KÄHLER MANIFOLDS WITH SLIGHTLY POSITIVE BISECTIONAL CURVATURE

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Abstract

Let M^n be an n-dimensional complete Kähler manifolds with everywhere nonnegative or everywhere nonpositive bisectional curvature. In this article, we address the question of when the universal covering space $\widetilde{M}$ of M is holomorphically isometric to $\mathbf{C}^{n-r} \times N^r$, where N is a Kähler manifold whose Ricci tensor is nondegenerate somewhere. It is now known that such is the case if M covers a compact Kähler manifold, its bisectional curvature is everywhere nonpositive, and the metric is real analytic. We prove in this article that, still with the real analytic assumption on the metric, such a splitting also takes place if the bisectional curvature of M is everywhere nonnegative but is positive only rarely.

Key Words: bisectional curvature, Ricci rank, Ricci kernel foliation, totally geodesic foliation, conullity operator, local splitting, de Rham decomposition.

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1. Background

The uniformization theory for Riemann surfaces states that the only simply-connected Riemann surfaces are $\mathbf{CP}^1$, $\mathbf{C}$, and the unit disc Δ. Any generalization to higher dimensions would likely require an understanding of the behavior and structure of compact (or complete) Kähler manifolds with positive (negative, nonpositive, nonnegative) bisectional curvature.

A condition on the bisectional curvature is weaker than the corresponding condition on sectional curvature. For complex geometry, however, the former is the more natural of the two not only because bisectional curvature is defined directly in terms of the complex structure, but also because sectional curvature conditions are often too restrictive.

A quick example to illustrate the difference between these two curvature conditions is the following. It is easy to show that any simply-connected, complete Kähler manifold with nonpositive sectional curvature must be Stein ([W1]; or [Z4], p. 182). ¿From the classic Cartan-Hadamard theorem we also know that it is diffeomorphic to $\mathbf{R}^{2n}$. So if M^n is a compact

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Kähler manifold with nonpositive sectional curvature, it must be a $K(\pi, 1)$ space and its universal cover is Stein. By contrast, it is still an open question whether a compact Kähler manifold with negative bisectional curvature must be simply-connected when the dimension is 2 or higher. Such a manifold is projective, so by the Lefschetz hyperplane section theorem and the fact that bisectional curvature decreases on complex submanifolds, this question boils down to whether there are examples of simply-connected compact Kähler surfaces with negative bisectional curvature. Note that the cotangent bundle of a manifold with negative bisectional curvature must be ample, and for any $n \geq 2$, Bun Wong [Wo] has constructed examples of simply-connected projective manifold with ample cotangent bundle.

A definite answer to this question, either affirmative or negative, would be very interesting. It involves the subtle difference between the global condition of an ample cotangent bundle and pointwise conditions such as negative bisectional curvature of a Hermitian or Kähler metric. It is worth pointing out that, for negative bisectional curvature, whether the metric is Hermitian or Kähler makes a difference. For instance, let M^2 be any *Kodaira fibration surface*, that is, M^2 is a holomorphic fibration without singular fibers over a compact Riemann surface which is not a holomorphic fiber bundle, in the sense that the fibers are not all isomorphic to each other as complex curves. In 1985, M. Schneider [S] proved that such a M^2 always has ample cotangent bundle and, more interestingly, I-Hsun Tsai [T] constructed in 1989 a Hermitian metric on M^2 which has negative bisectional curvature. On the other hand, such a surface M^2 cannot admit any Kähler metric with nonpositive bisectional curvature, by a result of Paul Yang [Ya]. The reason is that in the Kähler case, each complex curve (or submanifold) would minimize the volume within its homology class. So Yang's variational formula would force the fibration structure of M^2 to be a holomorphic fiber bundle.

The above discussion is aimed, in some sense, at justifying the focus of our attention on bisectional curvature, at least in the context of uniformization theory. But even under the stronger sectional curvature assumptions, there are still many questions in Kähler geometry that remains open. For instance, if M^n is a compact Kähler manifold with nonpositive sectional curvature, and is of general type, then is it always Kobayashi hyperbolic? For $n = 2$, the answer was found to be affirmative ([Z3]) by using a result of Lu and Yau [LY]. For $n > 2$, it is still unknown. Another example is whether a complete, noncompact, Kähler manifold of positive sectional curvature is biholomorphic to complex Euclidean space.

In (high dimensional) uniformization theory, a major objective is to understand complex manifolds in dimension $n \geq 2$ which admit complete Kähler metric with nonnegative (or nonpositive) bisectional curvature, especially the compact ones.

In the case of nonnegative bisectional curvature, the situation regarding compact manifolds is well understood as a result of the combined effort of

Frankel [Fk], Howard-Smyth-Wu [HSW], [W2], Mori [Mo], and Siu-Yau [SY], culminating in Mok's solution of the generalized Frankel conjecture [M]. It turns out that, given any compact Kähler manifold M^n with nonnegative bisectional curvature, there exists a finite cover M' of M such that the Albanese map $\pi : M' \to T^q$ of M' is a surjective holomorphic fiber bundle, whose fiber has $c_1 > 0$ and is in fact biholomorphic to a product of irreducible compact Hermitian symmetric spaces.

For noncompact Kähler manifolds, the situation regarding both everywhere nonpositive and everywhere nonnegative bisectional curvature remains a mystery. In dimension 2, the classification theory for surfaces makes possible a reduction, namely, if M^2 is a compact Kähler surface with nonpositive bisectional curvature, then either M^2 has ample canonical line bundle (with negative first Chern class), or there exists a finite cover of M^2 which is isometric to either a flat complex torus or the product of an elliptic curve with a curve of genus at least 2. Of course in the $c_1 < 0$ case we don't know much about this M^2, e.g., must $\pi_1(M)$ be infinite? must M^2 be Kobayashi hyperbolic? etc.

Dual to the splitting theorem of [HSW] and [W1] in the case of nonnegative bisectional curvature, we do have a conjecture for the nonpositive case. This conjecture is probably the common belief of many people in the field, but because we could not find it in written form in the literature, we will attribute it to S.-T. Yau, from whom the second-named author first learned of its existence.

Conjecture (Yau): *Let M^n be a compact Kähler manifold with nonpositive bisectional curvature. Then there exists a finite cover M' of M such that M' is a holomorphic and metric fiber bundle over N, a compact Kähler manifold with nonpositive bisectional curvature and $c_1(N) < 0$, and the fiber is a (flat) complex torus.*

Here as later, we will call fiber bundle $f : M \to N$ between two Riemannian manifolds a *metric* bundle, if for any $p \in N$, there is a neighborhood $p \in U \subseteq N$ such that the bundle over U is isometric to the product of the fiber with U.

In light of Demailly's notion of nef-ness for holomorphic vector bundles over a compact Kähler manifold, and the reduction theorem of Demailly, Peternell and Schneider [DPS], we propose the following conjecture which is a slight generalization of Yau's:

Generalized Yau's Conjecture: *Let M^n be a compact Kähler manifold with nef cotangent bundle in the sense of Demailly. Denote by κ its Kodaira dimension. Then there exists a finite cover M' of M, such that M' is a smooth fibration over a projective manifold N^κ of dimension κ, and each fiber is a complex torus, and $c_1(N) < 0$.*

In a recent article ([WZ2]), the authors gave a partial solution to Yau's original conjecture:

Theorem 1. *Yau's conjecture is true if the metric on M^n is assumed to be real analytic.*

We believe the additional real analyticity assumption is purely technical, but we do not know at this point and time how to remove it.

In §2 of this article, we will give an outline of the proof of Theorem 1 and, in the process, point out certain key steps of the proof which are valid under the assumption of either everywhere nonpositive or everywhere nonnegative bisectional curvature. This then sets the stage for the proof of our main theorem concerning manifolds of nonnegative bisectional curvature, to be stated presently. Recall that for compact Kähler manifolds of nonnegative bisectional curvature, the theorem of [HSW] settles the issue of when such a manifold is holomorphically isometric to a direct product in terms of the Ricci form. We now pose a conjecture which is the counterpart of [HSW] for noncompact complete manifolds.

Conjecture. *Let M^n be a complete Kähler manifold with nonnegative bisectional curvature. Then its universal covering manifold $\widetilde{M}$ is holomorphically isometric to $\mathbf{C}^{n-r} \times N^r$, where r is the Ricci rank, i.e., the maximum rank of the Ricci form.*

In other words, the Ricci tensor will be *quasi-positive* (i.e., positive definite somewhere and positive semi-definite everywhere) unless $\widetilde{M}$ has non-trivial flat de Rham factor.

Our main theorem proves the special case of this conjecture when the Ricci rank is at most 2 and the metric is real analytic. If the Ricci rank is 1, the conjecture is easily seen to be true (cf. the discussion in §2 below condition $(\star)$). We will therefore concentrate on the rank 2 case.

Theorem 2. *Let M^n be a complete Kähler manifold with nonnegative bisectional curvature and with Ricci rank $r = 2$. If the metric is real analytic, then the universal covering space of M^n is holomorphically isometric to a product $\mathbf{C}^{n-2} \times N^2$, where N^2 is a complete Kähler manifold of dimension 2 with quasi-positive Ricci tensor.*

If an n-dimensional Kähler manifold of nonnegative bisectional curvature has Ricci rank equal to ≤ 2, then at each point the bisectional curvature can be positive only within a 2-dimensional subspace; elsewhere it must be zero. This explains the title of the present article. It should be mentioned that if nonnegative *sectional* curvature instead of bisectional curvature is assumed, then the second-named author has proved elsewhere ([Z3]) that Theorem 2 holds with no restriction on the Ricci rank. If the analyticity of the metric is not assumed, then M has at least a local splitting $D^{n-2} \times N^2$ at each point, where D is an open subset of $\mathbf{C}^{n-2}$. The real analyticity assumption enters when we try to extend the local splitting to a global one. The idea here is to apply Nijenhuis' theorem on the identity of the local holonomy groups of a real analytic Riemannian manifold with the global holonomy group ([KN1], p. 101, Theorem 10.8). The proof of Theorem 2 occupies §3.

2. Discussion of the Proof of Theorem 1

The purpose of this discussion is to supplement, rather than repeat the technical proof given in [WZ2] of Theorem 1 in order to pave the way for the proof of Theorem 2 in §3. In the process, we also fix the notation and the terminology.

¿From now on, (M^n, g) will always be a Kähler manifold with everywhere nonpositive or everywhere nonnegative bisectional curvature. The basic reference here is [Z4].

Denote by ρ the Ricci $(1,1)$-form of M. It is (positive or negative) semi-definite everywhere on M. Denote by r the maximum of the (complex) rank of ρ, and by $U \subseteq M$ the open subset where ρ has rank r. We will say that M has *degenerate Ricci,* if $r < n$. That is, if the Ricci tensor is nowhere negative or positive definite. In this case, denote by $\mathcal{L}$ the distribution in U given by the kernel of ρ.

We claim that *the everywhere nonpositivity or everywhere nonnegativity of the bisectional curvature implies that $X \in \mathcal{L}$ if and only if $R(X, *, *, *) \equiv 0$, where R is the curvature tensor.* This fact is used without proof in [WZ2] in the case of nonpositive bisectional curvature, but we will give the general argument here. First, observe that if $X \in \mathcal{L}$, then $R(X, \overline{X}, T, \overline{T}) = 0$ for every vector field T of type $(1,0)$. Indeed, we may assume $|T| = 1$, and that locally there is a unitary frame $\{e_1, e_2, \cdots, e_n\}$ satisfying $e_1 = T$. Suppose the bisectional curvature is nonnegative, then by the definition of ρ, $\rho(X, \overline{X}) = 0$, so that $\sum_{a=1}^n R(X, \overline{X}, e_a, \overline{e_a}) = 0$. Since each $R(X, \overline{X}, e_a, \overline{e_a}) \geq 0$, we see that $\rho(X, \overline{X}) = 0$ iff $R(X, \overline{X}, e_a, \overline{e_a}) = 0$ for all $a = 1, \ldots n$. In particular, $R(X, \overline{X}, T, \overline{T}) = 0$. The argument under the assumption of nonpositive bisectional curvature is identical. Next, we show that if $X \in \mathcal{L}$, then $R(X, *, T, \overline{T}) = 0$ for any $(1,0)$ vector field T. Fixing T, we consider the Hermitian sesquilinear form $(U, V) \xrightarrow{F} R(U, \overline{V}, T, \overline{T})$ on each tangent space of M, where U, V are arbitrary vectors of type $(1,0)$. By the assumption of everywhere nonnegative or everywhere nonpositive bisectional curvature, $F(U, \overline{U}) \geq 0$ or $F(U, \overline{U}) \leq 0$, resp., for any U. So F is positive or negative semi-definite. But for a semi-definite Hermitian form φ on a vector space, a vector U is in the nullspace of φ iff $\varphi(U, \overline{U}) = 0$. Since we have seen that for any $X \in \mathcal{L}$, $F(X, \overline{X}) = 0$, therefore X is in the nullspace of F for every $X \in \mathcal{L}$, i.e., $R(X, *, T, \overline{T}) = 0$ for any $X \in \mathcal{L}$. Finally, if S, T are arbitrary $(1,0)$ vector fields, let $\lambda \in \mathbf{C}$. Then $R(X, *, T + \lambda S, \overline{T + \lambda S}) = 0$, i.e.,

$$\lambda R(X, *, S, \overline{T}) + \overline{\lambda} R(X, *, T, \overline{S}) = 0.$$

Differentiating with respect to λ immediately yields $R(X, *, S, \overline{T}) = 0$. This being true for all S and T, we have proved our claim.

So $\mathcal{L}$ is the kernel of the curvature tensor, or more precisely, the kernel of the mapping $X \to R(X, *, *, *)$ from the space of $(1,0)$ vectors to 3-fold covariant tensors. Thus $\mathcal{L}$ is a foliation whose leaves are totally geodesic, flat, complex submanifolds of U. By a theorem of Ferus [F], each leaf of $\mathcal{L}$ is complete if M is complete.

We will call r the *Ricci rank* and $\mathcal{L}$ the *Ricci kernel foliation* for M^n.

In general, the complex foliation $\mathcal{L}$ may not be holomorphic, i.e., even though the leaves of $\mathcal{L}$ are complex submanifolds, they may not vary holomorphically from leaf to leaf. More formally, we say a complex foliation $\mathcal{L}$ of dimension $n-r$ is *holomorphic* if locally there exist holomorphic vector fields $V_1, \ldots, V_{n-r}$ so that at each point p, the vectors $V_1(p), \ldots, V_{n-r}(p)$ form a basis of the tangent space to the leaf of $\mathcal{L}$ at p. Thus $\mathcal{L}$ is holomorphic iff locally there is a holomorphic coordinate system $\{z_1, \ldots, z_n\}$ so that the level sets $\{z_1 = \text{constant}, \ldots, z_r = \text{constant}\}$ define the leaves of $\mathcal{L}$.

We now give a third formulation of this concept on a Kähler manifold so that the verification of the holomorphicity of the Ricci kernel foliation to be given later will be reduced to a computation. We claim that a complex foliation $\mathcal{L}$ is holomorphic iff

$$(1) \qquad \nabla_{\overline{Y}} \mathcal{L} \subseteq \mathcal{L} \quad \text{for any vector field } Y \text{ of type } (1,0)$$

where, by abuse of notation, $\mathcal{L}$ is identified with the space of all vector fields of type $(1,0)$ tangent to $\mathcal{L}$, and ∇ is the Kähler connection. The fact that if $\mathcal{L}$ is holomorphic then (1) must be true is trivial. We prove the converse for the case of $n = 2$ and $r = 1$ for the sake of notation simplicity; it will be seen that the general case involves no new ideas.

So we must show that if (1) holds, then locally there is a nowhere zero holomorphic vector field T which is everywhere tangent to $\mathcal{L}$. Fix a point p and let z and w be a local complex coordinate system around p so that $z(p) = w(p) = 0$ and so that $\{w = 0\}$ defines the leaf of $\mathcal{L}$ passing through p. Let f and g be C^∞ real-valued functions defined in the same neighborhood of p so that $\{\Re z, \Im z, f, g\}$ (where $\Re z$ and $\Im z$ denote the real and imaginary parts of z) form a real coordinate system around p, $f(p) = g(p) = 0$, and the level sets $\{f = \text{constant}, g = \text{constant}\}$ define $\mathcal{L}$. In particular, $\{w = 0\} = \{f = 0, g = 0\}$ in this neighborhood. Because each leaf of $\mathcal{L}$ is a complex submanifold of M, it makes sense to speak of the space of $(1,0)$ vectors of $\mathcal{L}$. Let $T_q^{1,0}M$ and $T_q^{1,0}\mathcal{L}$ be the space of type $(1,0)$ vectors of M and $\mathcal{L}$, resp., at a point q in this coordinate neighborhood, and let $T_q\{\frac{\partial}{\partial w}\}$ be the subspace of $T_q^{1,0}M$ spanned by $\frac{\partial}{\partial w}$. Now at p, $T_p^{1,0}\mathcal{L}$ is the linear span of $\frac{\partial}{\partial z}(p)$, so we have a direct sum:

$$T_p^{1,0}M = T_p\{\frac{\partial}{\partial w}\} \oplus T_p^{1,0}\mathcal{L}.$$

By continuity, we may assume that the neighborhood of p is so small that for every q in this neighborhood, we continue to have a direct sum

$$T_q^{1,0}M = T_q\{\frac{\partial}{\partial w}\} \oplus T_q^{1,0}\mathcal{L}.$$

Thus there is a C^∞ $(1,0)$ vector field T tangent to $\mathcal{L}$, so that

$$\frac{\partial}{\partial z} = h\frac{\partial}{\partial w} + T.$$

We claim that T is holomorphic. Indeed, since (1) is true,

$$\nabla_{\frac{\partial}{\partial \overline{w}}} T = \varphi T, \quad \text{and} \quad \nabla_{\frac{\partial}{\partial \overline{z}}} T = \psi T$$

for some C^∞ function φ and ψ. But

$$\nabla_{\frac{\partial}{\partial \overline{w}}} T = \nabla_{\frac{\partial}{\partial \overline{w}}}\left(\frac{\partial}{\partial z} - h\frac{\partial}{\partial w}\right) = \frac{\partial h}{\partial \overline{w}}\frac{\partial}{\partial w},$$

whereas

$$\varphi T = \varphi\frac{\partial}{\partial z} - \varphi h\frac{\partial}{\partial w}.$$

Therefore, $\nabla_{\partial/\partial\overline{w}}T = \varphi T$ iff $\varphi = 0$ and $\partial h/\partial\overline{w} = 0$. Similarly, $\partial h/\partial\overline{z} = 0$. Therefore h is holomorphic and T, being equal to $\partial/\partial z - h(\partial/\partial w)$, is a holomorphic vector field. Moreover, T is by definition tangent to $\mathcal{L}$ everywhere and is obviously nowhere zero. The proof of our claim is complete.

It is not difficult to exhibit examples of Ricci kernel foliations whose leaves are all complex submanifolds but which are nevertheless not holomorphic. For instance, take any non-holomorphic complex foliation in an open neighborhood of the origin in $\mathbf{C}^2$. since the leaves are of complex codimension one, it is a Monge-Ampere foliation. So by Yau's result ([Y]), it is defined by the Ricci tensor of some Kähler metric. That is, it is a Ricci kernel foliation of some Kähler metric.

The first major step in the proof of Theorem 1 states that, *when M is complete, and the bisectional curvature is everywhere nonpositive or everywhere nonnegative, the Ricci kernel foliation $\mathcal{L}$ is always a holomorphic foliation.*

In case the leaves of $\mathcal{L}$ are one dimensional, this is implied by (the completeness of the leaves and) a result of Burns (Theorem 3.1 and Cor 3.2 in [B]), where he studied the more general Monge-Ampere foliations. When M has nonpositive sectional curvature, this was proved in [Z3] (Theorem 3, in this case the leaves of $\mathcal{L}$ actually *vary parallelly* in a sense to be defined below). The proof there also implies that the the same conclusion holds if the leaves of $\mathcal{L}$ are $(n-1)$ dimensional.

The proof in general of the holomorphicity of the Ricci kernel foliation on a complete Kähler manifold of nonpositive bisectional curvature is given in [WZ2], Theorem A. *It is important to note explicitly that the proof there is equally valid when the everywhere nonnegativity of the bisectional curvature is assumed instead.* This proof is intricate, so we will skip the technical details and give only an outline of the ideas. First we need to set up some notation that will be used also for the proof of Theorem 2 proper in §3.

We will set things up the same way as in Theorem 3 of [Z3]. Denote by $\mathcal{L}^{\perp}$ the distribution (of type $(1,0)$ tangent vectors) in U representing the orthogonal complement of $\mathcal{L}$. (Recall that U is the open set in which the Ricci form ρ has maximum rank r.) Also denote by $\mathcal{F}$ and $\mathcal{F}^{\perp}$ the underlying real distribution of $\mathcal{L}$ or $\mathcal{L}^{\perp}$. $\mathcal{F}$ is of real rank $2r$.

Recall that the *conullity operator* of a totally geodesic foliation $\mathcal{F}$ in a Riemannian manifold is defined by (cf. [A2], [DR])

$$C_T(X) = -(\nabla_X \widetilde{T})^{\perp}$$

where T and X are tangent vectors in $\mathcal{F}$ and $\mathcal{F}^{\perp}$, resp., and $\widetilde{T}$ is a local vector field in $\mathcal{F}$ extending T. Here $Y^{\perp}$ stands for the $\mathcal{F}^{\perp}$-component of Y. These operators are well defined tensors, and satisfy the Riccati type equations

$$\nabla_T C_S = C_S \circ C_T - C_{\nabla_T S} - \{R(T,\cdot)S\}^{\perp}$$

for any two vector fields T and S in $\mathcal{F}$. Under the assumption of either everywhere nonpositive or everywhere nonnegative bisectional curvature, the curvature term $R(T,\cdot)S$ vanishes because $R(T,*,*,*) = 0$. If we choose S to be parallel in each leaf of $\mathcal{F}$, then the above equation becomes

$$\nabla_T C_S = C_S \circ C_T$$

In particular, along any geodesic $\gamma(t)$ contained in a leaf F of $\mathcal{F}$, one has

$$\nabla_T C_T = (C_T)^2 \tag{2}$$

where $T = \gamma'(t)$.

Since each F is complete by Ferus' theorem [F], we know that *for any $T \in \mathcal{F}$, C_T can not have non-zero real eigenvalues at any point of U.* As pointed out in [DR], this fact is implicit in [A2]. Because of its importance for our purpose, we supply the simple proof here. So assuming the completeness of M, let $\gamma : \mathbf{R} \to F$ be an infinite geodesic and let $T = \gamma'(t)$ be as above. Let $\{e_i\}$ be a unitary basis in the tangent space to $\gamma(0)$ and let $\{e_i(t)\}$ be its parallel translate along $\gamma(t)$. Let the matrix of C_T relative to $\{e_i(t)\}$ along $\gamma(t)$ be $A(t)$. Then (2) implies that

$$A'(t) = A(t)^2.$$

Suppose C_T has a nonzero eigenvalue λ at $\gamma(0)$ corresponding to the unit eigenvector v. We may assume that $e_1(0) = v$. Then $A'(0)e_1 = \lambda e_1$, and λ is a nonzero real number. Let $A_1(t)$ be the first column of $A(t)$, say $A_1(t) = {}^t[a_1(t)\ a_2(t) \cdots a_n(t)]$. Then $a_1(0) = \lambda$ and $a_2(0) = \cdots = a_n(0) = 0$. If $\alpha(t)$ is the $(n-1) \times 1$ column matrix ${}^t[a_2(t)\ a_3(t) \cdots a_n(t)]$ and $A_0(t)$ is the $(n-1) \times n$ matrix obtained from $A(t)$ by deleting the first row, then the equation $A'(t) = A(t)^2$ implies that

$$\alpha'(t) = A_0(t)\alpha(t), \quad \text{and} \quad \alpha(0) = 0.$$

We therefore have a first order homogeneous system of ODE with zero initial data. By the uniqueness of solutions, $\alpha(t) \equiv 0$ for all t. Thus

$$\begin{bmatrix} a_1(t) & * \\ 0 & \\ \vdots & \ddots \\ 0 & \end{bmatrix}$$

Making use of the equation $A'(t) = A(t)^2$ again, we obtain $a_1'(t) = [a_1(t)]^2$, and this familiar Riccatti equation has an explicit solution $a_1(t) = \lambda/(1-\lambda t)$ (recall $a_1(0) = \lambda$). Therfore $a_1(t)$ is not C^∞ on $\mathbf{R}$, contradicting the fact that $A(t)$ is so. Hence there is no such eigenvalue λ.

For $T \in \mathcal{F}$, extend C_T linearly over $\mathbf{C}$ to the complexification

$$\mathcal{F}^\perp \otimes \mathbf{C} = \mathcal{L}^\perp \oplus \overline{\mathcal{L}^\perp}$$

Choose a local frame $\{e_i, \overline{e_i}\}_{i=1}^r$ such that each $e_i \in \mathcal{L}^\perp$. Write $C_T(e_i) = \sum_j A_{ij}e_j + B_{\bar{i}\bar{j}}\overline{e}_j$, then the matrix of C_T with respect to the basis $\{e, \overline{e}\}$ is

$$C_T = \begin{bmatrix} A & \overline{B} \\ B & \overline{A} \end{bmatrix}, \qquad \text{and} \quad C_{JT} = JC_T = \sqrt{-1} \begin{bmatrix} A & -\overline{B} \\ B & -\overline{A} \end{bmatrix}$$

where J is the almost complex structure of M. For any real numbers a and b, write $\lambda = a + \sqrt{-1}b$, we have

$$C_{aT+bJT} = \begin{bmatrix} \lambda A & \overline{\lambda B} \\ \lambda B & \overline{\lambda A} \end{bmatrix}$$

The above argument says that *the matrix C_{aT+bJT} has no non-zero real eigenvalue for any $\lambda = a + \sqrt{-1}b$ at any point of U.* We shall refer to this statement as *condition* $(\star)$ in the following discussion.

We claim that if $B = 0$, then $\mathcal{L}$ is a holomorphic foliation. This is the reason we are interested in the conullity operators in the present context. The claim follows easily from the formulation of the holomorphicity of a foliation in terms of (1) above. For, if $B = 0$, then for all i and for all vector field $T \in \mathcal{F}$,

$$C_T(\overline{e}_i) = \sum_j \overline{A}_{ij}\overline{e}_j \quad \text{and} \quad C_{JT}(\overline{e}_i) = -\sum_j \overline{A}_{ij}\overline{e}_j,$$

so that for any $(1,0)$ vector field $(T - \sqrt{-1}JT) \in \mathcal{L}$, we have

$$C_{T-\sqrt{-1}JT}(\overline{e}_i) = 0 \qquad \text{for all } i.$$

By the definition of the conullity operator, this means that for all vector fields Y of type $(1,0)$,

$$(\nabla_{\overline{Y}}\mathcal{L})^{\perp} = 0.$$

Because ∇ preserves type, $\nabla_{\overline{Y}}\mathcal{L}$ is a vector field of type $(1,0)$. Thus

$$\nabla_{\overline{Y}}\mathcal{L} \subseteq \mathcal{L}.$$

It follows from (1) that $\mathcal{L}$ is a holomorphic foliation.

(As a side remark, when $r = 1$, it is not hard to see that condition $(\star)$ already implies that $A = B = 0$, so $\mathcal{L}$ is holomorphic in this case (and its leaves vary parallelly). In particular, if M^n is a complete complex submanifold of $\mathbf{C}^N$ with Ricci rank (which equals the Gauss rank) 1, then it must be a cylinder, this is known as the Abe's cylinder theorem ([A1]), which is the complex version of the classical Hartman-Nirenberg cylinder theorem ([HN])).

The proof of the vanishing of B proceeds by studying the ODEs that the operators A and B must satisfy along a geodesic $\gamma(t)$ contained in a leaf of $\mathcal{L}$. Under the assumption that $B \neq 0$ somewhere, these ODEs can be partially solved to the extent that a contradiction to the above condition $(\star)$ is exhibited. This means that B must be identically zero and the foliation $\mathcal{L}$ is then holomorphic, by a previous remark. This completes the proof of the first major step in the proof of Theorem 1.

The seond major step in the proof of Theorem 1 is to use the holomorphicity of $\mathcal{L}$ to derive a splitting result in the compact case. Again, the bisectional curvature is allowed to be either everywhere nonpositive or everywhere nonnegative. To state this result, we introduce some terminology.

We say that the leaves of $\mathcal{L}$ *vary parallelly* if, within each connected component U_a of U (the open set where the Ricci form has maximum rank r), parallel translation from one point of M to another maps $\mathcal{L}$ onto itself. We would at times express this fact more informally by saying that *the leaves of $\mathcal{L}$ in U_a are parallel to each other.* ¿From the definition of the conullity operator, this is the same as saying that, within U_a, all the conullity operators of $\mathcal{L}$ vanish. In terms of the holonomy group of U_a, this is also equivalent to the fact that each fiber $\mathcal{L}_p$ is an invariant subspace of the (restricted) holonomy group of U_a. By the de Rham decomposition theorem, each point of U_a would have a neighborhood which splits holomorphically and isometrically as $L \times Y^r$ where L is flat and Y has dimension r if the leaves of $\mathcal{L}$ vary parallelly (cf. the proof of the de Rham decomposition theorem for Kähler manifolds on p. 172 of [KN2]).

Our principal observation is that if M^n as in Theorem 1 also has bounded curvature, then the leaves of $\mathcal{L}$ vary parallelly. However, we will prove something more general as it involves no extra effort. Introduce the following terminology: A function f on a complete Riemannian manifold M is said to have *sub-k growth,* if

$$\lim_{i\to\infty} \frac{|f(x_i)|}{d^k(x_i, x_0)} = 0$$

for any sequence $\{x_i\}$ in M with the distance $d(x_i, x_0)$ going to infinity. Here x_0 is a fixed point. If $k = 2$, then it is more common to say that f has *sub-quadratic growth.* The theorem we want to prove can now be stated. (In [WZ2], this is stated and proved for the case of nonpositive bisectional curvature as Theorem B.)

If M^n is a complete Kähler manifold with everywhere nonnegative or everywhere nonpositive bisectional curvature, with Ricci rank $r < n$, and with a scalar curvature s of sub-quadratic growth, then the leaves of the Ricci kernel foliation $\mathcal{L}$ vary parallelly.

The idea of the proof of this theorem is that if $l \geq 0$ be the smallest integer such that $C_T^{l+1} = 0$ for all conullity operator C_T, then we show that the scalar curvature s cannot have sub-$(2l)$ growth. In particular, if $l \geq 1$, then s cannot have sub-quadratic growth, contradicting the hypothesis. Therefore $l = 0$ and all conullity operators vanish. The leaves of $\mathcal{L}$ are thus parallel to each other within each component of U.

Technically, the proof uses the second Bianchi identity to obtain an equation expressing the covariant derivative of the curvature tensor along a complex line (to be denoted generically by $\mathbf{C}$) lying in $\mathcal{L}$. If z denotes the canonical variable in $\mathbf{C}$, then from $R(\partial/\partial z, *, *, *) = 0$, (which is guaranteed by either the everywhere nonpositivity or nonnegativity of the bisectional curvature), we obtained a system of ODE in ρ_{ij} (components of the Ricci form ρ) involving the conullity operator $C_{\partial/\partial z}$. This system of ODE turns out to be explicitly solvable and the solution then exhibits $s \equiv \text{trace}\, \rho$ as a function that grows at least like $|z|^{2l}$. But z dominates the distance function on M up to a positive constant, so we get the desired contradiction.

So far we have not invoked the assumption of real analyticity on the metric. Assume now the metric is real analytic in the preceding theorem. Then within each component U_a of U, the holonomy group H_a of U_a leaves each fiber of $\mathcal{L}$ in U_a invariant. Then the local holonomy group of M in U_a in the sense of Nijenhuis (cf. [KN1], p. 94) is reducible, so that — because the metric is real analytic — the holonomy group of M coincides with the local holonomy group in each U_a and therefore also leaves each fiber of $\mathcal{L}$ invariant (cf. Theorem 10.8 on p. 101 of [KN1]; this is a theorem of Nijenhuis). Therefore by the de Rham decomposition theorem for Kähler manifolds ([KN2], Theorem 8.1 on p. 172), we have proved:

Let M^n be a complete Kähler manifold with everywhere nonpositive (resp., everywhere nonnegative) bisectional curvature so that its scalar curvature is of sub-quadratic growth. Furthermore, let the Ricci rank of M be $r < n$. If the Kähler metric is real analytic, then the universal covering manifold $\widetilde{M}$ of M is holomorphically isometric to $\mathbf{C}^{n-r} \times N^r$, where N^r has quasi-negative (resp., quasi-positive) Ricci curvature.

Note in particular that if M is compact, its scalar curvature is bounded and therefore the scalar curvature of $\widetilde{M}$ must have sub-quadratic growth.

Finally we come to the proof of Theorem 1 proper. So let M be a compact Kähler manifold with nonpositive bisectional curvature (note that at this point we no longer allow the bisectional curvature to be nonnegative) and the metric is real anlaytic. By the preceding theorem, the universal covering manifold $\overline{M}$ splits holomorphically and isometrically as $\mathbf{C}^{n-r} \times N^r$, where N is an r-dimensional simply-connected Kähler manifold of quasi-negative Ricci curvature. Let $\pi : \overline{M} \to M$ be the covering map, then the critical argument here is to show that $\pi(N)$ is actually compact in M. As in the case of the Eberlein theory for Euclidean de Rham factors of compact Riemannian manifolds with nonpositive sectional curvature (e.g., [E1]-[E2]), the key is to establish the fact that the projection onto the N factor of the deck transformation group Γ of $\overline{M}$ is a discrete subgroup of the group of holomorphic isometries of N. In Eberlein's case, this was achieved by utilizing the properties of the action of Γ on the infinite space $\overline{M}(\infty)$, especially the so-called duality condition. In our case, since the assumption was not made on the sectional curvature, that argument cannot be applied or modified to be applicable here. We use instead the partial stability of the tangent bundle and the maximum principle to prove the discreteness of this projection. We refer the reader to [WZ2], Theorem E for the details.

3. Proof of Theorem 2

We use the terminology and notation of §2. From now on, we assume that M^n *is a simply-connected complete Kählar manifold with Ricci rank $r = 2 < n$, and with nonnegative bisectional curvature.*

In the open subset $U \subseteq M$ where the Ricci form ρ has maximum rank equal to 2, we have the holomorphic, totally geodesic foliation $\mathcal{L}$ whose leaves are complete and flat of codimension 2. We now recall that the conullity operators C_T are nilpotent matrices and that so is any linear combination of them ([A2], see also the discussion in [WZ2]). In our case, these conullity operators are 2×2 nilpotent matrices, which are unique up to scalar multiples at each point (since otherwise some combination would not be nilpotent). In particular, the subspace $\{C_T(\mathcal{L}^{\perp}) : T \in \mathcal{L}\}$ in the tangent space at each point is one-dimensional.

Denote by $V \subseteq U$ the open subset where $C_T \neq 0$ for some T. Note that $V = \phi$ would mean that U is (locally) isometrically and holomorphically a product, thus when the metric is real analytic, M itself would be a product $\mathbf{C}^{n-2} \times N^2$. (See the discussion of the proof of the second major step in §2 above.) So in the following we consider the case when $V \neq \phi$ and deduce a contradiction, which would then complete the proof of Theorem 2.

Denote by $\widetilde{\mathcal{L}}$ the distribution in V spanned by $\mathcal{L}$ and the image distribution $\{C_T(\mathcal{L}^{\perp}) : T \in \mathcal{L}\}$ of all the (nonzero) C_T's. We claim the following

Proposition 1. *Let M^n be a complete, simply-connected Kähler manifold with Ricci rank $r = 2 < n$ and with either nonpositive or nonnegative bisectional curvature. Denote U, $\mathcal{L}$, V, $\tilde{\mathcal{L}}$ as above. Then in V, $\tilde{\mathcal{L}}$ is a totally geodesic, holomorphic foliation with flat leaves.*

Proof. Fix any $p \in V$, take a local unitary frame $\{e_1, e_2, \ldots, e_n\}$ such that $e_\alpha \in \mathcal{L}$ for each $3 \leq \alpha \leq n$, the e_α's are parallel along $\mathcal{L}$, and e_1 is in the image space of C_T for some T. Note that e_1 is then in the kernel and the image space for any nonzero C_S. Also, e_1 and e_2 are unique up to scalar multiples (of norm 1).

Denote by $\{\varphi_1, \ldots, \varphi_n\}$ the dual coframe of $(1,0)$ forms. Denote by θ, Θ the matrices of connection and curvature under the frame e. For each $3 \leq \alpha \leq n$, the conullity operator $C^\alpha = C_{e_\alpha}$ satisfies

$$C^\alpha(e_2) = -\lambda_\alpha e_1, \quad C^\alpha(e_1) = 0$$

by our choice of the frame. Thus we have

$$\theta_{\alpha 2} = 0, \quad \theta_{\alpha 1} = \lambda_\alpha \varphi_2 \tag{3}$$

for any $3 \leq \alpha \leq n$. Note at least one λ_α will be nonzero.

We have the structure equations $d\theta_{ab} = \sum_{c=1}^n \theta_{ac} \wedge \theta_{cb} + \Theta_{ab}$ for all $a, b = 1, \cdots, n$, where Θ denotes the curvature form, i.e.,

$$\Theta_{ab} \equiv \sum_{c,d} R_{a\bar{b}c\bar{d}}\, \varphi_c \wedge \overline{\varphi}_d.$$

where

$$R_{a\bar{b}c\bar{d}} \equiv R(e_a, \overline{e}_b, e_c, \overline{e}_d).$$

Because $R(e_\alpha, *, *, *) = 0$, we have

$$\Theta_{\alpha *} = 0 \quad \text{for all } \alpha = 3, \cdots, n. \tag{4}$$

It follows that the structure equation

$$\Theta_{\alpha 2} = d\theta_{\alpha 2} - \sum_{b=1}^n \theta_{\alpha b} \wedge \theta_{b2}$$

simplifies to $\lambda_\alpha \varphi_2 \wedge \theta_{12} = 0$. Thus

$$\theta_{12} = \mu \varphi_2 \tag{5}$$

where μ is the complex-valued function $\theta_{12}(e_2) = \langle \nabla_{e_2}(e_1), \overline{e}_2 \rangle$. This says that $\widetilde{\mathcal{L}}$ is a holomorphic, totally geodesic foliation, for the following reason. Recall that $\widetilde{\mathcal{L}}$ is the linear span of $\mathcal{L}$ and e_1 at each point of V. Because $\nabla_Z e_1 = \sum_{a=1}^n \theta_{1a}(Z) e_a$ for any vector Z, (3) and (5) imply that

$$\nabla_{\widetilde{\mathcal{L}}} \widetilde{\mathcal{L}} \subseteq \widetilde{\mathcal{L}} \qquad \text{and} \qquad \nabla_{\overline{Y}} \widetilde{\mathcal{L}} \subseteq \widetilde{\mathcal{L}}$$

for any vector Y of type $(1,0)$. Therefore $\widetilde{\mathcal{L}}$ is totally geodesic and holomorphic (for the latter, see (1) in §2).

To see that the leaves of $\widetilde{\mathcal{L}}$ are flat, we only need to prove that $\widetilde{\mathcal{L}}$ has zero holomorphic (sectional) curvature. Because of (4), it suffices to prove that $R_{1\bar{1}1\bar{1}} = 0$. Let us use the second Bianchi identity

$$d\Theta_{\alpha b} = \sum_{c=1}^{n} (\theta_{\alpha c} \wedge \Theta_{cb} - \Theta_{\alpha c} \wedge \theta_{cb}).$$

Then for any $3 \leq \alpha \leq n$ and any $1 \leq b \leq n$, this equation simplifies drastically to the following on account of (3) and (4):

$$\varphi_2 \wedge \Theta_{1b} = 0.$$

Thus $R_{1\bar{a}1\bar{b}} = 0$ for any a, b. In particular $R_{1\bar{1}1\bar{1}} = 0$. This proves our claim. □

More generally, let us denote by $\mathcal{I} = \sum \mathrm{Im}(C_T)$ and $\mathcal{K} = \cap \ker(C_T)$ the total image space and the common kernel of all the conullity operators, resp., and analogously, let $V \subseteq U$ be the open subset where the dimension of $\mathcal{I}$ is maximum and the dimension of $\mathcal{K}$ is minimum. Then $\mathcal{I}$ and $\mathcal{K}$ are distributions in V. By an argument similar to the above, one can show that

If $\mathcal{I} \subseteq \mathcal{K}$, then $\tilde{\mathcal{L}} = \mathcal{L} \oplus \mathcal{I}$ is a holomorphic, totally geodesic foliation with flat leaves, and $\mathcal{L} \oplus \mathcal{K}$ is also a totally geodesic foliation.

This was observed in [WZ1] in the case when M is a complex submanifold of the complex Euclidean space, where $\mathcal{L} \oplus \mathcal{K}$ is called the ruling Gauss foliation and $\tilde{\mathcal{L}}$ is called the image distribution. Note the the condition $\mathcal{I} \subseteq \mathcal{K}$ is equivalent to $C_T C_S = 0$, namely, the composition of any two conullity operators is zero everywhere. When the (maximum) dimension of $\mathcal{I}$ is 1, this is the case since all C_T are nilpotent. This special case also includes the Ricci rank 2 case.

Now let us return to our discussion in the $r = 2$ case. A local unitary tangent frame $e = \{e_1, e_2, \ldots, e_n\}$ in an open subset $W \subseteq V$ is called an *adapted* frame, if e_1, $e_2 \in \mathcal{L}^{\perp}$ and $e_1 \in \widetilde{\mathcal{L}}$. e is called a *special adapted* frame, if it is adapted and

(6) $$\nabla_{e_i} e_j = \nabla_{\bar{e}_i} e_j = 0 \quad \text{for any} \quad i, j \in \{1, 3, \dots, n\}.$$

Note that given any $p \in V$, there always exists a specially adapted frame in a neighborhood of p. We can take a piece of holomorphic curve S through p that is transversal to $\widetilde{\mathcal{L}}$, and an adapted frame e along S , and then in a small neighborhood W of p, we extend e along the leaves of $\widetilde{\mathcal{L}}$ so that $\{e_1, e_3, \dots, e_n\}$ are parallel in each leaf in W.

Notice that for two special adapted frames e and e', $e'_1 = fe_1$ for some function f with $|f| = 1$ which is locally constant in each leaf of $\widetilde{\mathcal{L}}$.

Under an adapted frame e, we denote by φ, θ, Θ the dual coframe, matrix of connection or curvature as in the proof of Proposition 1. We have, from (3) and (5), that for each $3 \leq \alpha \leq n$,

$$\theta_{\alpha 2} = 0, \quad \theta_{\alpha 1} = \lambda_\alpha \varphi_2, \quad \theta_{12} = \mu \varphi_2.$$

Also write

$$B = R_{1\bar{1}2\bar{2}}, \quad \text{and} \quad \lambda = \sum_{\alpha=3}^{n} |\lambda_\alpha|^2.$$

We claim that B and λ are well defined, positive functions on V (μ on the other hand is only a locally defined function which depends on the choice of e_1.) The fact that B is well defined is because it is the bisectional curvature in the direction of e_1 and e_2. To see that λ is well-defined, recall that if φ is a linear map between inner product spaces S and T, then its L_2 norm is defined to be $\| \varphi \| \equiv \sqrt{\sum_i |\varphi(s_i)|^2}$, where $\{s_1, \dots, s_n\}$ is any orthonormal basis of S. Now we may regard, at each point p of V, the conullity operator C_T as a linear map Λ between inner product spaces $\mathcal{L}_p \longrightarrow \mathrm{End}(\mathcal{L}_p^\perp, \mathcal{L}_p^\perp)$, where $\mathrm{End}(\mathcal{L}_p^\perp, \mathcal{L}_p^\perp)$ denotes the space of endomorphism of $\mathcal{L}_p^\perp$ equipped with the obvious inner product, so that $\Lambda(T) = C_T$. Then λ is exactly the L_2 norm of Λ. So λ is also well-defined.

To see that B and λ are positive, first observe that in view of (4) and $R_{1\bar{1}1\bar{1}} = 0$, the Ricci curvature would be 0 if $B = 0$. This would contradict the fact that we are inside U (recall $V \subseteq U$). The fact that λ is positive is because V is the open set where at least one conullity operator is nonzero. The proof of the claim is complete.

Fixing a special adapted frame e in W, we claim that relative to e:

(7) $$e_1 B = -2\mu B \quad \text{and} \quad e_\alpha B = 0 \quad \text{for } 3 \leq \alpha \leq n.$$

We have the structure equations

$$d\varphi_a = -\sum_{b=1}^{n} \theta_{ba} \wedge \varphi_b$$

for $a = 1, \dots, n$. Because $\theta_{\alpha 2} = 0$ for $3 \leq \alpha \leq n$ (by (3)),

$$d\varphi_2 = -\theta_{12} \wedge \varphi_1 - \theta_{22} \wedge \varphi_2.$$

Using (5), we obtain

$$d\varphi_2 = \mu\varphi_1 \wedge \varphi_2 - \theta_{22} \wedge \varphi_2$$

so that

$$d(\varphi_2 \wedge \overline{\varphi}_2 = (\mu\varphi_1 + \overline{\mu\varphi_1}) \wedge \varphi_2 \wedge \overline{\varphi}_2.$$

By the second Bianchi identity,

$$d\Theta_{11} = \sum_{a=1}^{n} (\theta_{1a} \wedge \Theta_{a1} - \Theta_{1a} \wedge \theta_{a1})$$

so that

$$d\Theta_{11} = \theta_{12} \wedge \Theta_{21} - \Theta_{12} \wedge \theta_{21}, \tag{8}$$

where we have made use of (4). However, we also have

$$\Theta_{11} = R_{1\bar{1}2\bar{2}}\,\varphi_2 \wedge \overline{\varphi}_2 = B\,\varphi_2 \wedge \overline{\varphi}_2$$

because we have seen that $R_{1\bar{a}1\bar{b}} = 0$ (see end of proof of Proposition 1), and that $R_{1\bar{1}\alpha\bar{b}} = R_{\alpha\bar{1}1\bar{b}} = 0$ for $3 \leq \alpha \leq n$ (by (4)). Therefore,

$$\begin{aligned} d\Theta_{11} &= dB \wedge \varphi_2 \wedge \overline{\varphi}_2 + Bd(\varphi_2 \wedge \overline{\varphi}_2) \\ &= dB \wedge \varphi_2 \wedge \overline{\varphi}_2 + B(\mu\varphi_1 + \overline{\mu\varphi_1}) \wedge \varphi_2 \wedge \overline{\varphi}_2 \\ &= (dB + B[\mu\varphi_1 + \overline{\mu\varphi_1}]) \wedge \varphi_2 \wedge \overline{\varphi}_2 \end{aligned}$$

Moreover,

$$\Theta_{21} = R_{2\bar{1}1\bar{2}}\varphi_1 \wedge \overline{\varphi}_2 + R_{2\bar{1}2\bar{2}}\varphi_2 \wedge \overline{\varphi}_2,$$

because $R_{1\bar{a}1\bar{b}} = 0$ for all a, b, $R_{\alpha\bar{a}b\bar{c}} = 0$ for all a, b, c, and finally $R_{a\bar{b}c\bar{d}} = R_{c\bar{d}a\bar{b}} = \overline{R_{d\bar{c}b\bar{a}}}$ for all a, b, c, d. So

$$\Theta_{21} = B\varphi_1 \wedge \overline{\varphi}_2 + R_{2\bar{1}2\bar{2}}\varphi_2 \wedge \overline{\varphi}_2.$$

Substituting these expressions of $d\Theta_{11}$ and Θ_{21} into (8), while making use of (3), (5), and the fact that $\theta_{ab} = -\overline{\theta_{ba}}$ and $\Theta_{ab} = -\overline{\Theta_{ba}}$, we obtain:

$$dB \wedge \varphi_2 \wedge \overline{\varphi}_2 = -2B(\mu\varphi_1 + \overline{\mu\varphi_1}) \wedge \varphi_2 \wedge \overline{\varphi}_2.$$

Evaluating both sides on $(e_\alpha, e_2, \overline{e}_2)$ and $(e_1, e_2, \overline{e}_2)$ in succession yields (6).

Still referring to the special adapted frame e, we next claim:

$$e_1(|\lambda_\alpha|^2) = -\mu\,|\lambda_\alpha|^2 \quad \text{and} \quad e_\beta(|\lambda_\alpha|^2) = 0 \;\text{ for } 3 \leq \alpha, \beta \leq n. \tag{9}$$

Indeed, we have the structure equation

$$\begin{aligned} d\theta_{\alpha 1} &= \sum_{b=1}^{n} \theta_{\alpha b} \wedge \theta_{b1} + \Theta_{\alpha 1} \\ &= \sum_{b=1}^{n} \theta_{\alpha b} \wedge \theta_{b1} \quad \text{by (4))} \\ &= \lambda_\alpha \varphi_2 \wedge \theta_{11} + \sum_{\gamma=3}^{n} \lambda_\gamma \theta_{\alpha\gamma} \wedge \varphi_2 \quad \text{(by (3))} \end{aligned}$$

We also get by differentiating $\theta_{\alpha 1} = \lambda_\alpha \varphi_2$ (see (3) again) that $d\theta_{\alpha 1} = d\lambda_\alpha \wedge \varphi_2 + \lambda_\alpha d\varphi_2$. Replace the last $d\varphi_2$ by making use of the equation $d\varphi_2 = \mu\varphi_1 \wedge \varphi_2 - \theta_{22} \wedge \varphi_2$ obtained earlier, we arrive at a second expression for $d\theta_{\alpha 1}$:

$$d\theta_{\alpha 1} = d\lambda_\alpha \wedge \varphi_2 + \lambda_\alpha(\mu\varphi_1 \wedge \varphi_2 - \theta_{22} \wedge \varphi_2).$$

Equating these two expressions of $d\theta_{\alpha 1}$ yields:

$$\left(d\lambda_\alpha + \lambda_\alpha \mu \varphi_1 - \lambda_\alpha \theta_{22} + \lambda_\alpha \theta_{11} - \sum_\gamma \lambda_\gamma \theta_{\alpha\gamma} \right) \wedge \varphi_2 = 0.$$

Therefore, for some function f,

$$\lambda_\alpha + \lambda_\alpha \mu \varphi_1 - \lambda_\alpha \theta_{22} + \lambda_\alpha \theta_{11} - \sum_\gamma \lambda_\gamma \theta_{\alpha\gamma} = f\varphi_2. \tag{10}$$

Evaluating both sides of (10) on e_β for $3 \leq \beta \leq n$, we get

$$e_\beta(\lambda_\alpha) - \theta_{22}(e_\beta) + \lambda_\alpha \theta_{11}(e_\beta) - \sum_\gamma \lambda_\gamma \theta_{\alpha\gamma}(e_\beta) = 0.$$

But

$$\theta_{11}(e_\beta) = \langle \nabla_{e_\beta} e_1, \overline{e}_1 \rangle = 0 \quad \text{and} \quad \theta_{\alpha\gamma}(e_\beta) = \langle \nabla_{e_\beta} e_\alpha, \overline{e}_\gamma \rangle = 0$$

on account of (6), so $e_\beta(\lambda_\alpha) = \lambda_\alpha \theta_{22}(e_\beta)$. Hence,

$$e_\beta(\lambda_\alpha)\overline{\lambda_\alpha} = \theta_{22}(e_\beta)|\lambda_\alpha|^2.$$

Similarly, evaluating (10) on $\overline{e}_\beta$ gives $\overline{e}_\beta(\lambda_\alpha) = \lambda_\alpha \theta_{22}(\overline{e}_\beta)$, so that (noting $\overline{\theta}_{22} = -\theta_{22}$),

$$e_\beta(\overline{\lambda}_\alpha)\lambda_\alpha = -\theta_{22}(e_\beta)|\lambda_\alpha|^2.$$

It follows that $e_\beta(|\lambda_\alpha|^2) = 0$, thereby proving the second part of (9). If we now evaluate both sides of (10) on e_1, then the same reasoning leads to $e_1(|\lambda_\alpha|^2) = -|\lambda_\alpha|^2 \mu$. (9) is completely proved.

Finally, we claim that relative to e,

$$e_1 \mu = -\mu^2, \quad \text{and} \quad \overline{e}_1 \mu = -B \tag{11}$$

This is because

$$d\theta_{12} = \sum_{a=1}^n \theta_{1a} \wedge \theta_{a2} + \Theta_{12}.$$

But

$$\Theta_{12} = \sum_{a,b=1}^n R_{1\overline{2}a\overline{b}} \varphi_a \wedge \overline{\varphi}_b = B\varphi_2 \wedge \overline{\varphi}_1 + R_{1\overline{2}2\overline{2}} \varphi_2 \wedge \overline{\varphi}_2$$

for the usual reasons, so

$$d\theta_{12} = \theta_{11} \wedge \theta_{12} + \theta_{12} \wedge \theta_{22} + \sum_{\alpha=3}^n \theta_{1\alpha} \wedge \theta_{\alpha 2} + \Theta_{12}.$$

Because of (5) and $d\varphi_2 = \mu\varphi_1 \wedge \varphi_2 - \theta_{22} \wedge \varphi_2$, the left side equals $d\mu \wedge \varphi_2 + \mu^2\varphi_1 \wedge \varphi_2 - \mu\theta_{22} \wedge \varphi_2$. Hence we obtain $(d\mu + \mu^2\varphi_1 - \mu\theta_{11} + B\overline{\varphi}_1 + R_{1\overline{2}2\overline{2}}\overline{\varphi}_2) \wedge \varphi_2 = 0$, which means that for some function g,

$$d\mu + \mu^2\varphi_1 - \mu\theta_{11} + B\overline{\varphi}_1 + R_{1\overline{2}2\overline{2}}\overline{\varphi}_2 = g\varphi_2.$$

Evaluating both sides of this equation on e_1 and $\overline{e}_1$ in succession, we obtain (11).

Note that (11) (unlike (7) and (9)) is valid only relative to the chosen special adapted frame e.

We now put (7) and (9) to use in the following way. From (9), we get

$$e_1\lambda = -\mu\lambda \quad \text{and} \quad e_\alpha\lambda = 0 \quad \text{for} \quad 3 \leq \alpha \leq n,$$

where we recall that $\lambda = \sum_{\alpha=3}^{n} |\lambda|^2$. From (7) we also get

$$e_1 B = -2\mu B \quad \text{and} \quad e_\alpha B = 0 \quad \text{for} \quad 3 \leq \alpha \leq n.$$

Together, we see that in each leaf of $\widetilde{\mathcal{L}}$,

$$e_1\left(\frac{\lambda^2}{B}\right) = e_\alpha\left(\frac{\lambda^2}{B}\right) = 0 \quad \text{for} \quad 3 \leq \alpha \leq n.$$

It follows that in each leaf of $\widetilde{\mathcal{L}}$, λ^2/B is a positive constant c.

Now we know that $\widetilde{\mathcal{L}}$ is flat, and $\mathcal{L}$ is a holomorphic and totally geodesic foliation inside $\widetilde{\mathcal{L}}$. The orthogonal distribution of $\mathcal{L}$ in $\widetilde{\mathcal{L}}$ — which would be the linear span of e_1 at each point — is therefore also totally geodesic and holomorphic. Fix a point $p \in V$ and let Y be the leaf of the this orthogonal distribution in $\widetilde{\mathcal{L}}$ passing through p. Let D be the maximal star-shaped region around the origin in the tangent space to Y at p, $T_pY = \mathbf{C}$, (which is the linear span of e_1 in the tangent space of M at p,) so that the exponential map $\exp_p : D \to Y$ is defined. We claim that D is all of $\mathbf{C}$. Once this is proved, we shall deduce that, one way or another, there would be a contradiction. Therefore V must be empty and the the proof of Theorem 2 is concluded.

Suppose then $\gamma : [0, a] \to M$ is a unit speed geodesic such that $\gamma(0) = p$ and $\gamma([0, a)) \subset Y$. We want to show that at the end point $q = \gamma(a)$, the rank of the Ricci tensor is still 2 (thus $q \in U$), and some conullity operator is non-zero at q (thus $q \in V$), therefore $q \in Y$. This would show that, since M is complete, the whole semi-infinite geodesic $\gamma([0, \infty))$ lies in Y, thereby proving that γ is defined on all of $[0, \infty)$ and D is $\mathbf{C}$.

Let z be the standard complex Euclidean coordinate in D. We may assume that we have chosen a special adapted frame e in some neighborhood of $\exp_p(x)$ and $(\exp_p)_{*x}(\frac{\partial}{\partial z}) = e_1$. We can lift the functions λ, B and μ to D, where μ is the function defined by (5) relative to e, i.e., $\mu = \langle \nabla_{e_2} e_1, \overline{e}_2 \rangle$. Now we are in $D \subseteq \mathbf{C}$, with positive functions B and λ, and complex function μ. The functions B and λ are defined in D, but μ is defined only where the special adapted frame e is defined. We know that $\lambda^2 = cB$, $c > 0$ is a constant. Also, denoting the derivative with respect to z and $\overline{z}$ by a subscript as usual, we have from (7) and (11) that

$$B_z = -2\mu B, \quad \mu_z = -\mu^2, \quad \text{and} \quad \mu_{\overline{z}} = -B. \tag{12}$$

By the third equation of (12), we know that the open subset $D' = \{\mu \neq 0\} \subseteq D$ is dense in D. By the second equation of (12), we have $(\frac{1}{\mu} - z)_z = 0$, so that

$$\frac{1}{\mu} = z + \overline{f} \tag{13}$$

where $f(z)$ is a holomorphic function in D'. Note that $z + \overline{f}$ is never zero in D' because $\frac{1}{\mu}$ is never zero there. Let us write

$$Q = \frac{1}{\sqrt{B}} > 0.$$

By the first equation of (12), we have $\mu = (\log Q)_z$, so that by (13), $(\log Q)_z = 1/(z + \overline{f})$. So for any given point $x \in D'$, there exists a small neighborhood W_x in which

$$\frac{\partial}{\partial z}(\log Q - \log(z + \overline{f})) = 0.$$

Hence

$$\log Q = \log(z + \overline{f}) + \overline{g}$$

where g is a holomorphic function in W_x and the log on the right hand side is a branch of the logarithm function. From this we get

$$Q = z\,\overline{\alpha} - \overline{\beta}$$

in W_x, where α, β are holomorphic functions in W_x and α is nowhere zero. We shall subsequently exploit the fact that α is nowhere zero.

Now Q is a real-valued function. The following lemma shows that α and β must be linear polynomials.

Lemma. Suppose Q is a real-valued function defined in a neighborhood W of a point $z_0 \in \mathbf{C}$. Suppose further that $Q = z\,\overline{\alpha} - \overline{\beta}$ for some holomorphic functions α and β defined in W. Then α and β are linear polynomials and Q is a real quadratic polynomial.

Proof. Let $Q_0(z) \equiv Q(z + z_0)$. Then Q_0 is defined in a neighborhood $W_0 \equiv W - z_0$ of 0 so that

$$Q_0(z) = z\,\overline{\alpha_0(z)} - \overline{\beta_0(z)},$$

where $\alpha_0(z) = \alpha(z+z_0)$ and $\beta_0(z) = -\overline{z}_0\alpha(z+z_0)+\beta(z+z_0)$, an both α_0 and β_0 are holomorphic functions defined in W_0. $Q_0(z)$ is real-valued. Suppose we can prove that α_0 and β_0 are linear polynomials, then obviously so are

α and β. Therefore it suffices to prove the lemma for the case $z_0 = 0 \in \mathbf{C}$ and W is a neighborhood of 0. Assume henceforth that such is the case.

We may write

$$\alpha(z) = a_0 + a_1 z + z^2 f(z), \quad \beta(z) = b_0 + b_1 z + z^2 g(z),$$

where a_0, a_1, b_0, b_1 are complex constants and f, g are holomorphic functions in W. Then

$$Q(z) = (-\overline{b}_0) + (\overline{a}_0 z - \overline{b_1 z}) + (\overline{a}_1 |z|^2 + |z|^2\, \overline{zf(z)} - \overline{z^2 g(z)}).$$

We see that $-\overline{b}_0$ and $\overline{a}_0 z - \overline{b_1 z}$ are the zeroth and first order terms of the power series expansion of the *real* analytic function $Q(z)$ around 0. Each of $-\overline{b}_0$ and $\overline{a}_0 z - \overline{b_1 z}$ is therefore a *real*-valued polynomial for all z. It follows that b_0 is a real number. Moreover, letting $z = 1$ and $z = \sqrt{-1}$ in $\overline{a}_0 z - \overline{b_1 z}$ shows that both $\overline{a_0 - b_1}$ and $\overline{(a_0 + b_1)}\sqrt{-1}$ are real. This is possible iff $a_0 = -\overline{b}_1$. Thus

$$Q(z) = b + 2\Re(\xi z) + (\overline{a}_1 |z|^2 + |z|^2\, \overline{zf(z)} - \overline{z^2 g(z)}),$$

where $\Re$ denotes the real part of a complex number, $b \equiv -b_0$ is real, and $\xi \equiv \overline{a}_0 \in \mathbf{C}$.

Now consider the real analytic function $F(z) \equiv \overline{a}_1 |z|^2 + |z|^2\, \overline{zf(z)} - \overline{z^2 g(z)}$. We will show that in fact a_1 is real and $f = g = 0$. Once this is proved, we would be able to write (after a slight change of notation for future convenience):

$$\begin{array}{ll} \alpha(z) = \overline{\xi} + az & \quad (a \text{ is real}) \\ \beta(z) = -b + -\xi z & \quad (b \text{ is real}) \end{array}$$

so that

$$Q(z) = b + 2\Re(\xi z) + a|z|^2 \qquad (a,\, b \text{ real}).$$

The proof of the lemma would be complete.

To show $f = g = 0$, let

$$f(z) = \sum_n \overline{A}_n z^n \quad \text{and} \quad g(z) = \sum_n \overline{B}_n z^n$$

be the power series expansions of f and g around 0. Then the real anlaytic function $F(z)$ has the following power series expansion around 0:

$$\begin{aligned} F(z) &= \overline{a}_1 |z|^2 + \sum A_n |z|^2 \overline{z}^{n+1} - \sum B_n \overline{z}^{n+2} \\ &= \overline{a}_1 |z|^2 + B_0 \overline{z}^2 + \sum_{n=0}^{\infty} (A_n |z|^2 - B_{n+1} \overline{z}^2) \overline{z}^{n+1} \end{aligned}$$

Because the second order term $\overline{a}_1 |z|^2 + B_0 \overline{z}^2$ must be real for all z, letting $z = 1$ we get $a_1 - B_0$ is real, and letting $z = \sqrt{-1}$ we get $\overline{a}_1 - B_0$ is real. This is possible iff a_1 and B_0 are both real. Now let $z = \exp(-\sqrt{-1}(\pi/4))$, we get $a_1 - B_0\sqrt{-1}$ is real. Thus $B_0 = 0$.

Next we show that $A_n = B_{n+1} = 0$ for all $n \geq 0$. We know that $(A_n |z|^2 - B_{n+1} \overline{z}^2) \overline{z}^{n+1}$, being the term of order $(n+3)$ in the power series expansion of the real analytic function $F(z)$, must be real for all z. Letting $z = 1$ and $z = -\sqrt{-1}$ in succession in this expression yields that $A_n - B_{n+1}$ and $(A_n + B_{n+1})(\sqrt{-1})^{n+1}$ is real. Letting $z = \exp(-\sqrt{-1}\,\pi/(2n+2))$, we get that $\sqrt{-1}(A_n - B_{n+1}\exp(\sqrt{-1}\,\pi/(n+1))$ is also real. It is then elementary to show that, under the circumstance, $(A_n|z|^2 - B_{n+1}\overline{z}^2)\overline{z}^{n+1}$ can be real for all z iff $A_n = B_{n+1} = 0$ for all $n \geq 0$. Thus $f = g = 0$. $\square$

Now we return to the function $Q = 1/\sqrt{B} = z\,\overline{\alpha} - \overline{\beta}$ above. The lemma implies that for some constants k, a, b and ξ, we have $\alpha(z) = k + az$ and $\beta(z) = -b - \xi z$, so that

$$Q(z) = b + (\overline{k}z + \overline{\xi z}) + \overline{a}|z|^2.$$

Because Q is real, then arguing as in the proof of the Lemma leads to the fact that, necessarily, $\overline{k} = \xi$ and that both b and a must be real. Thus we have

$$Q(z) = b + 2\Re(\xi z) + a|z|^2 \qquad (a,\, b \text{ real}) \tag{14}$$

This expression for Q is valid in W_x, but the constants a, b and ξ being clearly independent of x, this expression exhibits Q globally as a quadratic function on all of D.

We are now in a position to prove that $D = \mathbf{C}$. Recall that we have a unit geodesic $\gamma : [0, a] \to M$, where $\gamma^0 \equiv \gamma([0, a)) \subseteq Y$. We can choose our frame e so that e_2 is parallel along γ^0, thus e_2 (and e_1) can be defined at $q = \gamma(a)$. Because when approaching any finite z, the quadratic function Q of (14) stays finite, so $B = 1/Q^2$ is not zero at q, meaning that the Ricci tensor has rank at least 2 at q, so $q \in U$. On the other hand, by the fact that $\lambda^2 = cB$ in D with $c > 0$ a constant, we know that $\lambda(q) > 0$ as well, that is, $q \in V$. Thus γ^0 can be extended beyond $[0, a]$ and therefore γ is defined on $[0, \infty)$. We have proved that $D = \mathbf{C}$.

We can now deduce the desired contradiction to close out the proof of Theorem 2. We have previously observed that the holomorphic function α in $Q(z) = z\,\overline{\alpha(z)} - \overline{\beta(z)}$ is zero-free. Since $D = \mathbf{C}$, Q is now a quadratic polynomial in $\mathbf{C}$, and so are α and β. But α, being a linear function, cannot be zero-free on $\mathbf{C}$ unless it is a constant. Thus $a = 0$ and $\alpha(z) = \overline{\xi}$ in the notation of (14). But then (14) shows that

$$Q(z) = b + 2\Re(\xi z),$$

so that $B = 1/Q^2 = 1/(b + 2\Re(\xi z))^2$. Since B is a curvature function and must be defined in all of $\mathbf{C}$ (i.e., all of Y), we must have $\xi = 0$. But then $\alpha = 0$, contradicting the fact that α is zero-free.

An alternate way to reach a contradiction is to observe that $\mu = Q_z/Q$, so that by (12),

$$-B = \mu_{\overline{z}} = \frac{Q_{z\overline{z}}}{Q} - \frac{Q_z Q_{\overline{z}}}{Q^2} = \frac{ab - |\xi|^2}{Q^2} \tag{15}$$

But by (14), $b + 2\Re(\xi z) + a|z|^2$ is a positive quadratic function on all of $\mathbf{C}$. Hence $ab - |\xi|^2 > 0$, which implies that $-B > 0$, or B is negative. Contradiction. □

Remark. The last argument points to the philosophical underpinning of why it is impossible to have $V \neq \phi$ when $D = \mathbf{C}$. For, another way to express (15) is $\Delta f = e^{2f}$, where $f = \log Q$ on all of $\mathbf{C}$. This implies that the Hermitian metric $e^{2f} dz d\overline{z}$ on $\mathbf{C}$ has Gaussian curvature -1, which is impossible. (This is classical, and one way to see it quickly is to realize that $\mathbf{C}$ is not Kobayashi hyperbolic.)

Incidentally, the proof we gave above concerning $\widetilde{\mathcal{L}}$ being totally geodesic and holomorphic is valid verbatim for the case of nonpositive bisectional curvature. We can elaborate on it slightly to arrive at

Proposition 2. *Let M^n be a complete Kähler manifold with nonpositive bisectional curvature and with Ricci rank $r = 2$. If the metric is real analytic, then either the universal covering space of M^n is holomorphically isometric to the product $\mathbf{C}^{n-2} \times N^2$, or it is foliated by a holomorphic, totally geodesic foliation with complete, flat, codimension 1 leaves.*

For the proof, notice that, in the real analytic case, the open subset V, if not empty, would be dense in M. So for any $q \in M \setminus V$, the limiting position of the leaves of $\widetilde{\mathcal{L}}$ would give flat, complete, totally geodesic complex hypersurface of M through q. Such a limit must be unique for dimensional considerations. Thus $\widetilde{\mathcal{L}}$ can be extended to M^n. In this case, the universal covering of M would be the total space of a holomorphic vector bundle over a Riemann surface (see [WZ1] for the extrinsic case). It would be highly desirable to have a precise description of the metric in this case, and we intend to pursue this on another occasion.

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