Existence and Multiplicity Results for Perturbations of the p-Laplacian

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Abstract. In this paper we apply the Morse theory to study the existence of nontrivial solutions of p-Laplacian type Dirichlet boundary value problems.
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1. Introduction and Main Results

Consider the Dirichlet boundary value problem

$$\begin{cases} -\triangle_p u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\triangle_p u = \operatorname{div}(|\bigtriangledown u|^{p-2}\bigtriangledown u)$ $(p>1)$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and satisfies the subcritical growth:

$$(f) \qquad |f(x,t)| \le c(1+|t|^{q-1}), \quad \text{a.e. } x\in\Omega, \quad t\in\mathbb{R}$$

for some $c>0$ and $q\in[1,\ p^*)$ where $p^*=\frac{Np}{N-p}$ if $1<p<N$ and $p^*=+\infty$ if $N\le p$.

Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space which is a Banach space endowed with the norm $\|u\| = (\int_\Omega |\bigtriangledown u|^p dx)^{\frac{1}{p}}$ for $u\in W_0^{1,p}(\Omega)$. Define the functional $J: W_0^{1,p}(\Omega)\to\mathbb{R}$ by

$$J(u) = \frac{1}{p}\int_\Omega |\bigtriangledown u|^p dx - \int_\Omega F(x,u)dx, \quad u\in W_0^{1,p}(\Omega) \tag{1.1}$$

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where $F(x,t) = \int_0^t f(x,s)ds$. Under the condition (f), J is well defined and is a C^1 functional with its derivative given by

$$\langle J'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx - \int_\Omega f(x,u)v dx, \qquad \forall\, u,\ v \in W_0^{1,p}(\Omega). \tag{1.2}$$

Hence the weak solutions of the problem (P) correspond to the critical points of the functional J.

Let $f(x,0) \equiv 0$, then the problem (P) has a trivial solution $u \equiv 0$. We are interested in finding nontrivial solutions of the problem (P). It follows from the Morse theory that comparing the critical groups of J at zero and at infinity may yield the existence of nontrivial solutions to the problem (P)(cf. [8, 22]). The critical groups depend mainly upon the behaviors of the perturbed function $f(x,t)$ or its primitive $F(x,t)$ near zero and near infinity, respectively.

In this paper we will apply the Morse theory to study the existence of nontrivial weak solutions of the problem (P) by imposed various conditions on $f(x,t)$ or its primitive $F(x,t)$ near the origin and near infinity.

Let λ_1 denote the first eigenvalue of the p-homogeneous boundary value problem

$$\begin{cases} -\triangle_p u = \lambda |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{P_0}$$

It is well-known that $\lambda_1 > 0$ is simple and can be characterized as

$$\lambda_1 = \inf\left\{ \int_\Omega |\nabla u|^p dx : \ u \in W_0^{1,p}(\Omega), \ \int_\Omega |u|^p dx = 1 \right\}. \tag{1.3}$$

It is also known that λ_1 is an isolated point of $\sigma(-\triangle_p)$, the spectrum of $-\triangle_p$, which contains at least an increasing eigenvalue sequence obtained by the Lusternik-Schnirlaman theory. The corresponding eigenfunction φ_1 may be taken positive in Ω. Putting

$$W := \left\{ w \in W_0^{1,p}(\Omega) : \ \int_\Omega |\varphi_1|^{p-2}\varphi_1\ w\ dx = 0 \right\}$$

and $V = \operatorname{span}\{\varphi_1\}$, we have from the simiplicity of λ_1 that $W_0^{1,p}(\Omega) = V \oplus W$. Since λ_1 is isolated, the number $\bar\lambda = \inf_{w \in W\setminus\{0\}} \frac{\int_\Omega |\nabla w|^p dx}{\int_\Omega |w|^p dx}$ exists and satisfies $\bar\lambda > \lambda_1$. In addition,

$$\int_\Omega |\bigtriangledown u|^p dx \geq \bar\lambda \int_\Omega |u|^p dx, \quad \text{for} \quad u \in W. \tag{1.4}$$

When $p = 2$, $\bar\lambda = \lambda_2$, the second eigenvalue of $-\triangle$ in $H_0^1(\Omega)$.

Now we state the assumptions and the main results in this paper. Near the origin we make the following assumptions:

(f_{01}). There exist $\delta > 0$ and $\mu \in (0,\ p)$ such that

$$f(x,t)t > 0, \qquad \text{for} \ \ x \in \Omega, \qquad 0 < |t| \leq \delta, \tag{1.5}$$

$$\mu F(x,t) - f(x,t)t \geq 0, \quad \text{for} \ \ x \in \Omega, \qquad |t| \leq \delta. \tag{1.6}$$

(f_{02}). There exist $\delta > 0$ and $\lambda \in (\lambda_1, \ \bar{\lambda})$ such that

$$\lambda_1 |t|^p \le pF(x,t) \le \lambda |t|^p, \qquad \text{for} \quad x \in \Omega, \quad |t| \le \delta. \tag{1.7}$$

(f_{03}). There exist $\delta > 0$ and $\lambda \in (0, \ \lambda_1)$ such that

$$pF(x,t) \le \lambda |t|^p, \qquad \text{for} \quad x \in \Omega, \quad |t| \le \delta. \tag{1.8}$$

Near infinity we make the following assumptions:

$(f_{\infty 1})$. There exist $M > 0$ and $\theta > p$ such that

$$0 < \theta F(x,t) \le f(x,t)t, \qquad \text{for} \ \ x \in \Omega, \quad |t| \ge \ M. \tag{1.9}$$

$(f_{\infty 2})$.

$$\lim_{|t| \to \infty} \frac{pF(x,t)}{|t|^p} = \lambda_1, \qquad \text{uniformly} \quad \text{in} \quad x \in \Omega, \tag{1.10}$$

$$\lim_{|t| \to \infty} (F(x,t) - \frac{1}{p}\lambda_1 |t|^p) = -\infty, \qquad \text{uniformly} \quad \text{in} \quad x \in \Omega. \tag{1.11}$$

$(f_{\infty 3})$. (1.10) and

$$\lim_{|t| \to \infty} (f(x,t)t - pF(x,t)) = -\infty, \qquad \text{uniformly} \quad \text{in} \quad x \in \Omega. \tag{1.12}$$

Our main results are the following:

Theorem 1.1. *Let the function f satisfy one of the following conditions:*

*(**a**). (f_{01}) and $(f_{\infty 2})$;* *(**b**). (f_{01}) and $(f_{\infty 3})$;*

*(**c**). (f_{03}) and $(f_{\infty 1})$;* *(**d**). (f_{03}) and $(f_{\infty 3})$.*

Then the problem (P) has at least one nontrivial weak solution in $W_0^{1,p}(\Omega)$.

Theorem 1.2. *Let f satisfy (f_{02}) and $(f_{\infty 2})$. Then the problem (P) has at least two nontrivial weak solutions in $W_0^{1,p}(\Omega)$.*

Remark 1.1. Now let us give some remarks about the conditions given above

(i). It is easy to see that the conditions (f_{01}) and $(f_{\infty 1})$ imply, respectively, that

$$\lim_{t \to 0} \frac{pF(x,t)}{|t|^p} = +\infty, \qquad \lim_{|t| \to \infty} \frac{pF(x,t)}{|t|^p} = +\infty. \tag{1.12}$$

In the case $p = 2$, (f_{01}) and $(f_{\infty 1})$ mean that the function f is sublinear near zero and superlinear at infinity.

(ii). The condition (f_{02}) means that the problem (P) is resonant near the zero at the first eigenvalue λ_1 from the right side. It is clear that (f_{02}) contains the case $\lim_{t \to 0} pF(x,t)/|t|^p = \lambda \in (\lambda_1, \bar{\lambda})$. But here we do not need to assume that the limit exists.

The condition (f_{03}) is a nonresonance condition. It should be pointed out that in the case $p=2$, we may allow that $\lambda=\lambda_1$ which means that the problem (P) is resonant near 0 at λ_1 from the left side.

(iii). $(f_{\infty 2})$ means the problem (P) is resonant near infinity at λ_1 from left side and $(f_{\infty 3})$ means the problem (P) is resonant near infinity at λ_1 from right side. In Theorem 1.2, the more interesting case is that near the zero and near infinity the problem (P) is resonant at the same eigenvalue λ_1.

Remark 1.2. The problem (P) has been studied by many authors. Most of them treated the problem (P) by using directly variational methods or the minimax method such as the well-known Saddle Point Theorem or Mountain Pass Theorem([4, 24]). Under various conditions imposed on $f(x,t)$ or $F(x,t)$, solvability results for one solution or one nontrivial solution were obtained. Let us mention some of previous results. In [3], the authors treated the resonance case at infinity when the function $g(x,t):=f(x,t)-\lambda_1|t|^{p-2}t$ was bounded and satisfied the well-known Landesman-Lazer condition(cf.[13]). In [10], the authors got the existence of at least one nontrivial solution for the case

$$\limsup_{t\to 0}\frac{pF(x,t)}{|t|^p}\le\alpha<\lambda_1<\beta\le\liminf_{|t|\to\infty}\frac{pF(x,t)}{|t|^p}\qquad \text{a.e. } x\in\Omega.$$

They also treated the resonance case at infinity under the condition:

$$(F_\tau^{\pm})\qquad \liminf_{|t|\to\infty}\Big(\pm\frac{f(x,t)t-pF(x,t)}{|t|^\tau}\Big)\ge a>0,\ \text{for some } \tau>0,\ \text{uniformly a.e. } x\in\Omega.$$

In our results, we do not assume the boundedness of $g(x,t)$ and it is easy to see that $(F_\tau^{\pm})$ implies (1.11) and (1.12). In [11], the authors applied the Mountain Pass Theorem to get the existence of one positive solution and one negative solution to (P) for the case $(f_{\infty 1})$ and (f_{03}). For other existence results obtained by minimax methods, one refers to [1, 2, 12, 15, 16, 25] and the references therein.

Theorem 1.2 is a slight generalization of a result in [20] where the Morse theory was applied to get the multiple solutions of the problem (P) when f satisfied (f_{02}), (1.10) and the condition $\lim_{|t|\to\infty}(f(x,t)t-pF(x,t))=+\infty$ which implies (1.11) by a directly calculation. In [19], the author applied the Morse theory to study the case where (f_{02}) and $(f_{\infty 1})$ held. When $p=2$, the conditions (f_{01}) and $(f_{\infty 1})$ were used in [21] and [26], respectively. Hence our existence results are new. These existence results are even new for the linear case $p=2$ because we only require that $f(x,t)$ is a Carathédory function and has subcritical growth. Under these conditions, the corresponding functional J defined by (1.1) is only of C^1 and no Morse indices are concerned.

As we have mentioned, we will use the Morse theory to prove our main existence results. Let us now collect some concepts and results that will be used below.

Let X be a real Banach space and $J\in C^1(X,\mathbb{R})$, $\mathcal{K}=\{u\in X: J'(u)=0\}$. Let $u\in\mathcal{K}$ be an isolated critical point of J with $J(u)=c\in\mathbb{R}$, and U be a neighborhood of u, containing the unique critical point, the group $C_q(J,u):=H_q(J^c\cap U, J^c\cap U\setminus\{u\}), q\in\mathbb{Z}$ is called the q-th critical group of J at u where $J^c=\{u\in X:\ J(u)\le c\}$ and $H_q(\cdot,\cdot)$ the q-th singular relative homology group with integer coefficients. Let $a<\inf J(\mathcal{K})$. We call the group $C_q(J,\infty):=H_q(X,J^a)$, $q\in\mathbb{Z}$ the critical groups of J at infinity.(cf [6, 8, 22]).

In Morse theory, the functional J is always required to satisfy the so-called deformation condition (D)([6, 9]).

Definition. *The functional J satisfies (D_c) at the level $c \in \mathbb{R}$ if for any $\bar{\varepsilon} > 0$ and any neighborhood $\mathcal{N}$ of $\mathcal{K}_c$, there are $\varepsilon > 0$ and a continuous deformation $\eta : X \times [0,1] \to X$ such that*

(i) $\eta(u,t) = u$ for either $t = 0$ or $u \notin J^{-1}[c - \bar{\varepsilon}, c + \bar{\varepsilon}]$;

(ii) $J(\eta(u,t))$ is non-increasing in t for any $u \in X$;

(iii) $\eta(J^{c+\varepsilon} \setminus \mathcal{N}) \subset J^{c-\varepsilon}$.

J satisfies (D) if J satisfies (D_c) for all $c \in \mathbb{R}$.

In applications, we always require the functional J to satisfy the following compactness conditions.

Definition. *The functional J satisfies the $(PS)_c$ condition at the the level $c \in \mathbb{R}$ if any sequence $\{u_n\} \subset X$ satisfying $J(u_n) \to c$, $J'(u_n) \to 0$ in X^*, the dual space of X, as $n \to \infty$, has a convergent subsequence. J satisfies (PS) if J satisfies $(PS)_c$ at any $c \in \mathbb{R}$.*

Definition. *The functional J satisfies the Cerami condition[7] at the the level $c \in \mathbb{R}$ (C_c in short) if any sequence $\{u_n\} \subset X$ satisfying that $J(u_n) \to c$, $(1+\|u_n\|)\|J'(u_n)\|_{X^*} \to 0$ as $n \to \infty$ has a convergent subsequence. J satisfies (C) if J satisfies $(C)_c$ at any $c \in \mathbb{R}$.*

We note that the (C) condition was introduced by Cerami[7] and is a weak version of (PS). If J satisfies the (PS) condition or the (C) condition, then J satisfies the deformation condition(cf. [5, 9]).

Let J satisfy the (D) condition and $u = 0$ is a critical point of J. The Morse theory[8, 22] tells us that if $\mathcal{K} = \{0\}$ then $C_q(J,\infty) \cong C_q(J,0)$ for all $q \in \mathbb{Z}$. It follows that if $C_q(J,\infty) \ncong C_q(J,0)$ for some $q \in \mathbb{Z}$ then J must have a nontrivial critical point. So one has to compute these groups to get the nontrivial critical point. This is the basic idea to be used to prove our main results.

We refer the readers to [6, 8, 22] for more information about the Morse theory. In Section 2, we compute the group $C_q(J,0)$. In Section 3, we verify the compactness condition and compute the group $C_q(J,\infty)$. The proof of main results will be given in Section 4.

2. Critical Groups at Zero

In this section we compute the critical groups of J at zero. Assume that the problem (P) has finitely many solutions. Hence $u = 0$ is an isolated critical point of J and the critical group of J at zero is defined.

Proposition 2.1. *If f satisfies (f_{01}), then we have*

$$C_q(J,0) \cong 0, \qquad \forall \ q \in \mathbb{Z}. \tag{2.1}$$

Proof. By definition we write $C_q(J,0) := H_q(B_\rho \cap J^0, B_\rho \cap J^0 \setminus \{0\})$, where $B_\rho(0) = \{u \in W_0^{1,p}(\Omega) : \|u\| \le \rho\}$ and $\rho > 0$ is to be chosen later. We will get (2.1) by constructing a deformation mapping for the topological pairs $(B_\rho, B_\rho \setminus \{0\})$ and $(B_\rho \cap J^0, B_\rho \cap J^0 \setminus \{0\})$. For this purpose we need to analyze the local properties of J near zero.

A direct calculation by using (1.5) and (1.6) shows that there exists a constant $c_0 > 0$ such that

$$F(x,t) \ge c_0\ |t|^\mu, \qquad \text{for} \;\; x \in \Omega, \quad |t| \le \delta. \tag{2.2}$$

It follows from (f) and (2.2) that

$$F(x,t) \ge c_0|t|^\mu - c_1|t|^q, \quad x \in \Omega, \quad t \in \mathbb{R} \tag{2.3}$$

for some $q \in (p, p^*)$ and $c_1 > 0$. Hence for $u \in W_0^{1,p}(\Omega)$ and $s > 0$, we have

$$\begin{aligned} J(su) = &\ \frac{1}{p} s^p \int_\Omega |\nabla u|^p dx - \int_\Omega F(x, su) dx \\ \le &\ \frac{s^p}{p} \|u\|^p - \int_\Omega (c_0 |su|^\mu - c_1 |su|^q) dx \\ \le &\ \frac{s^p}{p} \|u\|^p - c_0\ s^\mu\ \|u\|^\mu_{L^\mu} + c_1\ s^q\ \|u\|^q_{L^q}. \end{aligned} \tag{2.4}$$

Here and in the following we denote by $c_i, (i = 0, 1, \cdots)$ various positive constants. Since $\mu < p < q$, for given $u \in W_0^{1,p}(\Omega)$ with $u \ne 0$, there exists a $s_0 = s_0(u) > 0$ such that

$$J(su) < 0, \qquad \forall \;\; 0 < s < s_0. \tag{2.5}$$

Let $u \in W_0^{1,p}(\Omega)$ be such that $J(u) = 0$. Then

$$\begin{aligned} \frac{d}{ds} J(su)|_{s=1} = &\ \langle J'(su), u \rangle |_{s=1} \\ = &\ \int_\Omega |\nabla u|^p dx - \int_\Omega f(x,u) u dx \\ = &\ (1 - \tfrac{\mu}{p}) \int_\Omega |\nabla u|^p dx + \int_\Omega (\mu\ F(x,u) - f(x,u)u) dx \\ \ge &\ (1 - \tfrac{\mu}{p})\ \|u\|^p - c_3 \int_\Omega |u|^q dx \;\; (\text{for some} \;\; q \in (p, p^*) \text{ by } (f)) \\ \ge &\ (1 - \tfrac{\mu}{p}) \|u\|^p - c_4\ \|u\|^q \;\; (\text{by the embedding } W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)). \end{aligned} \tag{2.6}$$

One concludes that there exists a $\rho > 0$ such that

$$\frac{d}{ds} J(su)|_{s=1} > 0, \;\; \forall\ u \in W_0^{1,p}(\Omega) \;\; \text{with } J(u) = 0 \text{ and } 0 < \|u\| \le \rho. \tag{2.7}$$

Now we fix $\rho > 0$. Then it follows from (2.7) that

$$J(su) < 0, \;\; \text{for} \;\; s \in (0,1), \text{for} \;\; u \in W_0^{1,p}(\Omega) \;\; \text{with } J(u) < 0 \text{ and } \|u\| \le \rho. \tag{2.8}$$

In fact if $\|u\| \le \rho$ and $J(u) < 0$ then there exists a $\tau \in (0,1)$ such that $J(su) < 0$ for all $s \in (1-\tau, 1)$ by the continuity of J. Suppose that there is some $s_0 \in (0, 1-\tau]$ such that $J(s_0 u) = 0$ and $J(su) < 0$ as $s_0 < s < 1$. Denote $u_0 = s_0 u$. Then by (2.7) we have

$\frac{d}{ds}J(su_0)|_{s=1} > 0$. But $J(su) - J(s_0u) < 0$ implies that $\frac{d}{ds}J(su)|_{s=s_0} = \frac{d}{ds}J(su_0)|_{s=1} \leq 0$. This contradiction shows that (2.8) holds.

Now define a mapping $T : B_\rho(0) \to [0, 1]$ as

$$T(u) = \begin{cases} 1, & \text{for } u \in B_\rho(0) \text{ with } J(u) \leq 0, \\ s, & \text{for } u \in B_\rho(0) \text{ with } J(u) > 0,\ J(su) = 0,\ s < 1. \end{cases}$$

By (2.5), (2.7) and (2.8), the mapping T is well-defined and if $J(u) > 0$ then there exists an unique $T(u) \in (0, 1)$ such that

$$J(T(u)u) = 0, \quad J(su) < 0, \quad \forall\ s \in (0, T(u)) \quad \text{and} \quad J(su) > 0, \quad \forall\ s \in (T(u), 1). \quad (2.9)$$

It follows from (2.7), (2.9) and the Implicit Function Theorem that the mapping T is continuous in u. Define a mapping $\eta : [0, 1] \times B_\rho(0) \to B_\rho(0)$ by

$$\eta(s, u) = (1 - s)u + sT(u)u, \quad s \in [0, 1], \quad u \in B_\rho(0). \quad (2.10)$$

It is easy to see that the mapping η is a continuous deformation from $(B_\rho, B_\rho \setminus \{0\})$ to $(B_\rho \cap J^0, B_\rho \cap J^0 \setminus \{0\})$. By the homotopy invariance of homology group, we have

$$C_q(J, 0) = H_q(B_\rho \cap J^0, B_\rho \cap J^0 \setminus \{0\}) \cong H_q(B_\rho, B_\rho \setminus \{0\}) \cong 0, \quad \forall\ q \in Z$$

since $B_\rho(0) \setminus \{0\}$ is contractible. The proof is completed. □

Proposition 2.2. *Suppose that f satisfies (f_{02}). Then $C_1(J, 0) \ncong 0$.*

Proof. Using the condition (f_{02}) we can prove that the functional J has a local linking property at zero with respect to the splitting $W_0^{1,p}(\Omega) = V \oplus W$, that is, there exists $\rho > 0$ such that

$$J(u) \leq 0, \quad \text{for } \ u \in V \text{ with } \ \|u\| \leq \rho, \quad J(u) > 0 \quad \text{for } \ u \in W \text{ with } \ 0 < \|u\| \leq \rho. \ (2.11)$$

(See [20] for details of the proof.) Notice that $k = \dim V = 1$, it follows from [18] that $C_1(J, 0) \ncong 0$. □

Remark 2.1. Proposition 2.2 holds in one of the following cases:

(i). $\lim\limits_{t \to 0} \dfrac{f(x, t)}{|t|^{p-1}t} = \lambda \in (\lambda_1, \bar{\lambda})$ uniformly in $x \in \Omega$.

(ii). $\lim\limits_{t \to 0} \dfrac{pF(x, t)}{|t|^p} = \lambda \in (\lambda_1, \bar{\lambda})$ uniformly in $x \in \Omega$.

(iii). $\lim\limits_{t \to 0} \dfrac{F(x, t)}{|t|^p} = \lambda_1$ uniformly in $x \in \Omega$, $pF(x, t) \geq \lambda_1 |t|^p$ for $x \in \Omega$ and $|t|$ small.

If $p = 2$ then we can allow that $\bar{\lambda} = \lambda_2$, the second eigenvalue of $(-\triangle, H_0^1(\Omega))$. In fact in the case $p = 2$, we can replace (f_{02}) by

$$\lambda_k |t|^p \leq pF(x, t) \leq \lambda_{k+1} |t|^p, \quad \forall\ x \in \Omega, \quad \forall\ |t| \leq \delta.$$

where λ_k and λ_{k+1} are two consecutive eigenvalues of $(-\triangle, H_0^1(\Omega))$(cf. [20]).

Proposition 2.3. *Suppose that f satisfies (f_{03}). Then $C_q(J, 0) \cong \delta_{q,0}\mathbb{Z}$ for all $q \in \mathbb{Z}$.*

Proof. For any $u \in W_0^{1,p}(\Omega)$, it follows from (f_{03}), (f) and the Sobolev embedding theorem that

$$
\begin{aligned}
J(u) = & \frac{1}{p}\int_\Omega |\bigtriangledown u|^p dx - \int_\Omega F(x,u)dx \\
= & \frac{1}{p}\int_\Omega |\bigtriangledown u|^p dx - \int_{|u|\le\delta} F(x,u)dx - \int_{|u|>\delta} F(x,u)dx \\
\ge & \frac{1}{p}\int_\Omega |\bigtriangledown u|^p dx - \int_{|u|\le\delta} \frac{\lambda}{p}|u|^p dx - \int_{|u|>\delta} |F(x,u)|dx \\
\ge & \frac{1}{p}(1-\frac{\lambda}{\lambda_1})\|u\|^p - c\|u\|^q, \quad (p<q\le p^* = Np/(N-p)).
\end{aligned}
\tag{2.12}
$$

Hence $u=0$ is a local minimizer of J and so $C_q(J,0)\cong \delta_{q,0}\mathbb{Z}$ for all $q\in\mathbb{Z}$. This completes the proof. □

Remark. 2.2. For the case $p=2$, we can replace (f_{03}) by

$$pF(x,t)\le \lambda_1|t|^p, \quad \text{for } x\in\Omega, \quad |t|\le\delta.$$

In this situation, $u=0$ is still a local minimizer of the functional J. This fact can be proved by using the nice decomposition property of $H_0^1(\Omega)$(cf. [20]).

3. Compactness and Critical Groups at Infinity

In this section we verify the compactness condition for the functional J and compute the critical groups of J at infinity. Since the perturbed function $f(x,t)$ satisfies the subcritical growth condition (f), a standard argument(cf.[3, 20]) shows the following

Lemma 3.1. *Let f satisfy (f). Then any bounded sequence $\{u_n\}\subset W_0^{1,p}(\Omega)$ such that $J'(u_n)\to 0$ in $(W_0^{1,p}(\Omega))^*$ as $n\to\infty$, has a convergent subsequence.*

Proposition 3.1. *Let f satisfy $(f_{\infty 1})$. Then*

(i). The functional J satisfies the (PS) condition;

(ii). $C_q(J,\infty)\cong 0$, $\forall$ $q\in\mathbb{Z}$.

Proof. This proposition was proved in [26] for $p=2$ and in [19] for $p\ne 2$. Here we give the key steps of the proof for the reader's convenience.

(i). Let $\{u_n\}\subset W_0^{1,p}(\Omega)$ be such that

$$J(u_n)\to c\in\mathbb{R}, \quad J'(u_n)\to 0, \ \text{ in } (W_0^{1,p}(\Omega))^*, \text{ as } n\to\infty. \tag{3.1}$$

By (f), $(f_{\infty 1})$ and the embedding theorem, we have that

$$
\begin{aligned}
\theta c + \circ(1) + \circ(\|u_n\|) = & \ \theta J(u_n) + \langle J'(u_n), u_n\rangle \\
\ge & \ (\frac{\theta}{p}-1)\|u_n\|^p - C
\end{aligned}
\tag{3.2}
$$

for some constant $C \in \mathbb{R}$. Since $\theta > p > 1$, it follows from (3.2) that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. The (PS) condition follows from Lemma 3.1.

(ii). Let S^∞ be the unit sphere in $W_0^{1,p}(\Omega)$. By $(f_{\infty 1})$ we see that $J(su) \to -\infty$ as $s \to +\infty$ for any $u \in \mathsf{S}^\infty$. Now using $(f_{\infty 1})$ again and by a careful calculation, we have that there is a constant $A > 0$ such that for any $a > A$, if $J(su) \leq -a$ for some $s > 0$ and $u \in \mathsf{S}^\infty$ then $\frac{d}{ds}J(su) < 0$. Hence for any fixed $a > A$, there exists an unique $T := T(u) > 0$ such that $J(T(u)u) = -a$ for $u \in \mathsf{S}^\infty$. By the Implicit Function Theorem, T is a continuous function from S^∞ to R. Therefore the deformation retract $\eta : [0,1] \times (W_0^{1,p}(\Omega) \setminus \mathsf{B}^\infty) \to (W_0^{1,p}(\Omega) \setminus \mathsf{B}^\infty)$ defined by $\eta(s,u) = (1-s)u + sT(u)u$ satisfies $\eta(0,u) = u$, $\eta(1,u) \in J^{-a}$ for a large enough, where $\mathsf{B}^\infty = \{u \in W_0^{1,p}(\Omega) : \|u\| \leq 1\}$. It follows that

$$\begin{aligned} C_q(J,\infty) = \; & H_q(W_0^{1,p}(\Omega), J^{-a}) \cong H_q(W_0^{1,p}(\Omega), W_0^{1,p}(\Omega) \setminus \mathsf{B}^\infty)) \\ \cong \; & H_q(\mathsf{B}^\infty, \mathsf{S}^\infty) \cong 0, \quad \forall\, q \in \mathbb{Z} \end{aligned}$$

The proof is completed. □

Proposition 3.2. *Let f satisfy $(f_{\infty 2})$. Then*

(i). The functional J satisfies the (PS) condition;

(ii). $C_q(J,\infty) \cong \delta_{q,0}\mathbb{Z}$, $\forall \; q \in \mathbb{Z}$.

Proof. (i). We will show that under the condition $(f_{\infty 2})$ J is coercive on $W_0^{1,p}(\Omega)$, i.e. $J(u) \to +\infty$ as $\|u\| \to \infty$. Hence the (PS) sequence of J must be bounded. Denote

$$G(x,t) = F(x,t) - \frac{1}{p}\lambda_1 |t|^p.$$

Then (1.11) implies

$$\lim_{|t| \to \infty} G(x,t) = -\infty, \quad \text{uniformly in} \quad x \in \Omega. \tag{3.3}$$

Rewrite J as

$$J(u) = \frac{1}{p}\int_\Omega (|\bigtriangledown u|^p - \lambda_1 |u|^p)dx - \int_\Omega G(x,u)dx, \quad u \in W_0^{1,p}(\Omega). \tag{3.4}$$

Assume that J is not coercive on $W_0^{1,p}(\Omega)$, then there is a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that

$$\|u_n\| \to \infty, \quad \text{as} \quad n \to \infty \quad \text{but} \quad J(u_n) \leq \hat{C} \tag{3.5}$$

for some $\hat{C} \in \mathbb{R}$. Denote

$$v_n = \frac{u_n}{\|u_n\|}, \qquad n \in \mathbb{N}.$$

Then $\|v_n\| = 1$. We may assume that there is a $v_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v_0, & \text{weakly in } W_0^{1,p}(\Omega), \\ v_n \to v_0, & \text{strongly in } L^p(\Omega), \\ v_n(x) \to v_0(x), & \text{for a.e. } x \in \Omega. \end{cases} \tag{3.6}$$

Now by using (3.3) and (3.5) we deduce

$$\frac{\hat{C}}{\|u_n\|^p} \geq \frac{J(u_n)}{\|u_n\|^p} \geq \frac{1}{p}\int_\Omega (|\bigtriangledown v_n|^p - \lambda_1|v_n|^p)\ dx - \frac{C_1}{\|u_n\|^p} \tag{3.7}$$

for some $C_1 > 0$. It follows from (3.5)-(3.7) that

$$\limsup_{n\to\infty} \int_\Omega |\bigtriangledown v_n|^p dx \leq \lambda_1 \int_\Omega |v_0|^p dx. \tag{3.8}$$

On the other hand, we have by using the Poincaré inequality and the lower semicontinuity of the norm that

$$\lambda_1 \int_\Omega |v_0|^p dx \leq \int_\Omega |\bigtriangledown v_0|^p dx \leq \liminf_{n\to\infty} \int_\Omega |\bigtriangledown v_n|^p dx. \tag{3.9}$$

From (3.8), (3.9) and the uniform convexity of $W_0^{1,p}(\Omega)$ we have

$$v_n \to v_0 \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad \int_\Omega |\bigtriangledown v_0|^p dx = \lambda_1 \int_\Omega |v_0|^p dx. \tag{3.10}$$

Hence $\|v_0\| = 1$ and so $v_0 = \pm\varphi_1$. Take $v_0 = \varphi_1$, then $u_n(x) \to +\infty$ a.e. in Ω. (1.3), (3.3) and the Fatou lemma imply that

$$\begin{aligned} \hat{C} \geq\ & \frac{1}{p}\int_\Omega (|\nabla u_n|^p - \lambda_1|u_n|^p)dx - \int_\Omega G(x, u_n(x))dx \\ \geq\ & -\int_\Omega G(x, u_n(x))dx \to +\infty \quad \text{as} \quad n \to \infty. \end{aligned}$$

This contradiction shows that J is coercive on $W_0^{1,p}(\Omega)$ and then satisfies (PS).

(ii) Since J is coercive and is weakly lower semicontinuous, J is bounded from below. By (i) we have that $C_q(J,\infty) \cong \delta_{q,0}\mathbb{Z}$ for all $q \in \mathbb{Z}$. □

Proposition 3.3. *Let f satisfy $(f_{\infty 3})$. Then*

(i). The functional J satisfies the (C) condition;

(ii). $C_1(J,\infty) \ncong 0$.

Proof. (i) Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be such that

$$J(u_n) \to c \in \mathbb{R}, \qquad (1 + \|u_n\|)\|J'(u_n)\|_* \to 0, \ \text{ as } \ n \to \infty. \tag{3.11}$$

By Lemma 3.1 we only need to prove that $\{u_n\}$ is bounded. Assume that $\|u_n\| \to \infty$ as $n \to \infty$. We still denote $v_n = u_n/\|u_n\|$ and may assume that there is some $v_0 \in W_0^{1,p}(\Omega)$ satisfying (3.6). By (1.10) we see that for any given $\varepsilon > 0$ there is $M_\varepsilon > 0$ such that

$$|G(x,t)| < \frac{1}{p}\ \varepsilon\ |t|^p, \ \text{ uniformly in } \ x \in \Omega, \ \ |t| > M_\varepsilon.$$

Hence we have by (f) and (3.11) that

$$
\begin{aligned}
& \circ(1) + \frac{c}{\|u_n\|^p} \\
= \ & \frac{J(u_n)}{\|u_n\|^p} \\
= \ & \frac{1}{p}\int_\Omega (|\nabla v_n|^p - \lambda_1 |v_n|^p)dx - \frac{1}{\|u_n\|^p}\int_\Omega G(x,u_n)dx \\
= \ & \frac{1}{p}\int_\Omega (|\nabla v_n|^p - \lambda_1 |v_n|^p)dx - \frac{1}{\|u_n\|^p}\int_{|u_n(x)|>M_\varepsilon} G(x,u_n)dx \\
& -\frac{1}{\|u_n\|^p}\int_{|u_n(x)|\le M_\varepsilon} G(x,u_n)dx \\
\ge \ & \frac{1}{p}\int_\Omega (|\nabla v_n|^p - \lambda_1 |v_n|^p)dx - \frac{\varepsilon}{p\|u_n\|^p}\int_{|u_n(x)|>M_\varepsilon} |u_n|^p dx - \frac{C}{\|u_n\|^p} \\
\ge \ & \frac{1}{p}\int_\Omega (|\nabla v_n|^p - \lambda_1 |v_n|^p)dx - \frac{\varepsilon}{p\|u_n\|^p}\int_\Omega |u_n|^p dx - \frac{C}{\|u_n\|^p} \\
\ge \ & \frac{1}{p}\int_\Omega (|\nabla v_n|^p - \lambda_1 |v_n|^p)dx - \frac{\varepsilon}{p\lambda_1} - \frac{C}{\|u_n\|^p}.
\end{aligned} \tag{3.12}
$$

where $C > 0$ is a constant. Letting $n \to \infty$ and using (3.6), we see that

$$\limsup_{n\to\infty}\int_\Omega |\nabla v_n|^p dx \le \lambda_1 \int_\Omega |v_0|^p dx + \frac{\varepsilon}{\lambda_1}$$

which implies (3.8) because of the arbitrariness of ε. Now using(3.8) and (3.9), we still have (3.10). Hence $\|v_0\| = 1$ and $v_0 = \pm\varphi_1$. Take $v_0 = +\varphi_1$; then $u_n(x) \to +\infty$ a.e. $x \in \Omega$. It follows from (1.12) that

$$\lim_{n\to\infty}(f(x,u_n(x))u_n(x) - pF(x,u_n(x)) = -\infty, \qquad \text{uniformly in} \quad x \in \Omega. \tag{3.13}$$

which implies that

$$\int_\Omega (f(x,u_n)u_n - pF(x,u_n))dx \to -\infty \quad \text{as} \ \ n \to \infty. \tag{3.14}$$

On the other hand, (3.11) implies

$$pJ(u_n) - \langle J'(u_n), u_n\rangle \to pc \quad \text{as} \ \ n \to \infty.$$

Thus

$$\int_\Omega (f(x,u_n(x))u_n(x) - pF(x,u_n(x))dx \to pc \quad \text{as} \quad n \to \infty$$

which contradicts to (3.14). Hence $\{u_n\}$ is bounded.

(ii). We still write $G(x,t) = F(x,t) - \frac{1}{p}|t|^p$, $g(x,t) = f(x,t) - \lambda_1|t|^{p-2}t$. Then (1.10) and (1.12) imply that

$$\lim_{|t|\to\infty}\frac{pG(x,t)}{|t|^p} = 0 \tag{3.15}$$

and

$$\lim_{|t|\to\infty} (g(x,t)t - pG(x,t)) = -\infty \tag{3.16}$$

respectively. It follows from (3.16) that for every $M > 0$, there is $T > 0$ such that

$$g(x,t)t - pG(x,t)) \leq -M, \quad \forall\, t \in \mathbb{R}, \quad |t| \geq T, \text{ a.e. } x \in \Omega. \tag{3.17}$$

For $\tau > 0$, we have

$$\frac{d}{d\tau}\left[\frac{G(x,\tau)}{\tau^p}\right] = \frac{g(x,\tau)\tau - pG(x,\tau)}{\tau^{p+1}}. \tag{3.18}$$

Integrating (3.18) over $[t,\ s] \subset [T, +\infty)$, we obtain that

$$\frac{G(x,s)}{s^p} - \frac{G(x,t)}{t^p} \leq \frac{M}{p}\left(\frac{1}{s^p} - \frac{1}{t^p}\right). \tag{3.19}$$

Letting $s \to +\infty$ and using (3.15), we see that $G(x,t) \geq \frac{M}{p}$, for $t \in \mathbb{R}$, $t \geq T$, a.e. $x \in \Omega$. A similar way shows that $G(x,t) \geq \frac{M}{p}$, for $t \in \mathbb{R}$, $t \leq -T$, a.e. $x \in \Omega$. Hence

$$\lim_{|t|\to\infty} G(x,t) = +\infty \quad \text{a.e. } x \in \Omega. \tag{3.20}$$

By the variational characterization of λ_1 and (3.20) we get that

$$\begin{aligned} J(v) &= \frac{1}{p}\int_\Omega |\nabla v|^p dx - \int_\Omega F(x,v)dx \\ &= -\int_\Omega G(x,v)dx \to -\infty \ \text{ for } v \in V \ \text{ with } \|v\| \to \infty. \end{aligned} \tag{3.21}$$

It follows from (1.10) and (f) that there is some $\lambda \in [\lambda_1, \bar{\lambda})$ and $C > 0$ such that

$$F(x,t) \leq \frac{1}{p}\lambda|t|^p + C.$$

Hence for $w \in W$, we have by (1.4) that

$$\begin{aligned} J(w) &= \frac{1}{p}\int_\Omega |\nabla w|^p dx - \int_\Omega F(x,w)dx \\ &\geq \frac{1}{p}\int_\Omega |\nabla w|^p dx - \frac{1}{p}\lambda\int_\Omega |w|^p dx - C \\ &\geq \frac{1}{p}(1 - \lambda/\bar{\lambda})\ \|w\|^p - C \\ &\to +\infty \quad \text{for} \quad w \in W \ \text{ with } \|w\| \to \infty. \end{aligned} \tag{3.22}$$

Since J is weakly lower semicontinuous and is coercive on W, J is bounded from below on W. By Proposition 3.8 of [6], we have $C_1(J,\ \infty) \not\cong 0$ since $\dim V = 1$. The proof is completed. $\square$

4. Proof of Main Results

In this section we give the proofs of our main existence results.

Proof of Theorem 1.1. Case (**a**). By Proposition 3.2 we have that $C_q(J, \infty) \cong \delta_{q,0}\mathbb{Z}$, $\forall q \in \mathbb{Z}$ and so J has a critical point u^* with $C_q(J, u^*) \cong \delta_{q,0}\mathbb{Z}$. In fact u^* is a global minimizer of J. By Proposition 2.1 we have $C_q(J, 0) \cong 0$, $\forall q \in \mathbb{Z}$. Hence $u^* \neq 0$ and the problem (1.1) has at least one nontrivial solution $u^* \in W_0^{1,p}(\Omega)$.

Case (**b**) follows from Propositions 3.3 and 2.1.

Case (**c**) follows from Propositions 3.1 and 2.3.

Case (**d**) follows from Propositions 3.3 and 2.3.

The proof is completed. □

Proof of Theorem 1.2. Proposition 3.2 tells us that J has the deformation property, is bounded from below and has a global minimizer. By Proposition Since $C_1(J, 0) \neq 0$, 0 is not the minimizer of J and is homological nontrivial. It follows from Theorem 2.1 of [20] that J has at least two nontrivial critical points. The proof is completed. □

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