On Strong Convergence in Vortex Sheets Problem for 3-D Axisymmetric Euler Equations

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Abstract: We consider the three dimensional axisymmetric incompressible Euler equations without swirls with vortex-sheets initial data. Two kinds of approximate solutions generated by smoothing the initial data and viscous regularization respectively are discussed. It is proved that both kinds of approximate solutions converge strongly in $L^2([0,T]; L^2_{loc}(R^3))$ provided that they have strong convergence in the region away from the symmetry axis respectively, without the restriction on the signs of initial vorticity. This means that if there would appear singularity or energy lost in the process of limit for both approximate solutions respectively, it then must happen in the region away from the symmetry axis. Moreover, one sufficient condition to guarantee the strong convergence in the region away from the symmetry axis is given. And a higher decay rate for maximal vorticity function in the region away from the symmetry axis is obtained for non-negative initial vorticity. In order to exclude the possible concentrations on the symmetry axis, we use the special structure of the equations for axisymmetric flows and careful choice of test functions.

Key Words: 3-D axisymmetric Euler equations, strong convergence, weak solutions, existence

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1 Introduction

Consider the 3-D incompressible Euler equations in R^3

$$\begin{aligned} &\partial_t u + (u\cdot\nabla)u + \nabla p = 0, \quad (x,t)\in R^3\times(0,T),\\ &\text{div } u = 0, \quad |u|\to 0 \quad \text{as} \quad |x|\to+\infty, \end{aligned} \tag{1.1}$$

with initial conditions

$$u(x,t)\mid_{t=0} = u_0(x). \tag{1.2}$$

The equations (1.1) describe motions of incompressible homogeneous inviscid flows. The unknown functions here are the velocity vector $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ and the pressure $p(x,t)$. T is a fixed positive constant.

As is well-known, the global solvability of system (1.1)-(1.2) is still an outstanding open problem in the mathematical theory of fluid mechanics. The local existence and uniqueness of the classical solutions and the various criteria for development of singularities were shown in [1],[12] and [26] for $\Omega = R^3$ and bounded domain respectively. Even for the axisymmetric case, there is still no global solvability for (1.1)-(1.2), except for investigations of the possible development of singularities presented in [2], [3] and the references therein, which is similar to the general case.

We are concerned with the axisymmetric solutions of (1.1)-(1.2) in this paper. The cylindrical transformation in R^3 is usually defined as

$$\begin{aligned} &\pi : \bar{R}_+\times[0,2\pi)\times R \longrightarrow R^3,\\ &(r,\theta,z)\longmapsto(x_1,x_2,x_3),\\ &x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z. \end{aligned} \tag{1.3}$$

By axisymmetric solutions of (1.1), we mean that, in the cylindrical coordinate system, the unknown functions $u(x,t)$ and $p(x,t)$ do not depend on θ-variable, that is,

$$\begin{aligned} u(x,t) &= u^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + u^z(r,z,t)e_z,\\ p(x,t) &= p(r,z,t), \end{aligned}$$

where

$$e_r = (\cos\theta,\sin\theta,0), \quad e_\theta = (-\sin\theta,\cos\theta,0), \quad e_z = (0,0,1)$$

is the standard orthogonal bases in the coordinate system. In this case, the Euler equations (1.1) can be written as

$$\frac{\tilde{D}u^r}{Dt} - \frac{(u^\theta)^2}{r} + \partial_r p = 0, \tag{1.4}$$

$$\frac{\tilde{D}(ru^\theta)}{Dt} = 0, \tag{1.5}$$

$$\frac{\tilde{D}u^z}{Dt} + \partial_z p = 0. \tag{1.6}$$

$$\partial_r(ru^r) + \partial_z(ru^z) = 0. \tag{1.7}$$

In the equations (1.4)-(1.6) and in the following, we denote

$$\frac{\tilde{D}}{Dt} = \frac{\partial}{\partial t} + u^r\partial_r + u^z\partial_z, \quad r = (x_1^2 + x_2^2)^{1/2}.$$

When $u^\theta \equiv 0$, The equations (1.4)-(1.7) are called 3-D axisymmetric Euler equations without swirls. In this case, the vorticity of the velocity has a simple expression,

$$\omega = \nabla \times u = \omega^\theta e^\theta,$$

where $\omega^\theta = \partial_r u^z - \partial_z u^r$. It is an important fact that ω^θ/r satisfies the following transport equation,

$$\frac{\tilde{D}}{Dt}\left(\frac{\omega^\theta}{r}\right) = 0. \tag{1.8}$$

Due to this conservation property for ω^θ/r, global existence and uniqueness of regular solutions for 3-D axisymmetric Euler equations without swirls have been proved by Majda, Saint-Raymond, and Shirota and Yanagisava in [21], [25] and [27] respectively. For less regular initial data, i.e., if $u_0 \in L^2$, $\omega_0, \omega_0/r \in L^q \cap L^\infty$ for some $q < 3$, the global existence and uniqueness have also been established in [25]. If the initial data becomes more singular such as , $\omega_0, \omega_0/r \in L^{5/6} \cap L^p (p > 3)$ or even weaker, $u_0 \in L^2, \omega_0/r \in L^1 \cap L(\log^+ L)^\alpha (\alpha > 1/2)$, which is an Orlitz space, one only has global existence of weak solutions (see [4], [5] and the references therein).

Our main interest here is to study the solvability of (1.4)-(1.7) with vortex-sheets initial data. To this end, we first recall the definition of the weak solutions to the 3-D Euler equations (1.1) in R^3 with initial data (1.2):

Definition 1.1. Suppose that $u_0(x) \in L^2_{loc}(R^3)$. For all $T > 0$, $u(x,t) \in L^\infty([0,T]; L^2_{loc}(R^3))$ is called a weak solution to (1.1)-(1.2), if

i) $\forall \psi \in C_0^\infty(R^3 \times [0,T)), \operatorname{div} \ \psi = 0,$

$$\int_{R^3} \psi(x,0)u_0(x)dx + \int_0^T \int_{R^3} (\psi_t \cdot u + \nabla\psi : u \otimes u)dxdt = 0; \tag{1.9}$$

ii) $\forall \phi \in C_0^\infty(R^3 \times [0,T])$,

$$\int_0^T \int_{R^3} \nabla \phi \cdot u dx dt = 0;$$

iii)

$$u(x,t) \in Lip([0,T]; H_{loc}^{-s}(R^3)) \quad \text{for some } s > 0,$$

In (1.9), $u \otimes u$ means a matrix $(u_i u_j)$ and $A : B = \sum_{i,j} a_{ij} b_{ij}$ for two matrixes $A = (a_{ij})$ and $B = (b_{ij})$.

Vortex-sheets problem, roughly speaking, is the solvability problem for the Euler equations under the assumption that the initial data is a vortex-sheets data, *i.e.* , the initial vorticity is a finite Radon measure and the initial velocity is locally square integrable. More precisely, in the 2-D case, the problem is to obtain global existence of weak solutions and their structures to the Euler equations with initial vorticity $\omega_0(x) = curl u_0(x)$ satisfying

$$\omega_0 \in M(R^2) \cap H_{comp}^{-1}(R^2), \tag{1.10}$$

Here $M(R^2)$ is the space of finite Radon measures and $H_{comp}^{-1}(R^2)$ is the dual space of usual Hilbert space $H^1(R^2)$ with compact support. Such initial data arises in the evolution of vortex sheets (see [9], [21] and [23]). Vortex-sheets problem is a hard and open problem for both two-dimensional Euler equations and three-dimensional Euler equations. Only partial results are available. Indeed, observing the important fact that the key to the existence of classical weak solutions to the 2-D vortex-sheets problem is the vorticity concentrations instead of the energy concentration, Delort obtained the first existence of a global classical weak solution to the 2-D vortex-sheets problem with the additional assumption that the initial vorticity ω_0 is of one-sign by showing the convergence of the approximate solutions constructed by regularizing the initial data [6]. Furthermore, for the one-sign initial vorticity, the convergence to classical weak solutions of the vortex-sheets problem of the approximate solutions generated either by viscous regularization or vortex methods has been established respectively by Majda [22] and Liu-Xin in [19] and [20] by considering uniform decay of the vorticity maximal functions. In these contexts, the readers are referred to [11] and [28] for more simplifications and clarifications.

In the case that the vorticity may change signs, then the fluids with different directions of rotations may interact and interwine and thus produce complicated flow patterns, the only known results are due to Lopes Filho-Nussenzveig Lopes-Xin in [18], where they established the global existence of a classical weak solution to the 2-D vortex-sheets problem for initial vorticity with reflection symmetry. Their results are based on the fact that there are no vorticity-concentrations occuring even for flows which allow interactions of fluids with different signs of vorticity but no interwining [18].

The global existence of classical weak solutions to the 3-D Euler equations remains to be one of the biggest challenges in the mathematical theory of fluid dynamics. This is so even for smooth initial data. For the axisymmetric 3-D flows without swirls, the well-posedness theory and analysis are similar to those of 2-D flows for regular initial data due to the conservation property (1.8). However, surprisingly, this parallelness breaks down for vortex-sheets initial data in that for one-sign initial vorticity there is no concentration-cancellation occuring for axisymmetric flows, as was observed by Delort in [7]. Precisely, Delort proved that, if the vortex-sheets initial vorticity has distinguished sign, the sequence of approximate solutions obtained by smoothing the initial data either converges strongly in $L^2_{loc}(R^3 \times (0, +\infty))$ or converges weakly in $L^2_{loc}(R^3 \times (0, +\infty))$ to a limit which is not a classical weak solution to the Euler equations in the sense of distribution [7]. This is in sharp contrast to the 2-D theory. Still, the existence of classical weak solutions to the vortex-sheets problem for axisymmetric 3-D Euler equations without swirls has not been established yet.

Our main concerns in this paper are structures and convergence properties of the sequences of approximate solutions to the vortex-sheets problem for axisymmetric flows without swirls. We study approximate solutions generated by both regularizing the vortex-sheets initial data and solving Navier-Stokes equations respectively. Our main results in this paper show that for approximate solutions to the vortex-sheets problem, generated either by smoothing the initial data or by viscous approximation, if they converge strongly in L^2 over the region outside the symmetry axis, then they must converge strongly in $L^2([0, T]; L^2_{loc}(R^3))$. This implies that for the 3-D axisymmetric Euler equations without swirls, if there would appear energy concentration in the process of limit for both kinds of approximate solutions, the set of energy-concentration must contain points in the region outside the symmetry axis. It should be pointed out that there is no restriction on the signs of the initial vorticity in our theory. Our analysis is based on some concentration-compactness arguments. To exclude the possible energy concentrations of the approximate solutions to the vortex-sheets problem on the symmetry axis, we make full use of the integral equations satisfied by the weak limits of $|u^\varepsilon|^2$ in the sense of measure and carefully construct various special test functions. It should be noted that although similar concentration-compactness arguments were used in [13] and [17] for solving nonstationary axisymmetric and general stationary compressible Navier-Stokes equations respectively, their arguments are based on the a priori bound on the space-time integral of $|\nabla u^\varepsilon|^2$, which can not hold to be true for approximate solutions to the vortex-sheets problem. So the weak limit of $|u^\varepsilon|^2$ in $M(R^3 \times [0,T])$, the space of Radon measures, may have more generic and complicated structures on their singular part, compared with the counterparts in [13] and [17]. Our approach to overcome these difficulties is to make use of the special structures of the equations for axisymmetric flows, and take into account the assumption that no energy concentrations occur in the region away from the axis of symmetry, and then choose carefully the test functions to obtain more cancellation-combinations which yield the strong convergence in the whole space, for details, see Section 3.

It follows from our main results that the key to the existence of classical weak solutions to the vortex-sheets problem for axisymmetric flows without swirls is whether energy concentrations occur in the region outside the axis of symmetry in the limit of approximate solutions. Following the approach of DiPerna and Majda in [9] for the 2-D Euler equations, we derive a sufficient condition in terms of uniform decay of the vorticity maximal function to guarantee the strong convergence of the velocity fields in the L^2- norm in the region outside the axis of symmetry for both kinds of approximate solutions, see Theorem 4.1. We also obtain an uniform decay estimate for the vorticity maximal function of the approximate solutions to the vortex-sheets problem for axisymmetric flows without swirls with one-sign vorticity in the region away from the symmetry axis, which is,

$$\max_{x_0\in\Omega_\delta^K, 0\le t\le T} \int_{B_R(x_0)} |\omega^\varepsilon(x,t)|dx \le C(K,\delta) R \log(\frac{1}{R})^{-\frac{1}{2}}, \tag{1.11}$$

where $B_R(x_0) = \{x : |x - x_0| < R\}$, $0 < R << 1\}$, and $\Omega_\delta^K = \{x \in R^3 \mid 0 < \delta < \sqrt{x_1^2 + x_2^2},\ |x| < K < +\infty\}$. Although this decay rate is not higher enough to get the strong convergence of our approximations solutions sequences in the region away from the symmetry axis and it is obtained only for the vorticity with distinguished sign, we notice that it is higher than that in general case, which is due to the axisymmetric property of the solution. There is still a gap between this decay rate and the sufficient condition mentioned above.

The rest of this paper is organized as follows. In Section 2, we construct sequences of approximate solutions to vortex-sheets problem for axisymmetric flows without swirls by smoothing the initial data and by viscous approximations, which are solutions of corresponding Navier-Stokes equations. Some properties and preliminary uniform estimates on the approximate solutions are also given here. Section 3 contains the main arguments to prove the strong convergence in L^2 of the velocity fields of the approximate solutions in the whole space provided that they do converge in the region away from the symmetry axis. In Section 4, we give a sufficient condition in terms of the decay of the vorticity maximal function to guarantee the strong convergence in the region away from the symmetry axis. Finally, we obtain a decay rate for the vorticity maximal function in the region outside the symmetry axis and in the case of one-sign vorticity in Section 5.

2 Approximate Solutions

For the 3-D axisymmetric Euler equations without swirls, the vortex-sheets initial data with non-negative initial vorticity can be described in the following way (see [7]).

Assumptions (A): Suppose that $\omega_0^\theta = \omega_0^\theta(r,z)$ is a positive finite measure with compact support on $\{(r,z) \in R^2 \mid r \geq 0\}$, and that the initial vorticity $\omega_0(x), x \in R^3$, defined by $\omega_0 dx = \pi_*(r\omega_0^\theta(r,z)e_\theta d\theta)$, belongs to $H^{-1}(R^3)$, which guarantees that the initial velocity $u_0(x) = -\triangle^{-1}\nabla \times \omega_0(x)$ belongs to $L^2(R^3)$.

Here π_* is the measure image induced by π which is defined in (1.3), *i.e.,* assume that $d\mu$ is a measure on (r,θ,z)-space, then $\pi_* d\mu$ is a measure on (x_1,x_2,x_3)-space defined by

$$\int_\Omega g(x)(\pi_* d\mu) = \int_{\tilde{\Omega}} g \circ \pi d\mu, \tag{2.1}$$

where $g(x) \in C_0(\Omega)$ and $\Omega = \pi(\tilde{\Omega})$.

The initial data given in Assumptions (A) can be regularized through the usual way.

Proposition 2.1.(see [7]) (1) There exists a smooth sequences $\{(\omega_0^\theta)^\varepsilon(r,z)\}$ such that $(\omega_0^\theta)^\varepsilon \geq 0$, $(\omega_0^\theta)^\varepsilon \in C_0^\infty((0,+\infty)\times R)$, and

$$\int_{-\infty}^{+\infty}\int_0^{+\infty}(\omega_0^\theta)^\varepsilon dr dz \leq C,$$

(2) Let $\omega_0^\varepsilon(x) = (\omega_0^\theta)^\varepsilon e_\theta$. Then $\omega_0^\varepsilon \in C_0^\infty(R^3\backslash\{r=0\})$ and ω_0^ε is uniformly bounded in $H^{-1}(R^3)\cap L^1(R^3)$, and

$$\omega_0^\varepsilon \rightharpoonup \omega_0$$

weakly in $H^{-1}(R^3)$.

(3) Let $u_0^\varepsilon = -\nabla \times \triangle^{-1}\omega_0^\varepsilon$. Then $u_0^\varepsilon \in C^\infty(R^3)$, and u_0^ε is uniformly bounded in $L^2(R^3)$, and u_0^ε is axisymmetric. Furthermore, we have $(u_0^\varepsilon)^\theta \equiv 0$ and

$$u_0^\varepsilon \rightharpoonup u_0 = -\nabla \times \triangle^{-1}\omega_0$$

weakly in $L^2(R^3)$.

Proof. To be self-contained, we outline the proof here. The details can be found in [7].

Step 1. Cutting off ω_0^θ near the axis

Let $\Psi = \Psi(s) \in C_0^\infty(R)$ be a smooth function satisfying $0 \leq \Psi(s) \leq 1$ and $\Psi \equiv 1$ for $|s| \leq 1$. Let

$$\begin{aligned} \omega_0^\delta dx \ &= \pi_*((1-\Psi(\tfrac{r}{\delta}))\Psi(\delta r)\Psi(\delta z) r\omega_0^\theta e_\theta d\theta) \\ &(1-\Psi(\tfrac{x_1^2+x_2^2}{\delta}))\Psi(\delta(x_1^2+x_2^2))\Psi(\delta x_3)\omega_0 dx. \end{aligned}$$

Then it can be seen easily that $\omega_0^\delta(x)$ converges to $\omega_0(x)$ in the sense of distributions as $\delta \to 0$. Furthermore, it can be proved that ω_0^δ is bounded in $H^{-1}(R^3)$ and so $\omega_0^\delta(x)$ converges weakly in $H^{-1}(R^3)$ to $\omega_0(x)$.

Step 2. Regularizing the initial data

Suppose that $\rho \in C_0^\infty(R)$ is a smooth function satisfying $0 \leq \rho \leq 1$ and $\int \rho(t)dt = 1$. Suppose that $\rho_\varepsilon(s) = (1/\varepsilon)\rho(s/\varepsilon)$ is the standard mollifer for $s \in R$ and $0 < \varepsilon \leq 1$. Define

$$(\omega_0^\theta)^{\varepsilon,\delta}(r,z) = \frac{1}{\varepsilon^2}\int \rho\left(\frac{\ln r - \ln r'}{\varepsilon}\right)\rho(\frac{z-z'}{\varepsilon})(\frac{1}{r'})d\omega_0^\delta(r',z'),$$

$$\omega_0^{\varepsilon,\delta}dx = \pi_*(r(\omega_0^\theta)^{\varepsilon,\delta}e_\theta drdzd\theta).$$

Then it is clear that for any $\varepsilon, \delta > 0$, when $\varepsilon < \delta$ small enough, $(\omega_0^\theta)^{\varepsilon,\delta}$ belongs to $C_0^\infty(R^+ \times R)$ and is uniformly bounded in $L^1(R^+ \times R; drdz)$, and $\omega_0^{\varepsilon,\delta}$ is a smooth function with compact support in $R^3\backslash\{r=0\}$. Furthermore, we claim that for any $\delta > 0$ small, $\omega_0^{\varepsilon,\delta}dx$ is uniformly bounded in $H^{-1}(R^3)$ and converge to $(\omega_0^\theta)^\delta$ weakly in $H^{-1}(R^3)$ as $\varepsilon \to 0$ and this convergence is uniformly on δ. In fact, for any $\varphi \in C_0^\infty(R^3)$, it suffices to prove that for $0 < \varepsilon < \delta$ small enough,

$$|\int \varphi(x)\cdot(\omega_0^\theta)^{\varepsilon,\delta}(x)dx| \leq C\|\varphi\|_{H^1}, \tag{2.2}$$

where C does not depend on $(\omega_0^\theta)^{\varepsilon,\delta}$ and δ. To this end, we rewrite the left hand of (2.2) as

$$\int \varphi(x)\cdot(\omega_0^\theta)^{\varepsilon,\delta}(x)dx = \int \varphi^\varepsilon \cdot (\omega_0^\theta)^\delta,$$

where

$$\begin{aligned} &\varphi^\varepsilon(r'\cos\theta, r'\sin\theta, z') \\ &= \int_0^{+\infty}\int_R \frac{r}{r'^2}\rho\left(\frac{\ln r - \ln r'}{\varepsilon}\right)\rho(z-z')\varphi(r\cos\theta, r\sin\theta, z)drdz \\ &\equiv I^\varepsilon(r',z',\theta). \end{aligned}$$

Therefore, to get (2.2), it suffices to prove that the $L^2(r'dr'dz'd\theta)-$ norms of I^ε, $\partial_{r'}I^\varepsilon$, $(1/r')\partial_\theta I^\varepsilon$ and $\partial_{z'}I^\varepsilon$ are all bounded by $C\|\varphi\|_{H^1}$, where C is a constant independent of ω_0^δ and δ. These estimates are routine and we omit the details here.

It follows from the existence and uniqueness of the classical solutions to the 3-D axisymmetric Euler equations without swirls established in [21], [25] and [27], one can construct the approximate solutions by regularizing the initial data:

Proposition 2.2. (see [7]) Under the assumptions (A), there exist smooth approximate solutions $(u^\varepsilon, p^\varepsilon)$ to the 3-D Euler equations (1.1) with initial data presented in Proposition 2.1 such that for any $T > 0$,

(i)

$$u^\varepsilon(x,t) \text{ is uniformly bounded in } L^\infty([0,T]; L^2(R^3));$$

(ii) u^ε is axisymmetric, i.e.

$$u^\varepsilon(x,t) = (u^r)^\varepsilon e_r + (u^z)^\varepsilon e_z;$$

(iii)

$$\omega^\varepsilon = \nabla \times u^\varepsilon = (\omega^\theta)^\varepsilon e_\theta, \quad (\omega^\theta)^\varepsilon = \partial_r (u^z)^\varepsilon - \partial_z (u^r)^\varepsilon \geq 0;$$

(iv) $(\omega^\theta)^\varepsilon$ is uniformly bounded in $L^\infty(R; L^1(\bar{R}_+ \times R, (1+r^2)drdz))$ (Here $\bar{R}_+$ is the set of $[0,+\infty)$), that is

$$\max_{0\leq t\leq T} \int_{-\infty}^{+\infty}\int_0^{+\infty} (\omega^\theta)^\varepsilon drdz \leq C;$$
$$\max_{0\leq t\leq T} \int_{-\infty}^{+\infty}\int_0^{+\infty} (\omega^\theta)^\varepsilon r^2 drdz \leq C,$$

where C is a constant independent of ε and T.

Proof. Under Assumptions (A), the regular initial data are constructed in Proposition 2.1. Then for any $\varepsilon > 0$, there exists a unique smooth axisymmetric solutions $(u^\varepsilon, p^\varepsilon)$ of (1.1) with initial condition $u_0^\varepsilon(x)$ ([21], [25], [27]). And standard energy estimate gives (i) of the proposition. Since u^ε is axisymmetric, it can be expressed as

$$u^\varepsilon(x,t) = (u^r)^\varepsilon e_r + (u^z)^\varepsilon e_z.$$

So the vorticity $\omega^\varepsilon = \nabla \times u^\varepsilon$ has only one non-vanishing component $(\omega^\theta)^\varepsilon$, i.e., $\omega^\varepsilon = (\omega^\theta)^\varepsilon e^\theta$. Furthermore, $(\omega^\theta)^\varepsilon$ satisfies

$$\begin{cases} (\dfrac{\partial}{\partial t} + (u^r)^\varepsilon \partial_r + (u^z)^\varepsilon \partial_z)(\dfrac{(\omega^\theta)^\varepsilon}{r}) = 0, \\ (\omega^\theta)^\varepsilon \mid_{t=0} = (\omega_0^\theta)^\varepsilon, \end{cases}$$

from which, we get that $(\omega^\theta)^\varepsilon \geq 0$ and

$$\int_{-\infty}^{+\infty}\int_0^{+\infty}(\omega^\theta)^\varepsilon(r,z,t)drdz = \int\int(\omega_0^\theta)^\varepsilon drdz \leq C.$$

On the other hand, the following well-known conserved integral quantity for smooth solutions of the 3-D Euler equations (1.1) ([21])

$$\int_{R^3} x\times\omega(x,t)dx = \int_{R^3} x\times\omega_0(x)dx, \quad x\in[0,T],$$

implies

$$\int_{-\infty}^{+\infty}\int_0^{+\infty}\omega^\theta(r,z,t)r^2drdz = \int_{-\infty}^{+\infty}\int_0^{+\infty}\omega_0^\theta(r,z)r^2drdz, \quad t\in[0,T].$$

Noting that in our case, $(\omega_0^\theta)^\varepsilon$ has uniform compact surpport on ε, we obtain

$$\max_{0\leq t\leq T}\int_{-\infty}^{+\infty}\int_0^{+\infty}(\omega^\theta)^\varepsilon r^2drdz \leq C.$$

The proof of the proposition is finished.

For general vortex-sheet initial data, we may assume

Assumptions (A′): Suppose that $\omega_0^\theta = \omega_0^\theta(r,z)$ is a finite measure with compact support on $\{(r,z)\in R^2 \mid r\geq 0\}$, and that the initial vorticity $\omega_0(x), x\in R^3$, defined by $\omega_0 dx = \pi_*(r\omega_0^\theta(r,z)e_\theta d\theta)$, belongs to $H^{-1}(R^3)$, which guarantees the initial velocity $u_0(x) = -\triangle^{-1}\nabla\times\omega_0(x)$ belonging to $L^2(R^3)$.

In a similar way as for the proof of Proposition 2.1 and Proposition 2.2, one can show that

Proposition 2.1′. Under Assumptions (A′), the following statements hold

(1) There exists a smooth sequence $\{(\omega_0^\theta)^\varepsilon(r,z)\}$ such that $(\omega_0^\theta)^\varepsilon \in C_0^\infty((0,+\infty)\times R)$, and

$$\int_{-\infty}^{+\infty}\int_0^{+\infty}|(\omega_0^\theta)^\varepsilon|drdz \leq C.$$

(2) Let $\omega_0^\varepsilon(x) = (\omega_0^\theta)^\varepsilon e_\theta$. Then $\omega_0^\varepsilon \in C_0^\infty(R^3\backslash\{r=0\})$ and ω_0^ε is uniformly bounded in $H^{-1}(R^3)\cap L^1(R^3)$, and

$$\omega_0^\varepsilon \rightharpoonup \omega_0$$

weakly in $H^{-1}(R^3)$.

(3) Let $u_0^\varepsilon = -\nabla \times \triangle^{-1}\omega_0^\varepsilon$. Then $u_0^\varepsilon \in C^\infty(R^3)$, and u_0^ε is uniformly bounded in $L^2(R^3)$, and u_0^ε is axisymmetric. Furthermore, we have $(u_0^\varepsilon)^\theta \equiv 0$ and

$$u_0^\varepsilon \rightharpoonup u_0 = -\nabla \times \triangle^{-1}\omega_0$$

weakly in $L^2(R^3)$.

Proposition 2.2′. Under the assumptions (A′), there exist smooth approximate solutions $u^\varepsilon, p^\varepsilon$ of 3-D Euler equations (1.1) with initial data presented in Proposition 2.1′ such that for any $T > 0$,

(i)

$$u^\varepsilon(x,t) \text{ is uniformly bounded in } L^\infty([0,T]; L^2(R^3));$$

(ii) u^ε is axisymmetric, i.e.

$$u^\varepsilon(x,t) = (u^r)^\varepsilon e_r + (u^z)^\varepsilon e_z;$$

(iii)

$$\omega^\varepsilon = \nabla \times u^\varepsilon = (\omega^\theta)^\varepsilon e_\theta, \quad (\omega^\theta)^\varepsilon = \partial_r (u^z)^\varepsilon - \partial_z (u^r)^\varepsilon;$$

(iv) $(\omega^\theta)^\varepsilon$ is uniformly bounded in $L^\infty(R; L^1(\bar{R_+} \times R, drdz))$, that is

$$\max_{0\le t\le T} \int_{-\infty}^{+\infty}\int_0^{+\infty} |(\omega^\theta)^\varepsilon| drdz \le C;$$

where C is a constant independent of ε and T.

Remark 2.1 Since $(\omega^\theta)^\varepsilon$ may change sign in general, although we can obtain the uniform estimate of ω^ε/r in $L^\infty([0,T]; L^1(R^3))$ under Assumptions (A′), yet uniform estimate on $(\omega^\theta)^\varepsilon$ in the same space is not clear.

Now we construct the viscous approximations through Navier-Stokes equations. We start with the case of Assumptions (A), where the initial vorticity is non-negative. Consider the following problem

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \varepsilon \triangle u, \quad (x,t) \in R^3 \times (0,T), \\ \text{div } u = 0, \\ u(x,t) \mid_{t=0} = u_0^\varepsilon(x), \end{cases} \tag{2.3}$$

Here u_0^ε is the smooth initial data stated in Proposition 2.1. It is known that there exists a unique smooth solutions u, p to (2.3) (see [15]), and u is axisymmetric with $u^\theta \equiv 0$. Furthermore, the corresponding vorticity $\omega = \omega^\theta e_\theta$ satisfies

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u + \nabla p = \varepsilon\triangle\omega, \quad (x,t)\in R^3\times(0,T), \\ \text{div } u = 0, \\ \omega\mid_{t=0} = \omega_0^\varepsilon(x), \end{cases} \tag{2.4}$$

and

$$\begin{cases} \frac{\tilde{D}}{Dt}\left(\frac{\omega^\theta}{r}\right) = \varepsilon(\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r)\left(\frac{\omega^\theta}{r}\right) + \varepsilon\frac{2}{r}\partial_r\left(\frac{\omega^\theta}{r}\right), \quad (r,z)\in R^+\times R, \\ \frac{\omega^\theta}{r}\mid_{t=0} = \frac{(\omega_0^\theta)^\varepsilon}{r}, \end{cases} \tag{2.5}$$

Here ω_0^ε and $(\omega_0^\theta)^\varepsilon$ are also the smooth initial data given in Proposition 2.1. It follows easily then

$$\begin{aligned} &\frac{(\omega_0^\theta)^\varepsilon}{r} \geq 0, \\ &\frac{(\omega_0^\theta)^\varepsilon}{r} \longrightarrow 0 \quad \text{as} \quad r\to 0. \end{aligned} \tag{2.6}$$

Moreover, since $\omega = \omega^\theta e_\theta$ is the smooth solutions of (2.4), one can show easily that $\omega^\theta(r,z,t)\mid_{r=0} = 0$ and $\partial_r^k\omega^\theta(r,z,t)\mid_{r=0} = 0$ for any $k\geq 1$. Therefore, we can obtain that

$$\frac{\omega^\theta}{r} \longrightarrow 0 \quad \text{as} \;\; r\to 0, \tag{2.7}$$

(in fact, one can obtain that $\omega^\theta/r^k \to 0$ as $r\to 0$ for any $k\geq 1$). We denote by $\{u^\varepsilon, \omega^\varepsilon\}$ the smooth solutions to the problems (2.3) and (2.4), which are our viscous approximations for Cauchy problem of (1.1). Then we have

Proposition 2.3 The results of Proposition 2.2 still hold for the viscous approximate solutions u^ε and ω^ε, which solve the problems (2.3) and (2.4) respectively.

Proof. It suffices to prove the property (iv) presented in Proposition 2.2 and

$$(\omega^\theta)^\varepsilon = \partial_r(u^z)^\varepsilon - \partial_z(u^r)^\varepsilon \geq 0, \tag{2.8}$$

for our viscous approximations, since the other statements clearly hold true.

First (2.8) can be proved by the maximum principle. Indeed, if (2.8) is not true, then it follows from (2.6) and (2.7) that $(\omega^\theta)^\varepsilon/r$ must achieve its least negative value in an interior point $(r_0,z_0,t_0)\in(0,+\infty)\times(-\infty,+\infty)\times(0,T]\equiv\Omega_T$. Let

$$\frac{(\omega^\theta)^\varepsilon}{r} = \xi^\varepsilon e^{\lambda t},$$

where $\lambda > 0$ is some constant. Then $\xi^\varepsilon = \xi^\varepsilon(r, z, t)$ satisfies the following equation,

$$\frac{\tilde{D}}{Dt}\xi^\varepsilon + \lambda\xi^\varepsilon - \varepsilon(\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r)\xi^\varepsilon - \varepsilon\frac{2}{r}\partial_r\xi^\varepsilon = 0, \quad (r, z, t) \in \Omega_T. \qquad (2.9)$$

Noting that at the minimum point $(r_0, z_0, t_0) \in \Omega_T$,

$$\partial_t\xi^\varepsilon \leq 0, \quad \partial_r\xi^\varepsilon = 0, \quad \partial_z\xi^\varepsilon = 0,$$

and

$$\partial_r^2\xi^\varepsilon \geq 0, \quad \partial_z^2\xi^\varepsilon \geq 0,$$

We evaluate the equation (2.9) at the minimum point (r_0, z_0, t_0) to get a contradiction. This shows that there is no negative value of $(\omega^\theta)^\varepsilon/r$ in the interior point of Ω_T. In view of (2.6), (2.7), and the fact that $(\omega^\theta)^\varepsilon/r$ is also zero at infinity, the non-negativity of $(\omega^\theta)^\varepsilon/r$ is established.

Next, we prove the property (iv) in Proposition 2.2. It is noted that our solutions here are all smooth solutions and $(\omega^\theta)^\varepsilon/r$ tends to zero as $r \to 0$ and as $|x| \to +\infty$. So multiplying r on both side of the equation (2.5) and integrating on $(0, +\infty) \times (-\infty, +\infty)$ with respect to (r, z) variable, we easily get

$$\max_{0\leq t\leq T} \int_{-\infty}^{+\infty}\int_0^{+\infty} (\omega^\theta)^\varepsilon drdz \leq C,$$

where C is a constant independent of ε.

On the other hand, as in the case of 3-D Euler equations, we observes that the quantity $\int_{R^3} x \times (\omega)^\varepsilon dx$ is also conserved for our smooth solutions $\omega(x, t)$ of the equations (2.4). In fact, it follows from integration by parts and $divu = 0$ that

$$\frac{d}{dt}\int_{R^3} x \times \omega^\varepsilon dx = \int_{R^3} x \times ((\omega^\varepsilon \cdot \nabla)u^\varepsilon - (u^\varepsilon \cdot \nabla)\omega^\varepsilon + \varepsilon\triangle\omega^\varepsilon)dx = 0.$$

Therefore,

$$\int_{R^3} x \times \omega^\varepsilon dx = \int_{R^3} x \times \omega_0^\varepsilon dx.$$

In particular, for axisymmetric solutions without swirls, one obtains

$$\int_{-\infty}^{+\infty}\int_0^{+\infty} (\omega^\theta)^\varepsilon r^2 drdz = \int_{-\infty}^{+\infty}\int_0^{+\infty} (\omega_0^\theta)^\varepsilon r^2 drdz.$$

Since ω_0^θ has compact support in $\{(r, z) \in R^2 \mid r \geq 0\}$, so $(\omega_0^\theta)^\varepsilon$ has uniform compact support on ε. Moreover,

$$\int_{R^3} (\omega_0^\theta)^\varepsilon dx \leq |\omega_0|.$$

So we finally get

$$\int_{-\infty}^{+\infty}\int_0^{+\infty}\omega^\theta r^2 drdz \leq C,$$

where C does not depend on ε.

The proof of the Proposition is finished.

The conclusions of Proposition 2.2′ are true for the viscous approximations, i.e.,

Proposition 2.3′ The results of Proposition 2.2′ still hold for the viscous approximate solutions $\{u^\varepsilon, \omega^\varepsilon\}$, which are solutions to the problems (2.3) and (2.4) respectively.

Now we state some simple temporal estimates.

Proposition 2.4 Suppose that $u^\varepsilon(x,t)$ and $p^\varepsilon(x,t)$ are the approximate solutions stated in either Proposition 2.2 or Proposition 2.3. Then

$$\frac{\partial u^\varepsilon}{\partial t} \in L^\infty([0,T]; H^{-s}_{loc}(R^3))$$

for some positive number $s \geq 3$.

Proof. For the approximate solutions $\{u^\varepsilon, p^\varepsilon\}$ obtained by smoothing the initial data, the proof of the Proposition is given in [5]. The proof is based on the equation (1.1) and the representation of the pressure,

$$\triangle p^\varepsilon = -\text{div } (u^\varepsilon \cdot \nabla)u^\varepsilon, x \in R^3.$$

For the viscous approximate solutions generated by solving the Navier-Stokes equations, the proof is completely similar. So we omit the details of the proof here.

Corollary 2.5 Let u^ε be an approximate solution as in Proposition 2.4. The the vorticity $\omega^\varepsilon = \nabla \times u^\varepsilon$ admits the following estimates:

$$\begin{aligned} &\frac{\partial \omega^\varepsilon}{\partial t} \in L^\infty([0,T]; H^{-s-1}_{loc}(R^3)); \\ &\omega^\varepsilon \in Lip([0,T]; H^{-s-1}_{loc}(R^3)). \end{aligned} \tag{2.10}$$

Here $s \geq 3$ is some positive number.

The following is a proposition needed later, which is due to D. Chae and O. Y. Imanuvilov (see [4] for details).

Proposition 2.6 ([4]) Let $\{u^\varepsilon\}$ and $\{\omega^\varepsilon\}$ be the approximate solutions constructed in Proposition 2.2′. Then, for any $T > 0$,

$$\int_0^T \int_{R^3} \frac{1}{1+x_3^2}(\frac{u_r^\varepsilon}{r})^2 dxdt \leq C(\|u_0^\varepsilon\|_{L^2}^2 + \|\frac{\omega_0^\varepsilon}{r}\|_{L^1}), \tag{2.11}$$

where $C = C(T)$ is a constant which does not depend on ε.

Proof. For convenience, we omit the supscript ε here. The vorticity $\omega = \omega^\theta e_\theta$ satisfies the following equations

$$\begin{cases} \dfrac{\tilde{D}}{Dt}\left(\dfrac{\omega^\theta}{r}\right) = 0, \\ \dfrac{\omega^\theta}{r} \mid_{t=0} = \dfrac{\omega_0^\theta}{r}. \end{cases} \tag{2.12}$$

Set $\rho(x_3) = \int_{-\infty}^{x_3} \frac{1}{1+\tau^2} d\tau$.

Multiplying $2\pi r \rho(x_3)$ on both sides of (2.12) and integrating with (r, z, t) over $(0, +\infty) \times (-\infty, +\infty) \times [0, T]$, we obtain

$$\begin{aligned} 0 &= \int_{R^3} \frac{\rho\omega^\theta}{r} dx \mid_0^T - \int_0^T \int_H 2\pi \rho' u_3 \omega^\theta dr dx_3 dt \\ &= \int_{R^3} \frac{\rho\omega^\theta}{r} dx \mid_0^T - \int_0^T \int_H 2\pi \rho' u_3 (\partial_r u_3 - \partial_3 u_r) dr dx_3 dt \\ &= \int_{R^3} \frac{\rho\omega^\theta}{r} dx \mid_0^T + \int_0^T \int_{-\infty}^{+\infty} \pi \rho' u_3^2(0, x_3, t) dx_3 dt \\ &- \int_0^T \int_H 2\pi (\rho'' u_3 u_r + \rho' u_r \partial_3 u_3) dr dx_3 dt. \end{aligned} \tag{2.13}$$

Here $H = \{(r, z) \in R \times R \mid r \geq 0\}$. And in the above equation, we have used the integration by parts, which can be justified easily, for instance,

$$\begin{aligned} &\int_0^T \int_H 2\pi \rho' u_3 (\partial_r u_3 - \partial_3 u_r) dr dx_3 dt \\ &= \lim_{r_k \to +\infty} 2\pi \int_0^T \int_{-\infty}^{+\infty} \int_0^{r_k} \rho' u_3 \partial_r u_3 dr dx_3 dt \\ &- \lim_{b_k \to +\infty} 2\pi \int_0^T \int_{-b_k}^{b_k} \int_0^{\infty} \rho' u_3 \partial_3 u_r dr dx_3 dt \\ &= - \int_0^T \int_{-\infty}^{+\infty} \pi \rho' u_3^2(0, x_3, t) dx_3 dt + \lim_{r_k \to +\infty} \int_0^T \int_{-\infty}^{+\infty} \pi \rho' u_3^2(r_k, x_3, t) dx_3 dt \\ &- \lim_{b_k \to +\infty} \int_0^T \int_0^{+\infty} 2\pi \rho' u_3 u_r dr dt \mid_{-b_k}^{b_k} \\ &+ \lim_{b_k \to +\infty} \int_0^T \int_{-b_k}^{b_k} \int_0^{+\infty} 2\pi (\rho'' u_3 u_r + \rho' u_r \partial_3 u_3) dr dx_3 dt. \end{aligned}$$

Note that

$$\begin{aligned} \int_0^T \int_{R^3} |u|^2 dx dt &= 2\pi \int_0^{+\infty} (\int_0^T \int_{-\infty}^{+\infty} |u|^2 dx_3 dt) r dr \\ &= 2\pi \int_{-\infty}^{+\infty} (\int_0^T \int_0^{+\infty} |u|^2 r dr dt) dx_3 < +\infty, \end{aligned}$$

we can find a sequence $r_k \to +\infty$ and $b_k \to +\infty$ such that

$$\lim_{r_k\to+\infty}\int_0^T\int_{-\infty}^{+\infty}\rho' u_3^2(r_k,x_3,t)dx_3dt \to 0,$$
$$\lim_{b_k\to+\infty}\int_0^T\int_0^{+\infty}2\pi\rho' u_3u_r drdt\ |_{-b_k}^{b_k}\to 0.$$

Substituting

$$\partial_3 u_3 = -\frac{u_r}{r} - \partial_r u_r,$$

into (2.13), we have

$$\begin{aligned} 0\ &= \int_{R^3}\frac{\rho\omega^\theta}{r}dx\ |_0^T + \int_0^T\int_{-\infty}^{+\infty}\pi\rho' u_3^2(0,x_3,t)dx_3dt \\ &-\int_0^T\int_H 2\pi(\rho'' u_3u_r + \rho' u_r(-\frac{u_r}{r} - \partial_r u_r))drdx_3dt. \end{aligned} \tag{2.14}$$

Since $\rho'(x_3) > 0, |\rho(x_3)| < C$ for all $x_3 \in R^1$, applying integration by parts again, we obtain from (2.14) that

$$\begin{aligned} &2\pi\int_0^T\int_H\rho'\frac{(u_r)^2}{r}drdx_3dt \le 2\pi\int_0^T\int_H|\rho'' u_3u_r|drdx_3dt + C\|\frac{\omega_0}{r}\|_{L^1} \\ &\le 2\pi(\int_0^T\int_H\rho'\frac{(u_r)^2}{r}drdx_3dt)^{1/2}(\int_0^T\int_H u_3^2\frac{\rho''}{\rho'}rdrdx_3dt)^{1/2} + C\|\frac{\omega_0}{r}\|_{L^1}. \end{aligned}$$

By Cauchy-Schwartz inequality, we obtain (3.2) and the proof of the proposition is finished.

Finally, we point out that for the viscous approximations given in Proposition 2.3′, a similar result as Proposition 2.6 can be established, which is

Proposition 2.7 The results of Proposition 2.6 holds for the viscous approximations u^ε and $\omega^\varepsilon (0 < \varepsilon < 1)$ presented in Proposition 2.3′.

Proof. Similar to Proposition 2.6, we define $\rho(x_3) = \int_{-\infty}^{x_3}\frac{1}{1+\tau^2}d\tau$. Multiplying $2\pi r\rho(x_3)$ on both sides of (2.5), integrating with (r,z,t) over $(0,+\infty)\times(-\infty,+\infty)\times[0,T]$, noting that for our viscous approximations, the integration by parts make sense, we have

$$\int_0^T\int_H\partial_r(\frac{\omega^\theta}{r})\cdot 2\pi\rho(x_3)drdx_3dt = 0,$$
$$\int_0^T\int_H\partial_r^2(\frac{\omega^\theta}{r})\cdot 2\pi r\rho(x_3)drdx_3dt = 0,$$
$$\int_0^T\int_H\partial_3^2(\frac{\omega^\theta}{r})\cdot 2\pi r\rho(x_3)drdx_3dt = 2\pi\int_0^T\int_H\omega^\theta\rho''(x_3)drdx_3dt.$$

Applying the approach in Proposition 2.6, we obtain that for $0 < \varepsilon < 1$,

$$2\pi \int_0^T \int_H \rho' \frac{(u_r)^2}{r} dr dx_3 dt \leq 2\pi \int_0^T \int_H |\rho'' u_3 u_r| dr dx_3 dt + C \|\frac{\omega_0}{r}\|_{L^1}$$
$$\leq 2\pi (\int_0^T \int_H \rho' \frac{(u_r)^2}{r} dr dx_3 dt)^{1/2} (\int_0^T \int_H u_3^2 \frac{\rho''}{\rho'} r dr dx_3 dt)^{1/2} + C \|\frac{\omega_0}{r}\|_{L^1}.$$

By Cauchy-Schwartz inequality, we obtain (3.2) and the proof of the Proposition is finished.

3 Strong Convergence

For the approximate solutions $\{u^\varepsilon\}$ constructed in section 2, it follows from Propositions 2.2-2.4 that

$$u^{\varepsilon_j} \rightharpoonup u \;\; weakly\ in\ L^2([0,T]; L^2(R^3)) \tag{3.1}$$

for some subsequence $\{u^{\varepsilon_j}\}$ of $\{u^\varepsilon\}$. One of the main concerns is whether such a weak convergence becomes strong. Along this line, Delort gives the following interesting result.

Proposition 3.1 ([7]) Suppose that assumptions (A) hold. Let $\{u^\varepsilon\}$ be the approximate solutions constructed in Proposition 2.2. Then

$$u_1^{\varepsilon_j} u_3^{\varepsilon_j} \rightharpoonup u_1 u_3,$$
$$u_2^{\varepsilon_j} u_3^{\varepsilon_j} \rightharpoonup u_2 u_3,$$
$$(u_1^{\varepsilon_j})^2 + (u_2^{\varepsilon_j})^2 - (u_3^{\varepsilon_j})^2 \rightharpoonup (u_1)^2 + (u_2)^2 - (u_3)^2,$$

in the sense of distributions.

Furthermore, if the weak limit $u(x,t)$ in (3.1) is a solution of 3-D axisymmetric Euler equations in the sense of distributions, then u^{ε_j} converges strongly to u in $L^2([0,T]; L^2_{loc}(R^3))$.

Thus, by Proposition 3.1, the key to the existence of classical weak solutions to the vortex-sheets for 3-D axisymmetric flows lies in whether the weak convergence (3.1) becomes strong. Further investigations on the properties of the approximate solutions are desired. For axisymmetric flows, it is usually the case that the solutions are more singular near the axis of symmetry if there is any singularity at all. However, in this section, we will prove that if a sequence of approximate solutions converges strongly in the region away from the symmetry axis, then it has a subsequence which converges strongly in the whole space. This means that if there are energy defects in the limiting process of the approximate solutions, then they must appear in the region away from the axis. We note that this result is true for general initial vorticity, not only for one sign initial vorticity.

For concise presentations, we call the term $\alpha_1(u_1^\varepsilon)^2+\alpha_2(u_2^\varepsilon)^2+\alpha_3(u_3^\varepsilon)^2$ ($\alpha_i \in R, i=1,2,3$) the *cancellation combinations* of u^ε if

$$\alpha_1(u_1^\varepsilon)^2+\alpha_2(u_2^\varepsilon)^2+\alpha_3(u_3^\varepsilon)^2 \rightharpoonup \alpha_1 u_1^2+\alpha_2 u_2^2+\alpha_3 u_3^2$$

in the sense of distributions.

Now we consider the approximate solutions obtained through smoothing the initial data under Assumtions (A′), which are given by Proposition 2.2′.

Our main result of this section is stated as

Theorem 3.2 For the approximate solutions $\{u^\varepsilon\}$ constructed in Proposition 2.2′, if there exists a subsequence $\{u^{\varepsilon_j}\} \subset \{u^\varepsilon\}$ such that for any $Q \subset\subset R^3 \backslash \{x \in R^3 | r=0\}$,

$$u^{\varepsilon_j} \longrightarrow u \quad \text{strongly in} \quad L^2([0,T]; L^2(Q)), \tag{3.2}$$

then there exists a further subsequence of $\{u^{\varepsilon_j}\}$, still denoted by itself, such that, as $\varepsilon_j \to 0$,

$$u^{\varepsilon_j} \longrightarrow u \quad \text{strongly in} \quad L^2([0,T]; L^2_{loc}(R^3)). \tag{3.3}$$

Proof. The proof is decomposed into the following steps.

Step I. Test Functions and the Integral Equations for the Weak Limit

It follows from the assumption (3.2) that there exists a subsequence $\{u^{\varepsilon_j}\} \subset \{u^\varepsilon\}$ such that, for any $Q \subset\subset R^3\backslash\{r=0\}$,

$$u^{\varepsilon_j} \longrightarrow u \quad \text{strongly in} \quad L^2([0,T]; L^2(Q)), \ \text{as } \varepsilon_j \to 0, \tag{3.4}$$

where $u=u(x,t)$ is the weak limit of u^ε, satisfying $|u|^2 \in L^1(R^3 \times [0,T])$. We denote the subsequence of u^{ε_j} by itself in the following. Then it is clear that

$$u^{\varepsilon_j} \longrightarrow u \quad a.e.\ (x,t) \in R^3 \times [0,T], \ \text{as } \varepsilon_j \to 0, \tag{3.5}$$

which implies that there are no osillations in the limit process. On the other hand, using the energy estimate we directly get that $\{|u^{\varepsilon_j}|^2\}$ (its subsequence actually) converges weakly in the sense of measure, that is, as $\varepsilon_j \to 0$,

$$(u_i^{\varepsilon_j})^2 dxdt \rightharpoonup \mu_i \quad \text{weakly in} \quad M(R^3 \times [0,T]) \tag{3.6}$$

for $i=1,2,3$, where $M(R^3\times[0,T])$ is the space of finite Radon measures, $\mu_i\geq 0$. By the Lebesgue decomposition and the Radon-Nikodym theorem, there exist $f_i(x,t)\in L^1(R^3\times[0,T])$ and $\gamma_i\in M(R^3\times[0,T])$ such that

$$\mu_i=f_i(x,t)dxdt+\gamma_i,\quad i=1,2,3, \tag{3.7}$$

where $\gamma_i\perp dxdt$, i.e., γ_i and $dxdt$ are mutually orthogonal, and γ_i is the singular part of μ_i (i=1,2,3). Thanks to (3.4) and (3.5), one concludes that $f_i=|u_i|^2$ $(i=1,2,3)$ and the support of γ_i, denoted by $Supp\{\gamma_i\}$ $(i=1,2,3)$, is contained in the set $\{(x,t)\in R^3\times[0,T]\mid r=0\}$, which will be denoted by $\{r=0\}$ in the following. In other words, as $\varepsilon_j\to 0$, we have

$$\begin{aligned}(u_1^{\varepsilon_j})^2dxdt &\rightharpoonup u_1^2dxdt+\gamma_1,\\ (u_2^{\varepsilon_j})^2dxdt &\rightharpoonup u_2^2dxdt+\gamma_2,\\ (u_3^{\varepsilon_j})^2dxdt &\rightharpoonup u_3^2dxdt+\gamma_3,\end{aligned} \tag{3.8}$$

weakly in $M(R^3\times[0,T])$. In (3.8), $\gamma_i(i=1,2,3)$ are non-negative Radon measures satisfying

$$Supp\{\gamma_i\}\subseteq\{r=0\},\quad |\gamma_i|<+\infty,\quad i=1,2,3. \tag{3.9}$$

Here $|\gamma_i|$ means the total variations of $\gamma_i(i=1,2,3)$.

By construction,

$$\begin{aligned}&\int_{R^3}u_0^{\varepsilon_j}(x)\Phi(x,0)dx+\int_0^T\int_{R^3}(u^{\varepsilon_j}\Phi_t+u^{\varepsilon_j}\otimes u^{\varepsilon_j}:\nabla\Phi)dxdt\\ &=\int_{R^3}u^{\varepsilon_j}(x,T)\Phi(x,T)dx,\end{aligned} \tag{3.10}$$

for all $\Phi\in C_0^\infty(R^3\times[0,T])$ satisfying $div\Phi=0$.

Note that

$$\begin{aligned}|\int_{R^3}u^{\varepsilon_j}(x,T)\Phi(x,T)dx| &\leq(\int_{R^3}|u^{\varepsilon_j}|^2dx)^{1/2}(\int_{R^3}|\Phi(x,T)|^2dx)^{1/2}\\ &\leq C(\int_{R^3}|\Phi(x,T)|^2dx)^{1/2},\end{aligned}$$

where C is a constant independent of ε. This, together with (3.10), implies

$$\begin{aligned}&\int_{R^3}u_0^{\varepsilon_j}(x)\Phi(x,0)dx+\int_0^T\int_{R^3}(u^{\varepsilon_j}\Phi_t+u^{\varepsilon_j}\otimes u^{\varepsilon_j}:\nabla\Phi)dxdt\\ &\leq C(\int_{R^3}|\Phi(x,T)|^2dx)^{1/2}.\end{aligned} \tag{3.11}$$

Our first aim is to show that $(u_1^{\varepsilon_j})^2 + (u_2^{\varepsilon_j})^2 - 2(u_3^{\varepsilon_j})^2$ is a *cancellation combination* of u^ε. To this end, the major step is to construct the test function $\Phi = (\Phi_1, \Phi_2, \Phi_3) \in C_0^\infty(R^3 \times [0,T])$ satisfying $div\Phi = 0$ and

$$\max_{r\leq h, t\in[0,T], x_3\in R} |\partial_i\Phi_k(x,t)| \to 0, \ h \to 0, \tag{3.12}$$

for $i,k = 1,2,3, \quad i \neq k$.

For such test functions, we obtain from (3.8) and (3.11) that as $\varepsilon_j \to 0$,

$$\begin{aligned}
&|\int_0^T\int_{R^3} \partial_1\Phi_1 d\gamma_1 + \int_0^T\int_{R^3} \partial_2\Phi_2 d\gamma_2 + \int_0^T\int_{R^3} \partial_3\Phi_3 d\gamma_3| \\
&\leq |\int_{R^3} u_0(x)\Phi(x,0)dx| + |\int_0^T\int_{R^3}(u\Phi_t + u_1^2\partial_1\Phi_1 + u_2^2\partial_2\Phi_2 + u_3^2\partial_3\Phi_3 \\
&+u_1u_2\partial_1\Phi_2 + u_1u_3\partial_1\Phi_3 + u_2u_1\partial_2\Phi_1 + u_2u_3\partial_2\Phi_3 + u_3u_1\partial_3\Phi_1 \\
&+u_3u_2\partial_3\Phi_2)dxdt| + C(\int_{R^3}|\Phi(x,T)|^2dx)^{1/2}.
\end{aligned} \tag{3.13}$$

In the limit process above, we have used the following facts

$$\int_0^T\int_{R^3} u_i^{\varepsilon_j}u_k^{\varepsilon_j}\partial_i\Phi_k dxdt \longrightarrow \int_0^T\int_{R^3} u_iu_k\partial_i\Phi_k dxdt, \ \varepsilon_j \to 0 \tag{3.14}$$

for $i,k = 1,2,3$, and $i \neq k$. Indeed, for any $h > 0$, by the assumption (3.2), one has

$$\int_{\{r\geq h\}} u_i^{\varepsilon_j}u_k^{\varepsilon_j}\partial_i\Phi_k dxdt \longrightarrow \int_{\{r\geq h\}} u_iu_k\partial_i\Phi_k dxdt, \quad \varepsilon_j \to 0. \tag{3.15}$$

While, it follows from (3.12) that

$$|\int_{\{r\leq h\}} u_i^{\varepsilon_j}u_k^{\varepsilon_j}\partial_i\Phi_k dxdt| \leq C \max_{r\leq h, t\in[0,T], x_3\in R} |\partial_i\Phi_k(x,t)| \to 0, \ h \to 0, \tag{3.16}$$

for $i \neq k$ $(i,k = 1,2,3)$. Moreover, it is clear that

$$|\int_{\{r\leq h\}} u_iu_k\partial_i\Phi_k dxdt| \leq C\int_{\{r\leq h\}} |u|^2 dxdt \to 0, \ h \to 0, \tag{3.17}$$

since $u \in L^2_{loc}([0,T] \times R^3)$.

Thus (3.14) follows from (3.15), (3.16) and (3.17). We now separate the two cases depending on whether $\bigcup_{i=1}^3 Supp\{\gamma_i\}$ lies in a bounded set or not.

Step II. $\bigcup_{i=1}^3 Supp\{\gamma_i\}$ *Is Finite*

Now we suppose that $\bigcup_{i=1}^3 Supp\{\gamma_i\}$ has a compact support in $\{r = 0\}$. Let $\Omega = \{(x_3,t) \in (-\infty,+\infty) \times [0,T] \mid x_3^2 + t^2 < R\} \subset \{r = 0\}$ be an open set with smooth boundary satisfying

$$\bigcup_{i=0}^3 Supp\{\gamma_i\} \subseteq \bar{\Omega}.$$

In this case, we define the test functions $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ as follows

$$
\begin{aligned}
\Phi_1(x,t) &= -\tfrac{1}{2}x_1\chi(\tfrac{r}{\delta})\{[g(x_3,t) + (x_3 - x_3^0)\partial_{x_3}g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}} \\
&\quad +(x_3 - x_3^0)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\}, \\
\Phi_2(x,t) &= -\tfrac{1}{2}x_2\chi(\tfrac{r}{\delta})\{[g(x_3,t) + (x_3 - x_3^0)\partial_{x_3}g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}} \\
&\quad +(x_3 - x_3^0)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\}, \\
\Phi_3(x,t) &= [\chi(\tfrac{r}{\delta}) + \tfrac{r}{2\delta}\chi'(\tfrac{r}{\delta})](x_3 - x_3^0)g(x_3,t)\mathbf{I}_\Omega^{\tilde{\delta}}.
\end{aligned}
\tag{3.18}
$$

Here $\delta > 0$ is small and $\tilde{\delta} > 0$ is arbitrary. The function $\chi = \chi(s)$ is a smooth function satisfying $\chi(s) = 1$ for $|s| \leq 1$ and $\chi(s) = 0$ for $|s| \geq 2$. And $g = g(x_3,t)$ and $\mathbf{I}_\Omega^{\tilde{\delta}} = \mathbf{I}_\Omega^{\tilde{\delta}}(x_3,t)$ are also smooth functions which will be determined in the sequel.

Denote by $diam(\Omega)$ the diameter of Ω. For any large positive number $\tilde{\delta} > diam(\Omega)$, let $\mathbf{I}_\Omega^{\tilde{\delta}} = \mathbf{I}_\Omega^{\tilde{\delta}}(x_3,t)$ be a smooth function satisfying

$$
\begin{aligned}
&0 \leq \mathbf{I}_\Omega^{\tilde{\delta}} \leq 1, \\
&\mathbf{I}_\Omega^{\tilde{\delta}}(x_3,t) \equiv 1, \quad (x_3,t) \in \Omega, \\
&\mathbf{I}_\Omega^{\tilde{\delta}}(x_3,t) \equiv 0, \quad |x_3| > \tilde{\delta}, \\
&|\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}(x_3,t)| \leq \frac{C}{\tilde{\delta}},
\end{aligned}
\tag{3.19}
$$

where C is a constant independent of $\tilde{\delta}$.

For any $f(x_3,t) \in C_0^\infty(\bar{\Omega})$, we define a smooth function $g = g(x_3,t) \in C^\infty((-\infty,+\infty) \times [0,T])$ satisfying

$$
\begin{aligned}
&g + (x_3 - x_3^0)\partial_{x_3}g = f \quad \text{in } \ \Omega, \\
&g(x_3,t) = 0, \quad |x_3| > K
\end{aligned}
\tag{3.20}
$$

for large enough $K > diam(\Omega) > 0$. In (3.20), $x_3^0 \in (-\infty,+\infty)$ is any fixed real number such that $(x_3^0,t) \notin \Omega$ for $t \in [0,T]$.

It can be verified directly that the test functions given by (3.18), (3.19) and (3.20) satisfy

$$
\Phi = (\Phi_1, \Phi_2, \Phi_3) \in C_0^1(R^3 \times [0,T]), \quad div\Phi = 0,
$$

and the restrictions (3.12) are satisfied. Thus the integral inequality (3.13) holds for the test functions defined by (3.18).

Note that

$$
\begin{aligned}
\partial_1\Phi_1 &= [-\tfrac{1}{2}\chi(\tfrac{r}{\delta}) - \tfrac{x_1^2}{2\delta r}\chi'(\tfrac{r}{\delta})]\{[g(x_3,t)\\
&\quad +(x_3-x_3^0)\partial_{x_3}g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}} + (x_3-x_3^0)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\},\\
\partial_2\Phi_2 &= [-\tfrac{1}{2}\chi(\tfrac{r}{\delta}) - \tfrac{x_2^2}{2\delta r}\chi'(\tfrac{r}{\delta})]\{[g(x_3,t)\\
&\quad +(x_3-x_3^0)\partial_{x_3}g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}} + (x_3-x_3^0)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\},\\
\partial_3\Phi_3 &= [\chi(\tfrac{r}{\delta}) + \tfrac{r}{2\delta}\chi'(\tfrac{r}{\delta})]\{[g(x_3,t)\\
&\quad +(x_3-x_3^0)\partial_{x_3}g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}} + (x_3-x_3^0)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\}.
\end{aligned}
\tag{3.21}
$$

So

$$
\begin{aligned}
&\partial_1\Phi_1 \,|_{r=0,(x_3,t)\in Supp\{\gamma_1\}} = -1/2 f(x_3,t);\\
&\partial_2\Phi_2 \,|_{r=0,(x_3,t)\in Supp\{\gamma_2\}} = -1/2 f(x_3,t);\\
&\partial_3\Phi_3 \,|_{r=0,(x_3,t)\in Supp\{\gamma_3\}} = f(x_3,t).
\end{aligned}
\tag{3.22}
$$

Substitute these test functions $\Phi = (\Phi_1,\Phi_2,\Phi_3)$ into the integral inequality (3.13) to lead to

$$
\begin{aligned}
&|\int_0^T\int_{R^3} f(x_3,t)(-\frac{1}{2}d\gamma_1 - \frac{1}{2}d\gamma_2 + d\gamma_3)|\\
&\le |\int_{R^3} u_0(x)\Phi(x,0)dx| + |\int_0^T\int_{R^3}(u\Phi_t + u_1^2\partial_1\Phi_1 + u_2^2\partial_2\Phi_2 + u_3^2\partial_3\Phi_3\\
&u_1u_2\partial_1\Phi_2 + u_1u_3\partial_1\Phi_3 + u_2u_1\partial_2\Phi_1 + u_2u_3\partial_2\Phi_3 + u_3u_1\partial_3\Phi_1\\
&+u_3u_2\partial_3\Phi_2)dxdt| + C(\int_{R^3}|\Phi(x,T)|^2dx)^{1/2}\\
&\equiv I_1 + I_2 + I_3.
\end{aligned}
\tag{3.23}
$$

We now estimate the terms on the right hand side of (3.23). Noting that for any fixed $\tilde{\delta} > 0$, we have

$$
|[g(x_3,t) + (x_3-x_3^0)\partial_{x_3}g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}} + (x_3-x_3^0)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}| \le C(\tilde{\delta}).
$$

So we get the estimates of the right hand side of (3.23) as follows

$$
\begin{aligned}
|I_1| \le&\ C(\tilde{\delta})\int_{\{|r|\le 2\delta,|x_3|\le K\}}|u_0(x)|dx,\\
|I_2| \le&\ C(\tilde{\delta})[(\int_0^T\int_{\{|r|\le 2\delta,|x_3|\le K\}}|u|^2dxdt)^{1/2}\\
&+\int_0^T\int_{\{|r|\le 2\delta,|x_3|\le K\}}|u|^2dxdt],\\
|I_3| \le&\ C(\int_{\{|r|\le 2\delta,|x_3|\le K\}}|\Phi(x,T)|^2dx)^{1/2}.
\end{aligned}
\tag{3.24}
$$

For any fixed $\tilde{\delta} > 0$, letting $\delta \to 0$, we get

$$\int_0^T \int_{R^3} f(x_3, t)(-\frac{1}{2}d\gamma_1 - \frac{1}{2}d\gamma_2 + d\gamma_3) = 0, \tag{3.25}$$

for all $f(x_3, t) \in C_0^\infty(\bar{\Omega})$.

It is noted that the Radon measure $\gamma_i = \gamma_i \mid_{\bar{\Omega}} \in M(\bar{\Omega})$, where $\gamma_i \mid_{\bar{\Omega}}$ is the restriction of γ_i on $\bar{\Omega}$ $(i = 1, 2, 3)$. Since the conjugate space of $M(\bar{\Omega})$ is $C(\bar{\Omega})$, by approximation, we get from (3.25) that

$$\gamma_1 + \gamma_2 - 2\gamma_3 = 0. \tag{3.26}$$

Combing (3.26) with (3.8), we get

$$(u_1^{\varepsilon_j})^2 + (u_2^{\varepsilon_j})^2 - 2(u_3^{\varepsilon_j})^2 \rightharpoonup u_1^2 + u_2^2 - 2u_3^2, \tag{3.27}$$

weakly in $M(R^3 \times [0, T])$. This means that $(u_1^{\varepsilon_j})^2 + (u_2^{\varepsilon_j})^2 - 2(u_3^{\varepsilon_j})^2$ is a *cancellation combinations*.

Step III. $\bigcup_{i=1}^3 Supp\{\gamma_i\}$ *Is Infinite*

Now we consider the case that $\bigcup_{i=1}^3 Supp\{\gamma_i\}$ is infinite. Let

$$B_k = \{(x_3, t) \in (-\infty, +\infty) \times [0, T] \mid x_3^2 + t^2 \le k\}, \quad k \in N,$$

where N is the set of natural number. Let

$$\begin{aligned} Q_1 &= \textstyle\bigcup_{i=1}^3 Supp\{\gamma_i\} \cap B_1, \\ Q_2 &= \textstyle\bigcup_{i=1}^3 Supp\{\gamma_i\} \cap B_2 - Q_1, \\ &\cdots \\ Q_k &= \textstyle\bigcup_{i=1}^3 Supp\{\gamma_i\} \cap B_k - Q_{k-1}, \\ &\cdots \end{aligned}$$

Then

$$\bigcup_{i=1}^3 Supp\{\gamma_i\} \subseteq \cup_{k=1}^\infty Q_k, \quad Q_i \cap Q_j = \emptyset, \text{ for } i \ne j, i, j \in N.$$

Define

$$\gamma_i^{(k)} = \gamma_i \mid_{Q_k}, \quad k \in N, i = 1, 2, 3.$$

Then

$$\gamma_i^{(k_1)} \perp \gamma_i^{(k_2)}, \quad k_1, k_2 \in N, \quad k_1 \ne k_2, \quad i = 1, 2, 3,$$

and

$$\gamma_i = \sum_{k=1}^{\infty} \gamma_i^{(k)}, \quad i = 1, 2, 3.$$

Clearly, we have $\gamma_i^{(k)} = 0 (i = 1, 2, 3)$ if $Q_k = \emptyset$ for some $k \in N$. Then the integral inequality (3.13) becomes

$$\begin{aligned}
&\left| \sum_{k=1}^{\infty} \left(\int_0^T \int_{R^3} \partial_1 \Phi_1 d\gamma_1^{(k)} + \int_0^T \int_{R^3} \partial_2 \Phi_2 d\gamma_2^{(k)} + \int_0^T \int_{R^3} \partial_3 \Phi_3 d\gamma_3^{(k)} \right) \right| \\
&\leq | \int_{R^3} u_0(x) \Phi(x, 0) dx | + | \int_0^T \int_{R^3} (u \Phi_t + u_1^2 \partial_1 \Phi_1 + u_2^2 \partial_2 \Phi_2 + u_3^2 \partial_3 \Phi_3 \\
&+ u_1 u_2 \partial_1 \Phi_2 + u_1 u_3 \partial_1 \Phi_3 + u_2 u_1 \partial_2 \Phi_1 + u_2 u_3 \partial_2 \Phi_3 + u_3 u_1 \partial_3 \Phi_1 \\
&+ u_3 u_2 \partial_3 \Phi_2) dx dt | + C(\int_{R^3} |\Phi(x, T)|^2 dx)^{1/2}.
\end{aligned} \tag{3.28}$$

Suppose that $Q_l \neq \emptyset$ for some $l \in N$. Let $\Omega_l \subset \{r = 0\}$ be an open bounded set with smooth boundary satisfying $\cup_{k=1}^{l} Q_k \subseteq \bar{\Omega}_l$. For any $\tilde{\delta}_l > diam(\Omega_l) > 0$, define $\mathbf{I}_{\Omega_l}^{\tilde{\delta}_l}$ to be a smooth function satisfying

$$\begin{aligned}
&0 \leq \mathbf{I}_{\Omega_l}^{\tilde{\delta}_l} \leq 1, \\
&\mathbf{I}_{\Omega_l}^{\tilde{\delta}_l}(x_3, t) \equiv 1, \quad (x_3, t) \in \Omega_l, \\
&\mathbf{I}_{\Omega_l}^{\tilde{\delta}_l}(x_3, t) \equiv 0, \quad |x_3| > \tilde{\delta}_l, \\
&|\partial_{x_3} \mathbf{I}_{\Omega_l}^{\tilde{\delta}_l}(x_3, t)| \leq \frac{C}{\tilde{\delta}_l},
\end{aligned}$$

where C is a constant independent of $\tilde{\delta}_l$. Similar to the previous step, for any $f_l(x_3, t) \in C_0^{\infty}(\bar{\Omega}_l)$, we define a smooth function $g_l(x_3, t) \in C^{\infty}((-\infty, +\infty) \times [0, T])$ satisfying

$$\begin{aligned}
&g_l + (x_3 - x_3^l) \partial_{x_3} g_l = f_l \quad \text{in} \;\; \Omega_l, \\
&g_l(x_3, t) = 0, \quad |x_3| > K_l.
\end{aligned} \tag{3.29}$$

Here $K_l > diam(\Omega_l) > 0$ is a large positive number and $x_3^l \in (-\infty, +\infty)$ is any fixed number with $(x_3^l, t) \notin \Omega_l$ for $t \in [0, T]$.

Now we choose the test functions $\Phi(x, t) = (\Phi_1, \Phi_2, \Phi_3)$ in (3.18)with g and $\mathbf{I}_{\Omega}^{\tilde{\delta}}$ being replaced by g_l and $\mathbf{I}_{\Omega_l}^{\tilde{\delta}_l}$ respectively.

We also define

$$\tilde{\gamma}_i^{(l)} = \gamma_i - \gamma_i^{(l)}, \quad i = 1, 2, 3. \quad l \in N.$$

Then from (3.28), after letting $\delta \to 0$, we have

$$
\begin{aligned}
&|\int_0^T \int_{R^3} f_l(x_3,t) d(-\frac{1}{2}\gamma_1^{(l)} - \frac{1}{2}\gamma_2^{(l)} + \gamma_3^{(l)})| \\
&\leq |\int_0^T \int_{R^3} (x_3 - x_3^l) g_l(x_3,t) \partial_{x_3} \mathbf{I}_{\Omega_l}^{\tilde{\delta}_l} d(-\frac{1}{2}\tilde{\gamma_1}^{(l)} - \frac{1}{2}\tilde{\gamma_2}^{(l)} + \tilde{\gamma_3}^{(l)})|.
\end{aligned}
\tag{3.30}
$$

Note that $\gamma_i, \tilde{\gamma_i}^{(l)} (i = 1, 2, 3)$ are finite Radon measures. From (3.29), (3.30), it yields

$$
\begin{aligned}
&|\int_0^T \int_{R^3} f_l(x_3,t) d(-\frac{1}{2}\gamma_1^{(l)} - \frac{1}{2}\gamma_2^{(l)} + \gamma_3^{(l)})| \\
&\leq \frac{C}{\tilde{\delta}_l} (|\gamma_1| + |\gamma_2| + |\gamma_3|),
\end{aligned}
$$

where C is a constant independent of $\tilde{\delta}_l$. Due to the arbitrariness of $\tilde{\delta}_l > 0$, we have

$$
\int_0^T \int_{R^3} f_l(x_3,t) d(-\frac{1}{2}\gamma_1^{(l)} - \frac{1}{2}\gamma_2^{(l)} + \gamma_3^{(l)}) = 0
$$

for any $f_l(x_3,t) \in C_0^\infty(\bar{\Omega}_l)$. Consequently, one has

$$
\gamma_1^{(l)} + \gamma_2^{(l)} - 2\gamma_3^{(l)} = 0.
$$

Therefore,

$$
\sum_{l=1}^{+\infty} (\gamma_1^{(l)} + \gamma_2^{(l)} - 2\gamma_3^{(l)}) = 0,
$$

which leads to

$$
\gamma_1 + \gamma_2 - 2\gamma_3 = 0,
$$

So (3.27) holds true even when $\bigcup_{i=1}^3 Supp\{\gamma_i\}$ is infinite.

Step IV. Other Cancellation Combinations

In Step II and Step III, we have obtained that $(u_1^{\varepsilon_j})^2 + (u_2^{\varepsilon_j})^2 - 2(u_3^{\varepsilon_j})^2$ is a *cancellation combinations* of u^ε. In this section, we will use Proposition 2.6 to yield other types of *cancellation combinations*. In particular, we will show that $(u_1^{\varepsilon_j})^2 + (u_2^{\varepsilon_j})^2 - (u_3^{\varepsilon_j})^2$ is a *cancellation combinations* of u^ε. Here we just consider the case that $\cup_{i=1}^3 Supp\{\gamma_i\}$ is a compact set in $\{r = 0\}$. When $\cup_{i=1}^3 Supp\{\gamma_i\}$ is an infinite set, it can be dealt in a completely similar way as presented in Step III.

The key element in showing that $(u_1^{\varepsilon_j})^2 + (u_2^{\varepsilon_j})^2 - (u_3^{\varepsilon_j})^2$ is a *cancellation combinations* of u^ε is that $(u_1^\varepsilon)^2 + (u_2^\varepsilon)^2 = (u_r^\varepsilon)^2$ has a better estimate near the symmetry axis, *i.e.* Proposition 2.6. Indeed, we rearrange the integral equations (3.10) so that

$$
\begin{aligned}
&|\int_0^T\int_{R^3}(2(u_1^{\varepsilon_j})^2\partial_1\Phi_1+2(u_2^{\varepsilon_j})^2\partial_2\Phi_2+(u_3^{\varepsilon_j})^2\partial_3\Phi_3)dxdt| \\
&\le |\int_{R^3}u_0^{\varepsilon_j}(x)\Phi(x,0)dx|+|\int_0^T\int_{R^3}[u^{\varepsilon_j}\Phi_t++u_1^{\varepsilon_j}u_2^{\varepsilon_j}\partial_1\Phi_2+u_1^{\varepsilon_j}u_3^{\varepsilon_j}\partial_1\Phi_3 \\
&+u_2^{\varepsilon_j}u_1^{\varepsilon_j}\partial_2\Phi_1+u_2^{\varepsilon_j}u_3^{\varepsilon_j}\partial_2\Phi_3+u_3^{\varepsilon_j}u_1^{\varepsilon_j}\partial_3\Phi_1+u_3^{\varepsilon_j}u_2^{\varepsilon_j}\partial_3\Phi_2]dxdt| \\
&+|\int_0^T\int_{R^3}((u_1^{\varepsilon_j})^2\partial_1\Phi_1+(u_2^{\varepsilon_j})^2\partial_2\Phi_2)dxdt|+C(\int_{R^3}|\Phi(x,T)|^2dx)^{1/2}.
\end{aligned} \tag{3.31}
$$

Set

$$
G(x_3,t)=[g(x_3,t)+(x_3-x_3^0)\partial_{x_3}g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}}+(x_3-x_3^0)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}.
$$

Let the test functions be defined as in (3.18). In view of (3.21), we get by using Proposition 2.6 that

$$
\begin{aligned}
&|\int_0^T\int_{R^3}((u_1^{\varepsilon_j})^2\partial_1\Phi_1+(u_2^{\varepsilon_j})^2\partial_2\Phi_2)dxdt| \\
&\le \frac{1}{2}|\int_0^T\int_{R^3}[(u_1^{\varepsilon_j})^2+(u_2^{\varepsilon_j})^2]\chi(\frac{r}{\delta})G(x_3,t)dxdt| \\
&+|\int_0^T\int_{R^3}[(u_1^{\varepsilon_j})^2\cdot\frac{x_1^2}{2\delta r}+(u_2^{\varepsilon_j})^2\cdot\frac{x_2^2}{2\delta r}]\chi'(\frac{r}{\delta})G(x_3,t)dxdt| \\
&\le \frac{1}{2}|\int_0^T\int_{R^3}\frac{1}{1+x_3^2}(\frac{u_r^{\varepsilon_j}}{r})^2(1+x_3^2)r^2\chi(\frac{r}{\delta})G(x_3,t)dxdt| \\
&+|\int_0^T\int_{R^3}[(u_1^{\varepsilon_j})^2\cdot\frac{x_1^2}{2\delta r}+(u_2^{\varepsilon_j})^2\cdot\frac{x_2^2}{2\delta r}]\chi'(\frac{r}{\delta})G(x_3,t)dxdt| \\
&\le C(\tilde{\delta})\delta^2+|\int_0^T\int_{R^3}[(u_1^{\varepsilon_j})^2\cdot\frac{x_1^2}{2\delta r}+(u_2^{\varepsilon_j})^2\cdot\frac{x_2^2}{2\delta r}]\chi'(\frac{r}{\delta})G(x_3,t)dxdt|.
\end{aligned} \tag{3.32}
$$

Substituting (3.32), (3.22) and (3.24) into (3.31), letting $\varepsilon_j\to 0$, we have

$$
\begin{aligned}
&|\int_0^T\int_{R^3}f(x_3,t)(-d\gamma_1-d\gamma_2+d\gamma_3)| \\
&\le C(\tilde{\delta})\int_{\{|r|\le 2\delta,|x_3|\le K\}}|u_0(x)|dx \\
&+C(\tilde{\delta})[(\int_0^T\int_{\{|r|\le 2\delta,|x_3|\le K\}}|u|^2dxdt)^{1/2} \\
&+\int_0^T\int_{\{|r|\le 2\delta,|x_3|\le K\}}|u|^2dxdt] \\
&+C(\int_{\{|r|\le 2\delta,|x_3|\le K\}}|\Phi(x,T)|^2dx)^{1/2}+C(\tilde{\delta})\delta^2 \\
&+C(\tilde{\delta})\int_0^T\int_{\{|r|\le 2\delta,|x_3|\le K\}}|u|^2dxdt.
\end{aligned} \tag{3.33}
$$

For any fixed $\tilde{\delta} > 0$, letting $\delta \to 0$, we obtain

$$\int_0^T \int_{R^3} f(x_3,t)(-d\gamma_1 - d\gamma_2 + d\gamma_3) = 0,$$

for all $f(x_3,t) \in C_0^\infty(\bar{\Omega})$. This implies that

$$(u_1^{\varepsilon_j})^2 + (u_2^{\varepsilon_j})^2 - (u_3^{\varepsilon_j})^2 \rightharpoonup u_1^2 + u_2^2 - u_3^2, \tag{3.34}$$

weakly in $M(R^3 \times [0,T])$. So $(u_1^{\varepsilon_j})^2 + (u_2^{\varepsilon_j})^2 - (u_3^{\varepsilon_j})^2$ is a *cancellation combinations* of u^ε.

Combining (3.27) with (3.34), we obtain that $\gamma_1 = \gamma_2 = \gamma_3 = 0$, and so

$$|u^{\varepsilon_j}|^2 \rightharpoonup |u|^2,$$

weakly in $M(R^3 \times [0,T])$ as $\varepsilon_j \to 0$. As a result, we have

$$\|u^{\varepsilon_j}\|_{L^2_{loc}(R^3\times[0,T])} \longrightarrow \|u\|_{L^2_{loc}(R^3\times[0,T])}, \quad \varepsilon_j \to 0. \tag{3.35}$$

Recalling that

$$u^{\varepsilon_j} \rightharpoonup u$$

weakly in $L^2(R^3 \times [0,T])$, we obtain the desired strong convergence (3.3).

The proof of the theorem is proved.

Remark 3.1 Under the assumptions of the strong convergence in the region away from the axis, we can actually obtain more *cancellation combinations* by using the tricks of Step II-III. In fact we can prove that

$$\alpha_1(u_1^{\varepsilon_j})^2 + \alpha_2(u_2^{\varepsilon_j})^2 + \alpha_3(u_3^{\varepsilon_j})^2$$

are *cancellation combinations* for $\alpha_1 + \alpha_2 + \alpha_3 = 0, \alpha_i \in R (i = 1,2,3)$ by choosing the test functions in Step II-Step III as

$$\begin{aligned}
\Phi_1(x,t) &= -\alpha_1 x_1 \chi(\tfrac{r}{\delta})\{[g(x_3,t) + (x_3 - x_3^0)\partial_{x_3} g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}} \\
&\quad + (x_3 - x_3^0) g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\}, \\
\Phi_2(x,t) &= -\alpha_2 x_2 \chi(\tfrac{r}{\delta})\{[g(x_3,t) + (x_3 - x_3^0)\partial_{x_3} g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}} \\
&\quad + (x_3 - x_3^0) g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\}, \\
\Phi_3(x,t) &= [\alpha_3 \chi(\tfrac{r}{\delta}) + \tfrac{\alpha_1 x_1^2 + \alpha_2 x_2^2}{\delta r}\chi'(\tfrac{r}{\delta})](x_3 - x_3^0) g(x_3,t)\mathbf{I}_\Omega^{\tilde{\delta}}.
\end{aligned} \tag{3.36}$$

Remark 3.2 In Step II-Step III, choosing the test functions as, for example,

$$
\begin{aligned}
\Phi_1(x,t) &= -x_1^3\chi(\tfrac{r}{\delta})\{[g(x_3,t)+(x_3-x_3^0)\partial_{x_3}g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}} \\
&\quad +(x_3-x_3^0)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\}, \\
\Phi_2(x,t) &= -x_2\chi(\tfrac{r}{\delta})\{[g(x_3,t)+(x_3-x_3^0)\partial_{x_3}g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}} \\
&\quad +(x_3-x_3^0)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\}, \\
\Phi_3(x,t) &= \{[3x_1^2\chi(\tfrac{r}{\delta})+\chi(\tfrac{r}{\delta})]+\tfrac{x_1^4+x_2^2}{\delta r}\chi'(\tfrac{r}{\delta})\}(x_3-x_3^0)g(x_3,t)\mathbf{I}_\Omega^{\tilde{\delta}},
\end{aligned}
\tag{3.37}
$$

we can obtain that $\gamma_2=\gamma_3$. Similarly, we can also obtain that $\gamma_1=\gamma_3$. Therefore, using similar method given in Step II and Step III, we can obtain that $\gamma_1=\gamma_2=\gamma_3$ and $(u_1^{\varepsilon_j})^2-(u_2^{\varepsilon_j})^2$, $(u_2^{\varepsilon_j})^2-(u_3^{\varepsilon_j})^2$ and $(u_1^{\varepsilon_j})^2-(u_3^{\varepsilon_j})^2$ are all *cancellation combinations*.

Remark 3.3 We can also obtain that

$$
\begin{aligned}
u_1^{\varepsilon_j}u_2^{\varepsilon_j} &\to u_1u_2, \\
u_1^{\varepsilon_j}u_3^{\varepsilon_j} &\to u_1u_3, \\
u_2^{\varepsilon_j}u_3^{\varepsilon_j} &\to u_2u_3
\end{aligned}
\tag{3.38}
$$

in $M(R^3\times[0,T])$. Actually, from Remark 3.2, we have

$$
(u_1^{\varepsilon_j})^2-(u_2^{\varepsilon_j})^2\to u_1^2-u_2^2,
$$

in $M(R^3\times[0,T]$. So the first convergence in (3.38) can be obtained by rotation change. In order to obtain the second convergence in (3.38), we use the test functions defined by

$$
\begin{aligned}
\Phi_1(x,t) &= -x_1^3\chi(\tfrac{r}{\delta})\{[g(x_3,t)+(x_3-x_3^0)\partial_{x_3}g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}} \\
&\quad +(x_3-x_3^0)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\}, \\
\Phi_2(x,t) &= -\tfrac{1}{2}x_2^2\chi(\tfrac{r}{\delta})\{[g(x_3,t)+(x_3-x_3^0)\partial_{x_3}g(x_3,t)]\mathbf{I}_\Omega^{\tilde{\delta}} \\
&\quad +(x_3-x_3^0)g(x_3,t)\partial_{x_3}\mathbf{I}_\Omega^{\tilde{\delta}}\}, \\
\Phi_3(x,t) &= \{[3x_1^2\chi(\tfrac{r}{\delta})+x_2\chi(\tfrac{r}{\delta})]+\tfrac{x_1^4+x_2^3/2}{\delta r}\chi'(\tfrac{r}{\delta})\}(x_3-x_3^0)g(x_3,t)\mathbf{I}_\Omega^{\tilde{\delta}}.
\end{aligned}
$$

And the third convergence in (3.38) can be established similarly.

Remark 3.4 Using the tricks of Step IV, in which Proposition 2.6 has been used , we can prove that

$$
\alpha(u_1^{\varepsilon_j})^2+\alpha(u_2^{\varepsilon_j})^2+\beta(u_3^{\varepsilon_j})^2
$$

are *cancellation combinations* for $\alpha, \beta \in R$. This is enough to obtain the strong convergence of u^{ε_j} in $L^2_{loc}(R^3 \times [0,T])$. However, in Step II-III, it is not needed to use Proposition 2.6.

Remark 3.5 As described above, under the assumptions of Theorem 3.2, we have obtained more cancellation combinations of u^ε and more convergence in a different way, comparing with Proposition 3.1. It is also noted that Theorem 3.2 is proved for general initial vorticity, with distingushed sign or not.

As a direct corollary of Theorem 3.2, we have

Theorem 3.3 Under assumptions Theorem 3.2, there exists a global weak solutions of Cauchy problem for 3-D axisymmetric Euler equations without swirls.

Theorem 3.4 The results of Theorem 3.2 still hold for viscous approximations presented in Proposition $2.3'$.

Proof. Firstly, from the assumptions of Theorem 3.2, we know that there exits a subsequence $\{u^{\varepsilon_j}\}$ converges to u strongly in $L^2(Q \times [0,T])$ for $Q \subset\subset R^3 \backslash \{x \in R^3 | r = 0\}$ outside the axis. Then similar to Step I-Step III in the proof of Theorem 3.2, we obtain that $(u_1^{\varepsilon_j})^2 + (u_2^{\varepsilon_j})^2 - 2(u_3^{\varepsilon_j})^2$ is a *concellation combinations*. Finally, using Proposition 2.7, we can obtain that $(u_1^{\varepsilon_j})^2 + (u_2^{\varepsilon_j})^2 - (u_3^{\varepsilon_j})^2$ is a *concellation combinations*, similar to Step IV in Theorem 3.2. So one concludes that $\{u^{\varepsilon_j}\}$ (subsequence if necessary) converges to u strongly in $L^2_{loc}(R^3 \times [0,T])$. The proof of the theorem is finished and we omit the details here.

4 A Sufficient Condition Guaranteeing Strong Convergence in Region Away From the Axis

In this section, we give a sufficient condition which guaranteeing the strong convergence in the region away from the axis. A similar condition was given in [9] for 2-D Euler equations.

Set

$$\Omega_\delta \equiv \{x \in R^3 \mid \ r > \delta > 0\},$$

$$\Omega_\delta^K \equiv \{x \in R^3 | \ r > \delta > 0, \ |x| < K < +\infty\},$$

where $0 < 2\delta < K < +\infty$.

Our main result of this section is stated as follows

Theorem 4.1 Suppose that $\{u^\varepsilon\}$ and $\{\omega^\varepsilon\}$ are the approximate solutions constructed in Proposition 2.2$'$ or Proposition 2.3$'$, and there exists a number $\beta > 1$ such that

$$\max_{t\in[0,T]} \int_{|x-x_0|\le R} |\omega^\varepsilon(x,t)|dx \le C(K,\delta)R\left(\ln(\frac{1}{R})\right)^{-\beta}, \quad x_0 \in \Omega_\delta^K \tag{4.1}$$

for all $0 < R < \min\{\frac{1}{2}, \frac{\delta^2}{4}\}$. Then there exists a subsequence $\{u^{\varepsilon_j}\}$ of $\{u^\varepsilon\}$ such that for all $\rho(x,t) \in C_0^\infty(R^3\backslash\{r=0\}\times(0,T))$, we have

$$\int\int \rho|u^{\varepsilon_j}|^2 dxdt \longrightarrow \int\int \rho|u|^2 dxdt, \quad j\to+\infty. \tag{4.2}$$

To prove this theorem, we first introduce the potential functions and derive some estimates.

Using the vector identity

$$\nabla\times\nabla\times A = -\triangle A + \nabla(\nabla\cdot A),$$

one can introduce a potential vector $\Psi^\varepsilon(x,t)$ satisfying the following Possion equation

$$-\triangle\Psi^\varepsilon = \omega^\varepsilon, \tag{4.3}$$

and

$$\text{div } \Psi^\varepsilon = 0, \quad u^\varepsilon = \nabla\times\Psi^\varepsilon.$$

It then follows from the standard elliptic regularity and Corollary 2.5 that

$$\begin{aligned} &\Psi^\varepsilon \text{ is uniformly bounded in } L^\infty([0,T];H^1_{loc}(R^3)),\\ &\tfrac{\partial}{\partial t}\Psi^\varepsilon \text{ is uniformly bounded in } L^\infty([0,T];W^{-2,2}_{loc}(R^3)). \end{aligned} \tag{4.4}$$

Thus, there exists a subsequence $\{\Psi^{\varepsilon_j}\}$ of $\{\Psi^\varepsilon\}$ such that

$$\Psi^{\varepsilon_j}(x,t) \longrightarrow \Psi(x,t) \ \text{ strongly in } L^2([0,T];L^2_{loc}(R^3)), \tag{4.5}$$

and $\quad u = \nabla\times\Psi.$

Next, we show that Ψ^ε is uniformly continuous in space. For convenience, we omit ε in supscript of the approximate solutions in the following estimates.

Lemma 4.2 Suppose that the conditions in Theorem 4.1 are satisfied. Then, for any $x_1, x_2 \in \Omega_\delta^K$ and for any $t\in[0,T]$, it holds that

$$|\Psi(x_1,t)-\Psi(x_2,t)| \le C(K,\delta)\left(\ln(\frac{1}{|x_1-x_2|})\right)^{-\beta+1} \tag{4.6}$$

for $0<|x_1-x_2|<\min\{\frac{1}{2},\delta,\frac{\delta^2}{4}\}$.

Proof. The solution to (4.3) can be represented as

$$\Psi(x,t)=-\frac{1}{4\pi}\int_{\Omega_\delta^K}\frac{1}{|x-y|}\omega(y,t)dy+g(x,t),\quad x\in\Omega_\delta^K,$$

for any $t\in[0,T]$, where $g(x,t)$ is a harmonic function in x-variable in Ω_δ^K. So

$$\begin{aligned}|\Psi(x_1,t)-\Psi(x_2,t)| &\le \frac{1}{4\pi}\int_{\Omega_\delta^K}|\frac{1}{|x_1-y|}-\frac{1}{|x_2-y|}||\omega(y,t)|dy\\ &+|g(x_1,t)-g(x_2,t)|.\end{aligned}$$

It follows from the Minkowski's inequality that

$$\|\int_{\Omega_\delta^K}\frac{1}{|x-y|}\omega(y,t)dy\|_{L^2(\Omega_\delta^K)}\le(\int_{|x|\le 2K}\frac{1}{|x|^2}dx)^{1/2}\|\omega\|_{L^1(R^3)}\le C(K).$$

In view of (4.4), one has $g(x,t)\in L^\infty([0,T];L^2(\Omega_\delta^K))$. By the standard elliptic regularity arguments, it is easy to get that $g(x,t)\in L^\infty([0,T];H^3(\Omega_\delta^K))$ and therefore $g(x,t)\in L^\infty([0,T];C^\alpha(\Omega_\delta^K))$ for some $\alpha>0$. So in order to prove the lemma, it suffices to prove

$$I\equiv\int_{\Omega_\delta^K}|\frac{1}{|x_1-y|}-\frac{1}{|x_2-y|}||\omega(y,t)|dy\le C(K,\delta)\left(\ln(\frac{1}{|x_1-x_2|})\right)^{-\beta+1}. \tag{4.7}$$

To this end, we decompose I as

$$I=I_1+I_2+I_3,$$

where

$$\begin{aligned}I_1&=\int_{|x_1-y|\le\frac{|x_1-x_2|}{2}}|\frac{1}{|x_1-y|}-\frac{1}{|x_2-y|}||\omega(y,t)|dy;\\ I_2&=\int_{|x_2-y|\le\frac{|x_1-x_2|}{2}}|\frac{1}{|x_1-y|}-\frac{1}{|x_2-y|}||\omega(y,t)|dy;\\ I_3&=\int_{|x_1-y|\ge\frac{|x_1-x_2|}{2},|x_2-y|\ge\frac{|x_1-x_2|}{2}}|\frac{1}{|x_1-y|}-\frac{1}{|x_2-y|}||\omega(y,t)|dy.\end{aligned}$$

Step 1. $|x_1-y|>\frac{|x_1-x_2|}{2}$ and $|x_1-y|>\frac{|x_1-x_2|}{2}$.

In this case, we have

$$\left|\frac{1}{|x_1-y|}-\frac{1}{|x_2-y|}\right|\le\begin{cases}\frac{|x_1-x_2|}{|x_1-y|^2}, & \text{if}\quad |x_1-y|<|x_2-y|,\\ \frac{|x_1-x_2|}{|x_2-y|^2}, & \text{if}\quad |x_2-y|<|x_1-y|.\end{cases}$$

Without loss of generality, we consider the case of

$$\left|\frac{1}{|x_1-y|}-\frac{1}{|x_2-y|}\right|\le\frac{|x_1-x_2|}{|x_1-y|^2}.$$

The other case can be estimated similarly. Define

$$m(R,y,t)=\int_{|z|\le R}|\omega(z+y,t)|dz.$$

Then for any $y\in\Omega_\delta^K$ and $t\in[0,T]$, by assumptions, one has that

$$m(R,y,t)\le C(K,\delta)R\left(\ln(\frac{1}{R}\right)^{-\beta}\quad\text{for}\quad 0<R<\min\{\frac{1}{2},\frac{\delta^2}{4}\}.$$

In the following, we use $m(R)$ instead of $m(R,y,t)$ for simplicity. Choosing $0<\lambda=\lambda(\delta)<\delta$, one has

$$\begin{aligned}I_3 &\le|x_2-x_1|\int_{|x_1-y|>\frac{|x_1-x_2|}{2}}\frac{1}{|x_1-y|^2}|\omega(y,t)|dy\\ &\le\frac{4}{\lambda^2}|x_2-x_1|\|\omega^\theta(t)\|_{L^1}+|x_2-x_1|\int_{\frac{|x_1-x_2|}{2}<|x|<\frac{\lambda}{2}}\frac{|\omega(x_1+x,t)|}{|x|^2}dx\\ &=\frac{4}{\lambda^2}|x_2-x_1|\|\omega^\theta(t)\|_{L^1}+|x_2-x_1|\int_{\frac{|x_1-x_2|}{2}}^{\frac{\lambda}{2}}\frac{1}{R^2}dm(R)\\ &\le\frac{4}{\lambda^2}|x_2-x_1|\|\omega^\theta(t)\|_{L^1}+|x_2-x_1|\frac{m(R)}{R^2}\Big|_{|x_1-x_2|/2}^{\lambda/2}\\ &+2|x_2-x_1|\int_{|x_2-x_1|/2}^{\lambda/2}\frac{m(R)}{R^3}dR\\ &\le C(\delta)|x_2-x_1|+C(K,\delta)\left(\ln(\frac{2}{|x_1-x_2|})\right)^{-\beta}\\ &+C(K,\delta)|x_2-x_1|\int_{|x_2-x_1|/2}^{\lambda/2}\frac{1}{R^2}\left(\ln\frac{1}{R}\right)^{-\beta}dR.\end{aligned}\tag{4.8}$$

Note that

$$
\begin{aligned}
& |x_2-x_1| \int_{|x_2-x_1|/2}^{\lambda/2} \frac{1}{R^2}\left(\ln\frac{1}{R}\right)^{-\beta} dR \\
\leq\ & |x_2-x_1| \left| \int_{|x_2-x_1|/2}^{\lambda/2} \left(\ln\frac{1}{R}\right)^{-\beta} d\frac{1}{R} \right| \\
\leq\ & |x_2-x_1| \frac{1}{R}(\ln\frac{1}{R})^{-\beta} \,|_{|x_1-x_2|/2}^{\lambda/2} \\
+\ & \beta|x_1-x_2| \int_{|x_2-x_1|/2}^{\lambda/2} \frac{1}{R^2}\left(\ln\frac{1}{R}\right)^{-(\beta+1)} dR \\
\leq\ & C(\delta)|x_2-x_1| + 2\left(\ln(\frac{2}{|x_1-x_2|})\right)^{-\beta} \\
+\ & \beta|x_1-x_2| \left(\ln\frac{2}{\lambda}\right)^{-1} \int_{|x_2-x_1|/2}^{\lambda/2} \frac{1}{R^2}\left(\ln\frac{1}{R}\right)^{-\beta} dR.
\end{aligned}
$$

Choosing $0<\lambda=\lambda(\delta)<\delta$ small enough such that

$$
\beta\left(\ln\frac{2}{\lambda}\right)^{-1} \leq 1/2,
$$

we obtain

$$
\begin{aligned}
& |x_2-x_1| \int_{|x_2-x_1|/2}^{\lambda/2} \frac{1}{R^2}\left(\ln\frac{1}{R}\right)^{-\beta} dR \\
\leq\ & C(\delta)|x_2-x_1| + 4\left(\ln(\frac{2}{|x_1-x_2|})\right)^{-\beta}.
\end{aligned}
\tag{4.9}
$$

Substituting (4.9) into (4.8) leads to

$$
I_3 \leq C(K,\delta)\left[|x_2-x_1| + \left(\ln(\frac{2}{|x_1-x_2|})\right)^{-\beta}\right]. \tag{4.10}
$$

Step 2. $|x_1-y| \leq \frac{|x_1-x_2|}{2}$ or $|x_2-y| \leq \frac{|x_1-x_2|}{2}$.

When $|x_1 - y| \leq \frac{|x_1-x_2|}{2}$, then $|x_2 - y| \geq \frac{|x_1-x_2|}{2}$. I_1 is estimated as

$$
\begin{aligned}
I_1 &\leq |x_1 - x_2| \int_{|x_1-y|\leq \frac{|x_1-x_2|}{2}} \frac{1}{|x_1-y||x_2-y|}|\omega(y,t)|dy \\
&\leq 2\int_{|x_1-y|\leq \frac{|x_1-x_2|}{2}} \frac{1}{|x_1-y|}|\omega(y,t)|dy \\
&= 2\int_{|y|\leq \frac{|x_1-x_2|}{2}} \frac{1}{|y|}|\omega(x_1-y,t)|dy \\
&\leq C\int_0^{|x_1-x_2|/2} \frac{1}{R}dm(R) \qquad (4.11)\\
&\leq C\frac{m(R)}{R}\,|_0^{|x_1-x_2|} + C\int_0^{|x_1-x_2|/2} \frac{m(R)}{R^2}dR \\
&\leq C\left(\ln(\frac{1}{|x_1-x_2|})\right)^{-\beta} + C\left(\ln(\frac{1}{|x_1-x_2|})\right)^{-\beta+1} \\
&\leq C\left(\ln(\frac{1}{|x_1-x_2|})\right)^{-\beta+1},
\end{aligned}
$$

for $|x_1 - x_2| < 1/2$. Similar proof gives

$$I_2 \leq C\left(\ln(\frac{1}{|x_1-x_2|})\right)^{-\beta+1}. \qquad (4.12)$$

Thanks to (4.10), (4.11) and (4.12) , the estimate (4.7) is obtained. The proof of the lemma is finished.

Next, we show that the potential function is uniformly bounded, *i.e.*,
Lemma 4.3 For any $t \in [0,T]$ and $x \in \Omega_\delta^K$, we have

$$|\Psi(x,t)| \leq C(K,\delta). \qquad (4.13)$$

Proof. As before,

$$\Psi(x,t) = -\frac{1}{4\pi}\int_{\Omega_\delta^K} \frac{1}{|x-y|}\omega(y,t)dy + g(x,t) \equiv J(x) + g(x), t \in (0,T), x \in \Omega_\delta^K,$$

where $g(x,t)$ is a harmonic function in $x-$ variable in Ω_δ^K. Same arguments as in Lemma 4.2 for $g(x,t)$ and Sobolev embedding theorem give

$$|g(x,t)| \leq C(K),\ t \in [0,T], x \in \Omega_\delta^K. \qquad (4.14)$$

On the other hand, direct estimate gives

$$
\begin{aligned}
|J(x,t)| &\le \frac{1}{4\pi}\int_{|x-y|<\delta/2}\frac{1}{|x-y|}|\omega(y,t)|dy + \frac{1}{4\pi}\int_{|x-y|\ge\delta/2}\frac{1}{|x-y|}|\omega(y,t)|dy \\
&\le \frac{1}{4\pi}\int_{|\tilde{y}|<\delta/2}\frac{1}{|\tilde{y}|}|\omega(x-\tilde{y},t)|d\tilde{y} + \frac{1}{2\delta\pi}\|\omega(t)\|_{L^1} \\
&\le C(\delta)\left(1+(\ln\frac{2}{\delta})^{-\beta+1}\right)
\end{aligned}
\tag{4.15}
$$

for all $t\in[0,T]$ and $x\in\Omega_\delta^K$. In the last inequality above, similar way in Step 2 in the proof of Lemma 4.2 has been used. Combining (4.14) with (4.15), we get

$$|\Psi(x,t)|\le C(K,\delta),\quad t\in[0,T];x\in\Omega_\delta^K.$$

Proof of Theorem 4.1. It follows from the construction of the approximate solutions that

$$
\begin{aligned}
&u^{\varepsilon_j}(x,t)\rightharpoonup u(x,t) \quad \text{weakly in } L^2([0,T]\times R^3);\\
&u^{\varepsilon_j}(x,t)\rightharpoonup u(x,t) \quad \text{weak } -* \text{ in } L^\infty([0,T];L^2(R^3));\\
&\omega^{\varepsilon_j}(x,t)\rightharpoonup \omega(x,t) \quad \text{weakly in } M([0,T]\times R^3).
\end{aligned}
\tag{4.16}
$$

Here $T>0$ is any positive number. Furthermore, $\omega(x,t)\in L^\infty([0,T];M(R^3))$ and $\omega=\nabla\times u$.

Note that the vector potential $\Psi^\varepsilon(x,t)$ satisfies

$$-\triangle\Psi^\varepsilon=\omega^\varepsilon,\quad u^\varepsilon=\nabla\times\Psi^\varepsilon.$$

Integrating by parts yields

$$\int\int\rho|u^\varepsilon|^2dxdt=\int\int\rho\Psi^\varepsilon\cdot\omega^\varepsilon dxdt-\int\int(\nabla\rho\times\Psi^\varepsilon)\cdot u^\varepsilon dxdt. \tag{4.17}$$

From (4.5), it is clear that

$$\int\int(\nabla\rho\times\Psi^{\varepsilon_j})\cdot u^{\varepsilon_j}dxdt\longrightarrow\int\int(\nabla\rho\times\Psi)\cdot udxdt,\quad j\to+\infty. \tag{4.18}$$

where Ψ and ω are the vector potential and vorticity respectively corresponding to the velocity u.

Suppose that the supp$\{\rho(x,t)\}\subset\Omega\times[t_0,t_1]$, where $\Omega\subset\subset R^3\backslash\{x\in R^3|r=0\}$ is an open set and $t_0\ge0,t_1\le T$. We also assume that the distance between Ω and the axis $\{r=0\}$ is $\delta>0$. Then we utilize Lemma 4.2 and Lemma 4.3 to conclude that

$$\rho\Psi^\varepsilon \text{ is uniformly bounded in } \quad C([0,T];C_0^\gamma(\Omega)),$$

where $C^\gamma(\Omega)$ is the Hölder space of functions with modulus of continuity defined by $\gamma(|x-y|) = |\ln(\frac{1}{|x-y|})|^{-\frac{1}{2}}$. Since the injection $i: C_0^\gamma(\Omega) \to C_0(\Omega)$ is compact and $\rho\Psi^\varepsilon$ is uniformly Lipschitz in $H_{loc}^{-2}(R^3)$ (see (4.4)), by the Lions-Aubin lemma, we get

$$\rho\Psi^{\varepsilon_j} \longrightarrow \rho\Psi \text{ uniformly in } C([0,T], C_0(\Omega)), \text{ as } j \to +\infty.$$

In view of (4.16), it follows that

$$\int\int \rho\Psi^{\varepsilon_j} \cdot \omega^{\varepsilon_j} dxdt \longrightarrow \int\int \rho\Psi \cdot \omega dxdt, \quad j \to +\infty. \tag{4.19}$$

Substitute (4.18) and (4.19) into (4.17) to get

$$\int\int \rho|u^{\varepsilon_j}|^2 dxdt \longrightarrow \int\int \rho\Psi \cdot \omega dxdt - \int\int (\nabla\rho \times \Psi) \cdot \omega dxdt, \quad j \to +\infty. \tag{4.20}$$

Using standard approximate function $u^\tau(x,t) = \rho_\tau * u(x,t)$ of $u(x,t)$, where $\rho_\tau(x) = \frac{1}{\tau^3}\rho(\frac{x}{\tau})$, $\rho(x) \in C_0^\infty(R^3), \rho \geq 0$ and $\int \rho(x)dx = 1$, we can prove easily that

$$\int\int \rho|u|^2 dxdt = \int\int \rho\Psi \cdot \omega dxdt - \int\int (\nabla\rho \times \Psi) \cdot udxdt.$$

Due to (4.20), we finally get

$$\int\int \rho|u^{\varepsilon_j}|^2 dxdt \longrightarrow \int\int \rho|u|^2 dxdt, \quad j \to +\infty.$$

The proof of the theorem is completed.

As a direct corollary, we have

Corollary 4.4 Under the assumptions in Theorem 4.1, there exists a subsequence $\{u^{\varepsilon_j}\} \subset \{u^\varepsilon\}$ such that, for any $Q \subset\subset R^3 \backslash \{x \in R^3 | r = 0\}$,

$$u^{\varepsilon_j} \longrightarrow u \quad \text{strongly in} \quad L^2([0,T]; L^2(Q)), \text{ as } j \to +\infty.$$

We conclude this section by pointing out that the above procedure is applicable to the approximate solutions generated by the corresponding Navier-Stokes systems.

Theorem 4.5 The results of Theorem 4.1 and Corollary 4.4 still hold for the viscous approximations $\{u^\varepsilon\}$ and $\{\omega^\varepsilon\}$ constructed in Proposition 2.3′.

5 Decay Rate for Maximal Vorticity Function in Region Away From the Axis

Our purpose in this section is to get an estimate on the rate of decay rate of the maximum vorticity function defined as in the left side of (1.11). We still omit ε in the approximate solutions in our following estimates. Let

$$H = \{(r,z) \in \bar{R^+} \times R | \; r \geq 0\}; \quad H_\delta = \{(r,z) \in \bar{R^+} \times R | \; r \geq \delta > 0\}.$$

Also define

$$\zeta = (r,z), \quad \zeta_0 = (r_0, z_0), \quad \text{for} \quad (r,z), (r_0,z_0) \in \bar{R^+} \times R.$$

We start with the estimate of the decay rate in (r,z)-plane of the vorticity as an axisymmetric function, whose proof is similar to 2-D case indicated in [22], [28] and [19].

Lemma 5.1 Under the Assumptions (A), we have, for any $\delta > 0, \zeta_0 \in H_\delta$,

$$\max_{t\in[0,T]} \int\int_{|\zeta-\zeta_0|\leq R} \omega^\theta(\zeta,t) drdz \leq C(\delta) \left(\ln(\frac{1}{R})\right)^{-\frac{1}{2}} \tag{5.1}$$

for all $0 < R < \min\{\frac{1}{2}, \delta^2/4\}$.

Proof. Choose $0 < \eta$ with $\sqrt{\eta} < \min\{\frac{1}{2}, \delta/2\}$. Define $\chi_\eta(\zeta)$ as

$$\chi_\eta(\zeta) = \begin{cases} 1, & |\zeta| \leq \eta, \\ \dfrac{\ln(\sqrt{\eta}/|\zeta|)}{\ln(1/\sqrt{\eta})}, & \eta \leq |\zeta| \leq \sqrt{\eta}, \\ 0; & |\zeta| \geq \sqrt{\eta}. \end{cases}$$

Direct computation gives

$$\|\nabla_{(r,z)}\chi_\eta\|_{L^2} = \left(\int_0^\infty \int_{-\infty}^\infty |\nabla\chi_\eta(r,z)|^2 drdz\right)^{\frac{1}{2}} \leq \frac{C}{\sqrt{\ln(1/\eta)}}.$$

For any $\zeta_0 \in H_\delta$, noticing that $\omega^\theta = \partial_r u^z - \partial_z u^\theta$, we have

$$\begin{aligned} & \int\int \chi_\eta(\zeta-\zeta_0)\omega^\theta(\zeta)d\zeta \\ = & \left|\int\int (u^r(\zeta,t), u^z(\zeta,t)) \cdot (-\partial_z, \partial_r)\chi_\eta(\zeta-\zeta_0)d\zeta\right| \\ \leq & \left(\int\int_{|\zeta-\zeta_0|\leq\sqrt{\eta}} |\tilde{u}|^2 d\zeta\right)^{\frac{1}{2}} \|\nabla_{(r,z)}\chi_\eta\|_{L^2}, \end{aligned} \tag{5.2}$$

where $\tilde{u}(\zeta,t) = (u^r(\zeta,t), u^z(\zeta,t))$. From $|\zeta - \zeta_0| \le \sqrt{\eta}$, one has

$$\delta/2 \le r \le |r_0| + \sqrt{\eta}.$$

So

$$\int\int_{|\zeta-\zeta_0|\le\sqrt{\eta}} |\tilde{u}|^2 drdz \le \frac{2}{\delta}\int\int_{\{\delta/2\le r\le|r_0|+\sqrt{\eta}\}} |\tilde{u}|^2 rdrdz \le \frac{1}{\delta\pi}\int_{R^3}|u|^2dx.$$

Consequently, in view of (5.2) and $\omega^\theta(\zeta,t)\ge 0$ for $a.e.\ \ \zeta\in H$, one has

$$\int\int_{|\zeta-\zeta_0|\le\eta} \omega^\theta(\zeta,t)drdz \le C(\delta)\left(\ln(\frac{1}{\eta})\right)^{-\frac{1}{2}},$$

which is

$$\int\int_{|\zeta-\zeta_0|\le R} \omega^\theta(\zeta,t)drdz \le C(\delta)\left(\ln(\frac{1}{R})\right)^{-\frac{1}{2}}$$

for all $0 < R < \min\{\frac{1}{2}, \delta^2/4\}$.

The proof of the lemma is finished.

Then we obtain the decay rate of the maximal vorticity function on Ω_δ^K as follows

Theorem 5.2 Under the Assumptions (A), we have that for any $x_0 \in \Omega_\delta^K$,

$$\max_{t\in[0,T]}\int\int\int_{|x-x_0|\le R} |\omega(x,t)|dx \le C(K,\delta)R\left(\ln(\frac{1}{R}\right)^{-\frac{1}{2}} \tag{5.3}$$

for all $0 < R < \min\{\frac{1}{2}, \frac{\delta^2}{4}\}$.

Proof. For $x_0 = (x_{01}, x_{02}, x_{03}) \in \Omega_\delta^K$, $x = (x_1,x_2,x_3) \in R^3$, let $\zeta = (r,z)$ and $\zeta_0 = (r_0, z_0)$ be the corresponding values in (r,z)-plane respectively, where $r^2 = x_1^2 + x_2^2, r_0^2 = x_{01}^2 + x_{02}^2$. Then it is clear that the volume of the set $\{x \in R^3 \mid |\zeta-\zeta_0| \le R\}$ is

$$|\{x\in R^3 \mid |\zeta-\zeta_0|\le R\}| = \int\int\int_{|\zeta-\zeta_0|\le R} dx = 2\pi\int\int_{|\zeta-\zeta_0|\le R} rdrdz.$$

Noting that $r_0 \ge \delta$ and $R < \min\{\frac{1}{2}, \delta^2/4\}$, one gets easily that $r \ge \max\{\delta - 1/2, \delta - \delta^2/4\} \equiv m(\delta)$. So

$$|\{x\in R^3 \mid |\zeta-\zeta_0|\le R\}| \ge 2\pi m(\delta)\int\int_{|\zeta-\zeta_0|\le R} drdz = 2\pi^2 m(\delta)R^2.$$

While for $x_0 \in \Omega_\delta^K$, one has

$$|\{|x-x_0|\le R\}| = \frac{4}{3}\pi R^3.$$

So

$$\frac{|\{|x-x_0|\le R\}|}{|\{|\zeta-\zeta_0|\le R\}|}\le \frac{2R}{3\pi m(\delta)}.$$

Due to the axisymmetric property of $\omega(x,t)$, it deduces that

$$\begin{aligned}\int\int\int_{|x-x_0|\le R}|\omega(x,t)|dx &\le \frac{4R}{3m(\delta)}\int\int_{|\zeta-\zeta_0|\le R}\omega^\theta(r,z,t)rdrdz\\ &\le \frac{4KR}{3(\delta)}\int\int_{|\zeta-\zeta_0|\le R}\omega^\theta(r,z,t)drdz\\ &\le C(K,\delta)R\left(\ln(\frac{1}{R})\right)^{-\frac{1}{2}}.\end{aligned}$$

The proof of the theorem is finished.

Remark 5.1 In general, for axisymmetric flows without swirls in the whole space, one can obtain that

$$\max_{0\le t\le T, x_0\in R^3}\int_{|x-x_0|\le R}|\omega^\varepsilon(x,t)|dx\le CR^{-\beta} \tag{5.4}$$

for some $0<\beta<1/2$.

This can be deduced in a direct way. Choose a smooth function $\chi_\delta(x)\in C_0^\infty(B_{\delta^{1/s}}(0))$ $\quad(s>1, 0<\delta<1)$, where $B_{\delta^{1/s}}(0)$ is the ball centered at the origin with radius $\delta^{1/s}$, such that

$$\chi_\delta(x)\equiv 1,\quad x\in B_\delta(0);\quad \chi_\delta(x)\equiv 0, x\in R^3\backslash B_{\delta^{1/s}}(0),$$

and

$$|\nabla\chi_\delta(x)|\le \frac{C}{\delta^{1/s}-\delta}.$$

Then one has

$$\left(\int_{\delta\le|x-y|\le\delta^{1/s}}|\nabla_y\chi_\delta(x-y)|^2dy\right)^{1/2}\le C\delta^{\frac{1}{2s}}.$$

Noting that

$$\nabla\times\tilde{u^\varepsilon}=\omega^\theta e_\theta,$$

we get

$$\begin{aligned}&\int_{R^3}\chi_\delta(x-y)e_\theta(y)\cdot(\omega^\theta)^\varepsilon e_\theta(y)dy\\ =\ &\int_{R^3}\chi_\delta(x-y)e_\theta(y)\cdot\nabla\times u(\tilde{y})^\varepsilon dy\le\left|\int_{R^3}\nabla\times(\chi_\delta(x-y)e_\theta(y))\cdot u(\tilde{y})^\varepsilon dy\right|\\ \le\ &\left|\int_{R^3}\nabla\times\chi_\delta(x-y)\times e_\theta(y)\cdot u(\tilde{y})^\varepsilon dy\right|+\left|\int_{R^3}\chi_\delta(x-y)\nabla\times e_\theta(y)\cdot u(\tilde{y})^\varepsilon dy\right|\\ \le\ &\|\nabla\chi_\delta(x-y)\|_{L^2}\|u(\tilde{y})^\varepsilon\|_{L^2}\le C\delta^{\frac{1}{2s}},\end{aligned}$$

which leads to (5.4).

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