Local and global well-posedness results for generalized BBM-type equations

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Abstract The BBM or regularized long-wave equation was originally proposed as an alternative to the Korteweg-de Vries equation. It was shown in the paper of Benjamin *et al.* (1972) to be globally well-posed in $H^1(\mathbb{R})$, the class of square-integrable-functions whose derivative is also square-integrable. Recently, Bona and Tzvetkov (2002) have shown that the initial-value problem

$$\left.\begin{array}{ll} u_t + u_x + uu_x - u_{xxt} = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{array}\right\} \tag{0.0.1}$$

is globally well posed in $H^s(\mathbb{R})$ for any $s \geq 0$.

It is our purpose here to extend this well-posedness theory in weak spaces to some members of a more general class of evolution equations of the form

$$u_t + u_x + g(u)_x + Lu_t = 0 \tag{0.0.2}$$

where L is a Fourier-multiplier related to the linearized dispersion relation and g is a smooth, real-valued function of a real variable. Results are established analogous to those for equation (0.0.1), for (0.0.2) posed on the entire real axis. In addition, local and global well-posedness theory is established for bore-like or kink-like initial data, wherein u_0 has different limits as x tends to $\pm\infty$.

1 INTRODUCTION

The regularized long-wave equation or BBM-equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1.1.1}$$

was put forward by Peregrine (1964, 1967) and Benjamin *et al.* (1972) as an alternative model to the Korteweg-de Vries equation for small-amplitude, long wavelength surface water waves. In their paper, Benjamin and his co-workers discussed not only (1.1.1), but a class of evolution equations of the more general form

$$u_t + u_x + g(u)_x + Lu_t = 0 \tag{1.1.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth function (typically a polynomial in applications) and L is a Fourier multiplier operator with symbol α, say, so that

$$\widehat{Lv}(\xi) = \alpha(\xi)\widehat{v}(\xi)$$

for all wave numbers ξ, where the circumflex connotes the Fourier transform (with respect to the spatial variable x) of the function it surmounts and the symbol α of L is related to the dispersion suffered by infinitesimal waves. Thus, if $\omega(\xi)$ connotes the frequency corresponding to wavenumber ξ in the linearized theory, then at least for small values of wave number ξ (long waves)

$$\omega(\xi) = \frac{\xi}{1 + \alpha(\xi)}.$$

As pointed out by Benjamin *et al.* (1972) and many others, model equations of the form (1.1.2) arise in the description of waves in quite a number of physical situations.

In the analysis following the derivation of (1.1.1), Benjamin *et al.* (1972) showed (1.1.1) to be globally well posed in the Sobolev class $H^1(\mathbb{R})$ and in spaces such as $C_b^k(\mathbb{R}) \cap H^r(\mathbb{R})$ provided $r \geq 1$. In the Appendix to their paper, they sketched theory relating to the more general class of equations (1.1.2).

It is our purpose here to establish local and global well-posedness results for (1.1.2) in weaker L_p-based spaces for appropriate values of p. In this endeavor, we generalize some of the recent work of Bona & Tzvetkov (2002) concerned with (1.1.1). We also countenance bore-like initial data as well as L_p-based data.

The plan for the remainder of the paper is the following. In Section 2, the pure initial-value problem is converted into an integral equation. Local existence is then established for this integral equation by an application of the contraction-mapping principle in appropriate L_p-spaces. For a restricted class of the equations possessing a local well-posedness theory, an *a priori* bound is derived that leads to global well posedness. Section 3 follows a similar development relative to propagation of bore-like disturbances.

2 NOTATION AND LOCAL WELL-POSEDNESS

We begin with a brief synopsis of our notational conventions and function-space designations.

2.1 Notation

For $1 \leq r < \infty$, $L_r = L_r(\mathbb{R})$ connotes the r^{th}-power Lebesgue-integrable functions with the usual modification for the case $r = \infty$. The norm of a function $f \in L_r$ is written $|f|_r$. The Sobolev class $H^s(\mathbb{R})$ for $s \geq 0$ is the class of $L_2(\mathbb{R})$-functions whose Fourier transform $\widehat{f}$ has the property

$$\|f\|_s^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1+\xi^2)^s|\widehat{f}(\xi)|^2\,d\xi < +\infty$$

where $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,d\xi$. Note that $\|f\|_0 = |f|_2$ and we will in fact write the L_2-norm of f unadorned as simply $\|f\|$. If X is any Banach space and $T > 0$ given, $C(0,T;X)$ is the class of continuous functions from $[0,T]$ into X with its usual norm

$$\|u\|_{C(0,T;X)} = \max_{0\le t\le T}\|u(t)\|_X.$$

If $S \subset X$ is a subset, then $C(0,T:S)$ is the collection of elements u in $C(0,T;X)$ such that $u(t) \in S$ for $0 \le t \le T$. When $T = \infty$, $C(0,\infty;X)$ is a Fréchet space with defining set of semi-norms

$$p_n(u) = \max_{0\le t\le n}\|u(t)\|_X$$

for $n = 1, 2, \cdots$. The subspace $C_b(0,\infty;X)$ of elements of $C(0,\infty;X)$ which are uniformly bounded is a Banach space with norm

$$\|u\|_{C(0,\infty;X)} = \sup_{t\ge 0}\|u(t)\|_X.$$

The Banach space $C^1(0,T;X)$ is the subspace of $C(0,T;X)$ for which the limit

$$u'(t) = \lim_{h\to 0}\frac{u(t+h)-u(t)}{h}$$

exists in $C(0,T;X)$. It is equipped with the obvious norm. Inductively, one defines $C^k(0,T;X)$ and, by analogy, $C^k(0,\infty;X)$ and $C_b^k(0,\infty;X)$.

2.2 Associated Integral Equations

The theory begins by converting the original initial-value problem into an associated integral equation. For this, we operate formally and consider afterward the issue of whether or not solutions of the integral equation are solutions of the initial-value problem.

Write the evolution equation (1.1.2) posed on all of $\mathbb{R}$ in the form

$$(I+L)u_t = -\big(u+g(u)\big)_x \tag{2.2.1}$$

and take the Fourier transform with respect to the spatial variable x. Writing the Fourier transform of u with respect to x as $\widehat{u}$, there appears the formal relation

$$\big(1+\alpha(\xi)\big)\widehat{u}_t = -i\xi\big(\widehat{u}+\widehat{g(u)}\big).$$

Dividing by $1+\alpha$ and taking the inverse Fourier transform leads to the integral equation

$$u_t = K * \big(u+g(u)\big) \tag{2.2.2}$$

where the kernel K is the inverse Fourier transform of the function

$$\widehat{K}(\xi) = -i\xi / \big(1 + \alpha(\xi)\big).$$

Of course the convolution may have to be interpreted in the sense of tempered distributions. A formal integration in the temporal variable then leads to the BBM-type integral equation (Benjamin *et al.* 1972)

$$u(x,t) = u_0(x) + \int_0^t \int_{-\infty}^{\infty} K(x-y)\Big(u(y,s) + g(u(y,s))\Big)\,dy\,ds \tag{2.2.3}$$

where $u_0(x) = u(x,0)$ is the initial data. For classes of functions v defined on $\mathbb{R} \times [0,T]$ to be discussed presently, let $w = A(v)$ be the function obtained from v by replacing u by v on the right-hand side of (2.2.3). The equation (2.2.3) then takes the form

$$u = A(u). \tag{2.2.4}$$

In terms of the integral equation, a solution is thus seen to comprise a fixed point of the nonlinear operator A.

2.3 Local well-posedness with non-smooth initial data

To use (2.2.3) or (2.2.4) in a precise way, assumptions about the nonlinearity g and the dispersion operator L must be made and appropriate function classes put forward. As mentioned earlier, our goal is to work in relatively large function spaces.

Assumptions on g and the symbol α of L are now delineated.

(H1) The function $g : \mathbb{R} \to \mathbb{R}$ is C^1 and has the property that there is a $p > 1$ and a constant C_0 such that

$$\big|1 + g'(z)\big| \leq C_0(1 + |z|^{p-1})$$

for all $z \in \mathbb{R}$. This assumption will be referred to as the assumption of polynomial growth. Without loss of generality we take it that $g(0) = g'(0) = 0$.

(H2) The symbol α of L has the property that the tempered distribution K whose Fourier transform is $-i\xi/\big(1 + \alpha(\xi)\big)$ is given by a measurable function lying in $L_1(\mathbb{R}) \cap L_r(\mathbb{R})$ for some $r > 1$.

Examples: If $L = -\partial_x^2$, then

$$K(x) = \frac{1}{2}\,\mathrm{sgn}(x) e^{-|x|}$$

as one ascertains by a direct calculation using the Residue Theorem (see Benjamin *et al.* 1972). Clearly, this version of K satisfies (H2) for any positive value of r.

If $L = D^s$ with $s > 1$ where $\widehat{D^s h}(\xi) = |\xi|^s \widehat{h}(\xi)$, then

$$K(x) = \mathcal{F}^{-1}\Big\{\frac{i\xi}{1 + |\xi|^s}\Big\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\xi e^{ix\xi}}{1 + |\xi|^s}\,d\xi = -\frac{1}{\pi}\int_0^{\infty} \frac{\xi \sin(x\xi)}{1 + \xi^s}\,d\xi$$

where $\mathcal{F}$ connotes the Fourier transform in the spatial variable x and $\mathcal{F}^{-1}$ is its inverse. It thus follows that K is odd and in particular $K(0) = 0$. For any $x > 0$, integration by parts twice yields

$$\begin{aligned} K(x) &= -\frac{1}{\pi x}\int_0^\infty \frac{1-(s-1)\xi^s}{(1+\xi^s)^2}\cos(x\xi)\,d\xi \\ &= -\frac{1}{\pi x^2}\int_0^\infty \frac{s(s+1)\xi^{s-1}-s(s-1)\xi^{2s-1}}{(1+\xi^s)^3}\sin(x\xi)\,d\xi. \end{aligned}$$

It is thereby concluded that $K(x) = O(\frac{1}{x^2})$ as $x \to \infty$. On the other hand, for $x > 0$, $K(x)$ may also be represented in the form

$$\begin{aligned} K(x) &= -\frac{1}{\pi}\int_0^{\frac{1}{x}} \frac{\xi\sin(\xi x)}{1+\xi^s}\,d\xi - \frac{1}{\pi}\int_{\frac{1}{x}}^\infty \frac{\xi\sin(\xi x)}{1+\xi^s}\,d\xi \\ &= -\frac{1}{\pi}\int_0^{\frac{1}{x}} \frac{\xi\sin(\xi x)}{1+\xi^s}\,d\xi - \frac{x^s\cos 1}{\pi x^2(x^s+1)} - \frac{1}{\pi x}\int_{\frac{1}{x}}^\infty \frac{1-(s-1)\xi^s}{(1+\xi^s)^2}\cos(x\xi)\,d\xi \\ &= -\frac{1}{\pi}\int_0^{\frac{1}{x}} \frac{\xi\sin(\xi x)}{1+\xi^s}\,d\xi - \frac{x^s\cos 1}{\pi x^2(x^s+1)} - \frac{x^s}{\pi x^2}\int_1^\infty \frac{x^s-(s-1)y^s}{(x^s+y^s)^2}\cos y\,dy. \end{aligned}$$

It follows immediately that

$$|K(x)| \le \frac{1}{\pi}\int_0^{\frac{1}{x}} \frac{\xi^2 x}{1+\xi^s}\,d\xi + x^{s-2}(x^s+1).$$

Since the integrand $x\xi^2/(1+\xi^s)$ is bounded by $x\xi^{2-s}$, it follows that

$$|K(x)| = O(x^{s-2})$$

as $x \to 0$. These considerations imply $K \in L_1 \cap L_r$ for any $r < 1/(2-s)$ if $s < 2$, and for $r = \infty$ if $s \ge 2$.

As for the nonlinearity, if p is an integer, $p \ge 2$, and $g(z) = \sum_{j=2}^p c_j z^j$ is a polynomial of degree p, then

$$|1+g'(z)| = |1+\sum_{j=2}^p jc_j z^{j-1}| \le C_0(1+|z|^{p-1})$$

for a suitable constant C_0 depending only on the coefficients $c_2, \cdots, c_p$.

Here is a local existence theory based on (H1) and (H2).

THEOREM 2.1. *Consider the integral equation (2.2.3) and suppose the nonlinear function g and the integral kernel K satisfy hypotheses (H1) and (H2), respectively. Let q be such that*

$$q \ge \max\{p, (pr-r)/(r-1)\}$$

where p and r, specified in the hypotheses (H1) and (H2), represent the properties of the nonlinear function g and the integral kernel K, respectively. Then for any initial data $u_0 \in L_q$, there is a positive number $T = T(|u_0|_q)$ such that (2.2.3) has a unique solution u lying in the space $C(0,T;L_q)$. Moreover, the mapping $u_0 \mapsto u$ from the space L_q to $C(0,T;L_q)$ is continuous.

Recall that, for any $r_1, r_2 \geq 1$, if $u \in L_{r_1}$ and $v \in L_{r_2}$, then $u * v \in L_r$ and $|u*v|_r \leq |u|_{r_1}\,|v|_{r_2}$, where r is determined by $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} - 1$. With this inequality in mind, the tool for proving the proposition is the standard version of the contraction mapping theorem.

Proof. The remark just made implies that if $u \in L_q$, then

$$K * (g(u) + u) \in L_q$$

because

$$|g(u) + u| = \Big| \int_0^1 \Big(1 + g'(su)\Big)\, ds\, u \Big| \leq C_0 \Big(|u| + |u|^p\Big),$$

so,

$$|K * (g(u) + u)|_q \leq C_0 \big(|K|_1 |u|_q + |K|_{q/(q-p+1)} |u|_q^p\big). \tag{2.2.5}$$

Hence, the integral operator A may be considered as a map of $C(0,\infty; L_q)$ to itself. For any $\beta > 0$, let $B_\beta = \{u \in L_q : |u|_q \leq \beta\}$, and for any $T > 0$, let $X = X_{T,\beta} = C(0,T; B_\beta)$. It is asserted that if $\beta > 0$ is chosen properly and $T > 0$ is sufficiently small, then A maps X to itself and is a contraction mapping. In fact, for any $u \in X$,

$$\|Au\|_X \leq |u_0|_q + TC_0 \Big(|K|_1 \|u\|_X + |K|_{q/(q-p+1)} \|u\|_X^p\Big).$$

To estimate the norm of difference $Au - Av$ in the space X, the following sublemma will be useful.

SUBLEMMA 2.2. *If $u, v \in L_q$ and $|u|_q, |v|_q \leq \beta$, then*

$$\begin{aligned}
&\Big| K * \big(u - v + g(u) - g(v)\big) \Big|_q \\
&= \Big| K * \int_0^1 \big(1 + g'(v + s(u-v))\big)\, ds (u - v) \Big|_q \\
&\leq \Big| |K| * \int_0^1 C_0 (1 + |v + s(u-v)|^{p-1})\, ds |u - v| \Big|_q \\
&\leq C_0 \int_0^1 \Big\{ |K|_1 |u - v|_q + |K|_{q/(q-p+1)} |v + s(u-v)|_q^{p-1} |u - v|_q \Big\}\, ds \\
&\leq C_0 \Big\{ |K|_1 + |K|_{q/(q-p+1)} \beta^{p-1} \Big\} |u - v|_q
\end{aligned} \tag{2.2.6}$$

since $|v + s(u-v)|_q \leq \beta$ because u and v both lie in the ball of radius β about 0 in L_q.

Hence, it is straightforward to adduce the relation

$$\begin{aligned}
&\big|Au(\cdot,t) - Av(\cdot,t)\big|_q \\
&= \Big| \int_0^t K * \Big(u(\cdot,\tau) - v(\cdot,\tau) + g(u(\cdot,\tau)) - g(v(\cdot,\tau))\Big)\, d\tau \Big|_q \\
&\leq \int_0^t \Big| K * \Big(u(\cdot,\tau) - v(\cdot,\tau) + g(u(\cdot,\tau)) - g(v(\cdot,\tau))\Big)(\cdot,\tau) \Big|_q\, d\tau \\
&\leq C_0 \int_0^t \Big\{ |K|_1 + |K|_{q/(q-p+1)} \beta_q^{p-1} \Big\} |u(\cdot,\tau) - v(\cdot,\tau)|_q\, d\tau.
\end{aligned}$$

Taking the maximum in this inequality for $t \in [0, T]$ yields

$$||Au - Av||_X \le C_0 T(|K|_1 + |K|_{q/(q-p+1)} \beta^{p-1}) ||u - v||_X. \tag{2.2.7}$$

It follows readily that if we choose

$$\beta = 2|u_0|_q \quad \text{and} \quad T = \frac{1}{2C_0(|K|_1 + |K|_{q/(q-p+1)} \beta^{p-1})}, \tag{2.2.8}$$

then the operator A maps X to X and is contractive. Since X is a complete metric space, the contraction mapping theorem completes the proof. ∎

Remark: In fact, the result of Theorem 2.1 is true if the nonlinear function g is a C^α-function for some number $\alpha \in (0, 1]$ and there is a number $C_1 > 0$ such that for any $x, y \in \mathbb{R}$,

$$|g(x) - g(y)| \le C_1(|x|^{1-\alpha} + |y|^{1-\alpha} + |x|^{p-1} + |y|^{p-1})|x - y|^\alpha. \tag{2.2.9}$$

Furthermore, if there is a positive integer k such that $g \in C^{k+\alpha}$, and $g^{(k)}$ satisfies an inequality like (2.2.9) in which p is replaced by $p - k$, then the mapping $u_0 \mapsto u$ from the space L_q to $C(0, T; L_q)$ is k^{th}-order differentiable. Here we only outline the proof for $k = 1$. This case amounts to the assertion that the mapping taking u_0 to the associated solution u is differentiable. Write the solution with initial data u_0 as $u = u_{u_0}$. Fix initial data ϕ and a perturbation h in L_q and let $\delta \neq 0$ be an arbitrary real number. Clearly,

$$\frac{u_{\phi+\delta h} - u_\phi}{\delta} = h + \int_0^t K * \Big(\frac{u_{\phi+\delta h} - u_\phi}{\delta} + \frac{g(u_{\phi+\delta h}) - g(u_\phi)}{\delta} \Big) \, d\tau$$

and so, by the sublemma, it is thus implied that for any $T' > 0$ such that $u_{\phi+\delta h}$ and u_ϕ are well defined on $C(0, T'; L_q)$,

$$\Big\| \frac{u_{\phi+\delta h} - u_\phi}{\delta} \Big\|_{C(0,T';L_q)} \le |h|_q + C'(\delta, \phi) T' \Big\| \frac{u_{\phi+\delta h} - u_\phi}{\delta} \Big\|_{C(0,T';L_q)}$$

where $C'(\delta, \phi) = C_0(|K|_1 + |K|_{q/(q-p+1)})(\|u_\phi\|^{p-1}_{C(0,T';L_q)} + o(\delta))$ and the little o denotes terms for which $\lim_{\delta \to 0} o(\delta) = 0$. Notice that $C'T' \le \frac{1}{2}$ when $\delta \neq 0$ and T' are chosen sufficiently small. In these circumstances, it is not hard to verify that the difference quotient on the left-hand side of the last inequality is also Cauchy as $\delta \to 0$, and therefore one infers existence of the limit

$$\lim_{\delta \to 0} \frac{u_{\phi+\delta h} - u_\phi}{\delta}$$

in $C(0, T'; L_q)$. As differentiability is a local issue and the time interval is compact, this completes the proof of the remark. The argument for this result is similar in case $k > 1$. If g is analytic (e.g. if g is a polynomial), then the mapping is also analytic.

The L_q-results just established may be generalized in various directions. Here is an aspect of further regularity corresponding to enhanced smoothness of the initial data.

THEOREM 2.3. *(Regularity 1) Let $u \in C(0,T;L_q)$ be the solution of (2.2.3) described in Theorem 2.1. In addition, suppose $g \in C^k(\mathbb{R})$ for some $k \geq 1$ and $g^{(k)}$ is bounded by a polynomial with degree less than or equal to $p-k$. Then it follows that*

$$\frac{\partial u}{\partial t}, \cdots, \frac{\partial^k u}{\partial t^k} \in C(0,T;L_q).$$

Proof. Because of Theorem 2.1, u is given locally in time as the fixed point of the operator A as in (2.2.3). The fixed point of this contraction mapping may be obtained as the limit of the sequence $\{u_n\}_{n=1}^\infty$ generated by the iteration

$$u_1 = A(\theta), \cdots, u_{n+1} = A(u_n), \cdots, \tag{2.2.10}$$

commencing from the starting point θ which is the constant function equal to 0 everywhere. Thus, this iterated sequence $\{u_n\}_{n=1}^\infty$ is Cauchy in $C(0,T;L_q)$. Applying Sublemma 2.2, it follows that for any $n, m \geq 0$,

$$\begin{aligned} &\Big|\frac{\partial u_{n+m+1}(\cdot,t)}{\partial t} - \frac{\partial u_{n+1}(\cdot,t)}{\partial t}\Big|_q = |K * \big(u_{n+m} - u_n + g(u_{n+m}) - g(u_n)\big)(\cdot,t)|_q \\ &\leq C_0\{|K|_1 + |K|_{q/(q-p+1)} \max_{s\in[0,1]} \big\{|u_n + s(u_{n+m} - u_n)|_q^{p-1}\big\}|u_{n+m}(\cdot,t) - u_n(\cdot,t)|_q. \end{aligned} \tag{2.2.11}$$

Since $\{u_n\}_{n\geq 1}$ converges to u in $C(0,T;L_q)$, $\max_{s\in[0,1]}\big\{|u_n + s(u_{n+m} - u_n)|_q^{p-1}\big\}$ is bounded by a constant $C = C(\|u\|_{C(0,T;L_q)})$ dependent only on u. Taking supremum of this inequality for $t \in [0,T]$ yields

$$\Big\|\frac{\partial u_{n+m+1}}{\partial t} - \frac{\partial u_{n+1}}{\partial t}\Big\|_{C(0,T;L_q)} \leq C\|u_{n+m} - u_n\|_{C(0,T;L_q)}.$$

In consequence, the sequence $\{\frac{\partial u_n}{\partial t}\}_{n\geq 1} \subset C(0,T;L_q)$ is also a Cauchy sequence. Inductively, for $j = 0, 1, \cdots, k-1$ (where it is presumed that $\partial^0 u/\partial t^0 = u$), it is adduced that

$$\frac{\partial^{j+1} u_{n+1}}{\partial t^{j+1}} = K * \frac{\partial^j u_n}{\partial t^j} + K * \Big(g'(u_n)\frac{\partial^j u_n}{\partial t^j} + \cdots + g^{(j)}(u_n)(\frac{\partial u_n}{\partial t})^j\Big). \tag{2.2.12}$$

Arguing as above, the sequence $\big\{\frac{\partial^{j+1} u_n}{\partial t^{j+1}}\big\}_{n\geq 1}$ is also Cauchy in $C(0,T;L_q)$. The completeness of $C(0,T;L_q)$ implies that the sequence $\{\frac{\partial^{j+1} u_n}{\partial t^{j+1}}\}_{n\geq 1}$ is convergent to some function $w_{j+1} \in C(0,T;L_q)$, say, as $n \to \infty$. To prove $w_{j+1} = \partial^{j+1} u/\partial t^{j+1}$, simply let $n \to \infty$ in (2.2.12) to reach the conclusion

$$w_{j+1} = K * w_j + K * \big(g'(u)w_j + \cdots + g^{(j)}(u)w_1^j\big).$$

The uniqueness of the solution implies that $w_1 = \frac{\partial u}{\partial t}$, and inductively, it is seen that $w_{j+1} = \frac{\partial^{j+1} u}{\partial t^{j+1}}$. ∎

Remarks : If a function f has the property that $\widehat{f} \in L_p$ for some $p \in [1,2]$, then $f \in L_q$ and $|f|_q \leq |\widehat{f}|_p$, where $1/p + 1/q = 1$. For the integral equation (2.2.3), if the initial data u_0 has the property that $\widehat{u}_0 \in L_{q/(q-1)}$, which guarantees that

$u_0 \in L_q$, then the results of Theorem 2.1 hold true, and furthermore, at least in the case where g is a polynomial, the Fourier transform $\widehat{u}$ of the solution u with respect to the spatial variable lies in the space $C(0,T;L_{q/(q-1)})$. The proof is made by considering the Fourier transform of (2.2.3), namely

$$\widehat{u}(\xi,t) = \widehat{u_0}(\xi) + \frac{i\xi}{1+\alpha(\xi)} \int_0^t \Big(\widehat{u}(\xi,\tau) + \widehat{g(u)}(\xi,\tau)\Big)\, d\tau.$$

One shows that the mapping $\widetilde{A}$ defined by the right-hand side of the last formula is a contraction in $C(0,T;\widehat{B_\beta})$ for T and β chosen appropriately, where $\widehat{B_\beta}$ is the ball of radius β about 0 in $L_{q/(q-1)}$. For this step, we use the fact that g is a polynomial so $\widehat{g(u)}$ is a finite sum of the form ,

$$\sum_{k=2}^{N} a_k \widehat{u} * \widehat{u} * \cdots \widehat{u}$$

where the k^{th} term features a k-fold convolution of $\widehat{u}$ with itself. One thus infers existence of a solution of the above integral equation whose Fourier transform satisfies (2.2.3) and lies in $C(0,T;L_q)$.

Another point worth mention is the smoothing associated with a temporal derivative. Indeed, since $\widehat{u_t} = \frac{i\xi}{1+\alpha(\xi)}\big(\widehat{u} + \widehat{g(u)}\big)$, u_t is smoother than $u + g(u)$ if α grows super-linearly at infinity. For simplicity, let the nonlinear function g be homogeneous, say $g(z) = z^p$. For the dispersion α, suppose there is a positive number $s > 1 + \frac{p-1}{q}$ such that

$$\liminf_{|\xi|\to\infty} \frac{\alpha(\xi)}{|\xi|^s} > 0.$$

Then, for any ϵ in the range $[0, s-1-\frac{p-1}{q})$,

$$\begin{aligned}
\Big|(1+\xi^2)^{\frac{\epsilon}{2}}\widehat{u_t}(\xi,t)\Big|_{q/(q-1)} &= \Big|\frac{i\xi(1+\xi^2)^{\frac{\epsilon}{2}}}{1+\alpha(\xi)}\Big(\widehat{u}(\xi,t) + \widehat{u^p}(\xi,y)\Big)\Big|_{q/(q-1)} \\
&\le \gamma_1 |\widehat{u}(\cdot,t)|_{q/(q-1)} + \gamma_2 |\widehat{u^p}(\cdot,t)|_{q/(q-p)} \\
&\le \gamma_1 |\widehat{u}(\cdot,t)|_{q/(q-1)} + \gamma_2 |\widehat{u}(\cdot,t)|^p_{q/(q-1)}
\end{aligned}$$

where the numbers γ_1 and γ_2 are determined to be

$$\gamma_1 = \sup_{\xi\in\mathbb{R}} \frac{|\xi|(1+\xi^2)^{\frac{\epsilon}{2}}}{1+\alpha(\xi)} \quad \text{and} \quad \gamma_2^{\frac{q}{p-1}} = \int_{\mathbb{R}} \Big(\frac{|\xi|(1+\xi^2)^{\frac{\epsilon}{2}}}{1+\alpha(\xi)}\Big)^{\frac{q}{p-1}} d\xi.$$

Thus, $u_t \in C(0,T;W^{\epsilon,q})$ where $W^{\epsilon,q} = \{u \in L_q : (1+\xi^2)^{\frac{\epsilon}{2}}\widehat{u} \in L_q\}$. In particular, for the original BBM-equation where $s = 2$ and $p = 2$, if the initial data $u_0 \in L_2$, then the solution $u \in C(0,\infty;L_2)$ as proved by Bona and Tzvetkov 2001. The above ruminations and a bootstrap argument imply that the time derivative u_t lies in $C(0,\infty;H^1)$ and so is spatially smoother than u.

THEOREM 2.4. *(Regularity 2) Let $u \in C(0,T;L_q)$ be the solution whose existence is guaranteed in Theorem 2.1. Suppose in addition that $u_0 \in C_b^k$ and $g \in C^{k+1}$ for some $k \ge 0$ and $g^{(k+1)}$ is bounded by a polynomial of degree $p-k$. Then $u \in C(0,T;C_b^k \cap L_q)$.*

Proof. It is sufficient to prove that the sequence $\{\partial^j u_n/\partial x^j\}_{n\geq 1}$ for $j = 0, 1, \cdots, k$ is Cauchy in $C(0, T_0; C_b \cap L_q)$ for $T_0 > 0$ sufficiently small, where $\{u_n\}_{n\geq 1}$ is the iterated sequence defined in (2.2.10). A straightforward analysis based on Gronwall's inequality allows one to extend the result to any time interval $[0, T]$ for which the solution u of (2.2.3) is known to lie in $C(0, T; L_q)$.

Since $u_0 \in C_b^k$ and $k \geq 0$, define functions $u_j = u_j(x, t)$ in $C(0, \infty; C_b)$ as follows:

$$u_1 = u_0 + \int_0^t K * \Big(u_0 + g(u_0)\Big)(\cdot, \tau)\, d\tau,$$

and inductively, for $n = 1, 2 \cdots$,

$$u_{n+1} = u_0 + \int_0^t K * \Big(u_n + g(u_n)\Big)(\cdot, \tau)\, d\tau. \tag{2.2.13}$$

Naturally,

$$\begin{aligned}&u_{n+1}(\cdot, t) - u_n(\cdot, t)\\ &= \int_0^t K * \Big(\int_0^1 \big(1 + g'(u_{n-1} + s(u_n - u_{n-1})\big)(\cdot, \tau)\, ds(u_n - u_{n-1})\Big)\, d\tau\end{aligned}$$

and, as in Sublemma 2.2, it follows that

$$\begin{aligned}&\big|u_{n+1}(\cdot, t) - u_n(\cdot, t)\big|_\infty\\ &\leq \int_0^t C_0\Big(|K|_1 + |K|_{q/(q-p+1)} \max_{s\in[0,1]}\{|u_{n-1} + s(u_n - u_{n-1})|_q^{p-1}\}\Big)\\ &\qquad\qquad\qquad\qquad \big|u_n(\cdot, \tau) - u_{n-1}(\cdot, \tau)\big|_\infty\, d\tau.\end{aligned}$$

Since $\{u_n\}_{n\geq 1}$ is Cauchy in $C(0, T; L_q)$, it follows that for any $t \in [0, T]$ and $s \in [0, 1]$,

$$\limsup_{n\to\infty} |u_{n-1} + s(u_n - u_{n-1})(\cdot, t)|_q \leq |u(\cdot, t)|_q.$$

In consequence, the sequence $\{u_n\}_{n\geq 1} \subset C(0, T; C_b)$ is Cauchy. To prove that the limit is the solution u of (2.2.3), let $w = \lim_{n\to\infty} u_n$ and consider the limit as $n \to \infty$ in (2.2.13). There obtains the relation

$$w(\cdot, t) = u_0 + \int_0^t K * (w + g(w))(\cdot, \tau)\, d\tau,$$

whence,

$$\begin{aligned}&|w(\cdot, t) - u(\cdot, t)|_\infty\\ &= \Big|\int_0^t K * \big(w - u + g(w) - g(u)\big)(\cdot, \tau)\, d\tau\Big|_\infty\\ &\leq C_0 \int_0^t \Big(|K|_1 + |K|_{q/(q-p+1)}\big(|w|_q + |u|_q\big)^{p-1}\Big)|w(\cdot, \tau) - u(\cdot, \tau)|_\infty\, d\tau.\end{aligned}$$

It then follows from Gronwall's inequality that $w = u \in C(0, T; C_b \cap L_q)$.

Next consider the sequence $\{\partial u_n/\partial x\}_{n\geq 1}$. It is obvious that, for any $t \geq 0$,

$$\frac{\partial u_1}{\partial x} = u_0' + \int_0^t K * \Big((1+g'(u_0))u_0'\Big)\, d\tau$$

and the right-hand side of the last equation is in $C(0,\infty; C_b \cap L_q)$ because

$$\Big|\frac{\partial u_1(\cdot,t)}{\partial x}\Big|_\infty \le \big|u_0'\big|_\infty + C_0 \int_0^t \big(|K|_1 + |K|_{q/(q-p+1)}|u_0|_q^{p-1}\big)|u_0'|_\infty\, d\tau$$

and

$$\Big|\frac{\partial u_1(\cdot,t)}{\partial x}\Big|_q \le \big|u_0'\big|_q + C_0 \int_0^t \big(|K|_1 + |K|_{q/(q-p+1)}|u_0|_q^{p-1}\big)|u_0'|_q\, d\tau.$$

Induction on n yields that

$$\frac{\partial u_{n+1}}{\partial x} = u_0' + \int_0^t K * \Big(\big(1+g'(u_n)\big)\frac{\partial u_n}{\partial x}\Big)(\cdot,\tau)\, d\tau \in C(0,\infty; C_b \cap L_q) \qquad (2.2.14)$$

because

$$\Big|\frac{\partial u_{n+1}(\cdot,t)}{\partial x}\Big|_\infty \le \big|u_0'\big|_\infty + C_0 \int_0^t \Big(|K|_1 + |K|_{q/(q-p+1)}|u_n|_q^{p-1}(\cdot,\tau)\Big)\Big|\frac{\partial u_n(\cdot,\tau)}{\partial x}\Big|_\infty d\tau$$

and

$$\Big|\frac{\partial u_{n+1}(\cdot,t)}{\partial x}\Big|_q \le \big|u_0'\big|_q + C_0 \int_0^t \Big(|K|_1 + |K|_{q/(q-p+1)}|u_n|_q^{p-1}(\cdot,\tau)\Big)\Big|\frac{\partial u_n(\cdot,\tau)}{\partial x}\Big|_q d\tau.$$

Therefore, for t restricted to the interval $[0, T_0]$, where

$$T_0 = \min\Big\{T, \frac{1}{2C_0(|K|_1 + |K|_{q/(q-p+1)}|u|^{p-1}_{C(0,T;L_q)})}\Big\},$$

the sequence $\{\partial u_{n+1}/\partial x\}_{n\ge 1}$ is bounded in $C(0,T_0; C_b \cap L_q)$, and in fact for $t \in [0, T_0]$,

$$\Big|\frac{\partial u_{n+1}(\cdot,t)}{\partial x}\Big|_\infty \le 2|u_0'|_\infty \quad \text{and} \quad \Big|\frac{\partial u_{n+1}(\cdot,t)}{\partial x}\Big|_q \le 2|u_0'|_q.$$

Furthermore, for any $n \ge 0$

$$\begin{aligned}
&\frac{\partial u_{n+1}(\cdot,t)}{\partial x} - \frac{\partial u_n(\cdot,t)}{\partial x}\\
&= \int_0^t K * \Big(g'(u_n)\frac{\partial u_n}{\partial x} - g'(u_{n-1})\frac{\partial u_{n-1}}{\partial x} + \frac{\partial u_n}{\partial x} - \frac{\partial u_{n-1}}{\partial x}\Big)(\cdot,\tau)\, d\tau\\
&= \int_0^t K * \Big(\Big[g'(u_n) - g'(u_{n-1})\Big]\frac{\partial u_n}{\partial x} + \big(1+g'(u_{n-1})\big)\Big[\frac{\partial u_n}{\partial x} - \frac{\partial u_{n-1}}{\partial x}\Big]\Big)(\cdot,\tau)\, d\tau.
\end{aligned}$$

Notice that

$$g'(u_n) - g'(u_{n-1}) = \int_0^1 g''(u_{n-1} + s(u_n - u_{n-1}))\, ds (u_n - u_{n-1}),$$

and g'' is bounded by a polynomial of degree $p-2$, then there is a positive number C such that $|g''(x)| \leq C(1+|x|^{p-2})$. Thus applying Sublemma 2.2 yields

$$|K*(g'(u_n)-g'(u_{n-1}))| \leq C|K|*\int_0^1 (1+|u_{n-1}+s(u_n-u_{n-1})|^{p-2})\,ds|u_n-u_{n-1}|,$$

so,

$$\begin{aligned}&|K*(g'(u_n(\cdot,t))-g'(u_{n-1}(\cdot,t))|_\infty\\ \leq & C(|K|_1+|K|_{q/(q-p+2)})\int_0^1 |u_{n-1}+s(u_n-u_{n-1})|_q^{p-2}\,ds|u_n(\cdot,t)-u_{n-1}(\cdot,t)|_\infty\end{aligned}$$

and

$$\begin{aligned}&\Big|K*(1+g'(u_{n-1}))\Big(\frac{\partial u_n}{\partial x}-\frac{\partial u_{n-1}}{\partial x}\Big)(\cdot,t)\Big|_\infty\\ \leq & C_0(|K|_1+|K|_{q/(q-p+1)}|u_{n-1}(\cdot,t)|_q^{p-1})\Big|\frac{\partial u_n(\cdot,t)}{\partial x}-\frac{\partial u_{n-1}(\cdot,t)}{\partial x}\Big|_\infty.\end{aligned}$$

Hence, there is a number C' dependent only on the solution u such that

$$\Big|\frac{\partial u_{n+1}(\cdot,t)}{\partial x}-\frac{\partial u_n(\cdot,t)}{\partial x}\Big|_\infty \leq C'\int_0^t \Big|\frac{\partial u_n(\cdot,\tau)}{\partial x}-\frac{\partial u_{n-1}(\cdot,\tau)}{\partial x}\Big|_\infty\,d\tau.$$

In a same fashion, it follows that

$$\Big|\frac{\partial u_{n+1}(\cdot,t)}{\partial x}-\frac{\partial u_n(\cdot,t)}{\partial x}\Big|_q \leq C'\int_0^t \Big|\frac{\partial u_n(\cdot,\tau)}{\partial x}-\frac{\partial u_{n-1}(\cdot,\tau)}{\partial x}\Big|_q\,d\tau.$$

Thus, for $T_1>0$ chosen sufficiently small, $\{\frac{\partial u_n}{\partial x}\}_{n\geq 1}$ is Cauchy in $C(0,T_1;C_b\cap L_q)$. To prove that the limit of this sequence as $n\to\infty$ is equal to u_x for t restricted to $[0,T_1]$, denote by $w_1=\lim_{n\to\infty}\frac{\partial u_n}{\partial x}$, and let $n\to\infty$ in (2.2.14) to obtain

$$w_1 = u_0' + \int_0^t K*\Big((1+g'(u))w_1\Big)\,d\tau.$$

On the other hand, taking the derivative with respect to x in (2.2.3) leads to

$$u_x = u_0' + \int_0^t K*\Big((1+g'(u))u_x\Big)\,d\tau. \tag{2.2.15}$$

Forming the difference and estimating yields

$$|w_1(\cdot,t)-u_x(\cdot,t)|_\infty \leq C_0|K|_1\int_0^t (1+|u|_{C(0,T;C_b)}^{p-1})|w_1(\cdot,\tau)-u_x(\cdot,\tau)|_\infty\,d\tau.$$

Gronwall's inequality then implies $u_x=w_1$, which is to say, the solution u of (2.2.3) lies in $C(0,T_1;C_b^1\cap W_q^1)$. Furthermore, (2.2.15) implies that

$$|u_x|_\infty \leq |u_0'|_\infty + C_0\int_0^t (|K|_1+|K|_{q/(q-p+1)}|u(\cdot,\tau)|_q^{p-1})|u_x(\cdot,\tau)|_\infty\,d\tau$$

and

$$|u_x|_q \le |u_0'|_q + C_0 \int_0^t (|K|_1 + |K|_{q/(q-p+1)}|u(\cdot,\tau)|_q^{p-1})|u_x(\cdot,\tau)|_q \, d\tau.$$

Gronwall's inequality shows that the time interval over which the solution u can be extended is in fact the interval $[0,T]$ on which $u(\cdot,t)$ is known to lie in L_q, which is to say, $u \in C(0,T; C_b^1 \cap W_q^1)$. By induction, it can be shown that $u \in C(0,T; C_b^k \cap W_q^k)$. ∎

THEOREM 2.5. *In Theorem 2.1, suppose the relationship between* r, *the index appearing in (H2), and* p, *which governs the growth of the nonlinearity* g *in (H1) are further restricted by the relation*

$$r \ge \frac{p+1}{2}.$$

Then the integral equation (2.2.3) is locally well posed in $L_2 \cap L_{p+1}$, *so, if the initial data* $u_0 \in L_2 \cap L_{p+1}$, *then there is a* $T > 0$ *such that (2.2.3) has an unique solution* u *lying in* $C(0,T; L_2 \cap L_{p+1})$, *and the mapping* $u_0 \to u$ *is Lipschitz from the space* $L_2 \cap L_{p+1}$ *to* $C(0,T; L_2 \cap L_{p+1})$. *Moreover, its* L_2*-norm is bounded by*

$$||u(\cdot,t)||^2 \le e^{Ct}\|u_0\|^2 + 2\int_0^t e^{C(t-\tau)}|u(\cdot,\tau)|_{p+1}^{p+1} \, d\tau,$$

where $C = 2C_0|K|_1$.

Proof. The existence of the unique solution $u \in C(0,T; L_{p+1})$ of (2.2.3) for some $T > 0$ is a direct result of Theorem 2.1. Again, it is known that the sequence $\{u_n\}_{n\ge1}$ defined in (2.2.10) lies in $C(0,T; L_{p+1})$ and converges to u. We know further that

$$\begin{aligned}\|u_1(\cdot,t)\| &\le \|u_0\| + \int_0^t \|K * \big(u_0 + g(u_0)\big)\| \, d\tau \\ &\le \|u_0\| + C_0 \int_0^t \big(|K|_1\|u_0\| + |K|_{(2p+2)/(p+3)}|u_0|_{p+1}^p\big) \, d\tau,\end{aligned}$$

so, $u_1 \in L_2 \cap L_{p+1}$. Inductively, for $n \ge 1$,

$$\begin{aligned}\|u_{n+1}(\cdot,t)\| &\le \|u_0\| + \int_0^t \|K * u_n(\cdot,\tau) + K * g(u_n(\cdot,\tau))\| \, d\tau \\ &\le \|u_0\| + C_0 \int_0^t \big(|K|_1\|u_n(\cdot,\tau)\| + |K|_{(2p+2)/(p+3)}|u_n(\cdot,\tau)|_{p+1}^p\big) \, d\tau,\end{aligned}$$

which means $\{u_n(\cdot,t)\}_{n\ge1} \subset L_2 \cap L_{p+1}$ at least for $t \in [0,T]$. Moreover, for any $n > 0$,

$$\|u_{n+1}(\cdot,t) - u_n(\cdot,t)\| = \left\| \int_0^t K * \Big(u_n - u_{n-1} + g(u_n) - g(u_{n-1})\Big) \right\|,$$

since u_n converges to $u \in C(0,T;L_{p+1})$ as $n \to \infty$, for sufficiently large value of n, $\|u_n\|_{C(0,T;L_{p+1})} \le 2\|u\|_{C(0,T;L_{p+1})}$, and therefore, by Sublemma 2.2,

$$\begin{aligned}
&\|u_{n+1}(\cdot,t) - u_n(\cdot,t)\| \\
\le& C_0 \int_0^t \Big(|K|_1 + 2^{p-1}|K|_{(p+1)/2}\|u\|^{p-1}_{C(0,T;L_p+1)} \Big) \|u_n(\cdot,\tau) - u_{n-1}(\cdot,\tau)\| \, d\tau.
\end{aligned}$$

Thus, for $T_0 > 0$ chosen small, the sequence is Cauchy in $C(0,T_0;L_2)$, whence,

$$u \in C(0,T_0;L_2 \cap L_{p+1}).$$

Moreover, the solution u is continuously dependent on the initial data u_0. The regularity result of Theorem 2.3 allows us to formally multiply the equation (2.2.2) by $2u$ and integrate over $\mathbb{R}$ to reach the relations

$$\frac{d}{dt}\|u\|^2 = \int_{-\infty}^{\infty} 2uK * \big(u + g(u)\big)\, dx \le 2C_0|K|_1\|u\|^2 + 2C_0|K|_1|u|^{p+1}_{p+1}.$$

Gronwall's inequality gives

$$\|u(\cdot,t)\|^2 \le e^{Ct}\|u_0\|^2 + 2\int_0^t e^{C(t-\tau)}|u(\cdot,\tau)|^{p+1}_{p+1}\, d\tau$$

for any $t \in [0,T_0]$. The continuous dependence result allows us to conclude this relation holds for rough data. It is thereby implied that the time interval $[0,T_0]$ where $|u(\cdot,t)|_{(p+1)/2}$ stays finite can be extended to $[0,T]$. The theorem is complete.
■

LEMMA 2.6. *Let $u \in C(0,T;L_2 \cap L_{p+1})$ be a solution of (2.2.3). The functional*

$$\int_{-\infty}^{\infty} \big(F(u) + \frac{1}{2}u^2\big)\, dx$$

is invariant with respect to the time variable t, where F is the anti-derivative $F(z) = \int_0^z g(z)\, dz$ of g.

Proof. For smooth solutions, the following calculation is decisive:

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty} \big(F(u(x,t)) + \frac{1}{2}u^2(x,t)\big)\, dx \\
=& \int_{-\infty}^{\infty} (g(u) + u)u_t \, dx \\
=& -\int_{-\infty}^{\infty} (g(u) + u)(I+L)^{-1}\partial_x \big(g(u) + u\big)\, dx.
\end{aligned}$$

As $(I+L)^{-1}\partial_x$ is skew-adjoint, the right-hand side is obviously zero. For solutions in the advertised class, the result follows from the regularity theory, the continuous dependence of solutions on the initial data and density of, say, $\mathcal{D}(\mathbb{R})$ in $L_2 \cap L_{p+1}$.
■

COROLLARY 2.7. *In Lemma 2.6, if there is a positive number* $\underline{\gamma}$ *such that the function* F *satisfies* $2F(x)+x^2 > \underline{\gamma}(x^2+|x|^{p+1})$ *for any* $x \in \mathbb{R}$*, then it follows that, for all* $t>0$ *for which the solution* u *of (2.2.3) exists,*

$$\int_{-\infty}^{\infty} (u^2+|u|^{p+1})\,dx \le \frac{1}{\underline{\gamma}} \int_{-\infty}^{\infty} (2F(u)+u^2)\,dx.$$

In consequence of this a priori deduced estimate, it follows that the local existence result can be iterated to produce a solution u *of (2.2.3) which lies in* $C(0,\infty; L_2 \cap L_{p+1})$.

The next result is a special case of Theorem 2.1 and Corollary 2.7.

COROLLARY 2.8. *Let* $p \ge 1$ *be any integer. The generalized BBM-equation*

$$u_t + u_x + u^{p-1}u_x - u_{xxt} = 0, \quad x \in \mathbb{R},\ t>0,$$

is locally well-posed in L_q *for any* $q \ge p$*. That is, if the initial data* $u(\cdot,0)=u_0 \in L_q$*, then there exists a positive number* $T=T(|u_0|_q)$ *such that the above equation has an unique solution* $u \in C(0,T;L_q)$ *which is continuously dependent on* u_0*. If* $p \ge 3$ *is an odd integer and the initial data* $u_0 \in L_2 \cap L_{p+1}$*, then the solution* u *lies in* $C_b(0,\infty; L_2 \cap L_{p+1})$ *and so is globally defined.*

Remark: Unfortunately, except $p=2$, we don't have a global result in $L_2 \cap L_{p+1}$ for p an even integer greater than 1.

3 BORE-LIKE INITIAL DATA

The theory developed in Section 2 has concentrated on initial profiles that decay to zero at $\pm\infty$, at least in a weak sense. Attention is turned now to initial data that possesses different asymptotic states at $+\infty$ and $-\infty$. In the water wave context, this corresponds to bore propagation in field situations (see Peregrine 1964, 1967) and hydraulic surges in laboratory configurations. In other physical systems, such data is generated when a signal corresponding to a surge moves into an undisturbed stretch of the medium of propagation. Theoretical work on the bore problem in the context of the BBM-equation was initiated by Bona and Bryant (1973) (see also the paper of Bona, Rajopadhye and Schonbek 1994, where further theory was developed for both BBM and the Korteweg-de Vries equations).

In the present contribution, the assumptions on the initial data is weakened and the theory extended to the broader class of models featured in (1.1.2).

The mathematical problem amounts to being confronted with the prospect of solutions $u=u(x,t)$ satisfying the boundary conditions

$$\lim_{x\to-\infty} u(x,t) = l, \qquad \lim_{x\to\infty} u(x,t) = 0, \tag{3.3.1}$$

where $l>0$ is a constant. The question is, if the initial disturbance is bore-shaped, will the wave evolve in a bore-like pattern? If so, how long will this pattern last? Bona, Rajopadhye and Schonbek (1994) showed that the BBM-equation with bore-like initial data as in (3.3.1) is globally well posed and that the solution maintains the boundary behavior (3.3.1) for all time. In this section, the generalized BBM-type model equations (1.1.2) will be discussed in the bore context.

Consider the initial-value problem

$$\left.\begin{aligned} &u_t + u_x + g(u)_x + Lu_t = 0, \\ &u(x,0) = u_0(x), \end{aligned}\right\} \tag{3.3.2}$$

where the operator L and nonlinear function g are as described in Section 2 and the initial data u_0 satisfies the bore condition (3.3.1). Following the technique used by Bona, Rajopadhye and Schonbek (1994), u_0 can be decomposed into the sum of two parts v_0 and ϕ, say, where $\phi \in C^\infty(\mathbb{R})$ satisfies the bore condition (3.3.1) and its derivative ϕ' lies in H^∞, and v_0 is a measurable function on $\mathbb{R}$ whose smoothness is determined by the smoothness of u_0.

Introduce a new variable $v = v(x,t)$ by $u(x,t) = v(x,t) + \phi(x)$. Upon substitution of this form into (3.3.2), there follows the initial-value problem

$$\left.\begin{aligned} &(I+L)v_t + v_x + \big(g(v+\phi) - g(\phi)\big)_x = -\big(1 + g'(\phi)\big)\phi' \\ &v(x,0) = v_0 \end{aligned}\right\} \tag{3.3.3}$$

for v. Inverting the operator $I + L$ and then integrating with respect to t over $[0,t]$, there appears the integral equation

$$\begin{aligned} v =& v_0 + \int_0^t K * \{v + g(v+\phi) - g(\phi)\}(\cdot,\tau)\, d\tau \\ &+ tM * \big(1 + g'(\phi)\big)\phi' \end{aligned} \tag{3.3.4}$$

where the integral kernels K and M are determined via their Fourier symbols, *viz.*

$$\widehat{K}(\xi) = -i\xi/(1+\alpha(\xi)) \quad \text{and} \quad \widehat{M}(\xi) = -1/(1+\alpha(\xi)),$$

respectively. The following result is the analog in the bore context of Theorem 2.1.

THEOREM 3.1. *Suppose the nonlinear function g and the integral kernel K satisfy hypotheses (H1) and (H2), respectively. Moreover, suppose that*

$$\inf_{\xi\in\mathbb{R}} \alpha(\xi) > -1 \quad \textit{and} \quad \liminf_{|\xi|\to\infty} \frac{\alpha(\xi)}{|\xi|} > 0.$$

(This is true for most cases encountered in practice). For any q such that

$$q \geq \frac{pr - r}{r - 1},$$

if $v_0 \in L_q$, then there is a positive number $T = T(|\phi|_\infty, |\phi'|_q) > 0$ such that the integral equation (3.3.4) has an unique solution $v \in C(0,T;L_q)$ and, moreover, the mapping $v_0 \mapsto v$ is continuous from L_q to $C(0,T;L_q)$.

Proof. For any $v \in C(0,\infty;L_q)$, modify the definition of the operator A in Section 2 as follows:

$$Av = v_0 + tM * \Big(\big(1 + g'(\phi)\big)\phi'\Big) + \int_0^t K * \Big\{v + g(v+\phi) - g(\phi)\Big\}\, d\tau. \tag{3.3.5}$$

It is sufficient to prove that A has a fixed point in $C(0,T;L_q)$ for some $T>0$. Note as before that for any $v \in L_q$,

$$v + g(v+\phi) - g(\phi) = \int_0^1 \Big(1 + g'(\phi + sv)\Big)\, ds\, v.$$

In consequence, it follows that

$$\begin{aligned}|K * \big(v + g(v+\phi) - g(\phi)\big)|_q &\le C_0 \Big||K| * \Big(\big(1 + (|\phi| + |v|)^{p-1}\big)|v|\Big)\Big|_q \\ &\le C_0 |K|_1 |v|_q + C_0 \sum_{j=0}^{p-1} \binom{p-1}{j} |\phi|_\infty^j |K|_{q/(q+1-p+j)} |v|_q^{p-j}.\end{aligned}$$

Hence, it is seen that

$$K * \big(v + g(v+\phi) - g(\phi)\big) \in L_q.$$

Since $\phi \in C_b^\infty$, $\phi' \in H^\infty$, g is a C^1-function and the operator M is defined by its Fourier symbol $\frac{1}{1+\alpha(\xi)}$ where α has the decay property just described, it follows that

$$M * \Big(\big(1 + g'(\phi)\big)\phi'\Big) \in H^1 \subset L_q$$

because

$$\begin{aligned}&\int_{-\infty}^{\infty} (1+\xi^2) \Big|\mathcal{F}\Big(M * \big((1+g'(\phi))\phi'\big)\Big)(\xi)\Big|^2 d\xi \\ &= \int_{-\infty}^{\infty} \frac{1+\xi^2}{(1+\alpha(\xi))^2} \Big|\mathcal{F}\big((1+g'(\phi))\phi'\big)(\xi)\Big|^2 d\xi < \infty.\end{aligned}$$

So, A maps $C(0,\infty;L_q)$ to itself. Let B_β be, as before, the closed ball of radius $\beta > 0$ centered at the origin in L_q. For any $v, w \in C(0,\infty;L_q)$,

$$Av(\cdot,t) - Aw(\cdot,t) = \int_0^t K * \{v - w + g(v+\phi) - g(w+\phi)\}(\cdot,\tau)\, d\tau,$$

hence, if $v, w \in C(0,\infty;B_\beta)$, then applying Sublemma 2.2 yields

$$|Av(\cdot,t) - Aw(\cdot,t)|_q \le C \int_0^t \big(1 + (|\phi|_\infty + \beta)^{p-1}\big)|v(\cdot,\tau) - w(\cdot,\tau)|_q\, d\tau \qquad (3.3.6)$$

where the constant C may be taken to be

$$C = C_0 \max_{0 \le j \le p-1} \{|K|_{q/(q-j)}\}.$$

Following the line of argument laid down in the proof of Theorem 2.1, choose

$$\beta = 2|v_0|_q + 2\Big|M * \Big(\big(1 + g'(\phi)\big)\phi'\Big)\Big|_q$$

and

$$T = \min\big\{1, 1/\big(2C(|\phi|_\infty + \beta)^{p-1}\big)\big\}.$$

The operator A is then contractive on $C(0,T;B_\beta)$ and the stated results follow directly. ∎

THEOREM 3.2. *(Regularity 3) Let $v \in C(0,T;L_q)$ be the solution in Theorem 3.1. In addition, suppose for some $k \geq 1$, the nonlinear function $g \in C^k$ and $g^{(k)}$ is bounded by a polynomial of degree less than or equal to $p-k$. Then for $j = 1, \cdots, k$,*

$$\frac{\partial^j v}{\partial t^j} \in C(0,T;L_q).$$

Proof. Define a sequence $\{v_n\}_{n\geq 1}$ iteratively by

$$\begin{aligned} v_1 &= A\theta \\ &= v_0 + tM * \big((1+g'(\phi))\phi'\big) \end{aligned} \tag{3.3.7}$$

where θ is, as before, the zero-function and for $n \geq 1$,

$$\begin{aligned} v_{n+1} &= Av_n \\ &= v_0 + tM * \big((1+g'(\phi))\phi'\big) \\ &\quad + \int_0^t K * \{v_n + g(v_n+\phi) - g(\phi)\}(\cdot,\tau)\,d\tau. \end{aligned} \tag{3.3.8}$$

The solution v in Theorem 3.1 can be obtained as the limit of the sequence $\{v_n\}_{n\geq 1}$, so in particular, the sequence is Cauchy in $C(0,T;L_q)$. By the Fundamental Theorem of Calculus,

$$\begin{aligned} \partial v_1/\partial t &= M * \big((1+g'(\phi))\phi'\big) \\ &\vdots \\ \partial v_{n+1}/\partial t &= M * \big((1+g'(\phi))\phi'\big) + K * \{v_n + g(v_n+\phi) - g(\phi)\}. \end{aligned}$$

It is straightforward to see that $\{\partial v_n/\partial t\}_{n\geq 1} \subset C(0,T;L_q)$. As before, for $n > 1$,

$$\partial v_{n+1}/\partial t - \partial v_n/\partial t = K * \Big(v_n - v_{n-1} + g(\phi+v_n) - g(\phi+v_{n-1})\Big),$$

and thus, the Sublemma implies again that for sufficiently large values of n,

$$\begin{aligned} &|\partial v_{n+1}(\cdot,t)/\partial t - \partial v_n(\cdot,t)/\partial t|_q \\ \leq &C_1\Big(1 + \big[|\phi|_\infty + 2\|v\|_{C(0,T;L_q)}\big]^{p-1}\Big)|v_n(\cdot,t) - v_{n-1}(\cdot,t)|_q \end{aligned}$$

where $C_1 = C_0 \max_{0\leq j\leq p-1}\{|K|_{q/(q-j)}\}$. Since the sequence $\{v_n\}_{n\geq 1}$ is Cauchy in $C(0,T;L_q)$, so is $\{\partial v_n/\partial t\}_{n\geq 1}$. Denote its limit as $n \to \infty$ by w_1. The function w_1 satisfies the equation

$$w_1 = M * \big((1+g'(\phi))\phi'\big) + K * \{v + g(v+\phi) - g(\phi)\},$$

and by a uniqueness argument as expounded previously, $w_1 = v_t$. By induction on j, it follows that for any j with $1 \leq j \leq k$, $\{\partial^j v_n/\partial t^j\}_{n\geq 1}$ is Cauchy in $C(0,T;L_q)$ and the limit as $n \to \infty$ is equal to $\partial^j v/\partial t^j$. ∎

The following further regularity result is the analog of Theorem 2.4. As the proof is entirely similar, it is omitted.

THEOREM 3.3. *(Regularity 4) Let $v \in C(0,T;L_q)$ be the solution of (3.3.2) obtained in Theorem 3.1. Suppose in addition that for some $k \geq 1$, $v_0 \in C_b^{k-1}$ and $g \in C^k$ and its k^{th} derivative $g^{(k)}$ is bounded by a polynomial of degree less than or equal to $p-k$. Then $v \in C(0,T;C_b^{k-1} \cap W_q^{k-1})$.*

PROPOSITION 3.4. *In the above Theorem, if $p = 2n-1 > 1$ is an odd number and there are two positive numbers γ_1 and γ_2 such that the nonlinear function $g(z) \geq (\gamma_1 - 1)z + 2n\gamma_2 z^{2n-1}$ for all $z \geq 0$, then the equation (3.3.4) is well-posed in $L_2 \cap L_{2n}$ globally in time, in the sense that for any initial data $v_0 \in L_2 \cap L_{2n}$ and $\bar{T} > 0$, the solution v lies in $C(0,\bar{T};L_2 \cap L_{2n})$.*

Proof. Theorem 3.1 guarantees that there is $T > 0$ such that (3.3.4) has a unique solution $v \in C(0,T;L_{2n})$. As in the proof of Theorem 2.4, it can be shown that v also lies in $C(0,T;L_2)$. It is sufficient to show that the solution can be extended to times that are arbitrarily large.

Define a function F on $\mathbb{R}$ by $F(z) = \int_0^z g(z)\,dz$. Because of hypothesis (H2) and the restriction on g, it is easily deduced that $F(z) \geq \frac{1}{2}(\gamma_1 - 1)z^2 + \gamma_2 z^{2n}$ for some positive constants γ_1 and γ_2. Define a functional I by

$$I(v) = \int_{-\infty}^{\infty} \big(\frac{1}{2}v^2 + F(v)\big)\,dx.$$

If v is a solution of (3.3.4), then formally

$$\begin{aligned}
\frac{d}{dt}I(v) &= \int_{-\infty}^{\infty} (v + g(v))u_t\,dx \\
&= \int_{-\infty}^{\infty} (v + g(v))\big[K * (v + g(v)\big) + M * (1 + g'(\phi))\phi'\big]\,dx \\
&= \int_{-\infty}^{\infty} (v + g(v))\big(M * (1 + g'(\phi))\phi'\big)\,dx \\
&\leq C_0\|M * (1 + g'(\phi))\phi'\|\|v\| + C_0|M * (1 + g'(\phi))\phi'|_{p+1}|v|_{p+1}^p \\
&\leq C_1(\|v\|^2 + |v|_{2n}^p) + C_2 \\
&\leq \bar{\gamma}I(v) + C_2,
\end{aligned}$$

where C_1 and C_2 are constants dependent only on the quantities $|\phi|_\infty$ and $\|\phi'\|_1$ and $\bar{\gamma} = C_1/\min\{\frac{1}{2}\gamma_1, \gamma_2\}$. As before, this formal calculation is justified by the regularity theory combined with the continuous dependence result. A Gronwall-type inequality shows that for any $t > 0$,

$$I(v(\cdot,t)) \leq I(u_0)e^{\gamma_1 t} + \frac{C_2}{\gamma_1}(e^{\gamma_1 t} - 1).$$

This means that on any time interval $[0,\bar{T}]$, the L_2- and L_{2n}- norm of the solution v is finite. The standard extension argument then completes the proof. ∎

COROLLARY 3.5. *For the generalized BBM-equation*

$$u_t + u_x + u^{2n-2}u_x - u_{xxt} = 0,$$

where $n \geq 2$, if the initial data $u_0 = v_0 + \phi$ where ϕ is an infinitely smooth bore and $v_0 \in L_2 \cap L_{2n}$, then there is a unique solution $u = v + \phi$ where $v \in C(0,\infty;L_2 \cap L_{2n})$ which depends continuously on v_0.

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