Stability and Existence of Multidimensional Subsonic Phase Transitions

Ya-Guang Wang
Department of Mathematics, Shanghai Jiao Tong University, 200030 Shanghai
The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail: ygwang@sjtu.edu.cn

Zhouping Xin
The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail: zpxin@ims.cuhk.edu.hk

Abstract

The purpose of this paper is to prove the uniform stability of multidimensional phase transitions satisfying the viscosity-capillarity criterion in a van der Waals fluid, and further to establish the local existence of phase transition solutions.

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1. Introduction and Main Results

Phase transitions are important phenomena in physics, mechanics and fluid dynamics, such as in a van der Waals fluid and elastic-plastic materials. To study these phenomena, one should investigate boundary value problems for partial differential equations which change types. There has been rich literature devoted to the existence and stability of phase transitions in one space variable, cf. [1, 5, 6, 8, 9, 10, 14] and references therein. In particular, in [14], Slemrod proved that phase transitions are excluded in weak solutions of the one dimensional p-system only under the viscosity admissibility criterion, which is however sufficient to determine the classical shock wave solutions in the p-system when the presure law $p = p(\rho)$ is strictly increasing. Furthermore, he had studied the one-dimensional phase boundaries in a van der Waals fluid under the viscosity-capillarity criterion.

However, few rigorous result is known for multidimensional phase transitions except for Benzoni-Gavage's recent works. In [2], she had obtained the weakly linear stability of multidimensional subsonic phase transitions in a van der Waals fluid under the capillarity admissibility criterion, which is equivalent to a generalized equal area rule, and she observed there is a surface wave violating the uniformity of the stability. In her second paper [3], she further tried to study the influence of viscosity on the stability of phase transitions. By the mode analysis and some technique of complex analysis, she obtained a sufficient condition on the unperturbed planar phase transition to guarantee that the Lopatinski determinant associated to a linearized initial-boundary value problem derived from the planar phase transition is nonzero under the viscosity-capillarity criterion and some constraints on the states of phase transitions.

The purpose of this paper is to study the stability and existence of the subsonic phase transition under the viscosity-capillarity criterion ([3, 14]). By directly computing the associated Lopatinski determinant, we shall prove that the subsonic viscosity-capillarity admissible phase transition is uniformly stable in a van der Waals fluid without any restriction of Benzoni-Gavage on the states of the phase transition. These uniform stability estimates enable us to modify the techniques of Majda [12] to establish the local existence of multidimensional subsonic phase transitions under viscosity-capillarity criterion.

The precise statement of problems and main results will be given in next subsections. Detail proofs of the stability and existence of multidimensional phase transitions will be presented in §2 and §3 respectively.

1.1 Formulation of Problems

For simplicity of notations, we shall study problems only in two space variables, and it is easy to see that whole discussion is valid also in the case of higher dimensions.

Consider the following Euler equations:

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0 \\ (\rho u)_t + (\rho u^2 + p(\rho))_x + (\rho u v)_y = 0 \\ (\rho v)_t + (\rho u v)_x + (\rho v^2 + p(\rho))_y = 0 \end{cases} \tag{1.1}$$

where ρ and (u, v) represent the density and velocity of the fluid respectively, and the pressure law $p = p(\rho)$ is given by an equation of states.

Equations (1.1) can be rewritten as the following system form:

$$\partial_t F_0(U) + \partial_x F_1(U) + \partial_y F_2(U) = 0 \tag{1.2}$$

where $U = (\rho, u, v)^T$ and

$$F_0(U) = \begin{pmatrix} \rho \\ \rho u \\ \rho v \end{pmatrix}, \quad F_1(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho) \\ \rho u v \end{pmatrix}, \quad F_2(U) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p(\rho) \end{pmatrix}.$$

Denote by $\tau = \rho^{-1}$ the specific volume. The pressure law $P(\tau) := p(\frac{1}{\tau})$ in a van der Waals fluid reads

$$P(\tau) = \frac{RT}{\tau - b} - \frac{a}{\tau^2} \qquad (\tau > b) \tag{1.3}$$

where T denotes the temperature, which is assumed to be a positive constant, R is the perfect gas constant, and a, b are positive constants. When the temperature

$$\frac{a}{4bR} < T < \frac{8a}{27bR}$$

is fixed, there are $\tau_* < \tau^*$ such that

$$\begin{cases} P'(\tau) < 0, & \text{if } b < \tau < \tau_* \text{ or } \tau > \tau^* \\ P'(\tau) > 0, & \text{if } \tau_* < \tau < \tau^* \end{cases}. \tag{1.4}$$

Thus, the state of $\tau \in (b, \tau_*)$ represents the liquid phase while that of $\tau \in (\tau^*, +\infty)$ is the vapor phase. Generally, these two phases are likely to coexist and one may observe the propagation of phase boundaries.

As usual, the Maxwell equilibrium $\{\tau_\theta, \tau^\theta\}$ of a phase transition is defined by the equal area rule:

$$P(\tau_\theta) = P(\tau^\theta), \quad \int_{\tau_\theta}^{\tau^\theta} (P(\tau_\theta) - P(\tau))d\tau = 0 \tag{1.5}$$

and $\tau_\theta < \tau_*$, $\tau^\theta > \tau^*$.

It is obvious that there is a unique point $\tau_1 > \tau^\theta$ at which the tangent to the graph of $p = P(\tau)$ passes through τ_θ (see the Figure 1 below). Denote by

$$j_1^2 = -P'(\tau_1) \tag{1.6}$$

which equals to $(P(\tau_\theta) - P(\tau_1))/(\tau_1 - \tau_\theta)$.

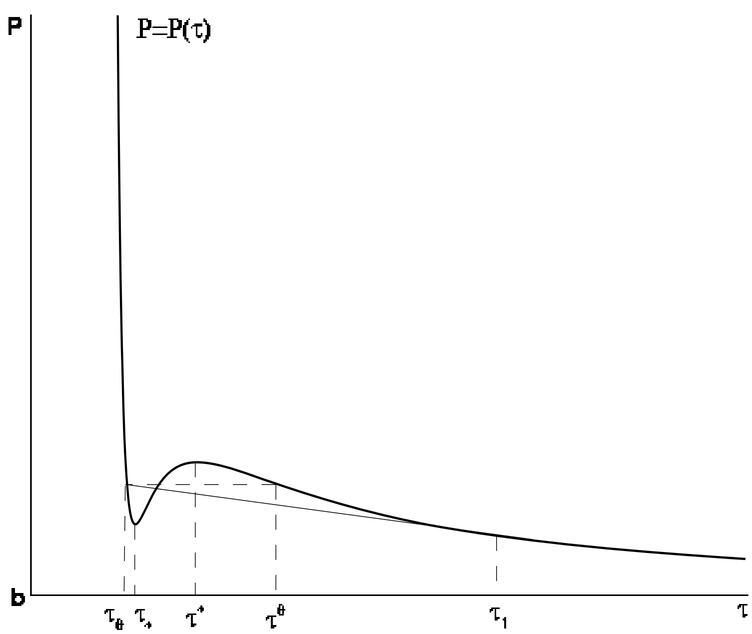

Figure 1

Suppose that

$$U(t,x,y) = \begin{cases} U_+(t,x,y), & x > \varphi(t,y) \\ U_-(t,x,y), & x < \varphi(t,y) \end{cases} \tag{1.7}$$

is a local phase transition solution to the system (1.2), which means that $\varphi \in C^2$, $U_+ \in C^1\{x > \varphi(t,y)\}$ and $U_- \in C^1\{x < \varphi(t,y)\}$ belong to different phases, satisfy (1.2) in $\{x > \varphi(t,y)\}$ and $\{x < \varphi(t,y)\}$ respectively, and the following Rankine-Hugoniot jumping condition:

$$\varphi_t[F_0(U)] - [F_1(U)] + \varphi_y[F_2(U)] = 0 \quad \text{on } x = \varphi(t,y) \tag{1.8}$$

holds. Here and after we always use $[\cdot]$ to denote the jump of related functions across the phase boundary $\{x = \varphi(t,y)\}$.

For simplicity, we assume that $\varphi(0,0) = \partial_y\varphi(0,0) = 0$, and outside a compact neighborhood of the origin, $U_\pm(t,x,y)$ are constant states and $\varphi(t,y) = \sigma t$ for a constant σ.

Furthermore, we assume that the phase transition (1.7) is subsonic, which means that at each point $(t,x,y)\in\Sigma=\{x=\varphi(t,y)\}$, if we denote by

$$c_\pm=(p'(\rho_\pm))^{\frac{1}{2}}$$

the sound speeds, and

$$M_\pm=\frac{1}{c_\pm}\left|\frac{u_\pm-\varphi_y v_\pm-\varphi_t}{(1+\varphi_y^2)^{\frac{1}{2}}}\right|$$

the Mach numbers evaluated at $U_\pm(t,x,y)=(\rho_\pm,u_\pm,v_\pm)^T$, then we have

$$M_\pm<1. \tag{1.9}$$

In particular, this shows that phase transitions we consider here are not the classical Lax shocks, they violate the Lax entropy inequalities. Several alternative criteria have been developed in order to determine the admissible phase fronts (cf. [1, 10, 14]). Here we are going to use the viscosity-capillarity criterion proposed by Slemrod in [14], which is also the one studied by Benzoni-Gavage in [3].

To introduce the viscosity-capillarity criterion precisely, let us denote by $E(\rho)$ the specific free energy such that $d_\rho E(\rho)=\frac{p(\rho)}{\rho^2}$, and $e(\rho)=\rho E(\rho)$ the free energy per unit volume. From the first component of the Rankine-Hugoniot condition (1.8), we have

$$\rho_+(u_+-\varphi_y v_+-\varphi_t)=\rho_-(u_--\varphi_y v_--\varphi_t)\qquad \text{on } x=\varphi(t,y).$$

We shall denote by

$$j=\rho_\pm(u_\pm-\varphi_y v_\pm-\varphi_t)/\sqrt{1+\varphi_y^2} \tag{1.10}$$

the mass transfer flux across the phase boundary.

As in [3, 14], we say that the phase transition (1.7) is viscosity-capillarity admissible if and only if for any fixed $P_0=(t_0,\varphi(t_0,y_0),y_0)\in\Sigma$, it has a viscosity-capillarity profile, which is a planar wave solution of the form

$$\begin{cases}\rho(t,x,y)=\rho(\frac{x-n_y y-\sigma t}{\epsilon})\\ (u,v)(t,x,y)=(u,v)(\frac{x-n_y y-\sigma t}{\epsilon})\\ \lim_{\xi\to\pm\infty}(\rho,u,v)(\xi)=(\rho_\pm,u_\pm,v_\pm)|_{P_0}\end{cases} \tag{1.11}$$

with $n_y=\varphi_y(1+\varphi_y^2)^{-1/2}|_{P_0}$ and $\sigma=\varphi_t(1+\varphi_y^2)^{-1/2}|_{P_0}$, to the following equations:

$$\begin{cases}\rho_t+(\rho u)_x+(\rho v)_y=0\\ (\rho u)_t+(\rho u^2+p(\rho))_x+(\rho uv)_y=\gamma\epsilon\Delta u-\epsilon^2\partial_x(\Delta(\rho^{-1}))\\ (\rho v)_t+(\rho uv)_x+(\rho v^2+p(\rho))_y=\gamma\epsilon\Delta v-\epsilon^2\partial_y(\Delta(\rho^{-1}))\end{cases}. \tag{1.12}$$

As in the discussion of Benzoni-Gavage in [3], if we denote by $\tau(\xi;j,\gamma)$ the viscosity-capillarity profile satisfying

$$\begin{cases}\tau''=\gamma j\tau'+\pi-P(\tau)-j^2\tau\\ \lim_{\xi\to-\infty}\tau=\frac{1}{\rho_-}|_{x=\varphi(t,y)},\quad \lim_{\xi\to+\infty}\tau=\frac{1}{\rho_+}|_{x=\varphi(t,y)}\end{cases} \tag{1.13}$$

with τ',τ'' being the first and the second order derivatives of τ with respect to ξ, $\pi=p(\rho_+)+\frac{j^2}{\rho_+}=p(\rho_-)+\frac{j^2}{\rho_-}$ valued at $x=\varphi(t,y)$ (This identity is a simple consequence of (1.8).), then for any $(t,x,y)\in\Sigma=\{x=\varphi(t,y)\}$, the viscosity-capillarity admissible $U_\pm(t,x,y)=(\rho_\pm,u_\pm,v_\pm)^T$ satisfies the following identity

$$[e'(\rho)+\frac{(u-\varphi_y v-\varphi_t)^2}{2(1+\varphi_y^2)}]=-\gamma j\int_{-\infty}^{\infty}\tau'^2(\xi;j,\gamma)d\xi\qquad \text{on } x=\varphi(t,y). \tag{1.14}$$

The existence of the viscosity-capillarity profile $\tau(\xi;j,\gamma)$ to (1.13) is obtained by Benzoni-Gavage in [3] when $0<\gamma\le\gamma_0$ for some small $\gamma_0>0$ and $0<j^2<j_1^2$ with j_1 being given in (1.6).

Moreover, Benzoni-Gavage showed in [3] that $U_\pm|_{x=\varphi(t,y)}$ depend smoothly only on (j,γ), and as a consequence,

$$a(j,\gamma)=j\int_{-\infty}^{\infty}\tau'^2(\xi;j,\gamma)d\xi \tag{1.15}$$

is a smooth function of (j,γ) in $\{0<j^2<j_1^2, 0<\gamma\le\gamma_0\}$.

In order to study the linearized stability of the phase transition (1.7), let us consider the perturbed family of subsonic phase transition solutions to (1.2),

$$U^\epsilon(t,x,y)=\begin{cases}U_+^\epsilon(t,x,y), & x>\varphi^\epsilon(t,y)\\ U_-^\epsilon(t,x,y), & x<\varphi^\epsilon(t,y)\end{cases} \tag{1.16}$$

satisfying

$$\begin{cases}\partial_t F_0(U_\pm^\epsilon)+\partial_x F_1(U_\pm^\epsilon)+\partial_y F_2(U_\pm^\epsilon)=0, & \pm(x-\varphi^\epsilon(t,y))>0\\ \varphi_t^\epsilon[F_0(U^\epsilon)]-[F_1(U^\epsilon)]+\varphi_y^\epsilon[F_2(U^\epsilon)]=0, & x=\varphi^\epsilon(t,y)\\ [e'(U_1^\epsilon)+\frac{(U_2^\epsilon-\varphi_y^\epsilon U_3^\epsilon-\varphi_t^\epsilon)^2}{2(1+(\varphi_y^\epsilon)^2)}]=-\gamma j^\epsilon\int_{-\infty}^{\infty}\tau'^2(\xi;j^\epsilon,\gamma)d\xi, & x=\varphi^\epsilon(t,y)\end{cases} \tag{1.17}$$

and

$$(U_+^\epsilon,U_-^\epsilon,\varphi^\epsilon)|_{\epsilon=0}=(U_+,U_-,\varphi) \tag{1.18}$$

where U_k^ϵ is the k–th component of U^ϵ (That is $U_1^\epsilon=\rho^\epsilon$, $U_2^\epsilon=u^\epsilon$, $U_3^\epsilon=v^\epsilon$, and these two notations for components of U will be used simultaneously in the remainder of this paper.),

$$j^\epsilon=U_{\pm,1}^\epsilon(U_{\pm,2}^\epsilon-\varphi_y^\epsilon U_{\pm,3}^\epsilon-\varphi_t^\epsilon)/\sqrt{1+(\varphi_y^\epsilon)^2}|_{x=\varphi^\epsilon(t,y)} \tag{1.19}$$

and $\tau(\xi;j^\epsilon,\gamma)$ is defined in a way similar to that in (1.13):

$$\begin{cases}\tau''=\gamma j^\epsilon\tau'+\pi^\epsilon-P(\tau)-(j^\epsilon)^2\tau\\ \lim_{\xi\to-\infty}\tau=\frac{1}{\rho_-^\epsilon}|_{x=\varphi^\epsilon(t,y)}, \quad \lim_{\xi\to+\infty}\tau=\frac{1}{\rho_+^\epsilon}|_{x=\varphi^\epsilon(t,y)}\end{cases} \tag{1.20}$$

with $\pi^\epsilon=p(\rho_\pm^\epsilon)+\frac{(j^\epsilon)^2}{\rho_\pm^\epsilon}$.

To differentiate (1.17) with respect to ϵ, as usual, let us transform this free boundary problem into a fixed one by introducing

$$\begin{cases}\tilde{x}=\pm(x-\varphi^\epsilon(t,y))\\ \tilde{y}=y\\ \tilde{t}=t\\ \tilde{U}_\pm^\epsilon(\tilde{t},\tilde{x},\tilde{y})=U_\pm^\epsilon(t,x,y)\end{cases} \tag{1.21}$$

for the plus and the minus parts of equations in (1.17) respectively. Then, from (1.17) we know that $\tilde{U}_\pm^\epsilon(\tilde{t},\tilde{x},\tilde{y})$ satisfy the following problem

$$\begin{cases}\partial_t U_\pm^\epsilon\pm(A_1(U_\pm^\epsilon)-\varphi_y^\epsilon A_2(U_\pm^\epsilon)-\varphi_t^\epsilon)\partial_x U_\pm^\epsilon+A_2(U_\pm^\epsilon)\partial_y U_\pm^\epsilon=0, & t,x>0\\ \varphi_t^\epsilon(F_0(U_+^\epsilon)-F_0(U_-^\epsilon))-(F_1(U_+^\epsilon)-F_1(U_-^\epsilon))+\varphi_y^\epsilon(F_2(U_+^\epsilon)-F_2(U_-^\epsilon))=0, & x=0\\ [e'(U_1^\epsilon)+\frac{(U_2^\epsilon-\varphi_y^\epsilon U_3^\epsilon-\varphi_t^\epsilon)^2}{2(1+(\varphi_y^\epsilon)^2)}]=-\gamma a(j^\epsilon,\gamma), & x=0\end{cases} \tag{1.22}$$

where we have dropped tildes of notations for simplicity, $a(j,\gamma)$ is defined in (1.15), and

$$A_1(U) = (F_0'(U))^{-1}F_1'(U) = \begin{pmatrix} u & \rho & o \\ \frac{c^2}{\rho} & u & 0 \\ 0 & 0 & u \end{pmatrix},$$

$$A_2(U_\pm) = (F_0'(U))^{-1}F_2'(U) = \begin{pmatrix} v & 0 & \rho \\ 0 & v & 0 \\ \frac{c^2}{\rho} & 0 & v \end{pmatrix}$$

with $c = (p'(\rho))^{\frac{1}{2}}$ being the sound speed.

Let $\tilde{U}_\pm(\tilde{t},\tilde{x},\tilde{y})$ be derived from $U_\pm(t,x,y)$ given in (1.7) under the same mapping $(t,x,y) \to (\tilde{t},\tilde{x},\tilde{y})$ as in (1.21) at $\epsilon = 0$, and we still denote by $U_\pm(t,x,y)$ the transformed functions without tildes for simplicity.

By setting

$$(\frac{dU_+^\epsilon}{d\epsilon}, \frac{dU_-^\epsilon}{d\epsilon}, \frac{d\varphi^\epsilon}{d\epsilon})|_{\epsilon=0} = (V_+, V_-, \phi), \tag{1.23}$$

differentiating (1.22) with respect to ϵ and letting $\epsilon = 0$, we obtain the linearized problem of (1.22) as follows:

$$\begin{cases} \partial_t V_\pm \pm (A_1(U_\pm) - \varphi_y A_2(U_\pm) - \varphi_t)\partial_x V_\pm + A_2(U_\pm)\partial_y V_\pm = f_\pm, & t,x>0 \\ B(V_+,V_-,\phi) := b_0\phi_t + b_1\phi_y + M_+V_+ + M_-V_- = g & \text{on } x=0 \\ (V_+,V_-,\phi)|_{t<0} \quad \text{vanish} \end{cases} \tag{1.24}$$

where

$$b_0 = \begin{pmatrix} [F_0(U)] \\ \frac{[U_2 - \varphi_y U_3]}{1+\varphi_y^2} + \gamma\partial_j a \frac{U_{-,1}}{\sqrt{1+\varphi_y^2}} \end{pmatrix},$$

$$b_1 = \begin{pmatrix} [F_2(U)] \\ \gamma\partial_j a \frac{U_{-,1}U_{-,3} + \varphi_y U_{-,1}(U_{-,2} - \varphi_t)}{(1+\varphi_y^2)^{3/2}} + \frac{[U_3(U_2 - \varphi_y U_3 - \varphi_t)]}{1+\varphi_y^2} + \frac{\varphi_y[(\varphi_t + \varphi_y U_3 - U_2)^2]}{(1+\varphi_y^2)^2} \end{pmatrix},$$

$$M_+ = \begin{pmatrix} \varphi_t F_0'(U_+) + \varphi_y F_2'(U_+) - F_1'(U_+) \\ l_+ \end{pmatrix}$$

and

$$M_- = \begin{pmatrix} F_1'(U_-) - \varphi_t F_0'(U_-) - \varphi_y F_2'(U_-) \\ l_- \end{pmatrix}$$

with $\partial_j a = \frac{\partial a}{\partial j}$,

$$l_+ = (-e''(U_{+,1}), \ \frac{\varphi_t - U_{+,2} + \varphi_y U_{+,3}}{1+\varphi_y^2}, \ \frac{\varphi_y(U_{+,2} - \varphi_t - \varphi_y U_{+,3})}{1+\varphi_y^2})$$

and

$$l_- = (\gamma\partial_j a \frac{\varphi_t - U_{-,2} + \varphi_y U_{-,3}}{\sqrt{1+\varphi_y^2}} + e''(U_{-,1}), \ -\gamma\partial_j a \frac{U_{-,1}}{\sqrt{1+\varphi_y^2}} + \frac{U_{-,2} - \varphi_y U_{-,3} - \varphi_t}{1+\varphi_y^2},$$

$$\varphi_y(\gamma\partial_j a \frac{U_{-,1}}{\sqrt{1+\varphi_y^2}} + \frac{\varphi_t + \varphi_y U_{-,3} - U_{-,2}}{1+\varphi_y^2})).$$

The form of the problem (1.24) is similar to the one derived by Majda [11] for multi-dimensional shock fronts. However, due to the subsonic property of the phase transition

(U_+, U_-, φ) and the viscosity-capillarity criterion (1.14), the well-posedness of the linearized problem (1.24) needs to be investigated.

The first goal of this paper is to study the uniform stability of the subsonic phase transition (U_+, U_-, φ), and the second one is to establish the existence of such a phase transition solution under certain compatibility conditions on initial data.

1.2 Main Results

To study the stability of the phase transition (1.7), for simplicity let us assume that it is the following planar case:

$$U(t,x,y) = \begin{cases} U_+ = (\rho_+, u_+, v_0), & x > \sigma t \\ U_- = (\rho_-, u_-, v_0), & x < \sigma t \end{cases} \tag{1.25}$$

with $(\rho_\pm, u_\pm, v_0, \sigma)$ being constants. Once we know that (1.25) is uniformly stable, then we can easily deduce that in a neighborhood of the origin, the general curved phase transition (1.7) is also uniformly stable by continuity.

In the case (1.25), the problem (1.24) is simplified as:

$$\begin{cases} \partial_t V_\pm \pm (A_1(U_\pm) - \sigma)\partial_x V_\pm + A_2(U_\pm)\partial_y V_\pm = f_\pm, \quad t, x > 0 \\ \overline{B}(V_+, V_-, \phi) := b_0\phi_t + b_1\phi_y + M_+V_+ + M_-V_- = g \\ (V_+, V_-, \phi)|_{t<0} \quad \text{vanish} \end{cases} \tag{1.26}$$

where

$$b_0 = ([\rho],\, [\rho u],\, v_0[\rho],\, [u] + \tilde{\gamma}\rho_-)^T,$$

$$b_1 = (v_0[\rho],\, v_0[\rho u],\, v_0^2[\rho] + [p],\, v_0([u] + \tilde{\gamma}\rho_-))^T,$$

$$M_+ = \begin{pmatrix} \sigma F_0'(U_+) - F_1'(U_+) \\ l_+ \end{pmatrix}, \quad M_- = \begin{pmatrix} F_1'(U_-) - \sigma F_0'(U_-) \\ l_- \end{pmatrix}$$

with

$$l_+ = (-e''(\rho_+),\, \sigma - u_+,\, 0), \quad l_- = (e''(\rho_-) - \tilde{\gamma}(u_- - \sigma),\, u_- - \sigma - \tilde{\gamma}\rho_-,\, 0).$$

Here $\tilde{\gamma} = \gamma(\alpha(j) + o(1))$ while $\gamma \to 0$ with $\alpha(j)$ satisfying

$$\alpha(j) = \lim_{\gamma\to 0} \frac{\partial}{\partial j} a(j,\gamma) \geq \alpha > 0$$

for a positive constant α (cf. [3]).

The first main result of this paper is

Theorem 1.1 (ONE-DIMENSIONAL STABILITY) *There is $\gamma_0 > 0$ such that for any $0 < \gamma \leq \gamma_0$, the subsonic phase transition (U_+, U_-, σ) is stable with respect to perturbations in the $x-$direction, which means that the problem (1.26) without terms of $y-$derivatives is well-posed.*

Denote by

$$V = (V_+, V_-)^T$$

and

$$\hat{V}(s,\omega,x) = \frac{1}{4\pi^2}\int_0^\infty \int_{-\infty}^\infty e^{-(st+i\omega y)} V(t,x,y)\,dy\,dt$$

the Laplace-Fourier transform of V in (t,y)—variables with $Res>0$. Then, from (1.26) we know that $\hat{V}$ satisfies

$$\frac{\partial\hat{V}}{\partial x}=B(s,\omega)\hat{V}+\hat{f} \tag{1.27}$$

where

$$B(s,\omega)=\begin{pmatrix} -(A_1(U_+)-\sigma I)^{-1}(sI+i\omega A_2(U_+)) & 0 \\ 0 & (A_1(U_-)-\sigma I)^{-1}(sI+i\omega A_2(U_-)) \end{pmatrix}$$

and $\hat{f}=((A_1(U_+)-\sigma I)^{-1}\hat{f}_+,(\sigma I-A_1(U_-))^{-1}\hat{f}_-)^T$.

Denote by $\{\lambda_j\}_{j=1}^l$ all distinct eigenvalues of $B(s,\omega)$ with multiplicity being m_j. Obviously, we have

$$\mathbb{C}^6=\bigoplus_{j=1}^{l}\ker[(\lambda_jI-B(s,\omega))^{m_j}].$$

Introduce

$$E^+(s,\omega)=\{w_j\in\ker[(\lambda_jI-B(s,\omega))^{m_j}]|\ Re\lambda_j<0, j=1,\ldots,l\} \tag{1.28}$$

the space of boundary values of all bounded solutions of the special form

$$\hat{V}(s,\omega,x)=\sum_{Re\lambda_j<0}e^{\lambda_jx}\sum_{p=0}^{m_j-1}\frac{x^p}{p!}(\lambda_jI-B(s,\omega))^pw_j$$

to (1.27) with $\hat{f}\equiv 0$. Then, the second main result of this paper is:

Theorem 1.2 (UNIFORM STABILITY) *There is $\gamma_0>0$ depending only on the bounds of $U_\pm$ given in (1.25), such that for any fixed $0<\gamma\le\gamma_0$, the viscosity-capillarity admissible phase transition (1.25) is uniformly stable, i.e. there is $\eta>0$ such that the estimate*

$$\inf_{\substack{Res>0\\ |s|^2+\omega^2=1}}|(b_0s+ib_1\omega)\mu+M_+V_++M_-V_-|^2\ge\eta^2(|V_+|^2+|V_-|^2+\mu^2) \tag{1.29}$$

holds for all $V=(V_+,V_-)\in E^+(s,\omega)$ and $\mu\in\mathbb{R}$.

Remark 1.3: *Notice that the above uniform stability result requires no constraint on the states of the phase transition, which constrasts to the conditions (63) in [3].*

The third main result concerns the local existence of single multidimensional phase transition.

Theorem 1.4: *For any fixed $s\ge 9$, suppose that the initial data*

$$(\rho,u,v)|_{t=0}=\begin{cases}(\rho_+^0,u_+^0,v_+^0), & x>\varphi_0(y)\\ (\rho_-^0,u_-^0,v_-^0), & x<\varphi_0(y)\end{cases} \tag{1.30}$$

satisfy $(\rho_\pm^0,u_\pm^0,v_\pm^0)\in H^{s+1}(\{\pm(x-\varphi_0(y))>0\})$, $\varphi_0\in H^{s+\frac{3}{2}}$ and certain compatibility conditions, which will be given precisely in §3, there is a subsonic phase transition solution

$$(\rho,u,v)=\begin{cases}(\rho_+,u_+,v_+), & x>\varphi(t,y)\\ (\rho_-,u_-,v_-), & x<\varphi(t,y)\end{cases} \tag{1.31}$$

to the problem (1.1)(1.30) locally in time.

Remark 1.5: The above regularity conditions on initial data can be weakened as

$$(\rho^0_\pm, u^0_\pm, v^0_\pm) \in H^{s+1/2}(\{\pm(x-\varphi_0(y)) > 0\}), \quad \varphi_0 \in H^{s+1/2}$$

for any fixed $s > 2$ by using the theory of paradifferential operators as in [7, 13]. But, in order to avoid much technique in this paper, we shall directly use Majda's theory ([12]) to study our nonlinear problems, that is why we have the condition $s \geq 9$.

2. The Stability of Phase Transitions

In this section, we shall prove Theorems 1.1 and 1.2, and conclude that the viscosity-capillarity admissible phase transition is uniformly stable.

2.1 The one-dimensional stability

In order to study the problem (1.26), from the expressions of M_+ and M_-, if we define

$$W_\pm = F_0'(U_\pm)V_\pm \tag{2.1}$$

then the equations in (1.26) are equivalent to

$$\frac{\partial W_\pm}{\partial t} \pm (\tilde{A}_1(U_\pm) - \sigma I)\frac{\partial W_\pm}{\partial x} + \tilde{A}_2(U_\pm)\frac{\partial W_\pm}{\partial y} = \tilde{f}_\pm \tag{2.2}$$

where $\tilde{A}_2(U_\pm) = F_2'(U_\pm)(F_0'(U_\pm))^{-1}$, $\tilde{f}_\pm = F_0'(U_\pm)f_\pm$ and

$$\tilde{A}_1(U_\pm) = F_1'(U_\pm)(F_0'(U_\pm))^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ c^2_\pm - u^2_\pm & 2u_\pm & 0 \\ -v_0 u_\pm & v_0 & u_\pm \end{pmatrix}$$

and the boundary condition in (1.26) is transformed into

$$b_0\phi_t + b_1\phi_y + \tilde{M}_+W_+ + \tilde{M}_-W_- = g \qquad \text{on } x = 0 \tag{2.3}$$

where b_0, b_1 are given in (1.26), and

$$\tilde{M}_+ = M_+(F_0'(U_+))^{-1} = \begin{pmatrix} \sigma I - \tilde{A}_1(U_+) \\ \tilde{l}_+ \end{pmatrix}$$

$$\tilde{M}_- = M_-(F_0'(U_-))^{-1} = \begin{pmatrix} \tilde{A}_1(U_-) - \sigma I \\ \tilde{l}_- \end{pmatrix}$$

with

$$\begin{cases} \tilde{l}_+ = (\frac{u_+^2 - \sigma u_+}{\rho_+} - e''(\rho_+), \frac{\sigma - u_+}{\rho_+}, 0)^T \\ \tilde{l}_- = (e''(\rho_-) - \frac{u_-^2 - \sigma u_-}{\rho_-} + \tilde{\gamma}\sigma, \frac{u_- - \sigma}{\rho_-} - \tilde{\gamma}, 0)^T \end{cases}.$$

To study the one dimensional stability in the x–variable for the problem (2.2)(2.3), let us study the following problem

$$\begin{cases} \partial_t W_\pm \pm (\tilde{A}_1(U_\pm) - \sigma I)\partial_x W_\pm = \tilde{f}_\pm, & t, x > 0 \\ b_0\phi_t + \tilde{M}_+W_+ + \tilde{M}_-W_- = g, & x = 0 \\ (W_+, W_-, \phi)|_{t<0} = 0 \end{cases}. \tag{2.4}$$

The eigenvalues of $\tilde{A}_1(U_\pm) - \sigma I$ are

$$\lambda_1^\pm = u_\pm - \sigma - c_\pm, \quad \lambda_2^\pm = u_\pm - \sigma, \quad \lambda_3^\pm = u_\pm - \sigma + c_\pm.$$

Denote by $r_j^\pm$ and $l_j^\pm$ the right and left eigenvectors of $\tilde{A}_1(U_\pm) - \sigma I$ associated with $\lambda_j^\pm$ with the normalization

$$l_j^\pm r_k^\pm = \delta_{jk} \quad (j,k = 1,2,3).$$

Since the mass transfer flux $j = \rho_+(u_+ - \sigma) = \rho_-(u_- - \sigma)$ is nonzero, first, we assume $j > 0$, which implies

$$\lambda_1^\pm < 0 < \lambda_2^\pm < \lambda_3^\pm \tag{2.5}$$

by using the subsonic property of the phase transition (1.25).

Denote by

$$W_+ = \sum_{j=1}^3 w_j^+ r_j^+ \quad \text{and} \quad W_- = \sum_{j=1}^3 w_j^- r_j^-$$

the decompositions of W_+ and W_- on the bases $\{r_j^+\}_{j=1}^3$ and $\{r_j^-\}_{j=1}^3$ respectively, i.e. $w_j^+ = l_j^+ W_+$ and $w_j^- = l_j^- W_-$. Then, one can rewrite the boundary condition in (2.4) as

$$(b_0, \tilde{M}_+ r_2^+, \tilde{M}_+ r_3^+, \tilde{M}_- r_1^-) \begin{pmatrix} \phi_t \\ w_2^+ \\ w_3^+ \\ w_1^- \end{pmatrix} = g - (\tilde{M}_+ r_1^+) w_1^+ - \sum_{j=2}^3 (\tilde{M}_- r_j^-) w_j^-. \tag{2.6}$$

Similar to [6, 11], due to (2.5), the necessary and sufficient condition for the well-posedness of the problem (2.4) is the following one-dimensional stability condition

$$\Delta := \det(b_0, \tilde{M}_+ r_2^+, \tilde{M}_+ r_3^+, \tilde{M}_- r_1^-) \neq 0. \tag{2.7}$$

Now, let us see whether the condition (2.7) is satisfied.

By a direct computation, one has,

$$r_1^- = (1, u_- - c_-, v_0)^T, \; r_2^+ = (0,0,1)^T, \; r_3^+ = (1, u_+ + c_+, v_0)^T \tag{2.8}$$

and

$$\Delta = \begin{vmatrix} [\rho] & 0 & -\lambda_3^+ & \lambda_1^- \\ [\rho u] & 0 & -\lambda_3^+(u_+ + c_+) & \lambda_1^-(u_- - c_-) \\ v_0[\rho] & -\lambda_2^+ & -\lambda_3^+ v_0 & \lambda_1^- v_0 \\ [u] + \tilde{\gamma}\rho_- & 0 & \tilde{l}_+ r_3^+ & \tilde{l}_- r_1^- \end{vmatrix}$$

$$= -\lambda_1^- \lambda_2^+ \lambda_3^+ \begin{vmatrix} [\rho] & 1 & 1 \\ [\rho u] & u_+ + c_+ & u_- - c_- \\ [u] + \tilde{\gamma}\rho_- & -\frac{\tilde{l}_+ r_3^+}{\lambda_3^+} & \frac{\tilde{l}_- r_1^-}{\lambda_1^-} \end{vmatrix}. \tag{2.9}$$

Denote by Δ_1 the determinant on the right side of (2.9).

It follows from a simple computation and $e''(\rho) = c^2/\rho$ that

$$\frac{\tilde{l}_+ r_3^+}{\lambda_3^+} = -\frac{c_+}{\rho_+} \quad \text{and} \quad \frac{\tilde{l}_- r_1^-}{\lambda_1^-} = -(\frac{c_-}{\rho_-} + \tilde{\gamma}). \tag{2.10}$$

Substituting (2.10) into (2.9), developing Δ_1 and using $\rho_+(u_+ - \sigma) = \rho_-(u_- - \sigma)$, one gets that

$$\Delta_1 = [u](u_- - c_- - u_+ - c_+) + [\rho]\left\{\frac{c_+}{\rho_+}(\sigma - u_- + c_-) - \frac{c_-}{\rho_-}(u_+ - \sigma + c_+)\right\} - \tilde{\gamma}(\rho_+ c_+ + \rho_- c_-)$$

$$= -[u]^2 - \frac{c_+c_-}{\rho_+\rho_-}[\rho]^2 - \tilde{\gamma}(\rho_+c_+ + \rho_-c_-)$$

which is negative for any $\tilde{\gamma} \geq 0$. Thus, we conclude

$$\det(b_0, \tilde{M}_+r_2^+, \tilde{M}_+r_3^+, \tilde{M}_-r_1^-) \neq 0. \tag{2.11}$$

For the case $j < 0$, we have $u_\pm - \sigma < 0$, which implies

$$\lambda_1^\pm < \lambda_2^\pm < 0 < \lambda_3^\pm$$

and the necessary and sufficient condition for the well-posedness of the problem (2.4) is:

$$\det(b_0, \tilde{M}_+r_3^+, \tilde{M}_-r_1^-, \tilde{M}_-r_2^-) \neq 0. \tag{2.12}$$

The condition (2.12) for any $\tilde{\gamma} \geq 0$ can be verified in a way similar to that for (2.11). Hence, we obtain the conclusion of Theorem 1.1 by noting $\tilde{\gamma} = \gamma(\alpha(j) + o(1))$ with

$$\alpha(j) = \lim_{\gamma \to 0} \frac{\partial}{\partial j}(j \int_{-\infty}^{\infty} \tau'^2(\xi; j, \gamma) d\xi) \geq \alpha > 0.$$

2.2 The uniform stability

The purpose of this subsection is to establish the uniform stability of subsonic phase transitions claimed in Theorem 1.2.

First, let us recall a lemma from Majda [12] as follows:

Lemma 2.1: *For the problem (1.26), and $s \in \mathbb{C}$, $\omega \in \mathbb{R}$, if we define $e^+(s,\omega) = sb_0 + i\omega b_1$ and the projection*

$$P(s,\omega)V = V - \frac{(V, e^+)}{|e^+|^2} e^+$$

for any $V \in \mathbb{C}^6$, then the problem (1.26) is uniformly well-posed if there exists $\eta > 0$, such that

$$\inf_{Res>0, |s|^2+\omega^2=1} |e^+(s,\omega)| \geq \eta \tag{2.13}$$

and

$$\inf_{Res>0, |s|^2+\omega^2=1} |P(s,\omega)(M_+V_+ + M_-V_-)| \geq \eta(|V_+| + |V_-|) \tag{2.14}$$

for all $V = (V_+, V_-) \in E^+(s,\omega)$ with $E^+(s,\omega)$ being defined in (1.28).

Set $\tilde{s} = s + i\omega v_0$. For the problem (1.26), we have

$$e^+(s,\omega) = (\tilde{s}[\rho], \tilde{s}[\rho u], \tilde{s}v_0[\rho] + i\omega[p], \tilde{s}([u] + \tilde{\gamma}\rho_-))^T$$

which satisfies (2.13) obviously.

To study (2.14), let us investigate the space $E^+(s,\omega)$. For simplicity, we shall only consider the case

$$j = \rho_\pm(u_\pm - \sigma) > 0$$

and the other case $j < 0$ can be studied similarly.

2.2.1 The space $E^+(s,\omega)$

For any $s = i\xi + \eta$ ($\eta > 0$) and $\omega \in \boldsymbol{R}$, denote by

$$\hat{U}(s,\omega,x) = (2\pi)^{-2}\int_0^{\infty}\int_{-\infty}^{\infty} e^{-(st+i\omega y)}U(t,x,y)dydt$$

the Laplace-Fourier transform of $U(t,x,y)$ satisfying $U|_{t\le 0} = 0$.

Taking the Laplace-Fourier transform on the equations of (1.26) with $f_\pm = 0$ yields

$$\frac{\partial \hat{V}_+}{\partial x} = \frac{1}{d_+^2}\begin{pmatrix} \tilde{s}(u_+ - \sigma) & -\tilde{s}\rho_+ & i\omega\rho_+(u_+ - \sigma) \\ -\frac{\tilde{s}c_+^2}{\rho_+} & \tilde{s}(u_+ - \sigma) & -i\omega c_+^2 \\ \frac{i\omega c_+^2 d_+^2}{\rho_+(\sigma - u_+)} & 0 & \frac{\tilde{s}d_+^2}{\sigma - u_+} \end{pmatrix}\hat{V}_+(s,\omega,x) \tag{2.15}$$

where

$$\tilde{s} = s + i\omega v_0 \quad \text{and} \quad d_+ = \sqrt{c_+^2 - (u_+ - \sigma)^2}.$$

As in [11], if we define $\hat{Z}_+ = (\hat{Z}_1^+, \hat{Z}_2^+, \hat{Z}_3^+)^T$ by

$$\begin{cases} \hat{Z}_1^+ = \hat{V}_{+,3} \\ \hat{Z}_2^+ = \frac{1}{\sqrt{2}}(\hat{V}_{+,2} + \frac{c_+}{\rho_+}\hat{V}_{+,1}) \\ \hat{Z}_3^+ = \frac{1}{\sqrt{2}}(-\hat{V}_{+,2} + \frac{c_+}{\rho_+}\hat{V}_{+,1}) \end{cases} \tag{2.16}$$

then (2.15) is equivalent to

$$\frac{\partial \hat{Z}_+}{\partial x} = N_+(s,\omega)\hat{Z}_+ \tag{2.17}$$

where

$$N_+(s,\omega) = \begin{pmatrix} -\frac{\tilde{s}}{u_+ - \sigma} & -\frac{i\omega c_+}{\sqrt{2}(u_+ - \sigma)} & -\frac{i\omega c_+}{\sqrt{2}(u_+ - \sigma)} \\ -\frac{i\omega c_+}{\sqrt{2}(u_+ - \sigma + c_+)} & -\frac{\tilde{s}}{u_+ - \sigma + c_+} & 0 \\ -\frac{i\omega c_+}{\sqrt{2}(u_+ - \sigma - c_+)} & 0 & -\frac{\tilde{s}}{u_+ - \sigma - c_+} \end{pmatrix}.$$

The eigenvalues of $N_+(s,\omega)$ with negative real parts for $Re s > 0$ are

$$\lambda_1^+ = -\frac{\tilde{s}}{u_+ - \sigma}, \quad \lambda_2^+ = \frac{\tilde{s}(u_+ - \sigma) - c_+(\tilde{s}^2 + \omega^2 d_+^2)^{\frac{1}{2}}}{d_+^2} \tag{2.18}$$

and the corresponding eigenvectors are

$$\begin{cases} \vec{e}_1^{\,+} = (\sqrt{2}\tilde{s},\ i\omega(u_+ - \sigma),\ -i\omega(u_+ - \sigma))^T \\ \vec{e}_2^{\,+} = (-i\sqrt{2}\omega c_+,\ \tilde{s} + \lambda_2^+(u_+ - \sigma - c_+),\ \tilde{s} + \lambda_2^+(u_+ - \sigma + c_+))^T \end{cases} \tag{2.19}$$

Similarly, for the V_- part in (1.26), taking the Laplace-Fourier transform gives

$$\frac{\partial \hat{V}_-}{\partial x} = \frac{1}{d_-^2}\begin{pmatrix} \tilde{s}(\sigma - u_-) & \tilde{s}\rho_- & i\omega\rho_-(\sigma - u_-) \\ \frac{\tilde{s}c_-^2}{\rho_-} & \tilde{s}(\sigma - u_-) & i\omega c_-^2 \\ \frac{i\omega c_-^2 d_-^2}{\rho_-(u_- - \sigma)} & 0 & \frac{\tilde{s}d_-^2}{u_- - \sigma} \end{pmatrix}\hat{V}_-(s,\omega,x) \tag{2.20}$$

where

$$d_- = \sqrt{c_-^2 - (u_- - \sigma)^2}.$$

Set $\hat{Z}_- = (\hat{Z}_1^-, \hat{Z}_2^-, \hat{Z}_3^-)^T$ as

$$\begin{cases} \hat{Z}_1^- &= \hat{V}_{-,3} \\ \hat{Z}_2^- &= \frac{1}{\sqrt{2}}(\hat{V}_{-,2} + \frac{c_-}{\rho_-}\hat{V}_{-,1}) \\ \hat{Z}_3^- &= \frac{1}{\sqrt{2}}(-\hat{V}_{-,2} + \frac{c_-}{\rho_-}\hat{V}_{-,1}). \end{cases} \tag{2.21}$$

It follows that (2.20) is equivalent to

$$\frac{\partial \hat{Z}_-}{\partial x} = N_-(s,\omega)\hat{Z}_- \tag{2.22}$$

where

$$N_-(s,\omega) = \begin{pmatrix} \frac{\tilde{s}}{u_- - \sigma} & \frac{i\omega c_-}{\sqrt{2}(u_- - \sigma)} & \frac{i\omega c_-}{\sqrt{2}(u_- - \sigma)} \\ \frac{i\omega c_-}{\sqrt{2}(u_- - \sigma + c_-)} & \frac{\tilde{s}}{u_- - \sigma + c_-} & 0 \\ \frac{i\omega c_-}{\sqrt{2}(u_- - \sigma - c_-)} & 0 & \frac{\tilde{s}}{u_- - \sigma - c_-} \end{pmatrix}.$$

The eigenvalue of $N_-(s,\omega)$ with a negative real part for $Res > 0$ is

$$\lambda_3^- = \frac{\tilde{s}(\sigma - u_-) - c_-(\tilde{s}^2 + \omega^2 d_-^2)^{\frac{1}{2}}}{d_-^2} \tag{2.23}$$

and the corresponding eigenvector is

$$\vec{e}_3^- = (-i\sqrt{2}\omega c_-,\ \tilde{s} - \lambda_3^-(u_- - \sigma - c_-),\ \tilde{s} - \lambda_3^-(u_- - \sigma + c_-))^T. \tag{2.24}$$

In order to regard the problem (1.26) as a system for $V = (V_+, V_-)^T$, we take the following natural extension of $(\vec{e}_1^+,\ \vec{e}_2^+,\ \vec{e}_3^-)$ in $\mathbb{C}^6$:

$$\begin{cases} \vec{e}_1^+ = (\sqrt{2}\tilde{s},\ i\omega(u_+ - \sigma),\ -i\omega(u_+ - \sigma),\ 0,\ 0,\ 0)^T \\ \vec{e}_2^+ = (-i\sqrt{2}\omega c_+,\ \tilde{s} + \lambda_2^+(u_+ - \sigma - c_+),\ \tilde{s} + \lambda_2^+(u_+ - \sigma + c_+),\ 0,\ 0,\ 0)^T \\ \vec{e}_3^- = (0,\ 0,\ 0,\ -i\sqrt{2}\omega c_-,\ \tilde{s} - \lambda_3^-(u_- - \sigma - c_-),\ \tilde{s} - \lambda_3^-(u_- - \sigma + c_-))^T \end{cases}. \tag{2.25}$$

For these vectors, we have

Proposition 2.2: *$(\vec{e}_1^+, \vec{e}_2^+, \vec{e}_3^-)$ are linearly independent except at $\{(\tilde{s},\omega) | \tilde{s} = \omega(u_+ - \sigma)$ or $\tilde{s} = -\omega(u_+ - \sigma),\ Re\tilde{s} > 0\}$.*

Proof: It suffices to verify that $\vec{e}_1^+$ and $\vec{e}_2^+$ are linearly independent when $(\tilde{s},\omega) \notin \{(\tilde{s},\omega) | \tilde{s} = \omega(u_+ - \sigma)$ or $\tilde{s} = -\omega(u_+ - \sigma),\ Re\tilde{s} > 0\}$.

On the contrary, obviously, $\vec{e}_1^+$ and $\vec{e}_2^+$ are linearly dependent if and only if

$$-\frac{\tilde{s}}{i\omega c_+} = \frac{i\omega(u_+ - \sigma)}{\tilde{s} + \lambda_2^+(u_+ - \sigma - c_+)} = \frac{i\omega(\sigma - u_+)}{\tilde{s} + \lambda_2^+(u_+ - \sigma + c_+)}$$

which is equivalent to

$$-\frac{\tilde{s}}{i\omega} = \frac{i\omega(u_+ - \sigma)(u_+ - \sigma + c_+)}{\tilde{s} + (\tilde{s}^2 + \omega^2 d_+^2)^{1/2}} = \frac{i\omega(u_+ - \sigma)(u_+ - \sigma - c_+)}{\tilde{s} - (\tilde{s}^2 + \omega^2 d_+^2)^{1/2}}.$$

From these identities, we immediately deduce either $\tilde{s} = \omega(u_+ - \sigma)$ for $\omega > 0$ or $\tilde{s} = -\omega(u_+ - \sigma)$ for $\omega < 0$.

Conversely, it is easy to verify that if $(\tilde{s},\omega) \in \{(\tilde{s},\omega) | \tilde{s} = \omega(u_+ - \sigma)$ or $\tilde{s} = -\omega(u_+ - \sigma),\ Re\tilde{s} > 0\}$, then $\vec{e}_1^+$ and $\vec{e}_2^+$ are linearly dependent. ¶

When $\tilde{s}=\omega(u_+-\sigma)$, we should have $\omega>0$ to guarantee $Re\tilde{s}>0$ from the assumption $j=\rho_\pm(u_\pm-\sigma)>0$. In this case, $\lambda_1^+=-\omega$ is the eigenvalue of multiplicity two of $N_+(s,\omega)$, and

$$\vec{e}_1^{\,+}=(\sqrt{2},\ i,\ -i)^T \tag{2.26}$$

is the corresponding eigenvector of $N_+(s,\omega)$.

Let us compute another generalized eigenvector of $N_+(s,\omega)$ with respect to $\lambda_1^+=-\omega$. The definition

$$(-\omega I-N_+(s,\omega))^2\vec{e}_2^{\,+}=0$$

gives rise to

$$\sqrt{2}(u_+-\sigma)\alpha+i(u_+-\sigma-c_+)\beta-i(u_+-\sigma+c_+)\gamma=0$$

for $\vec{e}_2^{\,+}=(\alpha,\beta,\gamma)$. Thus, we can take

$$\vec{e}_2^{\,+}=(u_+-\sigma+c_+,\ 0,\ -i\sqrt{2}(u_+-\sigma))^T \tag{2.27}$$

which is linearly independent of $\vec{e}_1^{\,+}$ given in (2.26).

When $\tilde{s}=-\omega(u_+-\sigma)$ with $\omega<0$, $\lambda_1^+=\omega$ is the eigenvalue of multiplicity two of $N_+(s,\omega)$. In a way similar to (2.27), we obtain that

$$\begin{cases}\vec{e}_1^{\,+}=(\sqrt{2},\ -i,\ i)^T\\ \vec{e}_2^{\,+}=(u_+-\sigma+c_+,\ 0,\ i\sqrt{2}(u_+-\sigma))^T\end{cases} \tag{2.28}$$

are two independent generalized eigenvectors of $N_+(s,\omega)$ with respect to $\lambda_1^+=\omega$.

As in (2.25), we still denote by $(\vec{e}_1^{\,+},\ \vec{e}_2^{\,+},\ \vec{e}_3^{\,-})$ the set of the following vectors in $\mathbb{C}^6$:

$$\vec{e}_3^{\,-}=(0,\ 0,\ 0,\ -i\sqrt{2}\omega c_-,\ \tilde{s}-\lambda_3^-(u_--\sigma-c_-),\ \tilde{s}-\lambda_3^-(u_--\sigma+c_-))^T \tag{2.29}$$

and

$$\begin{cases}\vec{e}_1^{\,+}=(\sqrt{2},\ i,\ -i,\ 0,\ 0,\ 0)^T\\ \vec{e}_2^{\,+}=(u_+-\sigma+c_+,\ 0,\ -i\sqrt{2}(u_+-\sigma),\ 0,\ 0,\ 0)^T\end{cases} \tag{2.30}$$

when $\tilde{s}=\omega(u_+-\sigma)$, or

$$\begin{cases}\vec{e}_1^{\,+}=(\sqrt{2},\ -i,\ i,\ 0,\ 0,\ 0)^T\\ \vec{e}_2^{\,+}=(u_+-\sigma+c_+,\ 0,\ i\sqrt{2}(u_+-\sigma),\ 0,\ 0,\ 0)^T\end{cases} \tag{2.31}$$

when $\tilde{s}=-\omega(u_+-\sigma)$.

Define the space for any $\omega\in\mathbb{R}$ and $s\in\mathbb{C}$ with $Res>0$:

$$E^+(s,\omega)=\begin{cases}span\{\vec{e}_1^{\,+},\vec{e}_2^{\,+},\vec{e}_3^{\,-}\}\ \text{ given in (2.25)}, & \text{if } \tilde{s}^2\neq\omega^2(u_+-\sigma)^2\\ span\{\vec{e}_1^{\,+},\vec{e}_2^{\,+},\vec{e}_3^{\,-}\}\ \text{ given in (2.29)(2.30)}, & \text{if } \tilde{s}=\omega(u_+-\sigma)\\ span\{\vec{e}_1^{\,+},\vec{e}_2^{\,+},\vec{e}_3^{\,-}\}\ \text{ given in (2.29)(2.31)}, & \text{if } \tilde{s}=-\omega(u_+-\sigma)\end{cases} \tag{2.32}$$

where $\tilde{s}=s+i\omega v_0$.

Next, let us study the critical case $\tilde{s}=s+i\omega v_0=0$, which implies $Res=0$ as well. For simplicity, we assume $\omega>0$.

It is easy to know that the eigenvalues of $N_+(s,\omega)$ are

$$\lambda_1^+=0,\quad \lambda_2^+=-\frac{\omega c_+}{d_+},\quad \lambda_3^+=\frac{\omega c_+}{d_+} \tag{2.33}$$

and the eigenvectors with respect to $\lambda_1^+=0$ and $\lambda_2^+=-\frac{\omega c_+}{d_+}<0$ are

$$\vec{e}_1^{\,+}=(0,\ 1,\ -1)^T,\quad \vec{e}_2^{\,+}=\left(1,\ \frac{i(c_+-u_++\sigma)}{\sqrt{2}d_+},\ -\frac{i(c_++u_+-\sigma)}{\sqrt{2}d_+}\right)^T. \tag{2.34}$$

The eigenvalue with a negative real part of $N_-(s,\omega)$ is

$$\lambda_3^- = -\frac{\omega c_-}{d_-} \tag{2.35}$$

and the corresponding eigenvector is

$$\vec{e_3^-} = \left(1,\ -\frac{i(c_- - u_- + \sigma)}{\sqrt{2}d_-},\ \frac{i(c_- + u_- - \sigma)}{\sqrt{2}d_-}\right)^T. \tag{2.36}$$

As before, we denote by $(\vec{e}_1^+,\ \vec{e}_2^+,\ \vec{e}_3^-)$ the following three vectors in $\mathbb{C}^6$

$$\begin{cases} \vec{e}_1^+ = (0,\ 1,\ -1,\ 0,\ 0,\ 0)^T \\ \vec{e}_2^+ = (1,\ \frac{i(c_+ - u_+ + \sigma)}{\sqrt{2}d_+},\ -\frac{i(c_+ + u_+ - \sigma)}{\sqrt{2}d_+},\ 0,\ 0,\ 0)^T \\ \vec{e}_3^- = (0,\ 0,\ 0,\ 1,\ -\frac{i(c_- - u_- + \sigma)}{\sqrt{2}d_-},\ \frac{i(c_- + u_- - \sigma)}{\sqrt{2}d_-})^T \end{cases}. \tag{2.37}$$

When $\tilde{s} = s + i\omega v_0 = 0$ and $\omega > 0$, we define

$$E^+(s,\omega) = span\{\vec{e}_1^+, \vec{e}_2^+, \vec{e}_3^-\} \quad \text{given by (2.37)} \tag{2.38}$$

which is a continuous extension of $E^+(s,\omega)$ defined in (2.32) from $\{Res > 0, \tilde{s} \neq \pm\omega(u_+ - \sigma)\}$ to $\{\tilde{s} = 0, \omega > 0\}$.

In order to use Lemma 2.1 to study the stability of the problem (1.26), let us investigate the boundary condition in (1.26) for the unknown function $\hat{Z} = (\hat{Z}_+, \hat{Z}_-)^T$ introduced in (2.16) and (2.21).

Taking the Laplace-Fourier transform on the boundary condition in (1.26) with $g = 0$ gives

$$(b_0 s + b_1 i\omega)\hat{\phi} + M_+\hat{V}_+ + M_-\hat{V}_- = 0 \quad \text{on } x = 0. \tag{2.39}$$

The first component in (2.39) can be expressed as

$$\tilde{s}\hat{\phi} = \frac{1}{[\rho]}((u_+ - \sigma)\hat{V}_{+,1} - (u_- - \sigma)\hat{V}_{-,1} + \rho_+\hat{V}_{+,2} - \rho_-\hat{V}_{-,2})$$

which is equivalent to the following one by using the transformations (2.16) and (2.21):

$$\begin{aligned} \tilde{s}\hat{\phi} = \frac{1}{\sqrt{2}[\rho]}\{ & \frac{\rho_+}{c_+}(u_+ - \sigma + c_+)\hat{Z}_2^+ + \frac{\rho_+}{c_+}(u_+ - \sigma - c_+)\hat{Z}_3^+ \\ & + \frac{\rho_-}{c_-}(\sigma - u_- - c_-)\hat{Z}_2^- + \frac{\rho_-}{c_-}(\sigma - u_- + c_-)\hat{Z}_3^-\}. \end{aligned} \tag{2.40}$$

(1) When $\tilde{s} \neq 0$, by substituting (2.40) into other components in (2.39), we deduce

$$\begin{cases} \frac{\rho_+}{c_+}[(u_+ - \sigma + c_+)^2\hat{Z}_2^+ + (c_+ + \sigma - u_+)^2\hat{Z}_3^+] \\ \qquad -\frac{\rho_-}{c_-}[(u_- - \sigma + c_-)^2\hat{Z}_2^- + (u_- - \sigma - c_-)^2\hat{Z}_3^-] = 0 \\ (\frac{\rho_+([u] + \tilde{\gamma}\rho_-)}{c_+[\rho]} - 1)(u_+ - \sigma + c_+)\hat{Z}_2^+ + (\frac{\rho_+([u] + \tilde{\gamma}\rho_-)}{c_+[\rho]} + 1)(u_+ - \sigma - c_+)\hat{Z}_3^+ \\ \qquad -(\frac{\rho_-([u] + \tilde{\gamma}\rho_+)}{c_-[\rho]} - 1)(u_- - \sigma + c_-)\hat{Z}_2^- - (\frac{\rho_-([u] + \tilde{\gamma}\rho_+)}{c_-[\rho]} + 1)(u_- - \sigma - c_-)\hat{Z}_3^- = 0 \\ \rho_+(u_+ - \sigma)\hat{Z}_1^+ + \rho_-(\sigma - u_-)\hat{Z}_1^- - \frac{i\omega[p]}{\sqrt{2}\tilde{s}[\rho]}(\frac{\rho_+}{c_+}(u_+ - \sigma + c_+)\hat{Z}_2^+ \\ \qquad + \frac{\rho_+}{c_+}(u_+ - \sigma - c_+)\hat{Z}_3^+ + \frac{\rho_-}{c_-}(\sigma - u_- - c_-)\hat{Z}_2^- + \frac{\rho_-}{c_-}(\sigma - u_- + c_-)\hat{Z}_3^-) = 0 \end{cases} \tag{2.41}$$

which can be formulated as a matrix form

$$B\hat{Z} = 0 \qquad \text{on } x = 0 \tag{2.42}$$

where the 3×6 matrix B is composed of coefficients in (2.41) explicitly.

(2) When $\tilde{s} = 0$ and $\omega \neq 0$, the third component of (2.39) is

$$\begin{aligned} i\omega[p]\hat{\phi} = & (u_+ - \sigma)v_0\hat{V}_{+,1} + \rho_+ v_0 \hat{V}_{+,2} + \rho_+(u_+ - \sigma)\hat{V}_{+,3} \\ & -v_0(u_- - \sigma)\hat{V}_{-,1} - \rho_- v_0 \hat{V}_{-,2} - \rho_-(u_- - \sigma)\hat{V}_{-,3} \end{aligned} \tag{2.43}$$

and other components of (2.39) are independent of $\hat{\phi}$, they can be expressed as follows:

$$\begin{cases} \frac{\rho_+}{c_+}[(u_+ - \sigma + c_+)\hat{Z}_2^+ + (u_+ - \sigma - c_+)\hat{Z}_3^+] \\ \qquad -\frac{\rho_-}{c_-}[(u_- - \sigma + c_-)\hat{Z}_2^- + (u_- - \sigma - c_-)\hat{Z}_3^-] = 0 \\ \frac{\rho_+}{c_+}[(u_+ + c_+)(u_+ - \sigma + c_+)\hat{Z}_2^+ + (u_+ - c_+)(u_+ - \sigma - c_+)\hat{Z}_3^+] \\ \qquad -\frac{\rho_-}{c_-}[(u_- + c_-)(u_- - \sigma + c_-)\hat{Z}_2^- + (u_- - c_-)(u_- - \sigma - c_-)\hat{Z}_3^-] = 0 \\ (u_+ - \sigma + c_+)\hat{Z}_2^+ - (u_+ - \sigma - c_+)\hat{Z}_3^+ \\ \qquad +(\frac{\tilde{\gamma}\rho_-}{c_-} - 1)(u_- - \sigma + c_-)\hat{Z}_2^- + (\frac{\tilde{\gamma}\rho_-}{c_-} + 1)(u_- - \sigma - c_-)\hat{Z}_3^- = 0 \end{cases} \tag{2.44}$$

by using the transformations (2.16) and (2.21) again. Certainly, (2.44) can be represented as

$$B\hat{Z} = 0 \qquad \text{on } x = 0 \tag{2.45}$$

with B being a 3×6 matrix.

In the remainder of this section, we want to compute the Lopatinski determinant for three cases $\tilde{s}^2 \neq \omega^2(u_+ - \sigma)^2$, $\tilde{s}^2 = \omega^2(u_+ - \sigma)^2$ and $\tilde{s} \neq 0$ separately.

2.2.2 The Lopatinski determinant for the case $\tilde{s}^2 \neq \omega^2(u_+ - \sigma)^2$ and $\tilde{s} \neq 0$.

For any $(\tilde{s}, \omega) \in \{\tilde{s}^2 \neq \omega^2(u_+ - \sigma)^2, Re\tilde{s} > 0, |\tilde{s}|^2 + \omega^2 = 1\}$, the Lopatinski determinant is

$$\Delta = \det(B(\vec{e}_1^{\,+}, \vec{e}_2^{\,+}, \vec{e}_3^{\,-})) \tag{2.46}$$

where B and $\{\vec{e}_1^{\,+}, \vec{e}_2^{\,+}, \vec{e}_3^{\,-}\}$ are given by (2.42) and (2.25) respectively. By direct computation, we obtain

$$\frac{\Delta}{4\sqrt{2}\rho_+^2\rho_-(u_+ - \sigma)} = \begin{vmatrix} 2i\omega(u_+ - \sigma) & \tilde{s}c_+ + (u_+ - \sigma)\sqrt{\tilde{s}^2 + \omega^2 d_+^2} & -(\tilde{s}c_- + (\sigma - u_-)\sqrt{\tilde{s}^2 + \omega^2 d_-^2}) \\ i\omega(\frac{[u] + \tilde{\gamma}\rho_-}{[\rho]} - \frac{u_+ - \sigma}{\rho_+}) & \frac{([u] + \tilde{\gamma}\rho_-)\sqrt{\tilde{s}^2 + \omega^2 d_+^2}}{[\rho]} - \frac{c_+\tilde{s}}{\rho_+} & \frac{([u] + \tilde{\gamma}\rho_+)\sqrt{\tilde{s}^2 + \omega^2 d_-^2}}{[\rho]} + \frac{c_-\tilde{s}}{\rho_-} \\ \tilde{s} + \frac{\omega^2[p]}{\tilde{s}[\rho]} & -i\omega(c_+(u_+ - \sigma) + \frac{[p]\sqrt{\tilde{s}^2 + \omega^2 d_+^2}}{\tilde{s}[\rho]}) & -i\omega(c_-(\sigma - u_-) + \frac{[p]\sqrt{\tilde{s}^2 + \omega^2 d_-^2}}{\tilde{s}[\rho]}) \end{vmatrix}. \tag{2.47}$$

By using the Rankine-Hugoniot condition (1.8), and a length computation for (2.47), we deduce

$$\frac{\Delta}{4\sqrt{2}\rho_+^2\rho_-(u_+ - \sigma)} = (\tilde{s}^2 - \omega^2(u_+ - \sigma)^2)(I + \tilde{\gamma}II) \tag{2.48}$$

where

$$I = \frac{[u]^2}{\tilde{s}[\rho]}\sqrt{(\tilde{s}^2+\omega^2 d_-^2)(\tilde{s}^2+\omega^2 d_+^2)} + \frac{[\rho]c_+c_-\tilde{s}}{\rho_+\rho_-} \tag{2.49}$$

and

$$II = \frac{\rho_+c_+}{[\rho]}\sqrt{\tilde{s}^2+\omega^2 d_-^2} + \frac{\rho_-c_-}{[\rho]}\sqrt{\tilde{s}^2+\omega^2 d_+^2} - \frac{\omega^2\rho_- d_+^2[u](\tilde{s}c_- + (u_- - \sigma)\sqrt{\tilde{s}^2+\omega^2 d_-^2})}{\tilde{s}[\rho](\tilde{s}c_+ + (u_+ - \sigma)\sqrt{\tilde{s}^2+\omega^2 d_+^2})}. \tag{2.50}$$

We claim that I is nonzero when $Re\tilde{s} > 0$. Indeed, if $I = 0$, then we have

$$(\tilde{s}^2+\omega^2 d_-^2)(\tilde{s}^2+\omega^2 d_+^2) = \left(\frac{c_+c_-}{(u_+-\sigma)(u_--\sigma)}\right)^2 \tilde{s}^4$$

which implies

$$\tilde{s}^2 = \frac{\omega^2(d_+^2+d_-^2) \pm \omega^2\sqrt{(d_+^2+d_-^2)^2 + 4d_+^2 d_-^2((M_+M_-)^{-2}-1)}}{2((M_+M_-)^{-2}-1)} \tag{2.51}$$

with $M_+ = \frac{u_+-\sigma}{c_+}$, $M_- = \frac{u_--\sigma}{c_-}$ being the Mach numbers. From the subsonic property of the phase transition (1.25), we have

$$0 < M_+, \quad M_- < 1.$$

Due to $Re\tilde{s} > 0$, we deduce that one should take the plus sign in (2.51), which is not the root of I obviously. Thus, I is always nonzero, which gives there exist constants $M_1 > 0$, and $\tilde{\gamma}_0 > 0$, such that for any $0 \le \tilde{\gamma} \le \tilde{\gamma}_0$, we have

$$\left|\frac{\Delta}{4\sqrt{2}\rho_+^2\rho_-(u_+-\sigma)(\tilde{s}^2-\omega^2(u_+-\sigma)^2)}\right| \ge M_1 > 0. \tag{2.52}$$

Therefore, we obtain

Proposition 2.3 *For any $(\tilde{s},\omega) \in \{\tilde{s}^2 \neq \omega^2(u_+-\sigma)^2, Re\tilde{s} > 0, |\tilde{s}|^2 + |\omega|^2 = 1\}$, there is $\gamma_0 > 0$ such that the Lopatinski determinant given in (2.46) does not vanish for any $0 < \gamma \le \gamma_0$.*

2.2.3 The Lopatinski determinant for the case $\tilde{s}^2 = \omega^2(u_+ - \sigma)^2$

Let us first consider the case $\tilde{s} = \omega(u_+ - \sigma)$ with $\omega > 0$.

For the boundary matrix B and the basis $\{\vec{e}_1^{\,+}, \vec{e}_2^{\,+}, \vec{e}_3^{\,-}\}$ of $E^+(s,\omega)$ given in (2.42) and (2.30)(2.29) respectively, the Lopatinski determinant is

$$\Delta = \det(B(\vec{e}_1^{\,+}, \vec{e}_2^{\,+}, \vec{e}_3^{\,-})) \tag{2.53}$$

which gives rise to

$$\frac{\Delta}{2\rho_+(u_+-\sigma)} = \left|\begin{matrix} i2\sqrt{2}\rho_+(u_+-\sigma) & -i\frac{\rho_+}{c_+}(u_+-\sigma)(u_+-\sigma-c_+)^2 & -\sqrt{2}\rho_-(\tilde{s}c_- - (u_--\sigma)b) \\ i\sqrt{2}(a_+-u_++\sigma) & -i(u_+-\sigma)(u_+-\sigma-c_+)(\frac{a_+}{c_+}+1) & \sqrt{2}(c_-\tilde{s}+a_-b) \\ \sqrt{2}(1+\frac{\rho_+}{\rho_-}) & u_+-\sigma+c_+-\frac{(u_--\sigma)(u_+-\sigma-c_+)}{c_+} & i\sqrt{2}(\omega c_- - b) \end{matrix}\right| \tag{2.54}$$

with $a_+ = \frac{\rho_+([u|+\tilde{\gamma}\rho_-)}{|\rho|}$, $a_- = \frac{\rho_-([u|+\tilde{\gamma}\rho_+)}{|\rho|}$ and $b = (\tilde{s}^2 + \omega^2 d_-^2)^{\frac{1}{2}}$.

By a direct computation and using the Rankine-Hugoniot condition (1.8), we get

$$\frac{\Delta}{4i\rho_+^2(u_+ - \sigma)^2} = I + \tilde{\gamma} II \tag{2.55}$$

where

$$I = 2c_+[u](\sqrt{\tilde{s}^2 + \omega^2 d_-^2} + \frac{\rho_-}{\rho_+}\omega c_-) \tag{2.56}$$

and II is a bounded term depending on $(u_\pm, \rho_\pm)$.

Therefore, it immediately follows:

Proposition 2.4: *For the case $\tilde{s} = \omega(u_+ - \sigma)$ with $\omega > 0$, there is $\gamma_0 > 0$ such that when $0 < \gamma \leq \gamma_0$, the Lopatinski determinant given in (2.53) does not vanish.*

Next, we study the case $\tilde{s} = -\omega(u_+ - \sigma)$ with $\omega < 0$.

As before, for the boundary matrix B and the basis $\{\vec{e}_1^+, \vec{e}_2^+, \vec{e}_3^-\}$ of $E^+(s,\omega)$ given in (2.42) and (2.31)(2.29) respectively, the Lopatinski determinant is

$$\Delta_1 = \det(B(\vec{e}_1^+, \vec{e}_2^+, \vec{e}_3^-)) \tag{2.57}$$

which gives rise to

$$\frac{\Delta_1}{2\rho_+(u_+ - \sigma)} = \begin{vmatrix} -i2\sqrt{2}\rho_+(u_+ - \sigma) & i\frac{\rho_\pm}{c_+}(u_+ - \sigma)(u_+ - \sigma - c_+)^2 & -\sqrt{2}\rho_-(\tilde{s}c_- - (u_- - \sigma)b) \\ -i\sqrt{2}(a_+ - u_+ + \sigma) & i(u_+ - \sigma)(u_+ - \sigma - c_+)(\frac{a_\pm}{c_+} + 1) & \sqrt{2}(c_-\tilde{s} + a_- b) \\ \sqrt{2}(1 + \frac{\rho_\pm}{\rho_-}) & u_+ - \sigma + c_+ - \frac{(u_- - \sigma)(u_+ - \sigma - c_+)}{c_+} & i\sqrt{2}(\omega c_- + b) \end{vmatrix} \tag{2.58}$$

with $a_+ = \frac{\rho_+([u|+\tilde{\gamma}\rho_-)}{|\rho|}$, $a_- = \frac{\rho_-([u|+\tilde{\gamma}\rho_+)}{|\rho|}$ and $b = (\tilde{s}^2 + \omega^2 d_-^2)^{\frac{1}{2}}$.

If setting $\tilde{\omega} = -\omega$, then, obviously we have

$$\Delta_1 = -\Delta$$

where Δ is the determinant given in (2.53) with $\tilde{s} = \tilde{\omega}(u_\pm - \sigma)$ and $b = (\tilde{s}^2 + \tilde{\omega}^2 d_-^2)^{\frac{1}{2}}$. Thus, from (2.55) and (2.56), we immediately conclude

$$\Delta_1 = -4i\rho_+^2(u_+ - \sigma)^2 I + O(\tilde{\gamma}) \tag{2.59}$$

with

$$I = 2c_+[u](\sqrt{\tilde{s}^2 + \tilde{\omega}^2 d_-^2} + \frac{\rho_-}{\rho_+}\tilde{\omega} c_-) \tag{2.60}$$

having the same sige as that of $[u]$. Therefore, we obtain:

Proposition 2.5: *For the case $\tilde{s} = -\omega(u_+ - \sigma)$ with $\omega < 0$, there is $\gamma_0 > 0$ such that when $0 < \gamma \leq \gamma_0$, the Lopatinski determinant given in (2.57) does not vanish.*

2.2.4 The Lopatinski determinant for the case $\tilde{s} = 0$

With the boundary matrix B and the basis $\{\vec{e}_1^+, \vec{e}_2^+, \vec{e}_3^-\}$ of $E^+(s,\omega)$ given in (2.45) and (2.37) respectively, the Lopatinski determinant is

$$\Delta = \det(\tilde{B}(\vec{e}_1^+, \vec{e}_2^+, \vec{e}_3^-)) = \begin{vmatrix} 2\rho_+ & \frac{i\sqrt{2}d_+\rho_+}{c_+} & \frac{i\sqrt{2}d_-\rho_-}{c_-} \\ 2\rho_+(2u_+ - \sigma) & \frac{i\sqrt{2}d_+\rho_+u_+}{c_+} & \frac{i\sqrt{2}d_-\rho_-u_-}{c_-} \\ 2(u_+ - \sigma) & 0 & -\frac{i\sqrt{2}\tilde{\gamma}d_-\rho_-}{c_-} \end{vmatrix}$$

which gives

$$\Delta = \frac{4d_+d_-\rho_+\rho_-}{c_+c_-}(u_+ - \sigma)([u] - \tilde{\gamma}\rho_+). \tag{2.61}$$

Thus, we immediately obtain

Proposition 2.6: *For the case $\check{s} = 0$, there is $\gamma_0 > 0$ such that when $0 < \gamma \leq \gamma_0$, the Lopatinski determinant (2.61) does not vanish.*

Together all results from Propositions 2.3, 2.4, 2.5 to 2.6, noting that the left hand side of (1.29) is homogeneous in (μ, V_+, V_-), and using Lemma 2.1 we obtain the conclusion of Theorem 1.2 immediately. ¶

Remark 2.7: From the above discussion, one sees that the constant γ_0 depends on the existence of the viscosity-capillarity profile $\tau(\xi; j, \gamma)$ $(0 < \gamma \leq \gamma_0)$ to the problem (1.13), which was given by Benzoni-Gavage in [3], and on the bounds of the viscosity-capillarity admissible phase transition $(U_\pm, \sigma)$ given in (1.25) such that Propositions 2.3, 2.4, 2.5 and 2.6 hold.

3. The Existence of Multidimensional Phase Transitions

In this section, we shall use the uniform stability obtained in §2 to estabilish the local existence of a subsonic phase transition. The main idea will follow the argument of Majda in [12] for the existence of multidimensional shock fronts, so we shall only sketch the main parts of the proof.

Consider the following Cauchy problem for the Euler equations (1.2) in two space variables:

$$\begin{cases} \partial_t F_0(U) + \partial_x F_1(U) + \partial_y F_2(U) = 0 \\ U(0,x,y) = \begin{cases} U_+^0(x,y), & x > \varphi_0(y) \\ U_-^0(x,y), & x < \varphi_0(y) \end{cases} \end{cases}. \tag{3.1}$$

We are going to construct a local solution to (3.1) in the form

$$U(t,x,y) = \begin{cases} U_+(t,x,y), & x > \varphi(t,y) \\ U_-(t,x,y), & x < \varphi(t,y) \end{cases} \tag{3.2}$$

which is a subsonic, viscosity-capillarity admissible phase transition.

To have such a phase transition solution, we make the following main assumptions on (3.1):

(MA1) for any fixed $(\underline{x}, \underline{y}) \in \Sigma_0 = \{x = \varphi_0(y)\}$, there exists $\sigma(\underline{y}) \in \mathbb{R}$ such that the problem (3.1) with the initial data frozen at $(\underline{x}, \underline{y})$, admits a planar subsonic phase transition:

$$\underline{U}(t,x,y) = \begin{cases} U_+^0(\varphi_0(\underline{y}), \underline{y}), & x > \varphi_0(\underline{y}) + \sigma(\underline{y})t \\ U_-^0(\varphi_0(\underline{y}), \underline{y}), & x < \varphi_0(\underline{y}) + \sigma(\underline{y})t \end{cases} \tag{3.3}$$

satisfying the viscosity-capillarity criterion.

It follows from Theorems 1.1 and 1.2 that the phase transition (3.3) is one-dimensional stable and multi-dimensional uniformly stable as well.

As in §1, the phase transition (U_+, U_-, φ) should solve the following problem

$$\begin{cases} \partial_t F_0(U_\pm) + \partial_x F_1(U_\pm) + \partial_y F_2(U_\pm) = 0, \quad \pm(x - \varphi(t,y)) > 0 \\ \varphi_t[F_0(U)] - [F_1(U)] + \varphi_y[F_2(U)] = 0, \quad \text{on} \quad x = \varphi(t,y) \\ [e'(U_1) + \frac{(U_2 - \varphi_y U_3 - \varphi_t)^2}{2(1+\varphi_y^2)}] = -\gamma j \int_{-\infty}^{\infty} \tau'^2(\xi; j, \gamma) d\xi, \quad \text{on} \quad x = \varphi(t,y) \\ U_\pm(0,x,y) = U_\pm^0(x,y), \quad \varphi(0,y) = \varphi_0(y) \end{cases} \tag{3.4}$$

where $j = U_{\pm,1}(U_{\pm,2}-\varphi_y U_{\pm,3}-\varphi_t)/\sqrt{1+\varphi_y^2}|_{x=\varphi(t,y)}$ is the mass transfer flux, and $\tau(\xi; j, \gamma)$ is the solution to the following problem for the profile equation:

$$\begin{cases} \tau'' = \gamma j \tau' + \pi - p(\tau^{-1}) - j^2\tau \\ \lim_{\xi\to-\infty} \tau = \frac{1}{U_{-,1}}|_{x=\varphi(t,y)}, \quad \lim_{\xi\to+\infty} \tau = \frac{1}{U_{+,1}}|_{x=\varphi(t,y)} \end{cases}$$

with $\pi = (p(U_{\pm,1}) + \frac{j^2}{U_{\pm,1}})|_{x=\varphi(t,y)}$.

To solve the problem (3.4), as in (1.21), we introduce

$$\begin{cases} \tilde{x} = \pm(x - \varphi(t,y)) \\ \tilde{y} = y \\ \tilde{t} = t \\ \tilde{U}_\pm(\tilde{t}, \tilde{x}, \tilde{y}) = U_\pm(t, x, y) \end{cases}, \tag{3.5}$$

so that $\tilde{U}_\pm(\tilde{t}, \tilde{x}, \tilde{y})$ solve the problem

$$\begin{cases} \partial_t U_\pm \pm (A_1(U_\pm) - \varphi_y A_2(U_\pm) - \varphi_t)\partial_x U_\pm + A_2(U_\pm)\partial_y U_\pm = 0, \quad t,\ x > 0 \\ \varphi_t(F_0(U_+) - F_0(U_-)) - (F_1(U_+) - F_1(U_-)) + \varphi_y(F_2(U_+) - F_2(U_-)) = 0, \quad x = 0 \\ [e'(U_1) + \frac{(U_2 - \varphi_y U_3 - \varphi_t)^2}{2(1+\varphi_y^2)}] = -\gamma j \int_{-\infty}^{\infty} \tau'^2(\xi; j, \gamma) d\xi, \quad x = 0 \\ U_\pm(0,x,y) = U_\pm^0(x,y), \quad \varphi(0,y) = \varphi_0(y) \end{cases} \tag{3.6}$$

where we have dropped tildes for simplicity, $A_1(U) = (F_0'(U))^{-1}F_1'(U)$ and $A_2(U) = (F_0'(U))^{-1}F_2'(U)$.

Notations: In the following discussion, we will always let the integer $s \geq 9$, ω be the part of a neighborhood of the origin in $\{t = 0, x > 0\}$, $I = \overline{\omega} \cap \{x = 0\}$, $\Omega \subset \{t, x > 0\}$ be a determinacy domain of ω with respect to the problem (3.6) when

$$\sup_{\overline{\Omega}_{T_0}} (|U_\pm - U_\pm^0| + |\varphi - \varphi_0| + |\partial_y(\varphi - \varphi_0)| + |\varphi_t - \sigma(y)|) \leq 1$$

with $\Omega_T = \Omega \cap \{t < T\}$, $\omega_\tau = \Omega \cap \{t = \tau\}$ and $b\Omega_T = \overline{\Omega}_T \cap \{x = 0\}$.

As usual, to obtain smooth solutions to the problem (3.6), it is necessary to impose certain compatibility conditions on initial data.

3.1 Compatibility conditions

The derivation of compatibility conditions follows the classical argument for standard hyperbolic mixed problems utilizing the formal Cauchy-Kowaleski computations. Here, we assume that there is a smooth solution (U_+, U_-, φ) to (3.6), and derive the formal compatibility conditions up to order $s-1$, s a given positive integer, which must be satisfied by the initial data. In the next subsection, we shall show there are large classes of initial data satisfying these compatibility conditions.

From (MA1) and (3.6), the *zero-th order compatibility condition* is that at $x = 0$, the initial data satisfy

$$\begin{cases} \sigma(y)(F_0(U_+^0) - F_0(U_-^0)) - F_1(U_+^0) + F_1(U_-^0) + \varphi_0'(y)(F_2(U_+^0) - F_2(U_-^0)) = 0 \\ e'(U_{+,1}^0) - e'(U_{-,1}^0) + [\frac{(U_2^0 - \varphi_0'(y)U_3^0 - \sigma(y))^2}{2(1+\varphi_0'^2(y))}] = -\gamma j_0 \int_{-\infty}^{\infty} \tau_0'^2(\xi; j_0, \gamma) d\xi \end{cases} \tag{3.7}$$

where

$$j_0 = U_{\pm,1}^0(U_{\pm,2}^0 - \varphi_0'(y)U_{\pm,3}^0 - \sigma(y))/\sqrt{1+\varphi_0'^2(y)}|_{x=0}$$

and $\tau_0(\xi; j_0, \gamma)$ satisfies

$$\begin{cases} \tau_0'' = \gamma j_0 \tau_0' + \pi_0 - p(\tau_0^{-1}) - j_0^2 \tau_0 \\ \lim_{\xi\to -\infty} \tau_0 = \frac{1}{U_{-,1}^0}|_{x=0}, \quad \lim_{\xi\to +\infty} \tau_0 = \frac{1}{U_{+,1}^0}|_{x=0} \end{cases}$$

with $\pi_0 = p(U_{\pm,1}^0) + \frac{j_0^2}{U_{\pm,1}^0}|_{x=0}$.

Next, let us derive relations among $\partial_t^{k+1}\varphi|_{t=0}$ and $\partial_x^k U_\pm^0|_{x=0}$ recursively.

Differentiating the second equation in (3.6) with respect to t, it follows

$$\partial_t^2\varphi(F_0(U_+^0) - F_0(U_-^0)) + F_0'(U_+^0)(\sigma(y) - A_1(U_+^0) + \varphi_0'(y)A_2(U_+^0))\partial_t U_+$$
$$-F_0'(U_-^0)(\sigma(y) - A_1(U_-^0) + \varphi_0'(y)A_2(U_-^0))\partial_t U_- = \sigma'(y)(F_2(U_-^0) - F_2(U_+^0)) \quad (3.8)$$

at $t = x = 0$.

Denote by

$$\alpha(j,\gamma) = \frac{\partial}{\partial j}(j\int_{-\infty}^{\infty} \tau'^2(\xi; j,\gamma)d\xi). \quad (3.9)$$

Differentiating the third equation in (3.6) with respect to t, we get

$$a_0(y)\partial_t^2\varphi + l_+^0(y)\partial_t U_+ + l_-^0(y)\partial_t U_- = g_1 \quad (3.10)$$

at $t = x = 0$, where

$$\begin{cases} a_0(y) = \varphi_0'(y)(U_{+,3}^0 - U_{-,3}^0) + U_{-,2}^0 - U_{+,2}^0 - \gamma\alpha(j_0,\gamma)U_{-,1}^0\sqrt{1+\varphi_0'^2(y)} \\ l_+^0(y) = ((1+\varphi_0'^2(y))e''(U_{+,1}^0),\ U_{+,2}^0 - \varphi_0'(y)U_{+,3}^0 - \sigma(y),\ \varphi_0'(y)(\varphi_0'(y)U_{+,3}^0 + \sigma(y) - U_{+,2}^0)) \\ l_-^0(y) = (\gamma\alpha(j_0,\gamma)\sqrt{1+\varphi_0'^2(y)}(U_{-,2}^0 - \varphi_0'(y)U_{-,3}^0 - \sigma(y)) - (1+\varphi_0'^2(y))e''(U_{-,1}^0), \\ \qquad \gamma\alpha(j_0,\gamma)U_{-,1}^0\sqrt{1+\varphi_0'^2(y)} + \varphi_0'(y)U_{-,3}^0 + \sigma(y) - U_{-,2}^0, \\ \qquad \varphi_0'(y)(U_{-,2}^0 - \varphi_0'(y)U_{-,3}^0 - \sigma(y) - \gamma\alpha(j_0,\gamma)U_{-,1}^0\sqrt{1+\varphi_0'^2(y)})) \end{cases}$$

and

$$g_1 = \gamma\alpha(j_0,\gamma)\sigma'(y)U_{-,1}^0(\sqrt{1+\varphi_0'^2(y)}U_{-,3}^0 + \frac{\varphi_0'(y)}{\sqrt{1+\varphi_0'^2(y)}}(U_{-,2}^0 - \varphi_0'(y)U_{-,3}^0 - \sigma(y)))$$
$$+\sigma'(y)[U_3^0(U_2^0 - \varphi_0'(y)U_3^0 - \sigma(y))] + \frac{\varphi_0'(y)\sigma'(y)}{1+\varphi_0'^2(y)}[(U_2^0 - \varphi_0'(y)U_3^0 - \sigma(y))^2].$$

On the other hand, from the equations of $U_\pm$ in (3.6) it follows

$$\partial_t U_\pm|_{t=0} = \pm(\sigma(y) + \varphi_0'(y)A_2(U_\pm^0) - A_1(U_\pm^0))\partial_x U_\pm^0 - A_2(U_\pm^0)\partial_y U_\pm^0. \quad (3.11)$$

Substituting (3.11) into (3.8) and (3.10), we obtain the following *first order compatibility condition* for the problem (3.6) at $t = x = 0$:

$$\begin{cases} (F_0(U_+^0) - F_0(U_-^0))\partial_t^2\varphi + F_0'(U_+^0)(\sigma(y) + \varphi_0'(y)A_2(U_+^0) - A_1(U_+^0))^2\partial_x U_+^0 \\ \qquad +F_0'(U_-^0)(\sigma(y) + \varphi_0'(y)A_2(U_-^0) - A_1(U_-^0))^2\partial_x U_-^0 = f_1^{(1)} \\ a_0(y)\partial_t^2\varphi + l_+^0(y)(\sigma(y) + \varphi_0'(y)A_2(U_+^0) - A_1(U_+^0))\partial_x U_+^0 \\ \qquad -l_-^0(y)(\sigma(y) + \varphi_0'(y)A_2(U_-^0) - A_1(U_-^0))\partial_x U_-^0 = f_2^{(1)} \end{cases} \quad (3.12)$$

where

$$\begin{cases} f_1^{(1)} = \sigma'(y)(F_2(U_-^0) - F_2(U_+^0)) + F_0'(U_+^0)(\sigma(y) + \varphi_0'(y)A_2(U_+^0) - A_1(U_+^0))A_2(U_+^0)\partial_y U_+^0 \\ \qquad -F_0'(U_-^0)(\sigma(y) + \varphi_0'(y)A_2(U_-^0) - A_1(U_-^0))A_2(U_-^0)\partial_y U_-^0 \\ f_2^{(1)} = g_1 + l_+^0(y)A_2(U_+^0)\partial_y U_+^0 + l_-^0(y)A_2(U_-^0)\partial_y U_-^0 \end{cases}. \quad (3.13)$$

Here we note that the right hand sides of (3.12), $f_1^{(1)}$ and $f_2^{(1)}$ depend only upon $U_\pm^0$ and their first order tangential derivatives at $\Sigma_0 = \{x = t = 0\}$.

Similarly, for any fixed $k \in I\!N$, by acting ∂_t^k on the boundary conditions in (3.6) and using the equations of $U_\pm$ in (3.6) to replace $\partial_t^k U_\pm$ by the normal and tangential derivatives of $U_\pm$ at Σ_0, it leads to the following *$k-th$ order compatibility conditions*:

$$\begin{cases} (F_0(U_+^0) - F_0(U_-^0))\partial_t^{k+1}\varphi + F_0'(U_+^0)(\sigma(y) + \varphi_0'(y)A_2(U_+^0) - A_1(U_+^0))^{k+1}\partial_x^k U_+^0 \\ \qquad + F_0'(U_-^0)(A_1(U_-^0) - \sigma(y) - \varphi_0'(y)A_2(U_-^0))^{k+1}\partial_x^k U_-^0 = f_1^{(k)} \\ a_0(y)\partial_t^{k+1}\varphi + l_+^0(y)(\sigma(y) + \varphi_0'(y)A_2(U_+^0) - A_1(U_+^0))^k \partial_x^k U_+^0 \\ \qquad + l_-^0(y)(A_1(U_-^0) - \sigma(y) - \varphi_0'(y)A_2(U_-^0))^k \partial_x^k U_-^0 = f_2^{(k)} \end{cases} \tag{3.14}$$

at $t = x = 0$, where $f_1^{(k)}$ and $f_2^{(k)}$ smoothly depend on $\{\partial_y^l \partial_t^j \varphi|_{t=0} : 0 \le j \le k, l+j \le k+1\}$ and $\{\partial_y^l \partial_x^j U_\pm^0|_{x=0} : 0 \le j \le k-1, l+j \le k\}$.

3.2 Large classes of initial data satisfying the compatibility conditions

Denote by $\lambda_k^\pm(y)$ $(k = 1, 2, 3)$ the eigenvalues of $(A_1(U_\pm^0) - \varphi_0'(y)A_2(U_\pm^0) - \sigma(y))|_{x=0}$. Then by simple computation, we have

$$\begin{cases} \lambda_1^\pm(y) = u_\pm^0 - \sigma(y) - \varphi_0'(y)v_\pm^0 - \sqrt{1+\varphi_0'^2(y)}c_\pm^0 \\ \lambda_2^\pm(y) = u_\pm^0 - \sigma(y) - \varphi_0'(y)v_\pm^0 \\ \lambda_3^\pm(y) = u_\pm^0 - \sigma(y) - \varphi_0'(y)v_\pm^0 + \sqrt{1+\varphi_0'^2(y)}c_\pm^0 \end{cases}$$

where $c_\pm^0 = (p'(\rho_\pm^0))^{\frac{1}{2}}$.

Without loss of generality, we assume that the initial mass transfer flux

$$j_0(y) = \rho_\pm^0(u_\pm^0 - \varphi_0'(y)v_\pm^0 - \sigma(y))/\sqrt{1+\varphi_0'^2(y)}|_{x=0} \tag{3.15}$$

is positive, then we have

$$\lambda_1^\pm(y) < 0 < \lambda_2^\pm(y) < \lambda_3^\pm(y) \tag{3.16}$$

by using the subsonic property of the planar phase transition given in (MA1).

Denote by $P^+(y)$ and $P^-(y)$ the smoothly varying projections onto the subspaces spanned by the eigenvectors associated with eigenvalues $\lambda_2^+(y)$, $\lambda_3^+(y)$ of

$$(A_1(U_+^0) - \varphi_0'(y)A_2(U_+^0) - \sigma(y))|_{x=0}$$

and with $\lambda_1^-(y)$ of

$$(A_1(U_-^0) - \varphi_0'(y)A_2(U_-^0) - \sigma(y))|_{x=0}.$$

Similar to the Lemma 2.1 of Majda in [12], we have the following result:

Lemma 3.1: *If $(v^+, v^-) \in I\!R^3 \times I\!R^3$ satisfies*

$$P^+(y)v^+(y) = v^+(y), \quad P^-(y)v^-(y) = v^-(y) \tag{3.17}$$

and β is a constant, then from the identity

$$M(\beta, v^+, v^-) = \beta \begin{pmatrix} [F_0(U^0)] \\ a_0(y) \end{pmatrix} + \begin{pmatrix} F_0'(U_+^0)(\sigma(y) + \varphi_0'(y)A_2(U_+^0) - A_1(U_+^0))^{k+1} \\ l_+^0(y)(\sigma(y) + \varphi_0'(y)A_2(U_+^0) - A_1(U_+^0))^k \end{pmatrix} v^+$$

$$+\begin{pmatrix} F_0'(U_-^0)(A_1(U_-^0)-\sigma(y)-\varphi_0'(y)A_2(U_-^0))^{k+1} \\ l_-^0(y)(A_1(U_-^0)-\sigma(y)-\varphi_0'(y)A_2(U_-^0))^k \end{pmatrix} v^- = 0 \tag{3.18}$$

we should have $(\beta, v^+, v^-) = 0$, where $a_0(y), l_+^0(y)$ and $l_-^0(y)$ are given in (3.10).

Proof: Let $r_k^\pm(y)$ $(k=1,2,3)$ be the eigenvectors of $(A_1(U_\pm^0)-\varphi_0'(y)A_2(U_\pm^0)-\sigma(y))|_{x=0}$ with respect to $\lambda_k^\pm(y)$, then the basis of the set

$$\{(\beta, v^+, v^-) |\ P^+(y)v^+(y) = v^+(y), P^-(y)v^-(y) = v^-(y)\} \tag{3.19}$$

is given by

$$(1,0,0)\cup(0,r_2^+(y),0)\cup(0,r_3^+(y),0)\cup(0,0,r_1^-(y)). \tag{3.20}$$

From definitions, obviously we have

$$M(1,0,0) = \begin{pmatrix} [F_0(U^0)] \\ a_0(y) \end{pmatrix}$$

$$M(0,r_2^+(y),0) = \begin{pmatrix} (-\lambda_2^+(y))^{k+1}F_0'(U_+^0)r_2^+(y) \\ (-\lambda_2^+(y))^k l_+^0(y)r_2^+(y) \end{pmatrix}$$

$$M(0,r_3^+(y),0) = \begin{pmatrix} (-\lambda_3^+(y))^{k+1}F_0'(U_+^0)r_3^+(y) \\ (-\lambda_3^+(y))^k l_+^0(y)r_3^+(y) \end{pmatrix}$$

$$M(0,0,r_1^-(y)) = \begin{pmatrix} (\lambda_1^-(y))^{k+1}F_0'(U_-^0)r_1^-(y) \\ (\lambda_1^-(y))^k l_-^0(y)r_1^-(y) \end{pmatrix}$$

which implies

$$det(M(1,0,0), M(0,r_2^+(y),0), M(0,r_3^+(y),0), M(0,0,r_1^-(y))$$

$$= (\lambda_2^+(y)\lambda_3^+(y)\lambda_1^-(y))^{k+1}\begin{vmatrix} [F_0(U^0)] & F_0'(U_+^0)r_2^+(y) & F_0'(U_+^0)r_3^+(y) & F_0'(U_-^0)r_1^-(y) \\ a_0(y) & -\frac{l_+^0(y)r_2^+(y)}{\lambda_2^+(y)} & -\frac{l_+^0(y)r_3^+(y)}{\lambda_3^+(y)} & \frac{l_-^0(y)r_1^-(y)}{\lambda_1^-(y)} \end{vmatrix}$$

does not vanish by using the one-dimensional stability (2.11).

Thus, we obtain the conclusion. ¶

First, we suppose that there are functions $(V_+^0(y), V_-^0(y), \sigma(y)) \in H^{s+\frac{1}{2}}(I)$ and $\varphi_0(y)) \in H^{s+\frac{3}{2}}(I)$satisfying the zero-th order compatibility condition (3.7).

Remark 3.2: From [3], we know that for any fixed $y \in I$, $0 < \gamma \le \gamma_0$ and $0 < |j_0(y)| < |j_1|$ for $j_1^2 = -P'(\tau_1)$ given in (1.6), there is a unique planar subsonic phase transition

$$\underline{U}^0(t,x,y) = \begin{cases} V_+^0(y), & x > \varphi_0(y) + \sigma(y)t \\ V_-^0(y), & x < \varphi_0(y) + \sigma(y)t \end{cases}$$

satisfying

$$j_0(y) = \rho_\pm^0(y)(u_\pm^0(y) - \varphi_0'(y)v_\pm^0(y) - \sigma(y))/\sqrt{1+\varphi_0'^2(y)}$$

and the viscosity-capillarity criterion (1.14) on $\{x = \varphi_0(y)+\sigma(y)t\}$. Then, $(V_+^0(y), V_-^0(y), \sigma(y))$ satisfies the zero-th order compatibility condition (3.7).

The next proposition shows that large classes of initial data can be generated so that the compatibility conditions up to order $s-1$ are satisfied.

Proposition 3.3: *Assume that $(V_+^0(y), V_-^0(y), \sigma(y)) \in H^{s+\frac{1}{2}}(I)$ and $\varphi_0(y)) \in H^{s+\frac{3}{2}}(I)$ satisfy the zero-th order compatibility condition (3.7), and $g_k^\pm(y) \in H^{s+1-k}(I)$ $(k \le s-1)$*

are arbitrary functions satisfying $P^{\pm}(y)g_k^{\pm}(y) = 0$. *Then there are* $(U_{\pm}^0(x,y), \varphi^0(t,y)) \in H^{s+1}(\omega) \times H^{s+2}((-\infty,\infty)\times I)$ *so that*

(1) $U_{\pm}^0(0,y) = V_{\pm}^0(y)$, $\varphi^0(0,y) = \varphi_0(y)$, $\partial_t\varphi^0(0,y) = \sigma(y)$, *and* $(I - P^{\pm}(y))\partial_x^k U_{\pm}^0|_{x=0} = g_k^{\pm}(y)$ *for* $1 \le k \le s-1$;

(2) $(U_{\pm}^0(x,y), \varphi^0(t,y))$ *satisfies the compatibility conditions (3.14) for any* $0 \le k \le s-1$.

Proof: From Lemma 3.1, we know there are uniquely determined functions $(h_k^{\pm}(y), \alpha_k(y)) \in H^{s+\frac{1}{2}-k}$ $(0 \le k \le s-1)$ such that $P^{\pm}(y)h_k^{\pm}(y) = h_k^{\pm}(y)$, such that the following identity:

$$\begin{aligned}
&\begin{pmatrix} [F_0(U^0)] \\ a_0(y) \end{pmatrix}\alpha_k(y) + \begin{pmatrix} F_0'(U_+^0)(\sigma(y) + \varphi_0'(y)A_2(U_+^0) - A_1(U_+^0))^{k+1} \\ l_+^0(y)(\sigma(y)+\varphi_0'(y)A_2(U_+^0) - A_1(U_+^0))^k \end{pmatrix} v_k^+(y) \\
&\quad + \begin{pmatrix} F_0'(U_-^0)(A_1(U_-^0) - \sigma(y) - \varphi_0'(y)A_2(U_-^0))^{k+1} \\ l_-^0(y)(A_1(U_-^0) - \sigma(y) - \varphi_0'(y)A_2(U_-^0))^k \end{pmatrix} v_k^-(y) \\
&\quad = f^{(k)}\left(\{\partial_y^l \alpha_j(y)\}_{\substack{0\le j\le k-1 \\ l+j\le k}}, \{\partial_y^l v_j^{\pm}(y)\}_{\substack{0\le j\le k-1 \\ l+j\le k}} \right)
\end{aligned} \tag{3.21}$$

holds for any $0 \le k \le s-1$, where $v_k^{\pm}(y) = h_k^{\pm}(y) + g_k^{\pm}(y)$ and $f^{(k)}(\cdot)$ are given in (3.14).

By using the trace inverse theorem, there exist

$$U_{\pm}^0(x,y) \in H^{s+1}(\omega), \quad \varphi^0(t,y) \in H^{s+2}((-\infty,\infty)\times I)$$

such that

$$\begin{cases} \partial_x^k U_{\pm}^0|_{x=0} = v_k^{\pm}(y), \quad 0 \le k \le s-1 \\ \varphi^0(0,y) = \varphi_0(y), \quad \partial_t\varphi^0(0,y) = \sigma(y), \quad \partial_t^k\varphi^0(0,y) = \alpha_{k-1} \quad (2 \le k \le s) \end{cases}.$$

¶

3.3 Proof of Theorem 1.4

3.3.1 Construction of an approximate solution

For simplicity of notations, denote the problem (3.6) by

$$\begin{cases} L^{\pm}(U_{\pm},\varphi)U_{\pm} = 0, \quad t,x>0 \\ G_1(U_+,U_-,\varphi_t,\varphi_y) = 0, \quad \text{on } x=0 \\ G_2(U_+,U_-,\varphi_t,\varphi_y) = 0, \quad \text{on } x=0 \\ U_{\pm}(0,x,y) = U_{\pm}^0(x,y), \quad \varphi(0,y) = \varphi_0(y) \end{cases} \tag{3.22}$$

where

$$L^{\pm}(U_{\pm},\varphi) = \partial_t \pm (A_1(U_{\pm}) - \varphi_y A_2(U_{\pm}) - \varphi_t)\partial_x + A_2(U_{\pm})\partial_y$$

and $G_1(\cdot) = 0$, $G_2(\cdot) = 0$ represent the Rankine-Hugoniot condition, the viscosity-capillarity admissibility criterion given in (3.6) respectively.

As in Theorem 1.4, we assume the following condition (MA2):

> for any fixed $s \ge 9$, $U_{\pm}^0 \in H^{s+1}(\omega)$, $\varphi_0 \in H^{s+\frac{3}{2}}(I)$ satisfy (MA1) and the compatibility condition (3.14) for any $0 \le k \le s-1$.

With $\varphi^0 \in H^{s+2}(b\Omega_{T_0})$ being given in Proposition 3.3, we are going to construct

$$\tilde{U}^0_\pm \in \cap_{j=0}^{s+1} C^j([0,T_0], H^{s+1-j}(\omega_t)) \tag{3.23}$$

such that

$$\begin{cases} L^\pm(\tilde{U}^0_\pm, \varphi^0)\tilde{U}^0_\pm = f^0_\pm, & t, x > 0 \\ G_1(\tilde{U}^0_+, \tilde{U}^0_-, \varphi^0_t, \varphi^0_y) = g^0_1, & \text{on } x = 0 \\ G_2(\tilde{U}^0_+, \tilde{U}^0_-, \varphi^0_t, \varphi^0_y) = g^0_2, & \text{on } x = 0 \\ \tilde{U}^0_\pm(0,x,y) = U^0_\pm(x,y), & \varphi^0(0,y) = \varphi_0(y) \end{cases} \tag{3.24}$$

and

$$\partial_t^j f^0_\pm|_{t=0} = 0, \quad \partial_t^j g^0_1|_{t=0} = \partial_t^j g^0_2|_{t=0} = 0 \tag{3.25}$$

for $0 \le j \le s-1$.

Denote by

$$m^j_\pm = \frac{\partial^j \tilde{U}^0_\pm}{\partial t^j}|_{t=0} \qquad (0 \le j \le s)$$

then from (3.24)(3.25), we obtain

$$m^j_\pm \in H^{s+1-j} \qquad (0 \le j \le s). \tag{3.26}$$

Let $P(\partial_t, \partial_x, \partial_y)$ be a scalar linear hyperbolic operator of order $s+1$, $\tilde{m}^j_\pm \in H^{s+1-j}(\mathbb{R}^2)$ be an appropriate extension of $m^j_\pm$ to $\{x < 0\}$, and $W^0_\pm \in \cap_{j=0}^{s+1} C^j([0,T_0], H^{s+1-j})$ be the unique solution to the following Cauchy problem:

$$\begin{cases} PW^0_\pm = 0, & t > 0 \\ \partial_t^j W^0_\pm|_{t=0} = \tilde{m}^j_\pm(x,y), & 1 \le j \le s \end{cases} \tag{3.27}$$

Then, the restriction

$$\tilde{U}^0_\pm(t,x,y) = W^0_\pm|_{x>0} \tag{3.28}$$

together with $\varphi^0(x,y)$ are the approximate solutions satisfying (3.24)(3.25). Indeed, since the initial data $(U^0_\pm(x,y), \varphi_0(y))$ satisfy the compatibility conditions up to order $s-1$, and from (3.27) we conclude (3.25). ¶

3.3.2 The iteration scheme

Similar to [12], first, for any $\eta \ge 1$ and integer $s \ge 0$, let us introduce notations as follows

$$< g >^2_{s,\eta,T} = \sum_{\alpha+\beta+\gamma=s} \int_0^T \int_{-\infty}^{+\infty} |\eta|^{2\alpha} e^{-2\eta t} |\partial_y^\beta \partial_t^\gamma g|^2 dy dt$$

$$|f|^2_{s,\eta,T} = \sum_{k=0}^{s} \int_0^\infty < \partial_x^k f >^2_{s-k,\eta,T} dx$$

$$\|V\|^2_{s,\eta,T} = < \varphi >^2_{s+1,\eta,T} + \sum_{j=0}^{s} (< \frac{\partial^j v_+}{\partial x^j} >^2_{s-j,\eta,T} + < \frac{\partial^j v_-}{\partial x^j} >^2_{s-j,\eta,T}) + \eta(|v_+|^2_{s,\eta,T} + |v_-|^2_{s,\eta,T})$$

where $V = (v_+, v_-, \varphi)$. We will simply denote $< \cdot >_{s,T}$, $|\cdot|_{s,T}$, $\|\cdot\|_{s,T}$ the cases when the above norms are independent of $\eta \ge 1$, and $< \cdot >_s$, $|\cdot|_s$, $\|\cdot\|_s$ the cases when $T = +\infty$.

Denote by E_T the extension operator given in the Lemma 3.1 of [12]. That is, for any fixed $0 < T \leq \frac{T_0}{2}$, $V = (v_+, v_-, \varphi)$ satisfies $\|V\|_{s,\eta,T} < \infty$ and

$$\begin{cases} \partial_t^j v_\pm|_{t=0} = 0 & (0 \leq j \leq s-1) \\ \partial_t^j \varphi|_{t=0} = 0 & (0 \leq j \leq s) \end{cases}$$

the extended function $E_T V$ satisfies

$$\begin{cases} E_T V = V \quad \text{for} \quad 0 \leq t \leq T \\ E_T V = 0 \quad \text{for} \quad t < 0 \quad \text{and} \quad t > T_0 \\ \|E_T V\|^2_{s_1,\eta,T_0} \leq C_s \|V\|^2_{s_1,\eta,T} \quad \text{for any} \quad 0 \leq s_1 \leq s \end{cases} \tag{3.29}$$

with a constant C_s depending only on s.

Now, we introduce the iteration scheme for the nonlinear problem (3.22). Let

$$(\tilde{U}^0_\pm(t,x,y), \varphi^0(t,y)) \in (\cap_{j=0}^{s+1} C^j([0,T_0], H^{s+1-j}(\omega_t))) \times H^{s+2}(b\Omega_{T_0})$$

be the approximate solution constructed in §3.3.1. Define the functions inductively as follows:

$$\begin{cases} U^n_\pm = \tilde{U}^0_\pm + E_{T_n} W^n_\pm \\ \varphi^n = \varphi^0 + E_{T_n} \phi^n \end{cases} \tag{3.30}$$

where

$$V^0 = (W^0_+, W^0_-, \phi^0) = (0,0,0) \tag{3.31}$$

and $V^n = (W^n_+, W^n_-, \phi^n)$ is the unique solution for $0 \leq t \leq T_n$ to the following problem provided $V^{n-1} = (W^{n-1}_+, W^{n-1}_-, \phi^{n-1})$ being known already for $0 \leq t \leq T_{n-1}$:

$$\begin{cases} L^\pm(U^{n-1}_\pm, \varphi^{n-1}) W^n_\pm = f^n_\pm \\ G'_{1,(U^{n-1}_+, U^{n-1}_-, \varphi^{n-1}_t, \varphi^{n-1}_y)}(W^n_+, W^n_-, \phi^n_t, \phi^n_y) = g^n_1 \\ G'_{2,(U^{n-1}_+, U^{n-1}_-, \varphi^{n-1}_t, \varphi^{n-1}_y)}(W^n_+, W^n_-, \phi^n_t, \phi^n_y) = g^n_2 \\ (W^n_+, W^n_+, \phi^n) \quad \text{vanish for} \quad t < 0 \end{cases} \tag{3.32}$$

where $G'_{j,(U_+,U_-,\varphi_t,\varphi_y)}(W_+, W_-, \phi_t, \phi_y)$ $(j = 1,2)$ denote the Fréchet derivatives of G_j with respect to their arguments at $(U_+, U_-, \varphi_t, \varphi_y)$,

$$f^n_\pm = \begin{cases} -L^\pm(U^{n-1}_\pm, \varphi^{n-1}) \tilde{U}^0_\pm, & t > 0 \\ 0, & t < 0 \end{cases}$$

and

$$g^n_j = G'_{j,(U^{n-1}_+, U^{n-1}_-, \varphi^{n-1}_t, \varphi^{n-1}_y)}(E_{T_{n-1}} W^{n-1}_+, E_{T_{n-1}} W^{n-1}_-, (E_{T_{n-1}} \phi^{n-1})_t, (E_{T_{n-1}} \phi^{n-1})_y)$$
$$- G_j(U^{n-1}_+, U^{n-1}_-, \varphi^{n-1}_t, \varphi^{n-1}_y).$$

Similar to [12], in the iteration scheme (3.30)(3.32), we have taken the Picard iteration scheme for the equations, and the Newton scheme for the boundary conditions in (3.22).

Before studying (3.32), let us first state a result for the linearized problem of (3.22).

Under the assumption (MA1) given at the beginning of this section on the initial data $(U^0_+(x,y), U^0_-(x,y), \varphi_0(y))$, there is $\delta > 0$ such that for any smooth functions $(U_+(t,x,y), U_-(t,x,y), \varphi(t,y))$ satisfying

$$\sup_{\overline{\Omega}_{T_0}} (|U_\pm - U^0_\pm| + |\varphi - \varphi_0| + |\varphi_t - \sigma(y)| + |\partial_y(\varphi - \varphi_0)|) < \delta, \tag{3.33}$$

the problem (1.24) as follows:

$$\begin{cases} \partial_t V_\pm \pm (A_1(U_\pm) - \varphi_y A_2(U_\pm) - \varphi_t)\partial_x V_\pm + A_2(U_\pm)\partial_y V_\pm = f_\pm, & t, x > 0 \\ B(V_+, V_-, \phi) := b_0\phi_t + b_1\phi_y + M_+V_+ + M_-V_- = g & \text{on } x = 0 \\ (V_+, V_-, \phi)|_{t<0} \quad \text{vanish} \end{cases} \tag{1.24}$$

with (b_0, b_1, M_+, M_-) being the same functions of $(U_\pm, \varphi)$ as in (1.24), is well-posed for (V_+, V_-, ϕ), which means:

Proposition 3.4: *Suppose that the assumption (MA1) is satisfied, and (3.33) holds for (U_+, U_-, φ). If we have*

$$\begin{cases} (f_+, f_-, g) \quad \text{vanishes for} \quad t < 0 \quad \text{and} \quad t > T_0 \\ |f_+|_0^2 + |f_-|_0^2 + < g >_0^2 \quad \text{is finite} \end{cases} \tag{3.34}$$

then there is a unique strong solution $V = (V_+, V_-, \phi)$ to (1.24), and the estimate

$$|||V|||_{0,\eta,T}^2 \le C_1 \left\{ \frac{1}{\eta}(|f_+|_{0,\eta,T}^2 + |f_-|_{0,\eta,T}^2) + < g >_{0,\eta,T}^2 \right\}, \qquad 0 < T \le T_0 \tag{3.35}$$

holds for $\eta \ge C_2$, where $C_1, C_2 > 0$ depend only upon the quantities $(\delta, |U_+, U_-, \varphi|_{s,T_0})$ for any fixed $s \ge 9$.

Additionally, if $|f_+|_s^2 + |f_-|_s^2 + < g >_s^2$ is finite for $s \ge 9$, and

$$\partial_t^j f_\pm|_{t=0} = \partial_t^j g|_{t=0} = 0$$

for any $0 \le j \le s-1$, then the solution (V_+, V_-, φ) belongs to $H^s \times H^s \times H^{s+1}$, and satisfies the estimate

$$|||V|||_{s,\eta,T}^2 \le C_1 \left\{ \frac{1}{\eta}(|f_+|_{s,\eta,T}^2 + |f_-|_{s,\eta,T}^2) + < g >_{s,\eta,T}^2 \right\}. \tag{3.36}$$

This result can be obtained in the same way as in [11, 12] by using the uniform stability result, Theorem 1.2 which we proved in §2.2.

Let us apply Proposition 3.4 to study the convergence of the iteration scheme (3.30)(3.32).

For any fixed $s \ge 9$, denote by C_s the Sobolev embedding constant satisfying

$$\|v_+\|_{L^\infty(\Omega_T)} + \|v_-\|_{L^\infty(\Omega_T)} + \|\varphi\|_{W^{1,\infty}(b\Omega_T)} \le C_s |||V|||_{s,T}$$

for any $V = (v_+, v_-, \varphi) \in H^s \times H^s \times H^{s+1}$, and $\epsilon_0 > 0$ a small quantity such that when (U_+, U_-, φ) satisfies

$$|||U_+ - U_+^0, U_- - U_-^0, \varphi - \varphi_0'(y) - t\sigma(y)|||_{s,T_0}^2 < \epsilon_0 \tag{3.37}$$

we have (3.33).

For the iteration scheme (3.32), let us define

$$T_n' = \min\{T : \ |||W_+^n, W_-^n, \phi^n|||_{s,T}^2 \ge \epsilon_0\}$$

and

$$T_n = \min(\frac{T_0}{2}, T_n'). \tag{3.38}$$

Proposition 3.5: (BOUNDEDNESS) *For any fixed $s \ge 9$, and $\epsilon_0 > 0$ being given as in (3.37), there are $\beta \in (0,1)$ and $T_* > 0$ such that the solution sequence $V^n = (W_+^n, W_-^n, \phi^n)$ defined by (3.32) satisfies*

$$|||V^n|||_{s,\eta(T_*),T_*}^2 < \epsilon_0 \qquad (\forall n \in \mathbb{N}) \tag{3.39}$$

where $\eta(T) = C_0 T^{-\beta}$.

Proof: The proof is the same as that of Proposition 4.1 in [12]. For completeness, let us briefly recall it. (3.39) shall be proved by induction on n. It is true for $n = 0$ obviously. Assume that (3.39) holds for the case $n-1$, we study the problem (3.32).

Employing Proposition 3.4 for (3.32), it follows

$$|||V^n|||^2_{s,\eta(T),T} \leq C_1 \left\{ \frac{1}{\eta(T)} (|f^n_+|^2_{s,\eta(T),T} + |f^n_-|^2_{s,\eta(T),T}) + < g^n_1, g^n_2 >^2_{s,\eta(T),T} \right\}. \tag{3.40}$$

Without loss of generality, we consider the case $0 < T_0 \leq 1$, which yields

$$e^{-2C_0} \leq e^{-2C_0 t T^{-\beta}} \leq 1$$

for any $0 \leq \beta \leq 1$ and $0 \leq t \leq T \leq T_0 \leq 1$.

From the definition, we can easily deduce

$$\begin{cases} |f^n_\pm|^2_{s,\eta(T),T} \leq C_2 \sum_{k=0}^{s} T^{-2\beta(s-k)} |f^n_\pm|^2_{k,T} \\ < g^n_1, g^n_2 >^2_{s,\eta(T),T} \leq C_2 \sum_{k=0}^{s} T^{-2\beta(s-k)} < g^n_1, g^n_2 >^2_{k,T} \\ |||V^n|||^2_{s,T} \leq C_2 |||V^n|||^2_{s,\eta(T),T} \end{cases}$$

with an absolute constant $C_2 > 0$ when $\eta(T) = C_0 T^{-\beta}$.

Therefore, from (3.40) we obtain

$$|||V^n|||^2_{s,T} \leq C_3 \left(T^\beta \sum_{k=0}^{s} T^{-2\beta(s-k)} (|f^n_+|^2_{k,T} + |f^n_+|^2_{k,T}) + \sum_{k=0}^{s} T^{-2\beta(s-k)} < g^n_1, g^n_2 >^2_{k,T} \right). \tag{3.41}$$

On other hand, from the classical interpolation inequality, we have

$$\begin{cases} |f^n_\pm|^2_{k,T} \leq \tilde{C}_s |f^n_\pm|^{\frac{2k}{s}}_{s,T} |f^n_\pm|^{2(1-\frac{k}{s})}_{0,T} \\ < g^n_1, g^n_2 >^2_{k,T} \leq \tilde{C}_s < g^n_1, g^n_2 >^{\frac{2k}{s}}_{s,T} < g^n_1, g^n_2 >^{2(1-\frac{k}{s})}_{0,T} \end{cases} \tag{3.42}$$

for any $0 \leq T \leq \frac{T_0}{2}$, $0 \leq k \leq s$ with $\tilde{C}_s$ depending only upon s.

From the assumption (MA2), and the induction assumption on V^{n-1}, we have a constant $C(\epsilon_0)$ depending only upon $\epsilon_0 > 0$ such that

$$|f^n_\pm|_{s,T_0} \leq C(\epsilon_0) \tag{3.43}$$

for $f^n_\pm$ given in (3.32).

Furthermore, by using $f^n_\pm|_{t<0} = 0$ and $\|f^n_\pm\|_{C^1(\Omega_{T_0})} \leq C$, we have

$$|f^n_\pm|^2_{0,T} \leq CT^3. \tag{3.44}$$

Substituting (3.43) and (3.44) into (3.42), it follows

$$|f^n_\pm|^2_{k,T} \leq CT^{3(1-\frac{k}{s})} \qquad (0 \leq k \leq s) \tag{3.45}$$

which implies

$$\sum_{k=0}^{s} T^{-2\beta(s-k)} (|f^n_+|^2_{k,T} + |f^n_-|^2_{k,T}) \leq C(1 + T^{\frac{3}{s} - 2\beta}) \tag{3.46}$$

when $0 \leq \beta \leq \frac{3}{2(s+1)}$.

Set $Z_0 = (\bar{U}^0_+, \bar{U}^0_-, \partial_t\varphi^0, \partial_y\varphi^0)$. From the property of the Newton iteration scheme, we have

$$g^n_j = -G_j(Z_0) + O(|E_{T_{n-1}}(W^{n-1}_+, W^{n-1}_-, \partial_t\phi^{n-1}, \partial_y\phi^{n-1})|^2)$$

which implies

$$< g^n_j >^2_{s,T} \leq C(< G_j(Z_0) >^2_{s,T} + |||V^{n-1}|||^4_{s,T}) \tag{3.47}$$

for $j = 1, 2$.

Similar to (3.44), we obtain

$$< g^n_1, g^n_2 >^2_{0,T} \leq CT^3 \tag{3.48}$$

by using $g^n_j|_{t=0} = 0$ and $\|g^n_j\|_{C^1(b\Omega_T)} \leq C$.

Substituting (3.47)(3.48) into (3.42), it follows

$$< g^n_1, g^n_2 >^2_{k,T} \leq CT^{3(1-\frac{k}{s})}(< G_1(Z_0), G_2(Z_0) >^2_{s,T} + |||V^{n-1}|||^4_{s,T})^{\frac{k}{s}}$$

which implies

$$\sum_{k=0}^{s} T^{-2\beta(s-k)} < g^n_1, g^n_2 >^2_{k,T} \leq C(T^{3-2\beta s} + < G_1(Z_0), G_2(Z_0) >^2_{s,T} + |||V^{n-1}|||^4_{s,T}). \tag{3.49}$$

Substituting (3.46) and (3.49) into (3.41), we conclude

$$|||V^n|||^2_{s,T} \leq C(T^{3-2\beta s} + T^{\beta} + T^{\frac{3}{s}-\beta} + < G_1(Z_0), G_2(Z_0) >^2_{s,T} + |||V^{n-1}|||^4_{s,T})$$

which implies that there exist $T_* > 0$, $\epsilon_0 > 0$ such that if $|||V^{n-1}|||^2_{s,T_*} < \epsilon_0$, we have

$$|||V^n|||^2_{s,T_*} < \epsilon_0. \tag{3.50}$$

¶

Proposition 3.6: (CONVERGENCE) *Under the same assumption as in Proposition 3.5, there are constants $C_1, C_2 > 0$ depending only on δ, such that for any $\eta > C_2$ and $T \leq T_*$, we have*

$$|||V^{n+1} - V^n|||^2_{0,\eta,T} \leq C_1(\frac{1}{\eta} + T^2)|||V^n - V^{n-1}|||^2_{0,\eta,T}. \tag{3.51}$$

Proof: When $T \leq T_*$, we can omit $E_{T_{n-1}}$ in the problem (3.32), from which we know that $V^{n+1} - V^n = (W^{n+1}_+ - W^n_+, W^{n+1}_- - W^n_-, \phi^{n+1} - \phi^n)$ satisfies the following problem

$$\begin{cases} L^{\pm}(U^n_{\pm}, \varphi^n)(W^{n+1}_{\pm} - W^n_{\pm}) = \tilde{f}^{n+1}_{\pm}, \quad t,\ x > 0 \\ G'_{j,(U^n_+, U^n_-, \varphi^n_t, \varphi^n_y)}(W^{n+1}_+ - W^n_+, W^{n+1}_- - W^n_-, \phi^{n+1}_t - \phi^n_t, \phi^{n+1}_y - \phi^n_y) = \tilde{g}^{n+1}_j \quad (j = 1, 2) \\ (W^{n+1}_+ - W^n_+, W^{n+1}_- - W^n_-, \phi^{n+1} - \phi^n) \quad \text{vanishes for} \quad t < 0 \end{cases} \tag{3.52}$$

where

$$\begin{cases} \tilde{f}^{n+1}_{\pm} = -L^{\pm}(U^n_{\pm}, \varphi^n)\bar{U}^0_{\pm} + L^{\pm}(U^{n-1}_{\pm}, \varphi^{n-1})\bar{U}^0_{\pm} + (L^{\pm}(U^{n-1}_{\pm}, \varphi^{n-1}) - L^{\pm}(U^n_{\pm}, \varphi^n))W^n_{\pm} \\ \tilde{g}^{n+1}_j = g^{n+1}_j - g^n_j + (G'_{j,(U^{n-1}_+, U^{n-1}_-, \varphi^{n-1}_t, \varphi^{n-1}_y)} - G'_{j,(U^n_+, U^n_-, \varphi^n_t, \varphi^n_y)})(W^n_+, W^n_-, \phi^n_t, \phi^n_y) \end{cases}. \tag{3.53}$$

Employing Proposition 3.4 for the problem (3.52), it yields

$$|||V^{n+1} - V^n|||^2_{0,\eta,T} \leq C_1 \left\{ \frac{1}{\eta}(|\tilde{f}^{n+1}_+|^2_{0,\eta,T} + |\tilde{f}^{n+1}_-|^2_{0,\eta,T}) + < \tilde{g}^{n+1}_1, \tilde{g}^{n+1}_2 >^2_{0,\eta,T} \right\}. \tag{3.54}$$

On the other hand, from (3.53) we easily deduce

$$\begin{cases} |\tilde{f}_\pm^{n+1}|^2_{0,\eta,T} \le C_2 |||V^n - V^{n-1}|||^2_{0,\eta,T} \\ < \tilde{g}_1^{n+1}, \tilde{g}_2^{n+1} >^2_{0,\eta,T} \le C_2 T^2 < V^n - V^{n-1} >^2_{0,\eta,T} \end{cases} \tag{3.55}$$

when $T \le T_*$ be using Proposition 3.5.

Thus, we immediately conclude (3.51) from (3.54) and (3.55). ¶

The proof of Theorem 1.3 follows standard arguments. In detail, from Proposition 3.6 we know that there are $T_{**} \in (0, T_*]$, $\alpha \in (0,1)$ and $\eta_0 > 0$ such that

$$|||V^{n+1} - V^n|||^2_{0,\eta_0,T_{**}} \le \alpha |||V^n - V^{n-1}|||^2_{0,\eta_0,T_{**}}. \tag{3.56}$$

From Proposition 3.5 we know that $V^n = (W_+^n, W_-^n, \phi^n)$ is bounded in $H^s \times H^s \times H^{s+1}$ for $0 \le t \le T_*$. Thus we obtain $V = (W_+, W_-, \phi) \in (H^s(\Omega_{T_{**}}))^2 \times H^{s+1}(b\Omega_{T_{**}})$ such that

$$\begin{cases} W_\pm^n \longrightarrow W_\pm & \text{in } H^r(\Omega_{T_{**}}) \\ \phi^n \longrightarrow \phi & \text{in } H^{r+1}(b\Omega_{T_{**}}) \end{cases} \quad (n \to +\infty)$$

for any $0 \le r < s$, and

$$U_\pm = \tilde{U}_\pm^0 + W_\pm, \quad \varphi = \varphi^0 + \phi$$

are solutions to (3.22). ¶

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