi_k) = \max_{H^n} u_k \geq 1 \tag{25}$$

We claim that $\{\xi_k\}$ is a bounded subset of H^n. This is our next lemma.

Lemma 2.5 The subset $\{\xi_k\}$ defined by (25) is a bounded subset of H^n.

Proof. For u_k is bounded subset of E, by some standard estimates and the Folland-Stein-Sobelev embedding theorem, there is a positive constant α, such that

$$\sup_{H^n} u_k \leq \alpha.$$

So there is a large enough $\beta > 0$ such that

$$-\Delta_H u_k + \beta u_k = u_k^p - (\beta - 1) u_k \leq 0 \tag{26}$$

Define function $v = ce^{-\delta\rho(x)}$, where c and δ are positive number which shall be determined.

For Δ_H is a 2 order operator. So $\triangle_H v$ is a -1 order function. Then there are large positive numbers R_0, and $\delta > 0$, such that for all $\xi, \rho(\xi) > R_0$, and large positive number β' such that

$$-\Delta_H v(\xi) + \beta' v(\xi) \geq 0. \tag{27}$$

Choose large positive number R_0, for all $\xi, \rho(\xi) = R_0$, we have

$$(v - u)(\xi) \geq 0 \tag{28}$$

Set $\beta'' = \max\{\beta, \beta'\},$, then by (26, 27, 28)we have

$$\begin{cases} -\Delta_{H^n}(v-u) + \beta''(v-u) \geq 0 & in H^n \backslash B_{R_0}(0), \\ v - u \geq 0, \quad on \partial(H^n \backslash B_{R_0}(0) \end{cases}$$

By the maximum principle, this implies that for all $\xi > R_0$, , for any k,

$$u_k \leq ce^{-\delta\rho(\xi)} \tag{29}$$

The inequality implies that ξ_k is bounded.

For the Folland-Stein-Sobolve spaces have similar embedding theorems with the Sobolev embedding and the Sub-Laplacin operator have similar characters with the Laplacin operator(see [5]), so by the method of Noussair, Ezzat S. and Swanson, Charles A(see [13]), we have the following lemma.

Lemma 2.6. There is a subsequence of u_k we still denote it by u_k, such that for any bounded domain Ω, $u_k \to \overline{u}$ in $C^{2+\alpha}(\Omega)$, where α is a positive number. That is $u_k \to \overline{u}$ in $C^{2+\alpha}_{loc}(H^n)$.

From Lemma 2.5 and Lemma 2.6, we have the following Lemma.

Lemma 2.7. The functional defined by (24)

$$\overline{u} \not\equiv 0.$$

Proof. For ξ_k is bounded, so we may assume that there is a $\xi_0 \in H^k$, such that $\xi_k \to \xi_0$. So we have

$$u_k(\xi_k) \to \overline{u}(\xi_0).$$

By the inequality (25), we have $u(\xi_0) \geq 1$. That is to say $u \not\equiv 0$. #

2.2 the second method

In this subsection, we shall use the constraint functional method to study the problem. First we defined the manifold

$$M = \{u \in E \Big| \int |u|^{p+1} dx = 1\} \tag{30}$$

On this manifold, define a functional

$$I(u) = \frac{1}{2} \int |\nabla_H u|^2 + u, \quad \forall u \in M \tag{31}$$

It is obviously that the functional I is bounded from below. We shall study whether the functional defined by (31) arrive its minimum on the manifold M. That is we want to find a $u_0 \in M$, such that

$$I(u_0) = \min_{u \in M} I(u) = \alpha \tag{32}$$

For the embedding $E \hookrightarrow L^{p+1}(H^n)$ lost compactness, so the functional I does not satisfy P.S condition. To overcome this difficult, we first transplant the Lion's concentration-compactness Lemma([6,7,11]) to Heisenberg group case.

Lemma 2.2.1 Let $(\rho_m)_{m \geq 1}$ be a sequence in $L^1(H^n)$ satisfying:

$$\rho_m \geq 0 \text{ in } H^n, \int\limits_{H^n} \rho_m = 1 \tag{33}$$

Then there exists a sequence $(\rho_{m_k})_{k \geq 1}$ satisfying one the following three possibilities:

(i) (Compactness) There exists a sequence $z_k \in H^n$ such that $\rho_{n_k}(z)$ is tight, i.e

$$\forall \varepsilon > 0, \exists R < \infty, \int\limits_{z_k + B_R} \rho_{n_k}(z) dz \geq 1 - \varepsilon; \tag{34}$$

(ii) (Vanishing) $\lim\limits_{k \to \infty} \sup\limits_{y \in B_R} \int \rho_{n_k}(z) dz = 0$,, for all $R < \infty$;

(iii) (Dichotomy) There exists $\alpha \in (0,1)$ such that for all $\varepsilon > 0$, there $k_0 \geq 1$ and $\rho_k^1, \rho_k^2 \in L_+^1(H^n)$ satisfying for $k \geq k_0$,

$$\begin{aligned} &\|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\|_{L^1} \leq \varepsilon \\ &|\int\limits_{H^n} \rho_k^1 dz - \alpha| \leq \varepsilon \end{aligned} \tag{35}$$

and $\text{dist}(\text{supp}\rho_k^1, \text{supp}\rho_k^2) \to +\infty, k \to +\infty$, where $dz = dxdydt$.

For the measure $dxdydt$ on H^n, it has translation invariant and it is a homogeneous on dilations δ_λ like the measure on $\mathbb{R}^{2n+1}$. That is for $u \in L^1(H^n), z_0 \in H^n$,

$$\begin{aligned} \int\limits_{H^n} u(z) dz &= \int\limits_{H^n} u(z \cdot z_0^{-1}) dz \\ \int\limits_{H^n} u(z_\lambda z) dz &= \lambda^{-Q} \int\limits_{H^n} u(z) dz \end{aligned} \tag{36}$$

Where $\lambda > 0$. So, just like P.L.Lions[6,7], we can prove this lemma. We omit its proof here.

Let $\{u_m\} \subset M, I(u) \to \min_{u \in M} I(u) = \alpha, m \to \infty$. By the Folland-Stein-Sobolev and there exists a constant $c > 0$, such that

$$\|u\|_p \leq c\|u\|, \quad \forall u \in E \tag{37}$$

So $\alpha = \min_{u \in M} I(u) > 0$.

Lemma 2.2.2 For the sequence $\{u_m\}$, there is a positive number $\{R_m\}$, for the function

$$\nu_m(z) = R_m^{-\frac{1}{q}} u_m(\delta_{\frac{1}{R_m}}(z) \tag{38}$$

such that

$$\sup_{z \in H^n} \int_{B_1(z)} |\nu_m|^q(w)dw = \frac{1}{2} = \int_{B_1(0)} |\nu_m|^q dw \tag{39}$$

Proof. For $u_m \in \{u_m\}, r > 0, z_m^r \in H^n$, we define

$$u_m^r = r^{n/q} u_m(\delta_{\frac{1}{r}}(z \cdot z_m^r)) \tag{40}$$

From (36), we have

$$\int_{H^n} \|u_m^r\|^q = r^{-n} \int_{H^n} |u_m(\delta_{\frac{1}{r}}(zz_m^r))|^q = \int_{H^n} |u_m|^q = 1 \tag{41}$$

So there exists a R_m, for every $z_m' \in H_n$,

$$\int_{B_1(z_m')} |u_m^{p+1}|^q dz = \int_{B_{R_m}(0)} |u_m|^q dz = \frac{1}{2} \tag{42}$$

Define $\nu_m(z) = R_m^{-2/n} u_m(\delta_{\frac{1}{R_m}} z)$. From the formula (42), we have

$$\sup_{z \in H^n} \int_{B_1(z)} |\nu_m|^q dx = \int_{B_1(0)} |\nu_m|^q dx = \frac{1}{2} \qquad \#.$$

Let $\rho_m = |\nu_m|^q$, then $\rho_m \in L^1(H^n)$, and $\int_{H^n} \rho_m = 1$. From Lemma 2.2.2, we know case (ii) in Lemma 2.2.1 can't occurs. We declare that the case (iii) can't also. That is our following lemma.

Lemma 2.2.3 For the function $\rho_m \in L^1(H^n)$ defined above, there is $z_m \in H^n$, such that $\rho_m(z \cdot z_m^{-1})$ is tight, i.e. there exists a number $R > 0$ large enough, such that

$$\int_{z_m \cdot B_R(0)} \rho_m(z)dz \geq 1 - \varepsilon \tag{43}$$

Proof. By th Lemma 2.2.1 and Lemma 2.2.2, we only need prove the case(iii) in Lemma 2.2.1 does't occur. On contrary, there is a number $\beta \in (0, \lambda)$ such that for all $\varepsilon > 0$, there exist $m_0 \geq 1$ and $\rho_m^1, \rho_m^2 \in L^1(H^n)$ satisfies for $m > m_0$,

$$\begin{aligned} &\|\rho_m - (\rho_m^1 + \rho_m^2)\|_m \leq \varepsilon \\ &|\int_{H^n} \rho_m^1 dz - \beta| \leq \varepsilon \\ &|\int_{H^n} \int \rho_m^2 dz - (1-\beta)| \leq \varepsilon \end{aligned} \tag{44}$$

and $\operatorname{dist}(\operatorname{supp}\rho_m^1, \operatorname{supp}\rho_m^2) \to +\infty$.

Choose $r_m > 0$, such that $\operatorname{supp}\rho_m^1 \subset B_{r_m}(0)$, $\operatorname{supp}\rho_m^2 \subset H^n \backslash B_{r_m}(0)$, and $r_m \to +\infty$ as $m \to +\infty$. Set $\varphi \in C_0^\infty(B_2(0))$ such that $\varphi \equiv 1$ in $B_1(0), 1 \leq \varphi \leq 1$ and let $\varphi_m(\frac{x}{r_m})$. Decompose

$$\nu_m = \varphi_m \nu_m + (1 - \varphi_m)\nu_m$$

Then

$$\begin{aligned} \int_{H^n} |\nabla_H \nu_m|^2 + |\nu_m|^2 = \; & \int_{H^n} |\nabla_H(\varphi_m \nu_m)|^2 + \int_{H^n} (\varphi_m \nu_m)^2 + \int_{H^n} |\nabla_H (1-\varphi_m)\nu_m)|^2 \\ & + \int_{H^n} (1-\varphi_m)\nu_m)^2 + 2\int_{H^n} \nabla_{H^n}(\varphi_m \nu_m) \cdot (1-\varphi_m)\nu_m) \\ & + 2\int_{H^n} \varphi_m \nu_m (1-\varphi_m)\nu_m \end{aligned} \tag{45}$$

Next we estimate the last two terms in formula (45) respectively.

$$\begin{aligned} & \int_{H^n} \nabla_H(\varphi_m \nu_m) \cdot \nabla_H((1-\varphi_m)\nu) \\ & \geq -\int_{H^n} |\nabla_H(\varphi_m \nu_m) \cdot \nabla_H (1-\varphi_m)\nu_m)| \\ & \geq -\int_{H^n} |\nabla_H(\varphi_m \nu_m)||\nabla_H((1-\varphi_m)\nu_m))| \\ & = -\int_{B_{2r_m}(0)\backslash B_{r_m}(0)} |\nabla_H(\varphi_m \nu_m)||\nabla_H((1-\varphi_m)\nu_m)| \\ & \geq \tfrac{1}{2}\Big[\int_{B_{2r_m}(0)\backslash B_{r_m}(0)} |\nabla_H(\varphi_m \nu_m)|^2 + \int_{B_{2r_m}(0)\backslash B_{r_m}(0)} |\nabla_H((1-\varphi_m)\nu_m)|^2\Big] \\ & = -\tfrac{1}{2}[\int_{B_{2r_m}(0)\backslash B_{r_m}(0)} \Big\{|\nabla_H(\varphi_m|^2 \nu_m^2 + 2\nabla_H \varphi_m \cdot \nabla_H \nu_m \cdot \varphi_m \nu_m + \varphi_m^2 |\nabla_m \nu_m|^2)| \\ & \quad + |\nabla_h \varphi_m|^2 \nu_m^2 - 2\nabla_H \varphi_m \cdot \nabla_H \nu_m \cdot \varphi_m \nu_m + (1-\varphi_m)^2 |\nabla_m \nu_m|^2\Big\} \\ & \geq -c \int_{B_{2r_m}(0)\backslash B_{r_m}(0)} \nu_m^2 + |\nabla_H \nu_m|^2 \\ & \geq c \int_{B_{2r_m}(0)\backslash B_{r_m}(0)} |\nu_m|^{p+1} \end{aligned}$$

Then we have

$$\int_{H^n} \nabla_H(\varphi_m\nu_m)\cdot((1-\varphi_m)\nu_m) + \int \varphi_m\nu_m(1-\varphi_m)\nu_m \geq -c \int_{B_{2r_m}(0)\backslash B_{r_m}(0)} |\nu_m|^{p+1} \quad (46)$$

From the proof of Lemma 2.2.1, we have $\forall\varepsilon > 0, \exists m_0$, such that for $m > m_0$,

$$\begin{aligned} &\int_{B_{r_m}(0)} |\nu_m|^{p+1} \leq \beta \\ &\int_{H^n\backslash B_{r_m}(0)} |\nu_m|^{p+1} \leq 1-\beta+\varepsilon \end{aligned} \quad (47)$$

So by the inequalities of (44) and (47), we have

$$\int_{B_{2r_m}(0)\backslash B_{r_m}(0)} |\nu_m|^{p+1} \leq c[\int_{H^n} |\nu_m|^{p+1} - \int_{H^n} (\rho_m^1 + \rho + m^2)] + \varepsilon \quad (48)$$

The inequality (48) means that

$$\int_{B_{2r_m}(0)\backslash B_{r_m}(0)} |\nu_m|^{p+1} = o(1) \quad (49)$$

where $o(1) \to 0, m \to +\infty$.

Combine the formula (49),(45) and the inequality (46) we have

$$\int_{H^n} |\nabla_H\nu_m|^2 + |\nu_m|^2 = \|\varphi_m\nu_m\|^2 + \|(1-\varphi_m)\nu_m\|^2 + o(1) \quad (50)$$

By the Folland-Stein-Sobolev embedding and formula (50), we have

$$\begin{aligned} \|\nu_m\|^2 &= \|\varphi_m\nu_m\|^2 + \|(1-\varphi_m)\nu_m\|^2 + o(1) \\ &\geq S(\|\varphi_m\nu_m\|_{L^{p+1}}^{\frac{2}{p+1}} + \|(1-\varphi_m)\nu_m\|^{\frac{2}{p+1}}) + o(1) \\ &\geq S(\int_{H^n} \rho_m^1)^{\frac{1}{p+1}} + (\int_{H^n} \rho_m^2)^{\frac{2}{p+2}}) + o(1) \\ &\geq S(\beta^{\frac{2}{p+1}} + (1-\beta)^{\frac{2}{p+1}}) + o(1) \end{aligned} \quad (51)$$

By the define of α and the independence of domain of the best Folland-Stein-Sobolev constant we know $S = \alpha$. And by the define ν_m we have $\|\nu_m\|^2 \to \alpha(m \to \infty)$. So by the inequality we have

$$\alpha \geq \alpha(\beta^{\frac{2}{p+1}} + (1-\beta)^{\frac{2}{p+1}}) \quad (52)$$

For $\frac{2}{p+1} < 1, 0 < \beta < 1$, we get $\alpha > \alpha$, that is a contradiction. So the case (iii) of Lemma 2.2.1 can't occur.

Theorem 2.2.1 There is a subsequence of $\{\nu_m\}$, we still denote it by $\{\nu_m\}$, there exists a point $u_0 \in M$, such that $\nu_m \to \nu_0$ in E, and

$$I(\nu_0) = \alpha$$

Proof: For $\{\nu_m\}$ is bounded in E, we have a subsequence of it, and we still denote it by $\{\nu_m\}$, and there exists a $\varepsilon, \nu_0 \in E$ such that $\nu_m \to \nu_0$. For $\varepsilon < \frac{1}{2}$, and Lemma 2.1.2, we get $(z_m \cdot B_R(0)) \cap B_1(0) \neq \emptyset$, so the points sequence $\{z_m\}$ is a bounded set. That is implies that tere is a subsequence of $\{z_m\}$, we still denote it by $\{z_m\}$ and a point $z_0 \in H^n$, such that $z_m \to z_0, (m \to \infty)$. Then we have

$$\int_{z_0 \cdot B_{1+2R}(0)} |\nu_m|^{p+1} > 1 - \varepsilon \tag{53}$$

From the Folland-Stein-Sobolev emedding, we know there is a subsequence $\{\nu_m\}$ we still denote it by $\{\nu_m\}$, such that $\nu_m \to \nu_0, m \to \infty$ in $H^{1,2}(B_{1+2R}(0))$ and

$$\int_{z_0 \cdot B_{1+2R}(0)} |\nu_0|^{p+1} > 1 - \varepsilon \tag{54}$$

From the Fatou Lemma we know

$$\int_{H^n} |\nu_0|^{p+1} \leq \lim_{m \to \infty} \int_{H^n} |\nu_m|^{p+1} = 1 \tag{55}$$

Combine the inequalities of (54) and (55), we have

$$\int_{H^n} |\nu_0|^{p+1} = 1 \tag{56}$$

This implies that $\nu_m \to \nu_0, m \to \infty$ in E. So we have

$$I(\nu_0) = \lim_{m \to \infty} I(\nu_m) = \alpha$$

By the Lagrange multiplier, there is a positive number, such that

$$\Delta_{H^n} \nu_0 - \nu_0 + \lambda \nu_0^p = 0$$

Set $u_0 = \lambda^{p-1} u$, then we have

$$\Delta_{H^n} u_0 - u_0 + u_0^p = 0$$

By the maximum principle, we get our our main theorem 1.1.

3 The critical exponent case

In this section, we set $p = \frac{Q+2}{Q-2}$. In this case, even in the bounded domain case, the Folland-Stein-Sobolev embedding is not compact. So the first method of the subcritical exponent case does not work.

Like the second of the subcritical exponent case, denote $M = \{u \in \Big| \int\limits_{H^n} |u|^{\frac{2Q}{Q-2}} = 1\}$ and $I(u) = \int\limits_{H^n} |\nabla_h u|^2 + u^2$. Define

$$\alpha = \inf_{u \in M} I(u) \tag{57}$$

Assume $\{u_m\} \subset M$, and

$$I(u_m) \to \alpha, \quad m \to +\infty \tag{58}$$

Like the Euclidean case, we have the following limit case of concentration-compactness Lemma. Next we shall use $\{u_m\}$ to construct a solution of the problem (1).

Lemma 3.1 Suppose $u_m \to u$ weakly in E and $\mu_m = \nabla_H u_m|^2 dx + u_m^2 \to \mu, \nu_m = |u_m|^2 dx \to \nu$ weakly in the sence of measures μ and ν are bound non-negative measures on H^n. Then we have

(i) There exists at most countable set J, a family $\{x^{(j)}; j \in J\}$, distinct points in H^n, and a family $\{\nu^{(j)}; j \in J\}$ of positive numbers such that

$$\nu = \|u\|^q dx + \sum_{j \in J} \nu^{(j)} \delta_{x^j} \tag{59}$$

Where δ_x is the Dirac mass of mass 1 concentrated at $x \in \boldsymbol{R}^n$.

(ii) In addition, we have

$$\mu \geq |\nabla_H u|^2 dx + \sum_{j \in J} \mu^{(j)} \delta_{x^{(j)}} + |u|^2 \tag{60}$$

for some family numbers $\{\mu^{(j)}; i \in J\}, \mu^{(j)} > 0$ satisfying

$$\alpha(\nu^{(j)})^{\frac{(Q-2)}{2Q}} \leq \infty, \quad \text{for all } j \in J \tag{61}$$

In particular, $\sum\limits_{j \in J} (\nu^{(j)})^{\frac{1}{q} \frac{(Q-2)}{2Q}} < \infty$.

This Lemma's proof is just like the proof of the corresponding Lemma of Lion's ([8,9,11]), So we omit it here. The only need to say is that in the proof, we must use the invariant Harr measures of H^n to institute Legbesgue measure of $\boldsymbol{R}^n$, and carefully to estimate the term u_m^2 in the measure μ_m.

Thanks to this Lemma, we have our theorem.

Theorem 3.1 In the manifold M, the minimum can be arrived.

Proof. For the sequence $\{u_m\}$ in (58) satisfies

$$\int |u_m|^{\frac{2Q}{Q-2}} dx = 1$$

and notice the proof of Lemma 2.2.2 and Lemma 2.2.1 don't dependent of p, so we can define a sequence $\{\nu_m\}$ and a sequence $\{z_m\}$ like in the proof of Lemma 2.2.1, such that

$$\int\limits_{z_m \cdot B_R(0)} |\nu_m|^{\frac{2Q}{Q-2}} \geq 1 - \varepsilon \quad \forall \varepsilon > 0, \exists R < \infty \tag{62}$$

and

$$\int\limits_{B_1(0)} \rho |\nu_m|^{\frac{2Q}{Q-2}} = \frac{1}{2} \tag{63}$$

From (60) and (61), for $\varepsilon < \frac{1}{2}$, $(z_m \cdot B_R(0) \cap B_1(0) \neq \emptyset$. So we have

$$\int\limits_{B_R(0)} |\nu_m|^{\frac{2Q}{Q-2}} > 1 - \varepsilon \tag{64}$$

From the construction of $\{\nu_m\}$, we have

$$\begin{array}{l} I(\nu_m) \to \alpha, m \to +\infty \\ \int\limits_{H^n} |\nu_m|^{\frac{2Q}{Q-2}} dz = 1 \end{array} \tag{65}$$

So the measure $|\nu_m|^{\frac{2Q}{Q-2}}, |\nabla_H \nu_m|^2$ are bounded and $\{\nu_m\}$ are bounded in E. Then we know that there are nonegtative measure ν, μ are $\nu_0 \in E$ such that

$$\begin{array}{l} |\nu_m|^{\frac{2Q}{Q-2}} \rightharpoonup \nu, \ m \to \infty, \text{as measure} \\ |\nabla_H \nu_m|^2 \rightharpoonup \mu, \ m \to \infty, \text{as measure} \\ \nu_m \rightharpoonup \nu_0 \text{ in } E \end{array} \tag{66}$$

By the estimate (62), we have

$$\int\limits_{H^n} \nu = 1 \tag{67}$$

From Lemma 3.1 we know, there exists $\{x^j \in H^n, j \in J\}$, J is at most countable, and two families $\{\nu^{(j)}, j \in J\}, \{\mu^{(j)}, j \in J\}$, such that

$$\begin{array}{l} \nu = |\nu_0|^q dx + \sum\limits_{j \in J} \nu^{(j)} \delta_x(j) \\ \mu \geq |\nabla_H \nu_0|^2 dx + \sum\limits_{j \in J} \mu^{(j)} \delta_x(j) + |\nu_0|^2 \end{array}$$

and $\mu^{(j)}, \nu^{(j)}$ satisfies the estimate (61).

From (65), we know

$$\begin{aligned}\alpha + o(1) &= \int |\nabla_H \nu_m|^2 + \nu_m^2 \\ &= \int_{H^n} \mu_m \\ &= \int_{H^n} \mu + +o(1) \\ &\geq \|\nabla_H \nu\|^2 + \sum_{j\in J} \mu^{(j)} + o(1) \\ &\geq \alpha(\|\nu\|^2_{L^{\frac{2Q}{Q-2}}} + \sum_{j\in J} (\nu^{(j)})^{\frac{2(Q-2)}{2Q}}) + o(1) \\ &\geq \alpha(\|\nu\|^{\frac{2Q}{Q-2}}_{L^{\frac{2Q}{Q-2}}} + \sum_{j\in J} |\nu^{(j)}|)^{\frac{2(Q-2)}{2Q}} + o(1) \\ &= \alpha \int_{R^n} \nu_0 + o(1) = \alpha + o(1)\end{aligned}$$

where $o(1) \to 0$ as $m \to \infty$. From the above and the formula (67), then we have

$$\|\nu\|^2_{L^{\frac{2Q}{Q-2}}} + \sum_{j\in J} (\nu^{(j)})^{\frac{2(Q-2)}{2Q}} = (\|\nu\|^{\frac{2Q}{Q-2}}_{L^{\frac{2Q}{Q-2}}} + \sum_{j\in J} \nu^{(j)})^{\frac{2(Q-2)}{Q-2}} \tag{68}$$

It is well known that the equality is right if and only there is only one of the terms $\|\nu\|_{\frac{2Q}{Q-2}}, \nu^{(j)}, j \in J$ is nonzero. but from the construction ν_m we know that

$$\nu^{(j)} \leq \frac{1}{2} \text{ for all } j \in J \tag{69}$$

So $\nu^{(j)} = 0, \forall j \in J$, and

$$\|\nu\|_{L^{\frac{2Q}{Q-2}}} = 1 \tag{70}$$

This implies that $\nu_m \to \nu$ strongly in E.

For the definition of ν_m, we knows $\nu_m \in M$, and $I(\nu_m) \to \min_{u\in M} I(u)$, so we have

$$I(\nu) = \min_{u\in M} I(u).$$

Then by the Lagrange multipler, we get our proof of Theorem 1.1 in the case of critical exponent case.

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