Self-similar Solutions to the Stokes Approximation Equations for Two Dimensional Compressible Flows

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Abstract

The existence of forward self-similar solution is established to the Stokes approximation equations for two dimensional compressible flows. We obtain it by considering the Cauchy problem of its corresponding approximation system in some homogeneous Besov spaces with small date. Our result also holds for three dimensional case.

1 Introduction

As pointed out in[1], self-similar solutions with suitable homogeneity often play a crucial role in the theory of regularity and asymptotic stability of nonlinear problems, which are physically or geometrically interesting. The pure existence and structure of such solution often reflects the intrinct dynamics of the underlying nonlinearity and different length scales. This has been manifested in many interesting problems, such as the regularity theory of harmonic maps, minimal surfaces and heat flows. In the mathematical theory of fluid dynamics, the idea that the self-similar solutions are building blocks both locally and globally for general inviscid flows, is well known for two centenaries and has been the central idea of the mathematical theory of shock wave.For viscous flows, the importance of self-similar solutions in understanding the interactions between initial force and dissipation had been recognized long time ago. Indeed, in his seminal paper in 1934, Leray [2]raised the question of existence of self-similar solutions to the 3-dimensional incompressible Navier-Stokes equations as soon as he established the existence of weak solutions for such system. The existence of forward self-similar solution often reflects the dynamical dissipation mechanics of the underlying nonlinear problem, while the existence of backward self-similar solution shows the dominance of the nonlinearity over the dissipation and yields clues to construct general singular solutions.

For the incompressible Navier-stokes system, where the major nonlinearity is due to the nonlinear convection and thus the scale laws are similar to that for heat equations,

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the existence of forward self-similar solutions in various space(such as Morrey space, weak L^P-space and Besov space,etc.)have been obtained by many authors(see[3], [4], [5],[6]). The much difficult problem is the existence of the backward self-similar solutions,which was originally raised by Leray. The blow-up estimate in theory of partial regularity (see[7], [8]) implies that there are no backward self-similar solutions with small local energy. The Leary's problem was completely solved by Nečas,Růžička and Šverák In 1995([9]), where they showed,among other things that the only backward self-similar solutions satisfying the global energy estimates is zero. This important result was generalized by Tai-Peng Tsai in [10] showing that there are no backward self-similar solutions with even finite local energy. These results implies the complexity of possible singularities of solutions for 3-dimensional incompressible Navier-Stokes system.

On the other hand, these have been few studies on the self-similar solutions to the compressible Navier-Stokes system partially due to the complicated nonlinearities arising from both the nonlinear convection and the pressures and their interactions. For general equation of states,even the scaling law is not clear. Recently, Ershkov and Shchennikov [11] showed that the equation of self-similar solutions to the complete system of Navier-Stokes equations for steady axially symmetric swirling viscous compressible gas flow can be represented as a system of Riccati-type differential equations. However, the question how to do solve these system of Riccati-type differential equations is still open. The multi-dimensional problem for the viscous compressible flows are much more difficult. One of the reason is due to the great complexity in nonlinearities and their interactions. To isolate the difficulties, we consider the model of Stokes approximation equations for two dimensional compressible system, where the pressure plays dominant role over the nonlinear convection. We first establish the existence theory to Cauchy problem for this Stokes approximation system in a class of homogeneous Besov spaces with negative degrees with small data. As a consequence, we obtained the existence of small forward self-similar solutions to the 2-dimensional Stokes approximate equations in some Besov spaces.

More precisely, we will study the following model which corresponds to a stokes-like approximation to the momentum equations of the system of compressible isentropic Navier-Stokes equation. This is, we consider

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0 \\ \bar{\rho} u_t - \mu \triangle u - \xi \nabla \mathrm{div} u + a \nabla \rho^\gamma = 0 \end{cases}$$

where $\bar{\rho} > 0, a > 0, \mu > 0, \mu + \xi > 0$ and $\gamma \geq 1$.

This system is a good approximation for compressible flow which has been investigated by many authors. For multi-dimensional flows, P. L. Lions [12] proved the existence of weak solution under the assumptions of $\gamma \geq 1$ in the Dirichlet boundary conditions when Ω is bounded or $\gamma > 1$ in the periodic case when $\Omega = \mathbb{R}^N$ if $N = 2$;$\gamma \geq \frac{2N}{N+2}$ in the Dirichlet boundary conditions when Ω is bounded or in the periodic case when $\Omega = \mathbb{R}^N$ if $N \geq 3$. In [13], the existence and uniqueness of weak solution to the potential flow has been obtained in the periodic case when $\gamma = 1$. On the other situation, when $\rho \to \bar{\rho} = 1$ as $|x| \to \infty$,Lu Min, Alexandre V. Kazhikhov and Seiji Ukai [14] proved the global existence of weak and classical solutions to the Cauchy problem in $\mathbb{R}^2$ with large smooth initial data.

In the case of $\bar{\rho} = 1, \mu = 1, \xi = 0$ and $a = 1$, the system becomes

$$\rho_t + \mathrm{div}(\rho u) = 0 \tag{1.1}$$

$$u_t - \triangle u + \nabla \rho^\gamma = 0 \tag{1.2}$$

with $x \in \mathbb{R}^2, \gamma \geq 1$.

Our focus will be to find forward self-similar solutions to the system(1.1)-(1.2). Here forward self-similar solutions $(\rho(x,t), u(x,t))$ are ones satisfying

$$\rho(x,t) = \lambda^{\frac{2}{\gamma}}\rho(\lambda x, \lambda^2 t), u(x,t) = \lambda u(\lambda x, \lambda^2 t) \tag{1.3}$$

for $\lambda > 0$. This kind of solution is related to the large-time asymptotic behavior of the global solutions of our system. It is hoped that the study of self-similar solutions to (1.1)-(1.2) can shed some light on the regularity and structures of solutions to the compressible Navier-Stokes equations.

Before going into the details of the construction of self-similar solution, we first explain the main difficulties arising here. Note that(1.3) implies $\rho(x, t = 0) = \rho_0(x)$ and $u(x, t = 0) = u_0(x)$ must satisfy

$$\rho_0(\lambda x) = \lambda^{-\frac{2}{\gamma}}\rho_0(x), u_0(\lambda x) = \lambda^{-1}u_0(x). \tag{1.4}$$

This means that $(\rho_0(x), u_0(x))$ is homogeneous with degree $(-\frac{2}{\gamma}, -1)$ and every initial data that gives a self-similar solution must verify this property. Unfortunately, those functions do not belong to the usual Sobolev or Hölder spaces. We shall therefore replace them by other functional spaces that contain homogeneous function of degree $(-\frac{2}{\gamma}, -1)$, e.g., Besov spaces.

The existence of solutions to the system(1.1)-(1.2) in Besov spaces will be obtained by using vanishing viscosity method. The main difficulty is to get the passage to limit which need some new a priori estimates.

Our main result reads as follows:

Theorem 1.1. *For any* $(\rho_0(x), u_0(x))$ *such that* $\rho_0(\lambda x) = \lambda^{-\frac{2}{\gamma}}\rho_0(x)$ *and* $u_0(\lambda x) = \lambda^{-1}u_0(x)$,*for* $\lambda > 0$,*if*

$$\|\rho_0\|_{\dot{B}^{-2+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)} \leq \delta, \|\rho_0\|_{L_M(\mathbb{R}^2)} \leq C_1, \|\rho_0\|_{L_1(\mathbb{R}^2)} \leq C_2, \ \text{if } \gamma = 1 \ \text{with } 2 < p < 4, 2 < q < 4;$$
$$\|\rho_0\|_{\dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)} \leq \delta, \|\rho_0\|_{L^\gamma(\mathbb{R}^2)} \leq C_3, \ \text{if } \gamma = 2;$$
$$\|\rho_0\|_{\dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)} \leq \delta, \|\rho_0\|_{L^\gamma(\mathbb{R}^2)} \leq C_4, \|\rho_0\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2)} \leq C_5, \ \text{if } 2 < \gamma \leq 4;$$
$$\text{with } p > 2\gamma, q > 2, \ \text{for } 2 \leq \gamma \leq 4.$$
$$\|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}(\mathbb{R}^2)} \leq \eta, \|u_0\|_{L^2(\mathbb{R}^2)} \leq C_6.$$

Where δ, η *are small and* $C_i, i = 1, ..., 6$ *are absolute constants. Then there exists a self-similar solution* $(\rho(x,t), u(x,t))$ *to the problem* (1.1)-(1.2)*which satisfies*

$$\rho(x,t) = \frac{1}{t^{\frac{\gamma}{2}}}Q(\frac{x}{\sqrt{t}}); u(x,t) = \frac{1}{\sqrt{t}}U(\frac{x}{\sqrt{t}}), \forall 0 < t < T.$$

where

$$Q(x) \in \dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2); U(x) \in \dot{B}^{-1+\frac{2}{q}}_{q,\infty}(\mathbb{R}^2).$$

The present paper is structured as follows.

In section 2, we recall some basic facts about Littlewood-Paley decomposition and Besov spaces, which will be frequently used in our analysis later.

In section 3, we will obtain the existence of solutions to the system(1.1)-(1.2) in Besov spaces for the $\gamma = 1$ case.

The passage to limit in this case is investigated in section 4.

In section 5, we will treat the general case of γ of the existence of solutions to the system(1.1)-(1.2) in Besov spaces. Finally, we prove Theorem 1.1.

Notation: Throughout the paper, C or C_j stands for a "harmless" constant, and $C(T)$ stands for the constant C only dependent on T. $L^p(\mathbb{R}^2)$ and $H^p(\mathbb{R}^2)$ are standard Sobolev spaces.

2 Littlewood-Paley theory and Besov spaces

To understand homogeneous Besov spaces more clearly, we will give some details about them.

At first, let us introduce a dyadic partition of unity. We can use for instance any spherically symmetric Bump function $\hat{\varphi}(\xi) \in C_c^\infty(\mathbb{R}^2)$ satisfies

$$0 \le \hat{\varphi}(\xi) \le 1, \hat{\varphi}(\xi) = \begin{cases} 1, |\xi| \le 1; \\ 0, |\xi| \ge 2. \end{cases}$$

then $\varphi(x) \in S(\mathbb{R}^2)$, which is the Fourier transform of $\hat{\varphi}(\xi)$. Set

$$\begin{cases} \psi(x) = 2\varphi(2x) - \varphi(x); \\ \varphi_j(x) = 2^{2j}\varphi(2^j x), j \in \mathbb{N}; \\ \psi_j(x) = 2^{2j}\psi(2^j x), j \in \mathbb{N}; \\ S_j f = \varphi_j(x) * f, \triangle_j f = \psi_j(x) * f. \end{cases}$$

Obviously, $\{S_j, \triangle_j\}$ is the classical Littlewood-Paley decomposition, and homogeneous Besov spaces can be defined as

$$\dot{B}^{\alpha}_{p,\infty} = \{f \in S'(\mathbb{R}^2), \|f\|_{\dot{B}^{\alpha}_{p,\infty}} = \sup_{j\in\mathbb{Z}} 2^{j\alpha}\|S_j f\|_{L^p} < \infty\}, \alpha \in \mathbb{R}, p \ge 1.$$

On the other hand, for a Banach space χ and for any non-zero function ϕ such that $\|\phi(\lambda\cdot)\|_{\dot{\chi}}$ is a homogeneous function of λ with $\lambda > 0$, we introduce the following definition on smoothness degree of Banach space χ(see [15]).

Definition 2.1. *Let χ be a Banach space. The smoothness degree of χ is defined and denoted by*

$$deg(\chi) := \log_\lambda(\Lambda(\lambda)),$$

where $\Lambda(\lambda) = \frac{\|\phi(\lambda\cdot)\|_{\dot{\chi}}}{\|\phi(\cdot)\|_{\dot{\chi}}}$ and ϕ is a nonzero function in χ.

Remark 2.1. *(1) It is easy to see that the definition of $deg(\chi)$ is independent of the choice of ϕ;(2)Clearly, $deg(\chi_1) \ge deg(\chi_2)$, if $\chi_1 \subset \chi_2$; and*

$$deg(L^p(\mathbb{R}^N)) = -\frac{N}{p}, 1 \le p \le \infty; deg(W^{s,p}(\mathbb{R}^N)) = s - \frac{N}{p}, 1 \le p \le \infty, s \in \mathbb{R};$$
$$deg(\dot{B}^s_{p,q}(\mathbb{R}^N)) = s - \frac{N}{p}, 1 \le p, q \le \infty, s \in \mathbb{R}; deg(M^p_q(\mathbb{R}^N)) = -\frac{N}{p}, 1 \le q \le p < \infty;$$
$$deg(F^s_{p,q}(\mathbb{R}^N)) = s - \frac{N}{p}, 1 \le p < \infty, 1 \le q \le \infty, s \in \mathbb{R}.$$

By the above definition, the following properties hold(see[16],[18]).

(1) Let $1 \le p \le \infty, \alpha > 0$, then $\dot{B}^{-\alpha}_{p,\infty}$ has the following equivalent norms:

$$\sup_{j\in\mathbb{Z}} 2^{-j\alpha}\|\triangle_j u\|_{L^p} \sim \sup_{j\in\mathbb{Z}} 2^{-j\alpha}\|S_j u\|_{L^p} \sim \sup_{t\ge 0} t^{\frac{\alpha}{2}}\|S(t)u\|_{L^p}$$
$$\sim \sup_{t\ge 0}\|S(t)u\|_{\dot{B}^{-\alpha}_{p,\infty}}, u \in \dot{B}^{-\alpha}_{p,\infty}.$$

where $S(t)$ stands for the Poisson semigroup.

(2) By the Littlewood-Paley decomposition of Besov space, i.e.,

$$\|u\|_{\dot{B}^{-\alpha}_{p,\infty}} = \sup_{j\in\mathbb{Z}} 2^{-j\alpha}\|\triangle_j u\|_{L^p}, u \in \dot{B}^{-\alpha}_{p,\infty}$$

we have

$$L^2(\mathbb{R}^2) \hookrightarrow \dot{B}^{-\alpha_1}_{p_1,\infty}(\mathbb{R}^2) \hookrightarrow \dot{B}^{-\alpha_2}_{p_2,\infty}(\mathbb{R}^2)$$
$$\alpha_j = 1 - \frac{2}{p_j}, j = 1, 2, 2 \le p_1 \le p_2 \le \infty.$$

(3) More generally, by the Sobolev embedding, we have

$$L^p(\mathbb{R}^N) \hookrightarrow \dot{B}^{-\alpha}_{q,\infty}(\mathbb{R}^N);$$
$$W^{s,k}(\mathbb{R}^N) \hookrightarrow \dot{B}^{-\beta}_{l,\infty}(\mathbb{R}^N).$$

where

$$\deg(L^p(\mathbb{R}^N)) = \deg(\dot{B}^{-\alpha}_{q,\infty}(\mathbb{R}^N)); \deg(W^{s,k}(\mathbb{R}^N)) = \deg(\dot{B}^{-\beta}_{l,\infty}(\mathbb{R}^N)),$$

with

$$-\frac{N}{p} = -\alpha - \frac{N}{q}; s - \frac{N}{k} = -\beta - \frac{N}{l};$$
$$s \in \mathbb{R}, 1 \le p, q, k, l \le \infty.$$

In particular,

$$L^\gamma(\mathbb{R}^2) \hookrightarrow \dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{q,\infty}(\mathbb{R}^2), \text{ with } p > \gamma.$$

Remark 2.2. *Notes that*

$$|x|^{-\frac{2}{\gamma}} \in \dot{B}^{-\alpha}_{p,\infty}, \alpha = \frac{2}{\gamma} - \frac{2}{p}; |x|^{-1} \in \dot{B}^{-\beta}_{q,\infty}, \beta = 1 - \frac{2}{q}.$$

with $p > \gamma, q > 2$. *But*

$$|x|^{-\frac{2}{\gamma}} \notin L^\gamma(\mathbb{R}^2); |x|^{-1} \notin L^2(\mathbb{R}^2).$$

3 The Cauchy problem to the approximation system in Besov spaces

The existence of self-similar solution to the problem (1.1)-(1.2) up to a smallness condition on the initial data, will be obtained as a limit of solution to the following approximation system:

$$\rho_t + \mathrm{div}(\rho u) = \epsilon \triangle \rho \tag{3.1}$$

$$u_t - \triangle u + \nabla \rho^\gamma = 0 \tag{3.2}$$

where $x \in \mathbb{R}^2, \gamma \geq 1, \epsilon > 0$ is small, and $\rho \geq 0$.

Note that the approximation system (3.1)-(3.2) is invariant under scaling

$$(\rho(x,t), u(x,t)) = (\lambda^{\frac{2}{\gamma}} \rho(\lambda x, \lambda^2 t), \lambda u(\lambda x, \lambda^2 t).$$

In the sequel, we will first treat the case $\gamma = 1$ and then turn to the general case of $\gamma > 1$.

For the sake of simplicity in presentation, we drop the dependence on ϵ in the system (3.1)-(3.2) for $\gamma = 1$, thus consider the following Cauchy problem:

$$\rho_t + \mathrm{div}(\rho u) = \triangle \rho \tag{3.3}$$

$$u_t - \triangle u + \nabla \rho = 0 \tag{3.4}$$

$$\rho(x,0) = \rho_0(x), u(x,0) = u_0(x) \tag{3.5}$$

Furthermore, it is assumed that

$$\rho_0(x) \in \dot{B}^{-2+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2), u_0(x) \in \dot{B}^{-1+\frac{2}{q}}_{q,\infty}(\mathbb{R}^2) \tag{3.6}$$

where $p > 1, q > 2$.

From (3.4), we can represent $u(x,t)$ as

$$\begin{aligned} u(x,t) &= S(t)u_0(x) - \int_0^t S(t-\tau)\nabla\rho(\tau,x)d\tau \\ &:= S(t)u_0(x) - \mathcal{C}[\rho](t,x) \end{aligned} \tag{3.7}$$

So, a mild solution of (3.3) can be defined as

$$\begin{aligned} \rho(x,t) &= S(t)\rho_0(x) - \int_0^t S(t-\tau)\mathrm{div}(\rho(\tau,x)S(\tau)u_0(x))d\tau \\ &+ \int_0^t S(t-\tau)\mathrm{div}(\rho\mathcal{C}[\rho])(\tau,x)d\tau \\ &:= S(t)\rho_0(x) - \mathcal{A}[\rho,u_0](t,x) + \mathcal{B}[\rho,\rho](t,x) \end{aligned} \tag{3.8}$$

Where $S(t)$ denotes the heat operator. It follows from the classical regularity estimates of the heat semigroup(see e.g. [16],[15]), one easily sees that for $t > 0$

$$\|S(t)\varphi\|_{L^p} \leq Ct^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{p})}\|\varphi\|_{L^r}, 1 \leq r \leq p \leq \infty; \tag{3.9}$$

$$\|(-\triangle)^{\frac{d}{2}} S(t)\varphi\|_{L^p} \leq Ct^{-\frac{d}{2}-\frac{N}{2}(\frac{1}{r}-\frac{1}{p})}\|\varphi\|_{L^r}, d \geq 0, 1 \leq r \leq p \leq \infty, r \neq \infty. \tag{3.10}$$

Now, we will prove the following basic existence result:

Theorem 3.1. *For any $\rho_0(x) \in \dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)$, and $u_0(x) \in \dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2)$, if*

$$\|\rho_0\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)} \le \delta, \|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2)} \le \eta \tag{3.11}$$

with $2 < p < 4, 2 < q < 4$ and $\delta, \eta > 0$ both are small. Then there exists a unique solution $\rho(x,t)$ of (3.8) which satisfies:

$$\rho(t) \in C_w([0,\infty); \dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)); t^{1-\frac{1}{p}}\rho(t) \in C([0,\infty); L^p(\mathbb{R}^2)); \tag{3.12}$$

$$\rho(t) - S(t)\rho_0(x) \in C([0,\infty); L^1(\mathbb{R}^2)) \tag{3.13}$$

$$\sup_{t\ge 0} \|\rho\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)} + \sup_{t\ge 0} t^{1-\frac{1}{p}}\|\rho\|_{L^p(\mathbb{R}^2)} \le C, \tag{3.14}$$

where C depends only on $\|\rho_0\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)}$, and $\|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2)}$.

Proof. Set

$$Y = \{\rho(t); \rho(t) \in C_w([0,\infty); \dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2))\} \cap \{\rho(t); t^{1-\frac{1}{p}}\rho(t) \in C([0,\infty); L^p(\mathbb{R}^2))\}$$

endowed with the norm

$$\|\cdot\|_Y = \sup_{t\ge 0} \|\cdot\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)} + \sup_{t\ge 0} t^{1-\frac{1}{p}}\|\cdot\|_{L^p(\mathbb{R}^2)}$$

where $v(t) \in C_w([0,\infty); E)$ means that $v(t) \in C((0,\infty); E)$ and $v(t)$ is continuous at $t = 0$ in the weak topology $\sigma(E, E')$, that is

$$\lim_{t\to 0}\langle v(x,t) - v_0(x), \psi\rangle = 0, \psi(x) \in E'$$

where E' is the dual space of E.

Notes that(see[15]):

$$\sup_{t\ge 0} \|\cdot\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)} = \sup_{t\ge 0} t^{1-\frac{1}{p}}\|S(t)\cdot\|_{L^p(\mathbb{R}^2)}$$

then we have easily

$$\begin{aligned}\|S(t)\rho_0\|_Y &\le \sup_{t\ge 0} \|S(t)\rho_0\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)} + \sup_{t\ge 0} t^{1-\frac{1}{p}}\|S(t)\rho_0\|_{L^p(\mathbb{R}^2)} \\ &\le 2\|S(t)\rho_0\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)}.\end{aligned} \tag{3.15}$$

The existence result will be proved by applying the fixed point argument, so we need some basic estimates.

Step 1: Estimates of $\mathcal{A}[\rho, u_0]$

It follows from the definitions that

$$\mathcal{A}[\rho, u_0](t,x) = \int_0^t S(t-\tau)\mathrm{div}(\rho(\tau,x)S(\tau)u_0(x))d\tau$$

$$\|\mathcal{A}[\rho, u_0]\|_Y = \sup_{t\ge 0} \|\mathcal{A}[\rho, u_0]\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)} + \sup_{t\ge 0} t^{1-\frac{1}{p}}\|\mathcal{A}[\rho, u_0]\|_{L^p(\mathbb{R}^2)}$$

By the Sobolev embedding $L^1(\mathbb{R}^2) \hookrightarrow \dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)$ and (3.9)-(3.10), since $0 < \frac{1}{p} + \frac{1}{q} < 1$, one has

$$
\begin{aligned}
&\|\mathcal{A}[\rho, u_0]\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}} \le C\|\mathcal{A}[\rho, u_0]\|_{L^1} \\
&\le C \sup_{t\ge 0} \int_0^t |t-\tau|^{-\frac{1}{2}-(\frac{1}{p}+\frac{1}{q}-1)} \|\rho S(\tau) u_0(x)\|_{L^{\frac{pq}{p+q}}} d\tau \\
&= C \sup_{t\ge 0} \int_0^t |t-\tau|^{\frac{1}{2}-\frac{1}{p}-\frac{1}{q}} \|\rho S(\tau) u_0(x)\|_{L^{\frac{pq}{p+q}}} d\tau \\
&\le C \sup_{t\ge 0} \int_0^t |t-\tau|^{\frac{1}{2}-\frac{1}{p}-\frac{1}{q}} \|\rho\|_{L^p} \|S(\tau) u_0(x)\|_{L^q} d\tau \\
&= C \sup_{t\ge 0} \int_0^t \tau^{-\frac{3}{2}+\frac{1}{p}+\frac{1}{q}} |t-\tau|^{\frac{1}{2}-\frac{1}{p}-\frac{1}{q}} (\tau^{1-\frac{1}{p}} \|\rho\|_{L^p})(\tau^{\frac{1}{2}-\frac{1}{q}} \|S(\tau) u_0(x)\|_{L^q}) d\tau \\
&\le C \sup_{t\ge 0} \int_0^t \tau^{-\frac{3}{2}+\frac{1}{p}+\frac{1}{q}} |t-\tau|^{\frac{1}{2}-\frac{1}{p}-\frac{1}{q}} d\tau \|\rho\|_Y \|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}} \\
&= C \int_0^1 |1-s|^{\frac{1}{2}-\frac{1}{p}-\frac{1}{q}} s^{-\frac{3}{2}+\frac{1}{p}+\frac{1}{q}} ds \|\rho\|_Y \|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}} \\
&\le C\|\rho\|_Y \|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}},
\end{aligned}
$$

where we have used the fact that for $2 < p < 4, 2 < q < 4$, it holds that $-\frac{3}{2} + \frac{1}{p} + \frac{1}{q} > -1$, and $\frac{1}{2} - \frac{1}{p} - \frac{1}{q} > -1$, so

$$
\int_0^1 |1-s|^{\frac{1}{2}-\frac{1}{p}-\frac{1}{q}} s^{-\frac{3}{2}+\frac{1}{p}+\frac{1}{q}} ds
$$

is a finite constant. Therefore

$$
\sup_{t\ge 0} \|\mathcal{A}[\rho, u_0]\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}} \le C\|\rho\|_Y \|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}}. \tag{3.16}
$$

Next, we can also reduce from (3.9)-(3.10) that

$$
\begin{aligned}
&\sup_{t\ge 0} t^{1-\frac{1}{p}} \|\mathcal{A}[\rho, u_0]\|_{L^p(\mathbb{R}^2)} \le \sup_{t\ge 0} t^{1-\frac{1}{p}} \int_0^t |t-\tau|^{-\frac{1}{2}-(\frac{1}{p}+\frac{1}{q}-\frac{1}{p})} \|\rho S(\tau) u_0\|_{L^{\frac{pq}{p+q}}} d\tau \\
&\le \sup_{t\ge 0} t^{1-\frac{1}{p}} \int_0^t \tau^{-\frac{3}{2}+\frac{1}{p}+\frac{1}{q}} |t-\tau|^{-\frac{1}{2}-\frac{1}{q}} (\tau^{1-\frac{1}{p}} \|\rho\|_{L^p})(\tau^{\frac{1}{2}-\frac{1}{q}} \|S(\tau) u_0\|_{L^q}) d\tau \\
&\le \sup_{t\ge 0} \int_0^t t^{1-\frac{1}{p}} |t-\tau|^{-\frac{1}{2}-\frac{1}{q}} \tau^{-\frac{3}{2}+\frac{1}{p}+\frac{1}{q}} d\tau \|\rho\|_Y \|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}} \\
&= \int_0^1 |1-s|^{-\frac{1}{2}-\frac{1}{q}} s^{-\frac{3}{2}+\frac{1}{p}+\frac{1}{q}} ds \|\rho\|_Y \|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}} \\
&\le C\|\rho\|_Y \|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}},
\end{aligned}
$$

since $\int_0^1 |1-s|^{-\frac{1}{2}-\frac{1}{q}} s^{-\frac{3}{2}+\frac{1}{p}+\frac{1}{q}} ds$ is finite constant for $2 < p < 4, 2 < q < 4$ also. Thus we have

$$
\sup_{t\ge 0} t^{1-\frac{1}{p}} \|\mathcal{A}[\rho, u_0]\|_{L^p(\mathbb{R}^2)} \le C\|\rho\|_Y \|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}}. \tag{3.17}
$$

Combining (3.16) with (3.17), we have

$$\|\mathcal{A}[\rho, u_0]\|_Y \le C_0 \|\rho\|_Y \|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}}. \tag{3.18}$$

Step 2: Estimates of $\mathcal{B}[\rho,\rho]$
Exactly as step 1, we know

$$\mathcal{B}[\rho,\rho](t,x) = \int_0^t S(t-\tau)\mathrm{div}(\rho\mathcal{C}[\rho])(\tau,x)d\tau$$
$$\mathcal{C}[\rho] = \int_0^t S(t-\tau)\nabla\rho d\tau$$
$$\|\mathcal{B}[\rho,\rho]\|_Y = \sup_{t\ge0}\|\mathcal{B}[\rho,\rho]\|_{\dot{B}^{-2+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)} + \sup_{t\ge0} t^{1-\frac{1}{p}}\|\mathcal{B}[\rho,\rho]\|_{L^p(\mathbb{R}^2)}$$

Using the Sobolev embedding $L^1(\mathbb{R}^2) \hookrightarrow \dot{B}^{-2+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)$ and(3.9)-(3.10)again, one can obtain

$$\begin{aligned}
&\|\mathcal{B}[\rho,\rho]\|_{\dot{B}^{-2+\frac{2}{p}}_{p,\infty}} \le C\|\mathcal{B}[\rho,\rho]\|_{L^1}\\
&\le C\sup_{t\ge0}\int_0^t |t-\tau|^{-\frac{1}{2}-(\frac{2}{p}-1)}\|\rho\mathcal{C}[\rho]\|_{L^{\frac{p}{2}}}d\tau\\
&\le C\sup_{t\ge0}\int_0^t |t-\tau|^{\frac{1}{2}-\frac{2}{p}}\|\rho\|_{L^p}\|\mathcal{C}[\rho]\|_{L^p}d\tau\\
&= C\sup_{t\ge0}\int_0^t |t-\tau|^{\frac{1}{2}-\frac{2}{p}}\tau^{-1+\frac{1}{p}}(\tau^{1-\frac{1}{p}}\|\rho\|_{L^p})\|\mathcal{C}[\rho]\|_{L^p}d\tau\\
&\le C\sup_{t\ge0}\int_0^t |t-\tau|^{\frac{1}{2}-\frac{2}{p}}\tau^{-1+\frac{1}{p}}(\int_0^\tau |\tau-s|^{-\frac{1}{2}}\|\rho\|_{L^p}ds)d\tau\|\rho\|_Y\\
&= C\sup_{t\ge0}\int_0^t |t-\tau|^{\frac{1}{2}-\frac{2}{p}}\tau^{-1+\frac{1}{p}}(\int_0^\tau |\tau-s|^{-\frac{1}{2}}s^{-1+\frac{1}{p}}(s^{1-\frac{1}{p}}\|\rho\|_{L^p})ds)d\tau\|\rho\|_Y\\
&\le C\sup_{t\ge0}\int_0^t |t-\tau|^{\frac{1}{2}-\frac{2}{p}}\tau^{-1+\frac{1}{p}}(\int_0^\tau |\tau-s|^{-\frac{1}{2}}s^{-1+\frac{1}{p}}ds)d\tau\|\rho\|_Y^2\\
&\le C\sup_{t\ge0}\int_0^t |t-\tau|^{\frac{1}{2}-\frac{2}{p}}\tau^{-\frac{3}{2}+\frac{2}{p}}d\tau\|\rho\|_Y^2\\
&= C\int_0^1 |1-\theta|^{\frac{1}{2}-\frac{2}{p}}\theta^{-\frac{3}{2}+\frac{2}{p}}d\theta\|\rho\|_Y^2\\
&= C\|\rho\|_Y^2, \text{with} 2 < p < 4,
\end{aligned}$$

where we have used

$$\int_0^\tau |\tau-s|^{-\frac{1}{2}}s^{-1+\frac{1}{p}}ds = \tau^{-\frac{1}{2}+\frac{1}{p}}\int_0^1 |1-\eta|^{-\frac{1}{2}}\eta^{-1+\frac{1}{p}}d\eta = C\tau^{-\frac{1}{2}+\frac{1}{p}}$$

and the fact that both

$$\int_0^1 |1-\theta|^{\frac{1}{2}-\frac{2}{p}}\theta^{-\frac{3}{2}+\frac{2}{p}}d\theta \text{ and } \int_0^1 |1-\eta|^{-\frac{1}{2}}\eta^{-1+\frac{1}{p}}d\eta$$

are finite constants since $2 < p < 4$. Therefore,

$$\sup_{t\geq 0} \|\mathcal{B}[\rho,\rho]\|_{\dot{B}^{-2+\frac{2}{p}}_{p,\infty}} \leq C\|\rho\|_Y^2. \tag{3.19}$$

Similarly ,

$$\begin{aligned}
&\sup_{t\geq 0} t^{1-\frac{1}{p}}\|\mathcal{B}[\rho,\rho]\|_{L^p} \leq \sup_{t\geq 0}\int_0^t t^{1-\frac{1}{p}}|t-\tau|^{-\frac{1}{2}-(\frac{2}{p}-\frac{1}{p})}\|\rho\mathcal{C}[\rho]\|_{L^{\frac{p}{2}}}d\tau\\
&\leq \sup_{t\geq 0}\int_0^t t^{1-\frac{1}{p}}|t-\tau|^{-\frac{1}{2}-\frac{1}{p}}\tau^{-1+\frac{1}{p}}(\tau^{1-\frac{1}{p}}\|\rho\|_{L^p})\|\mathcal{C}[\rho]\|_{L^p}d\tau\\
&\leq \sup_{t\geq 0}\int_0^t t^{1-\frac{1}{p}}|t-\tau|^{-\frac{1}{2}-\frac{1}{p}}\tau^{-1+\frac{1}{p}}\|\mathcal{C}[\rho]\|_{L^p}d\tau\|\rho\|_Y\\
&\leq \sup_{t\geq 0}\int_0^t t^{1-\frac{1}{p}}|t-\tau|^{-\frac{1}{2}-\frac{1}{p}}\tau^{-1+\frac{1}{p}}(\int_0^\tau|\tau-s|^{-\frac{1}{2}}s^{-1+\frac{1}{p}}(s^{1-\frac{1}{p}}\|\rho\|_{L^p})ds)d\tau\|\rho\|_Y\\
&\leq \sup_{t\geq 0}\int_0^t t^{1-\frac{1}{p}}|t-\tau|^{-\frac{1}{2}-\frac{1}{p}}\tau^{-\frac{3}{2}+\frac{2}{p}}d\tau\|\rho\|_Y^2\\
&= \int_0^1 |1-\theta|^{-\frac{1}{2}-\frac{1}{p}}\theta^{-\frac{3}{2}+\frac{2}{p}}\|\rho\|_Y^2\\
&= C\|\rho\|_Y^2, \text{with} 2 < p < 4.
\end{aligned}$$

This, together with (3.19) implies the desired estimate

$$\|\mathcal{B}[\rho,\rho]\|_Y \leq C_1\|\rho\|_Y^2. \tag{3.20}$$

Step 3: Fixed point argument

Consider now

$$Y_\delta = \{\rho(t)\in Y, \|\rho\|_Y \leq 2C\delta\};$$
$$d(\rho_1,\rho_2) = \sup_{t\geq 0} t^{1-\frac{1}{p}}\|\rho_1-\rho_2\|_{L^p}, \forall \rho_1,\rho_2\in Y_\delta,$$

and

$$\mathcal{T}\rho = S(t)\rho_0 - \mathcal{A}[\rho,u_0] + \mathcal{B}[\rho,\rho].$$

Similarly as in step 1 and step 2, we have

$$t^{1-\frac{1}{p}}\|\mathcal{A}[\rho,u_0]\|_{L^p} \leq C(\sup_{0\leq\tau\leq t}\tau^{1-\frac{1}{p}}\|\rho\|_{L^p}) \tag{3.21}$$

$$t^{1-\frac{1}{p}}\|\mathcal{B}[\rho,\rho]\|_{L^p} \leq C(\sup_{0\leq\tau\leq t}\tau^{1-\frac{1}{p}}\|\rho\|_{L^p})^2 \tag{3.22}$$

$$\|\mathcal{A}[\rho,u_0]\|_{L^1} \leq C(\sup_{0\leq\tau\leq t}\tau^{1-\frac{1}{p}}\|\rho\|_{L^p}) \tag{3.23}$$

$$\|\mathcal{B}[\rho,\rho]\|_{L^1} \leq C(\sup_{0\leq\tau\leq t}\tau^{1-\frac{1}{p}}\|\rho\|_{L^p})^2 \tag{3.24}$$

Obviously, if $\rho_0 \in \dot{B}^{-2+\frac{2}{p}}_{p,\infty}$, then

$$S(t)\rho_0 \in \dot{B}^{-2+\frac{2}{p}}_{p,\infty}, t^{1-\frac{1}{p}}S(t)\rho_0 \in C([0,\infty);L^p)$$

and

$$\langle S(t)\rho_0-\rho_0,\psi(x)\rangle=\langle S(t)\rho_0,\psi(x)\rangle-\langle\rho_0,\psi(x)\rangle\to 0,\ \text{ as } t\to 0.$$

This means $S(t)\rho_0\in C_w([0,\infty);\dot{B}^{-2+\frac{2}{p}}_{p,\infty})$. So it follows from (3.23)-(3.24) and $t^{1-\frac{1}{p}}\rho\in C([0,\infty);L^p)$ that

$$\begin{aligned}\mathcal{T}\rho-S(t)\rho_0&=-\mathcal{A}[\rho,u_0]+\mathcal{B}[\rho,\rho]\in C([0,\infty);L^1);\\ \forall\psi\in\dot{B}^{2-\frac{2}{p}}_{p',1},\langle\mathcal{T}\rho-S(t)\rho_0-\rho_0,\psi\rangle&\le\|\mathcal{T}\rho-S(t)\rho_0-\rho_0\|_{\dot{B}^{-2+\frac{2}{p}}_{p,\infty}}\|\psi\|_{\dot{B}^{2-\frac{2}{p}}_{p',1}};\\ &\le C\|\mathcal{T}\rho-S(t)\rho_0-\rho_0\|_{L^1}\|\psi\|_{\dot{B}^{2-\frac{2}{p}}_{p',1}}\to 0,\ \text{ as } t\to 0.\end{aligned}$$

Consequently,

$$\mathcal{T}\rho-S(t)\rho_0\in C_w([0,\infty);\dot{B}^{-2+\frac{2}{p}}_{p,\infty}),\mathcal{T}\rho\in C_w([0,\infty);\dot{B}^{-2+\frac{2}{p}}_{p,\infty}).$$

Moreover, since $t^{1-\frac{1}{p}}\rho\in C([0,\infty);L^p)$ and from (3.21)-(3.22), we deduce that

$$t^{1-\frac{1}{p}}\mathcal{T}\rho\in C([0,\infty);L^p).$$

On the other hand, it follows from step 1 and step 2 that

$$\begin{aligned}\|\mathcal{T}\rho\|_{Y_\delta}&\le 2\|\rho_0\|_{\dot{B}^{-2+\frac{2}{p}}_{p,\infty}}+C_0\|\rho\|_{Y_\delta}\|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}}+C_1\|\rho\|^2_{Y_\delta}\\ &\le 2\delta+2C_0C\eta\delta+4C_1C^2\delta^2=\delta(2+2C_0C\eta+4C_1C^2\delta).\end{aligned}$$

Now then we can choose $C\ge 3$ fixed, such that for η and δ small enough,$2C_0\eta\le\frac{1}{2},4C_1C^2\delta\le C$. Therefore,

$$\|\mathcal{T}\rho\|_{Y_\delta}\le 2C\delta.$$

Next, note that

$$\begin{aligned}d(\mathcal{T}\rho_1,\mathcal{T}\rho_2)&\le\sup_{t\ge0}t^{1-\frac{1}{p}}\|\int_0^t S(t-\tau)\mathrm{div}[(\rho_1(\tau,x)-\rho_2(\tau,x))S(\tau)u_0(x)]d\tau\|_{L^p}\\ &\quad+\sup_{t\ge0}t^{1-\frac{1}{p}}\|\int_0^t S(t-\tau)\mathrm{div}(\rho_1\mathcal{C}[\rho_1]-\rho_2\mathcal{C}[\rho_2])(\tau,x)d\tau\|_{L^p}\\ &=I_1+I_2.\end{aligned}$$

we can estimate I_1 and I_2 as

$$\begin{aligned}I_1&\le C\sup_{t\ge0}t^{1-\frac{1}{p}}\int_0^t|t-\tau|^{-\frac{1}{2}-\frac{1}{q}}\|(\rho_1-\rho_2)S(\tau)u_0\|_{L^{\frac{pq}{p+q}}}d\tau\\ &\le C\sup_{t\ge0}t^{1-\frac{1}{p}}\int_0^t|t-\tau|^{-\frac{1}{2}-\frac{1}{q}}\|\rho_1-\rho_2\|_{L^p}\|S(\tau)u_0\|_{L^q}d\tau\\ &\le C\sup_{t\ge0}t^{1-\frac{1}{p}}\int_0^t|t-\tau|^{-\frac{1}{2}-\frac{1}{q}}\tau^{-\frac{3}{2}+\frac{1}{p}+\frac{1}{q}}d\tau\|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}}d(\rho_1,\rho_2)\\ &=C\|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}}d(\rho_1,\rho_2),\end{aligned}$$

and

$$
\begin{aligned}
I_2 &\le C \sup_{t\ge 0} t^{1-\frac{1}{p}} \int_0^t |t-\tau|^{-\frac{1}{2}-\frac{1}{p}} \|\rho_1 \mathcal{C}[\rho_1] - \rho_2 \mathcal{C}[\rho_2]\|_{L^{\frac{p}{2}}} d\tau \\
&\le C \sup_{t\ge 0} t^{1-\frac{1}{p}} \int_0^t |t-\tau|^{-\frac{1}{2}-\frac{1}{p}} \big(\|\rho_1-\rho_2\|_{L^p} \|\mathcal{C}[\rho_1]\|_{L^p} + \|\rho_2\|_{L^p} \|\mathcal{C}[\rho_1-\rho_2]\|_{L^p} \big) d\tau \\
&\le C \sup_{t\ge 0} t^{1-\frac{1}{p}} \int_0^t |t-\tau|^{-\frac{1}{2}-\frac{1}{p}} \tau^{-\frac{3}{2}+\frac{2}{p}} d\tau (\|\rho_1\|_Y + \|\rho_2\|_Y) d(\rho_1,\rho_2) \\
&\le C(\|\rho_1\|_Y + \|\rho_2\|_Y) d(\rho_1,\rho_2).
\end{aligned}
$$

So

$$
\begin{aligned}
d(\mathcal{T}\rho_1, \mathcal{T}\rho_2) &\le C(\|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}} + \|\rho_1\|_Y + \|\rho_2\|_Y) d(\rho_1,\rho_2) \\
&\le C(\eta + 4C\delta) d(\rho_1,\rho_2).
\end{aligned}
$$

Thus,for δ and η small enough, $\mathcal{T}$ is a contraction mapping from Y_δ into itself. So the Banach contraction mapping principle implies that there exists a unique solution $\rho \in Y_\delta$ to the problem (3.8). This proves our theorem. $\square$

With $\rho(x,t)$ at hand, one obtains $u(x,t)$ by the equation (3.7), i.e.

$$
\begin{aligned}
u(x,t) &= S(t)u_0(x) - \int_0^t S(t-\tau)\nabla\rho(\tau,x) d\tau \\
&:= S(t)u_0(x) - \mathcal{C}[\rho].
\end{aligned}
\tag{3.25}
$$

Furthermore, $u(x,t)$ can be estimated easily as follows. First,

$$
\|u\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}} \le \|S(t)u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}} + \|\mathcal{C}[\rho]\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}}. \tag{3.26}
$$

From (3.9), we deduce easily

$$
\begin{aligned}
\|S(t)u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}} &= \sup_{t\ge 0} t^{\frac{1}{2}-\frac{1}{q}} \|S(t)(S(t)u_0)\|_{L^q} \\
&\le C \sup_{t\ge 0} t^{\frac{1}{2}-\frac{1}{q}} \|S(t)u_0\|_{L^q} = \|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}}.
\end{aligned}
\tag{3.27}
$$

In a similar way as in step 1, using the Sobolev embedding $L^2(\mathbb{R}^2) \hookrightarrow \dot{B}^{-1+\frac{2}{q}}_{q,\infty}(\mathbb{R}^2)$ and (3.10), we can compute:

$$
\begin{aligned}
&\|\mathcal{C}[\rho]\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}} \le C\|\mathcal{C}[\rho]\|_{L^2} \\
&\le C \sup_{t\ge 0} \int_0^t |t-\tau|^{-\frac{1}{2}+\frac{1}{2}-\frac{1}{p}} \|\rho\|_{L^p} d\tau \\
&= \sup_{t\ge 0} \int_0^t |t-\tau|^{-\frac{1}{p}} \tau^{-1+\frac{1}{p}} (\tau^{1-\frac{1}{p}} \|\rho\|_{L^p}) d\tau \\
&\le C \sup_{t\ge 0} \int_0^t |t-\tau|^{-\frac{1}{p}} \tau^{-1+\frac{1}{p}} d\tau \|\rho\|_Y \\
&= C \int_0^1 |1-s|^{-\frac{1}{p}} s^{-1+\frac{1}{p}} ds \|\rho\|_Y \le C\|\rho\|_Y.
\end{aligned}
$$

Thus from Theorem 3.1 and (3.27), one has

$$\sup_{t\geq 0}\|u\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}}\leq C(\|\rho_0\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)},\|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2)}). \tag{3.28}$$

So combining with Theorem 3.1, we have

Theorem 3.2. *For any* $\rho_0(x)\in\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)$ *and* $u_0(x)\in\dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2)$*,if*

$$\|\rho_0\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)}\leq\delta,\|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2)}\leq\eta$$

with $2<p<4, 2<q<4$*, and* $\delta,\eta>0$ *are small. Then there exists a unique solution* $(\rho(x,t),u(x,t))$ *of* (3.3)-(3.5) *which satisfies:*

$$\begin{aligned}&\rho(t)\in C_w([0,\infty);\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2));u(t)\in C_w([0,\infty);\dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2));\\&\rho(t)-S(t)\rho_0(x)\in C([0,\infty);L^1(\mathbb{R}^2));t^{1-\frac{1}{p}}\rho(t)\in C([0,\infty);L^p(\mathbb{R}^2));\\&\sup_{t\geq 0}\|\rho\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)}+\sup_{t\geq 0}t^{1-\frac{1}{p}}\|\rho\|_{L^p(\mathbb{R}^2)}\leq C;\sup_{t\geq 0}\|u\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}}\leq C,\end{aligned}$$

where C *depends only on* $\|\rho_0\|_{\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2)}$ *and* $\|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2)}$.

4 The passage to limit

In this section, we would like to pass the limit $\epsilon\to 0^+$ in the approximate system

$$\rho_t+\operatorname{div}(\rho u)=\epsilon\triangle\rho, \tag{4.1}$$

$$u_t-\triangle u+\nabla\rho=0 \tag{4.2}$$

to obtain the desired solution to (1.1)-(1.2) with(3.5). From Theorem 3.1 and Theorem 3.2, we know there exist a unique solution $(\rho^\epsilon,u^\epsilon)$ in $C_w([0,\infty);\dot{B}_{p,\infty}^{-2+\frac{2}{p}})\times C_w([0,\infty);\dot{B}_{q,\infty}^{-1+\frac{2}{q}})$.

Note that the homogeneous Besov spaces$\dot{B}_{p,\infty}^{-\alpha}$ are not separable, so the convergence will be taken in the weak topology $\sigma(\dot{B}_{p,\infty}^{-\alpha},\dot{B}_{p',1}^{\alpha})$ (where p' is the conjugate exponent of p). Observe that an equivalent convergence condition in these spaces is given by the following definition(see [3]):

Definition 4.1. *Let* B *is a Banach functional space, Then a sequence* f_j *of vectors in* B *converges weakly to* $f\in B$*, if the sequence* $\|f_j\|_B$ *is bounded and* $f_j\rightharpoonup f$ *in the sense of distributions.*

Since we cannot obtain any energy bounds about homogeneous space, the above principle will be difficult to be used in the passage to limit. However, we can do it in the subset of our homogeneous space which are Sobolev spaces and Orlicz spaces. Indeed, we know

$$L^2(\mathbb{R}^2)\hookrightarrow\dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2);L_M(\mathbb{R}^2)\hookrightarrow H^{-1}(\mathbb{R}^2)\hookrightarrow\dot{B}_{p,\infty}^{-2+\frac{2}{p}}(\mathbb{R}^2). \tag{4.3}$$

with $p>1$ and $q>2$. Where $L_M(\mathbb{R}^2)$ denotes the Orlicz space defined over $\mathbb{R}^2$ with $M=M(s)=(1+s)\log(1+s)-s$.

To continue, we need some a priori estimates.

Lemma 4.1. *Let (ρ, u) be smooth solution of (4.1)-(4.2), and set*

$$E(t) = \int_{\mathbb{R}^2} [\frac{1}{2}u^2 + (\rho+1)\log(1+\rho)](x,t)dx.$$

Then the following energy inequality holds:

$$E(t) + \int_0^t \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \epsilon \frac{|\nabla \rho|^2}{1+\rho} dx d\tau \leq E(0)\exp(\frac{T}{2}) \tag{4.4}$$

for any $t \in [0, T], 0 < T < \infty$.

Proof. Direct computations using (4.1)-(4.2) give

$$\frac{dE}{dt} = \int_{\mathbb{R}^2} [u\triangle u - u\nabla\rho + \epsilon\triangle\rho\log(1+\rho) - \mathrm{div}(\varrho u)\log(1+\rho)]dx$$
$$= \int_{\mathbb{R}^2} [-|\nabla u|^2 - \epsilon\frac{|\nabla\rho|^2}{1+\rho} - \frac{u\nabla\rho}{1+\rho}]dx.$$

Thus

$$\frac{dE}{dt} + \int_{\mathbb{R}^2} |\nabla u|^2 + \epsilon\frac{|\nabla\rho|^2}{1+\rho}dx \leq \int_{\mathbb{R}^2} |\mathrm{div}u|\log(1+\rho)dx. \tag{4.5}$$

On the other hand, using Hölder inequality and Young's inequality, one has

$$\int_{\mathbb{R}^2} |\mathrm{div}u|\log(1+\rho)dx \leq (\int_{\mathbb{R}^2} |\mathrm{div}u|^2 dx)^{\frac{1}{2}}(\int_{\mathbb{R}^2} (\log(1+\rho))^2 dx)^{\frac{1}{2}}$$
$$\leq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} (\log(1+\rho))^2 dx$$
$$\because \log(1+\rho) \leq 1+\rho, \forall \rho \geq 0$$
$$\therefore \int_{\mathbb{R}^2} |\mathrm{div}u|\log(1+\rho)dx \leq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} (1+\rho)\log(1+\rho)dx.$$

That is

$$\int_{\mathbb{R}^2} |\mathrm{div}u|\log(1+\rho)dx \leq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2}E(t). \tag{4.6}$$

So from (4.5) and (4.6), we know

$$\frac{dE}{dt} + \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \epsilon\frac{|\nabla\rho|^2}{1+\rho}dx \leq \frac{1}{2}E(t).$$

Then by Gronwall's lemma, (4.4)holds. □

Remark 4.1. *The above lemma holds on the following restriction of the initial data:*

$$\|u_0\|_{L^2(\mathbb{R}^2)} \leq C; \|\rho_0\|_{L^1(\mathbb{R}^2)} \leq C, \|\rho_0\|_{L^M(\mathbb{R}^2)} \leq C.$$

and consequently, $\rho(x,t) \in L_M(\mathbb{R}^2)$.

Now, from lemma 4.1 and (4.5), we know

$$\sup_{0\le t\le T}\|u^\epsilon\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}} \le C\sup_{0\le t\le T}\|u^\epsilon\|_{L^2}\le C_1; \tag{4.7}$$

$$\sup_{0\le t\le T}\|\rho^\epsilon\|_{\dot{B}^{-2+\frac{2}{p}}_{p,\infty}} \le C\sup_{0\le t\le T}\|\rho^\epsilon\|_{H^{-1}}\le C\sup_{0\le t\le T}\|\rho^\epsilon\|_{L_M}\le C_2, \tag{4.8}$$

where C_1, C_2 are independent of ϵ. In addition,

$$\|\nabla u^\epsilon\|_{L^2([0,T];L^2(\mathbb{R}^2))}\le C_3,$$

C_3 is independent of ϵ also. So we can extract a subsequence of $(\rho^\epsilon, u^\epsilon)$, still denoted by $(\rho^\epsilon, u^\epsilon)$, such that

$$\rho^\epsilon \rightharpoonup \rho, \text{weak-* in } L^\infty([0,T];L_M(\mathbb{R}^2)); \tag{4.9}$$

$$u^\epsilon \rightharpoonup u, \text{for some } u, \text{weakly in } L^2([0,T];H^1(\mathbb{R}^2)). \tag{4.10}$$

Where $T>0$ is an arbitrary but fixed constant. Moreover, by the lower semi-continuity of weak convergence, $\nabla u\in L^2([0,T];L^2(\mathbb{R}^2))$.

In order to prove that (ρ,u) obtained in (4.9)-(4.10) is indeed a weak solution of (1.1)-(1.2) and (3.5), we need the following key compactness lemma, which gives some compactness concerning H^1 and L_M. It can be found in [20]:

Lemma 4.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, and p,q be conjugate numbers. Assume that $\{g^\epsilon(t,x)\}$ and $\{v^\varepsilon(t,x)\}$ are bounded uniformly in ϵ in $L^p([0,T];L_M(\Omega))$ and $L^q([0,T];H^{\frac{N}{2}}(\Omega))$ respectively and that, as $\epsilon\to 0, g^\epsilon$ and v^ϵ converge weakly to g and v in $L^p([0,T];L_M(\Omega))$ and $L^q([0,T];H^{\frac{N}{2}}(\Omega))$. Moreover, if $\{\partial_t g^\epsilon\}$ is uniformly bounded in $L^\lambda([0,T];W^{-m,1}(\Omega))$ for some $\lambda>1$ and $m>0$, then $g^\epsilon v^\epsilon$ converges weakly to gv in the sense of $\mathcal{D}'((0,T)\times\Omega)$ as $\epsilon\to 0$.*

For any $K\subset\mathbb{R}^2$, the estimates (4.4)imply that u^ϵ is bounded from above in$L^2([0,T];H^1(K))$ uniformly in ϵ provided that ϵ is sufficiently small, while by equations (4.1) and (4.4) we find that $\partial_t\rho^\epsilon$ is uniformly bounded in $L^\infty([0,T];W^{-1,1}(K))$. Thus, applying lemma 4.2, we obtain

$$\rho^\epsilon u^\epsilon \rightharpoonup \rho u, \text{in } \mathcal{D}'([0,T]\times K) \tag{4.11}$$

Letting $\epsilon\to 0$ in (4.1)-(4.2), and using (4.9)-(4.11), we see that (ρ,u) satisfies (1.1)-(1.2) in the sense of distribution. Thus we have the following theorem:

Theorem 4.1. *For any $\rho_0(x)\in\dot{B}^{-2+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)$ and $u_0(x)\in\dot{B}^{-1+\frac{2}{q}}_{q,\infty}(\mathbb{R}^2)$,if*

$$\|\rho_0\|_{\dot{B}^{-2+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)}\le\delta, \|\rho_0\|_{L_M(\mathbb{R}^2)}\le C_1, \|\rho_0\|_{L^1(\mathbb{R}^2)}\le C_2;$$

$$\|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}}\le\eta, \|u_0\|_{L^2(\mathbb{R}^2)}\le C_3.$$

with $\gamma=1, 2<p<4, 2<q<4, \delta$ and η are small, C_1, C_2 and C_3 are absolute constants. Then there exists a solution $(\rho(x,t),u(x,t))$ of (1.1)-(1.2) in the space

$$C_w([0,T];\dot{B}^{-2+\frac{2}{p}}_{p,\infty})\times C_w([0,T];\dot{B}^{-1+\frac{2}{q}}_{q,\infty}), \forall 0<T<\infty.$$

5 The general case

In this section, we will consider

$$\rho_t + \mathrm{div}(\rho u) = 0 \tag{5.1}$$
$$u_t - \triangle u + \nabla \rho^\gamma = 0 \tag{5.2}$$
$$\rho(x,0) = \rho_0(x), u(x,0) = u_0(x) \tag{5.3}$$

with $x \in \mathbb{R}^2, \gamma > 1$.

It has been proved in[14] that there exist a unique smooth solution to (5.1)-(5.3).

Obviously, if $(\rho, u)(x,t)$ solves (5.1)-(5.3), then the re-scaled pair $(\rho_\lambda, u_\lambda)(x,t)$, defined by

$$\rho_\lambda(x,t) = \lambda^{\frac{2}{\gamma}} \rho(\lambda x, \lambda^2 t), u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

is also a solution for each $\lambda > 0$. So the initial function $(\rho_0(x), u(x))$ must belong to homogeneous space $(\dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}, \dot{B}_{q,\infty}^{-1+\frac{2}{q}})$ with $p > \gamma, q > 2, \gamma > 1$.

The existence of solution for (5.1)-(5.3) in homogenous space will be obtained by using similar ideas as in last section. Thus, consider the following approximation problem:

$$\rho_t + \mathrm{div}(\rho u) = \epsilon \triangle \rho \tag{5.4}$$
$$u_t - \triangle u + \nabla \rho^\gamma = 0 \tag{5.5}$$
$$\rho(x,0) = \rho_0(x), u(x,0) = u_0(x) \tag{5.6}$$

with $x \in \mathbb{R}^2, \gamma > 1$.

Remark 5.1. *Notes that the approximation system* (5.4)-(5.6) *is invariant under scaling* $(\rho(x,t), u(x,t)) = (\lambda^{\frac{2}{\gamma}} \rho(\lambda x, \lambda^2 t), \lambda u(\lambda x, \lambda^2 t)$*, so the system* (5.1)-(5.3) *can be approximated by the system* (5.4)-(5.6) *in homogeneous spaces.*

As in the case of $\gamma = 1$, we have the following the existence result on the problem (5.4)-(5.6)(we also assume $\epsilon = 1$):

Theorem 5.1. *For any* $\rho_0(x) \in \dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}(\mathbb{R}^2)$ *and* $u_0(x) \in \dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2)$*, if*

$$\|\rho_0\|_{\dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}(\mathbb{R}^2)} \le \delta \text{ and } \|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2)} \le \eta \tag{5.7}$$

with $p > 2\gamma, \gamma \ge 2, q > 2$*, for* δ *and* $\eta > 0$ *suitably small. Then there exists a unique solution* $(\rho(x,t), u(x,t))$ *to the problem* (5.4)-(5.6) *satisfying:*

$$\rho(t) \in C_w([0,\infty); \dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}(\mathbb{R}^2)); t^{\frac{1}{\gamma}-\frac{1}{p}} \rho(t) \in C([0,\infty); L^p(\mathbb{R}^2)); \tag{5.8}$$
$$\sup_{t\ge 0} \|\rho\|_{\dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}(\mathbb{R}^2)} + \sup_{t\ge 0} t^{\frac{1}{\gamma}-\frac{1}{p}} \|\rho\|_{L^p(\mathbb{R}^2)} \le C_1; \tag{5.9}$$
$$\rho(t) - S(t)\rho_0(x) \in C([0,\infty); L^\gamma(\mathbb{R}^2)); \sup_{t\ge 0} \|u\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}} \le C_2. \tag{5.10}$$

Where C_1 *and* C_2 *are constants which depend only on* $\|\rho_0\|_{\dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}(\mathbb{R}^2)}$ *and* $\|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2)}$.

Proof. Indeed, we only need to replace the space Y in the proof of Theorem 3.1 by

$$Y' = \{\rho(t); \rho(t) \in C_w\big([0,\infty); \dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}(\mathbb{R}^2)\big)\} \cap \{\rho(t); t^{\frac{1}{\gamma}-\frac{1}{p}}\rho(t) \in C\big([0,\infty); L^p(\mathbb{R}^2)\big)\} \quad (5.11)$$

endowed with the norm

$$\|\cdot\|_{Y'} = \sup_{t\geq 0}\|\cdot\|_{\dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}(\mathbb{R}^2)} + \sup_{t\geq 0} t^{\frac{1}{\gamma}-\frac{1}{p}}\|\cdot\|_{L^p(\mathbb{R}^2)}$$

The approach to prove theorem is similar to that of Theorem 3.1, so we only sketch some key estimates. We start from

$$\mathcal{A}[\rho,u_0](t,x) = \int_0^t S(t-\tau)\mathrm{div}(\rho(\tau,x)S(\tau)u_0(x))d\tau$$

$$\|\mathcal{A}[\rho,u_0]\|_{Y'} = \sup_{t\geq 0}\|\mathcal{A}[\rho,u_0]\|_{\dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}(\mathbb{R}^2)} + \sup_{t\geq 0} t^{\frac{1}{\gamma}-\frac{1}{p}}\|\mathcal{A}[\rho,u_0]\|_{L^p(\mathbb{R}^2)}$$

By the Sobolev embedding $L^\gamma(\mathbb{R}^2) \hookrightarrow \dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}(\mathbb{R}^2)$ and using (3.9)-(3.10) for $0 < \frac{1}{p}+\frac{1}{q} < 1$, we deduce that

$$\begin{aligned}
&\|\mathcal{A}[\rho,u_0]\|_{\dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}} \leq C\|\mathcal{A}[\rho,u_0]\|_{L^\gamma}\\
&\leq C\sup_{t\geq 0}\int_0^t |t-\tau|^{-\frac{1}{2}-(\frac{1}{p}+\frac{1}{q}-\frac{1}{\gamma})}\|\rho S(t)u_0(x)\|_{L^{\frac{pq}{p+q}}}d\tau\\
&\leq C\sup_{t\geq 0}\int_0^t |t-\tau|^{\frac{1}{\gamma}-\frac{1}{2}-\frac{1}{p}-\frac{1}{q}}\|\rho\|_{L^p}\|S(t)u_0(x)\|_{L^q}d\tau\\
&= C\sup_{t\geq 0}\int_0^t \tau^{-\frac{1}{2}-\frac{1}{\gamma}+\frac{1}{p}+\frac{1}{q}}|t-\tau|^{\frac{1}{\gamma}-\frac{1}{2}-\frac{1}{p}-\frac{1}{q}}(\tau^{\frac{1}{\gamma}-\frac{1}{p}}\|\rho\|_{L^p})(\tau^{\frac{1}{2}-\frac{1}{q}}\|S(t)u_0(x)\|_{L^q})d\tau\\
&\leq C\sup_{t\geq 0}\int_0^t \tau^{-\frac{1}{\gamma}-\frac{1}{2}+\frac{1}{p}+\frac{1}{q}}|t-\tau|^{\frac{1}{\gamma}-\frac{1}{2}-\frac{1}{p}-\frac{1}{q}}d\tau\|\rho\|_{Y'}\|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}}\\
&= C\int_0^1 |1-s|^{\frac{1}{\gamma}-\frac{1}{2}-\frac{1}{p}-\frac{1}{q}}s^{-\frac{1}{2}-\frac{1}{\gamma}+\frac{1}{p}+\frac{1}{q}}ds\|\rho\|_{Y'}\|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}}\\
&\leq C\|\rho\|_{Y'}\|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}},
\end{aligned}$$

where since $p > 2\gamma, \gamma \geq 2, q > 2$, so

$$\int_0^1 |1-s|^{\frac{1}{\gamma}-\frac{1}{2}-\frac{1}{p}-\frac{1}{q}}s^{-\frac{1}{2}-\frac{1}{\gamma}+\frac{1}{p}+\frac{1}{q}}ds$$

is a finite constant. Thus

$$\sup_{t\geq 0}\|\mathcal{A}[\rho,u_0]\|_{\dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}} \leq C\|\rho\|_{Y'}\|u_0\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}}.$$

Next,we have

$$\begin{aligned}
&\sup_{t\geq 0} t^{\frac{1}{\gamma}-\frac{1}{p}}\|\mathcal{A}[\rho,u_0]\|_{L^p(\mathbb{R}^2)} \leq \sup_{t\geq 0} t^{\frac{1}{\gamma}-\frac{1}{p}}\int_0^t |t-\tau|^{-\frac{1}{2}-(\frac{1}{p}+\frac{1}{q}-\frac{1}{p})}\|\rho S(t)u_0\|_{L^{\frac{pq}{p+q}}}d\tau \\
&\leq \sup_{t\geq 0} t^{\frac{1}{\gamma}-\frac{1}{p}}\int_0^t \tau^{-\frac{1}{2}-\frac{1}{\gamma}+\frac{1}{p}+\frac{1}{q}}|t-\tau|^{-\frac{1}{2}-\frac{1}{q}}(\tau^{\frac{1}{\gamma}-\frac{1}{p}}\|\rho\|_{L^p})(\tau^{\frac{1}{2}-\frac{1}{q}}\|S(t)u_0\|_{L^q})d\tau \\
&\leq \sup_{t\geq 0}\int_0^t t^{\frac{1}{\gamma}-\frac{1}{p}}|t-\tau|^{-\frac{1}{2}-\frac{1}{q}}\tau^{-\frac{1}{2}-\frac{1}{\gamma}+\frac{1}{p}+\frac{1}{q}}d\tau\|\rho\|_{Y'}\|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}} \\
&= \int_0^1 |1-s|^{-\frac{1}{2}-\frac{1}{q}}s^{-\frac{1}{2}-\frac{1}{\gamma}+\frac{1}{p}+\frac{1}{q}}ds\|\rho\|_{Y'}\|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}} \\
&\leq C\|\rho\|_{Y'}\|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}},\ \text{ with } p>2\gamma, \gamma\geq 2, q>2.
\end{aligned}$$

It follows that

$$\sup_{t\geq 0} t^{\frac{1}{\gamma}-\frac{1}{p}}\|\mathcal{A}[\rho,u_0]\|_{L^p(\mathbb{R}^2)} \leq C\|\rho\|_{Y'}\|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}}.$$

Thus

$$\|\mathcal{A}[\rho,u_0]\|_{Y'} \leq C\|\rho\|_{Y'}\|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}}.$$

For the estimate of $\mathcal{B}[\rho,\rho](t,x)$,we need to make some changes. It becomes

$$\mathcal{B}[\rho,\rho](t,x) = \int_0^t S(t-\tau)\mathrm{div}(\rho\mathcal{C}[\rho])(\tau,x)d\tau,$$

with

$$\mathcal{C}[\rho] = \int_0^t S(t-\tau)\nabla\rho^\gamma d\tau.$$

To estimate

$$\|\mathcal{B}[\rho,\rho]\|_{Y'} = \sup_{t\geq 0}\|\mathcal{B}[\rho,\rho]\|_{\dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)} + \sup_{t\geq 0} t^{\frac{1}{\gamma}-\frac{1}{p}}\|\mathcal{B}[\rho,\rho]\|_{L^p(\mathbb{R}^2)},$$

one uses the Sobolev embedding $L^\gamma(\mathbb{R}^2)\hookrightarrow \dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)$ and(3.9)-(3.10) again for $p>2\gamma>$

$\gamma + 1$ to obtain

$$
\begin{aligned}
&\|\mathcal{B}[\rho,\rho]\|_{\dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}} \leq C\|\mathcal{B}[\rho,\rho]\|_{L^{\gamma}} \\
&\leq C\sup_{t\geq 0}\int_0^t |t-\tau|^{-\frac{1}{2}-(\frac{\gamma+1}{p}-\frac{1}{\gamma})}\|\rho\mathcal{C}[\rho]\|_{L^{\frac{p}{\gamma+1}}}d\tau \\
&\leq C\sup_{t\geq 0}\int_0^t |t-\tau|^{-\frac{1}{2}+\frac{1}{\gamma}-\frac{\gamma+1}{p}}\|\rho\|_{L^p}\|\mathcal{C}[\rho]\|_{L^{\frac{p}{\gamma}}}d\tau \\
&\leq C\sup_{t\geq 0}\int_0^t |t-\tau|^{-\frac{1}{2}+\frac{1}{\gamma}-\frac{\gamma+1}{p}}\tau^{-\frac{1}{\gamma}+\frac{1}{p}}(\int_0^{\tau}|\tau-s|^{-\frac{1}{2}}\|\rho^{\gamma}\|_{L^{\frac{p}{\gamma}}}ds)d\tau\|\rho\|_{Y'} \\
&\leq C\sup_{t\geq 0}\int_0^t |t-\tau|^{-\frac{1}{2}+\frac{1}{\gamma}-\frac{\gamma+1}{p}}\tau^{-\frac{1}{\gamma}+\frac{1}{p}}(\int_0^{\tau}|\tau-s|^{-\frac{1}{2}}s^{-1+\frac{\gamma}{p}}ds)d\tau\|\rho\|_{Y'}^{\gamma+1} \\
&\leq C\sup_{t\geq 0}\int_0^t |t-\tau|^{-\frac{1}{2}+\frac{1}{\gamma}-\frac{\gamma+1}{p}}\tau^{-\frac{1}{2}-\frac{1}{\gamma}+\frac{\gamma+1}{p}}d\tau\|\rho\|_{Y'}^{\gamma+1} \\
&= C\int_0^1 |1-\theta|^{-\frac{1}{2}+\frac{1}{\gamma}-\frac{\gamma+1}{p}}\theta^{-\frac{1}{2}-\frac{1}{\gamma}+\frac{\gamma+1}{p}}d\theta\|\rho\|_{Y'}^{\gamma+1} \\
&= C\|\rho\|_{Y'}^{\gamma+1}.
\end{aligned}
$$

Where one has used

$$
\int_0^{\tau}|\tau-s|^{-\frac{1}{2}}s^{-1+\frac{\gamma}{p}}ds = \tau^{-\frac{1}{2}+\frac{\gamma}{p}}\int_0^1|1-\eta|^{-\frac{1}{2}}\eta^{-1+\frac{\gamma}{p}}d\eta = C\tau^{-\frac{1}{2}+\frac{\gamma}{p}}
$$

and the fact that for $p > 2\gamma, \gamma \geq 2$ and $q > 2$,

$$
\int_0^1 |1-\theta|^{-\frac{1}{2}+\frac{1}{\gamma}-\frac{\gamma+1}{p}}\theta^{-\frac{1}{2}-\frac{1}{\gamma}+\frac{\gamma+1}{p}}d\theta \text{ and } \int_0^1 |1-\eta|^{-\frac{1}{2}}\eta^{-1+\frac{\gamma}{p}}d\eta
$$

both are finite constants. Therefore,

$$
\sup_{t\geq 0}\|\mathcal{B}[\rho,\rho]\|_{\dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}} \leq C\|\rho\|_{Y'}^{\gamma+1}.
$$

Similarly ,

$$
\begin{aligned}
&\sup_{t\geq 0} t^{\frac{1}{\gamma}-\frac{1}{p}}\|\mathcal{B}[\rho,\rho]\|_{L^p} \leq \sup_{t\geq 0}\int_0^t t^{\frac{1}{\gamma}-\frac{1}{p}}|t-\tau|^{-\frac{1}{2}-(\frac{\gamma+1}{p}-\frac{1}{p})}\|\rho\mathcal{C}[\rho]\|_{L^{\frac{p}{\gamma+1}}}d\tau \\
&\leq \sup_{t\geq 0}\int_0^t t^{\frac{1}{\gamma}-\frac{1}{p}}|t-\tau|^{-\frac{1}{2}-\frac{\gamma}{p}}\tau^{-\frac{1}{\gamma}+\frac{1}{p}}(\tau^{\frac{1}{\gamma}-\frac{1}{p}}\|\rho\|_{L^p})\|\mathcal{C}[\rho]\|_{L^{\frac{p}{\gamma}}}d\tau \\
&\leq \sup_{t\geq 0}\int_0^t t^{\frac{1}{\gamma}-\frac{1}{p}}|t-\tau|^{-\frac{1}{2}-\frac{\gamma}{p}}\tau^{-\frac{1}{\gamma}+\frac{1}{p}}\|\mathcal{C}[\rho]\|_{L^{\frac{p}{\gamma}}}d\tau\|\rho\|_{Y'} \\
&\leq \sup_{t\geq 0}\int_0^t t^{\frac{1}{\gamma}-\frac{1}{p}}|t-\tau|^{-\frac{1}{2}-\frac{\gamma}{p}}\tau^{-\frac{1}{2}-\frac{1}{\gamma}+\frac{\gamma+1}{p}}d\tau\|\rho\|_{Y'}^{\gamma+1} \\
&= \int_0^1 |1-\theta|^{-\frac{1}{2}-\frac{\gamma}{p}}\theta^{-\frac{1}{2}-\frac{1}{\gamma}+\frac{\gamma+1}{p}}\|\rho\|_{Y'}^{\gamma+1} \\
&= C\|\rho\|_{Y'}^{\gamma+1}, \text{ with } p > 2\gamma, \gamma \geq 2.
\end{aligned}
$$

Consequently,

$$\|\mathcal{B}[\rho,\rho]\|_{Y'} \le C\|\rho\|_{Y'}^{\gamma+1}.$$

The other parts of the estimates and the scheme of the existence proof are similar to those for Theorem 3.1, so we omit the details. □

Remark 5.2. *If* $1 < \gamma < 2$ *and*

$$2\gamma < p < \frac{2\gamma(\gamma+1)}{2-\gamma}, q > 2$$

the above theorem holds also.

Remark 5.3. *In the 3-dimensional case, the conclusion in above theorem also holds under the following restriction of* p, q*:*

$$\begin{aligned} p > 3\gamma, q > 3, \ \textit{if } \gamma \ge 2; \\ 3\gamma < p < \frac{3\gamma(\gamma+1)}{2-\gamma}, q > 3, \ \textit{if } 1 \le \gamma < 2, \end{aligned}$$

and $(\rho_0, u_0) \in (\dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{3}{p}}, \dot{B}_{q,\infty}^{-1+\frac{3}{q}})$.

To obtain the desired solution to the problem (5.1)-(5.3), we would like to take the limit $\epsilon \to 0^+$ in (5.4)-(5.6) .To this end, certain compactness on the sequence $(\rho^\epsilon, u^\epsilon)$ is required. This will be provided by the following key a priori estimates.

First, note that due to the property of homogeneous spaces, we can assume without lose of generality that $(\rho(x,t), u(x,t))$ vanishes as $x \to \infty$. The first a priori estimate is given in the following lemma:

Lemma 5.1. *Let* (ρ, u) *be a smooth solution to* (5.4)-(5.5)*,and set*

$$E(t) = \int_{\mathbb{R}^2} [\frac{1}{2}u^2 + \frac{\rho^\gamma}{\gamma - 1}]dx.$$

Then the following energy inequality holds:

$$E(t) + \int_0^t \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{4\epsilon}{\gamma}|\nabla(\rho^{\frac{\gamma}{2}})|^2 dx d\tau \le E(0) \tag{5.12}$$

for any $t \in [0,T], \gamma > 1$.

Proof. Multiplying (5.5) by u, integrating it over $\mathbb{R}^2$,and using (5.4), we deduce:

$$\frac{dE(t)}{dt} + \int_{\mathbb{R}^2} |\nabla u(x,t)|^2 + \epsilon\gamma|\nabla\rho|^2\rho^{\gamma-2}dx = 0.$$

Thus (5.12) follows by integration. □

Another necessary a priori estimate we need is the following

Lemma 5.2. *Let (ρ, u) be a smooth solution to the problem (5.1)-(5.3), then the following energy inequality holds:*

$$\int_{\mathbb{R}^2}(\frac{1}{2}u^2+\frac{\rho^\gamma}{\gamma-1}+\frac{2\rho^{\frac{\gamma}{2}}}{\gamma-2})dx+\int_0^t\int_{\mathbb{R}^2}\frac{1}{2}|\nabla u|^2+\frac{4\epsilon}{\gamma}|\nabla(\rho^{\frac{\gamma}{2}})|^2+\frac{8\epsilon}{\gamma}|\nabla(\rho^{\frac{\gamma}{4}})|^2dxd\tau$$
$$\leq E_1(0)\exp(\frac{\gamma-1}{2}T) \tag{5.13}$$

for any $t\in[0,T], \gamma>2$. where

$$E_1(t)=\int_{\mathbb{R}^2}(\frac{1}{2}u^2+\frac{\rho^\gamma}{\gamma-1}+\frac{2\rho^{\frac{\gamma}{2}}}{\gamma-2})dx \tag{5.14}$$

Proof. First, it follows from lemma 5.1 and $\gamma>2$ that $\rho\in L^2_{loc}((0,T)\times\mathbb{R}^2)$. Thus the equation(5.4) can be satisfied also in the sense of renormalized solutions introduced by Diperna and Lions [25]. More precisely,

$$b(\rho)_t+\mathrm{div}(b(\rho)u)+(b'(\rho)\rho-b(\rho))\mathrm{div}=\epsilon b'(\rho)\triangle\rho \tag{5.15}$$

holds for any $b\in C^1(\mathbb{R})$ and convex. Therefore,

$$\frac{\partial\rho^{\frac{\gamma}{2}}}{\partial t}+\mathrm{div}(\rho^{\frac{\gamma}{2}}u)=\frac{\gamma\epsilon}{2}\rho^{\frac{\gamma}{2}-1}\triangle\rho+(1-\frac{\gamma}{2})\rho^{\frac{\gamma}{2}}\mathrm{div}u \tag{5.16}$$

Integrating (5.16) over $\mathbb{R}^2$ yields

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^2}\frac{2\rho^{\frac{\gamma}{2}}}{\gamma-2}dx+\int_{\mathbb{R}^2}\frac{8\epsilon}{\gamma}|\nabla(\rho^{\frac{\gamma}{4}})|^2dx\leq\int_{\mathbb{R}^2}\rho^{\frac{\gamma}{2}}|\mathrm{div}u|dx. \tag{5.17}$$

On the other hand, it follows from the proof of lemma 5.1 that

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^2}[\frac{1}{2}u^2+\frac{\rho^\gamma}{\gamma-1}]dx+\int_{\mathbb{R}^2}|\nabla u|^2+\frac{4\epsilon}{\gamma}|\nabla(\rho^{\frac{\gamma}{2}})|^2dx=0. \tag{5.18}$$

Therefore, combining (5.17) with (5.18) and using Hölder inequality and Young's inequality, one has

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^2}[\frac{1}{2}u^2+\frac{2\rho^{\frac{\gamma}{2}}}{\gamma-2}+\frac{\rho^\gamma}{\gamma-1}]dx+\int_{\mathbb{R}^2}[|\nabla u|^2+\frac{8\epsilon}{\gamma}|\nabla(\rho^{\frac{\gamma}{4}})|^2+\frac{4\epsilon}{\gamma}|\nabla(\rho^{\frac{\gamma}{2}})|^2]dx$$
$$\leq\int_{\mathbb{R}^2}\rho^{\frac{\gamma}{2}}|\mathrm{div}u|dx\leq(\int_{\mathbb{R}^2}\rho^\gamma dx)^{\frac{1}{2}}(\int_{\mathbb{R}^2}|\nabla u|^2dx)^{\frac{1}{2}}$$
$$\leq\frac{1}{2}\int_{\mathbb{R}^2}\rho^\gamma dx+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2dx\leq\frac{\gamma-1}{2}E_1(t)+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2dx.$$

Thus

$$\frac{\partial E_1(t)}{\partial t}+\int_{\mathbb{R}^2}[\frac{1}{2}|\nabla u|^2+\frac{8\epsilon}{\gamma}|\nabla(\rho^{\frac{\gamma}{4}})|^2+\frac{4\epsilon}{\gamma}|\nabla(\rho^{\frac{\gamma}{2}})|^2]dx$$
$$\leq\frac{\gamma-1}{2}E_1(t)$$

So this gives the desired estimate (5.13)using Gronwall's inequality. □

Remark 5.1. *The above lemma holds on the following restriction of the initial data.*

$$\|u_0(x)\|_{L(\mathbb{R}^2)} \leq C_1; \|\rho_0(x)\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2)} \leq C_2, \|\rho_0(x)\|_{L^\gamma(\mathbb{R}^2)} \leq C_3.$$

where C_1, C_2 and C_3 are absolute constants.

It follows from lemma 5.1 and lemma 5.2 that the following conclusion holds:

Corollary 5.1. *Let (ρ, u) be a smooth solution to the problem (5.4)-(5.5). Then the following inequality holds:*

$$\epsilon \int_0^T \int_{\mathbb{R}^2} |\nabla \rho|^2 dxdt \leq C. \tag{5.19}$$

for $2 \leq \gamma \leq 4$. Where C is independent of ϵ.

Proof. Obviously, if $\gamma = 2$,lemma 5.1 implies (5.19). For $2 < \gamma \leq 4$, from lemma 5.2, we know

$$\epsilon \int_0^T \int_{\mathbb{R}^2} |\nabla \rho^{\frac{\gamma}{2}}|^2 dxdt \leq C_1, \tag{5.20}$$

$$\epsilon \int_0^T \int_{\mathbb{R}^2} |\nabla \rho^{\frac{\gamma}{4}}|^2 dxdt \leq C_2. \tag{5.21}$$

Thus

$$\begin{aligned}
&\epsilon \int_0^T \int_{\mathbb{R}^2} |\nabla \rho|^2 dxdt = \epsilon \int_0^T \int_{\mathbb{R}^2} \rho^{(\frac{\gamma}{2}-1)(\frac{8}{\gamma}-2)} |\nabla \rho|^{\frac{8}{\gamma}-2} \cdot \rho^{(\frac{\gamma}{4}-1)(4-\frac{8}{\gamma})} |\nabla \rho|^{4-\frac{8}{\gamma}} dxdt \\
&= C\epsilon \int_0^T \int_{\mathbb{R}^2} |\nabla \rho^{\frac{\gamma}{2}}|^{\frac{8}{\gamma}-2} |\nabla \rho^{\frac{\gamma}{4}}|^{4-\frac{8}{\gamma}} dxdt \\
&\leq C\epsilon \int_0^T \Big(\int_{\mathbb{R}^2} |\nabla \rho^{\frac{\gamma}{2}}|^2 dx\Big)^{\frac{4}{\gamma}-1} \Big(\int_{\mathbb{R}^2} |\nabla \rho^{\frac{\gamma}{4}}|^2 dx\Big)^{2-\frac{4}{\gamma}} dt \\
&\leq C\epsilon \Big(\int_0^T \int_{\mathbb{R}^2} |\nabla \rho^{\frac{\gamma}{2}}|^2 dxdt\Big)^{\frac{4}{\gamma}-1} \cdot \Big(\int_0^T \int_{\mathbb{R}^2} |\nabla \rho^{\frac{\gamma}{4}}|^2 dxdt\Big)^{2-\frac{4}{\gamma}} \\
&\leq C\Big(\epsilon \int_0^T \int_{\mathbb{R}^2} |\nabla \rho^{\frac{\gamma}{2}}|^2 dxdt\Big)^{\frac{4}{\gamma}-1} \cdot \Big(\epsilon \int_0^T \int_{\mathbb{R}^2} |\nabla \rho^{\frac{\gamma}{4}}|^2 dxdt\Big)^{2-\frac{4}{\gamma}} \\
&\leq C,
\end{aligned}$$

where we have used $\frac{4}{\gamma} - 1, 2 - \frac{4}{\gamma} \in [0, 1]$ for $2 < \gamma \leq 4$, (5.20) and (5.21). ∎

Note that

$$L^2(\mathbb{R}^2) \hookrightarrow \dot{B}^{-1+\frac{2}{q}}_{q,\infty}(\mathbb{R}^2); L^\gamma(\mathbb{R}^2) \hookrightarrow \dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2) \tag{5.22}$$

with $p > 2\gamma, \gamma \geq 2, q > 2$,and

$$L^{\frac{\gamma}{2}}(\mathbb{R}^2) \hookrightarrow \dot{B}^{-\frac{2}{\gamma}+\frac{2}{l}}_{l,\infty}(\mathbb{R}^2) \hookrightarrow \dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)$$

where for any $\frac{1}{2} \le r \le p < \infty$. It follows from lemma 5.1, lemma 5.2 and (5.22)that

$$\sup_{0\le t\le T} \|u^\epsilon\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}} \le C \sup_{0\le t\le T} \|u^\epsilon\|_{L^2} \le C_1; \tag{5.23}$$

$$\sup_{0\le t\le T} \|\rho^\epsilon\|_{\dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}} \le C \sup_{0\le t\le T} \|\rho^\epsilon\|_{L^\gamma} \le C_2, \tag{5.24}$$

where C_1, C_2 are independent of ϵ. In addition,

$$\|\nabla u^\epsilon\|_{L^2([0,T];L^2(\mathbb{R}^2))} \le C_3,$$

C_3 is independent of ϵ also. So we can extract a subsequence of $(\rho^\epsilon, u^\epsilon)$, still denoted by $(\rho^\epsilon, u^\epsilon)$, such that

$$\rho^\epsilon \rightharpoonup \rho, \text{weak-* in } L^\infty([0,T]; L^\gamma(\mathbb{R}^2)); \tag{5.25}$$

$$u^\epsilon \rightharpoonup u, \text{for some } u, \text{weakly in } L^2([0,T]; H^1(\mathbb{R}^2)), \tag{5.26}$$

where $T > 0$ is an arbitrary but fixed constant. Moreover, by the lower semi-continuity of weak convergence, $\nabla u \in L^2([0,T]; L^2(\mathbb{R}^2))$.

In order to prove that (ρ, u) obtained in (5.4)-(5.6)is indeed a weak solution of (5.1)-(5.3), we need the following high space-time regularity estimate for the density:

Lemma 5.3. *Let (ρ, u) be a weak solution to the problem* (5.4)-(5.6). *Then for any $\gamma \ge 2$, the following estimate holds:*

$$\int_0^T \int_K \rho^{\frac{3\gamma}{2}} dxdt \le C, \forall T > 0, \tag{5.27}$$

where $K \subset \mathbb{R}^2$ is bounded, and C depends on T, E_0, K and E_1.

Proof. Consider the function:

$$\varphi^i(t,x) = \psi(t)\phi(x)\mathcal{A}_i[\rho^{\frac{\gamma}{2}}], i = 1,2. \tag{5.28}$$

Where

$$\phi \in \mathcal{D}'(\mathbb{R}^2), |\nabla\phi| \le M \text{ on } \mathbb{R}^2; \psi \in \mathcal{D}'(0,T), |\psi'| \le M, \tag{5.29}$$

for some finite positive constant M, and $\mathcal{A}_i$ are pseudodifferential operators defined by means of the fourier multiplies with the symbols:

$$\widehat{\mathcal{A}_j}(\xi) = \frac{-i\xi_j}{|\xi|^2}, j = 1,2;$$

$$i.e. \mathcal{A}_j[h] = (-\triangle)^{-1}\partial_j h, j = 1,2.$$

By virtue of the Marcinkiewicz multiplier theorem [27] and the classical Sobolev embedding, we have

$$\begin{aligned}
&\|\mathcal{A}_i[h]\|_{W^{1,s}(\mathbb{R}^2)} \le C(s)\|h\|_{L^s(\mathbb{R}^2)}, 1 < s < \infty, \text{ in particular },\\
&\|\mathcal{A}_i[h]\|_{L^q(\mathbb{R}^2)} \le C(q,s)\|h\|_{L^s(\mathbb{R}^2)}, q \text{ finite, provided } \frac{1}{q} \ge \frac{1}{s} - \frac{1}{2},\\
&\|\mathcal{A}_i[h]_{L^\infty(\mathbb{R}^2)} \le C(s)\|h\|_{L^s(\mathbb{R}^2)}, \text{ if } s > 2.
\end{aligned} \tag{5.30}$$

So one can calculate that

$$
\begin{aligned}
&\varphi_t^i(t,x) = \psi'(t)\phi(x)\mathcal{A}_i[\rho^{\frac{\gamma}{2}}] + \psi(t)\phi(x)\mathcal{A}_i[-\mathrm{div}(\rho^{\frac{\gamma}{2}}u) \\
&+\frac{\gamma\epsilon}{2}(\rho^{\delta})^{\frac{\gamma}{2}-1}\triangle\rho + (1-\frac{\gamma}{2})\rho^{\frac{\gamma}{2}}\mathrm{div}u], \qquad (5.31)\\
&\partial_j\varphi^i(t,x) = \psi(t)\big(\phi\partial_j\mathcal{A}_i[\rho^{\frac{\gamma}{2}}] + \partial_j\phi\mathcal{A}_i[\rho^{\frac{\gamma}{2}}]\big), i,j=1,2.
\end{aligned}
$$

Thus

$$
\sum_{i=1}^{2}\partial_i\varphi^i = \psi(t)\big(\phi\rho^{\frac{\gamma}{2}} + \sum_{i=1}^{2}\partial_i\phi\mathcal{A}_i[\rho^{\frac{\gamma}{2}}]\big), \qquad (5.32)
$$

where we have used (5.16). Now due to lemma 5.1, lemma 5.2 and the regularity properties of ρ, one can justify easily the choice of $\varphi^i(t,x)$ as test functions for (5.5). It follows(5.31)and (5.32)that

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^2}\psi(t)\phi(x)\rho^{\frac{3\gamma}{2}}dxdt \\
&\le \sum_{i=1}^{2}\int_0^T\int_{\mathbb{R}^2}\rho^{\gamma}\psi\partial_i\phi\mathcal{A}_i[\rho^{\frac{\gamma}{2}}]dxdt + \sum_{i=1}^{2}\int_0^T\int_{\mathbb{R}^2}\psi\partial_i\phi|\nabla u^i|\mathcal{A}_i[\rho^{\frac{\gamma}{2}}]dxdt \\
&+\sum_{i=1}^{2}\int_0^T\int_{\mathbb{R}^2}\psi\phi|\nabla u^i|\partial_i\mathcal{A}_i[\rho^{\frac{\gamma}{2}}]dxdt + \sum_{i=1}^{2}\int_0^T\int_{\mathbb{R}^2}\psi'\phi|u^i|\mathcal{A}_i[\rho^{\frac{\gamma}{2}}]dxdt \\
&+\sum_{i=1}^{2}\int_{\mathbb{R}^2}\psi\phi|u^i|\mathcal{A}_i[\rho^{\frac{\gamma}{2}}]dx(t=0,T) + \sum_{i=1}^{2}\int_0^T\int_{\mathbb{R}^2}\psi\phi|u^i|\mathcal{A}_i[\mathrm{div}(\rho^{\frac{\gamma}{2}}u)]dxdt \\
&\sum_{i=1}^{2}\int_0^T\int_{\mathbb{R}^2}\psi\phi|u^i|\mathcal{A}_i[(\frac{\gamma}{2}-1)\rho^{\frac{\gamma}{2}}\mathrm{div}u]dxdt + \sum_{i=1}^{2}\int_0^T\int_{\mathbb{R}^2}\psi\phi|u^i|\mathcal{A}_i[\frac{\gamma}{2}\rho^{\frac{\gamma}{2}-1}\epsilon\triangle\rho]dxdt \\
&=\sum_{k=1}^{8}I_k. \qquad (5.33)
\end{aligned}
$$

In the following, we will estimate $I_k, k=1,2,...,8$ separately.

1. (estimate of I_1)

$$
\begin{aligned}
I_1 &\le \int_0^T(\|\rho^{\gamma}\|_{L^{\frac{3}{2}}}\|\mathcal{A}_i[\rho^{\frac{\gamma}{2}}]\|_{L^3})dt \le C\int_0^T(\|\rho^{\gamma}\|_{L^{\frac{3}{2}}}\|\rho^{\frac{\gamma}{2}}\|_{L^2})dt \\
&\le C\mathrm{ess\,sup}_{0\le t\le T}\|\rho\|_{L^{\gamma}}^{\frac{\gamma}{2}}\int_0^T\|\rho\|_{L^{\frac{3\gamma}{2}}}^{\gamma}dt \le CE_0^{\frac{1}{2}}\int_0^T\|\rho\|_{L^{\frac{3\gamma}{2}}}^{\gamma}dt \\
&\le \nu\|\rho\|_{L^{\frac{3\gamma}{2}}((0,T)\times\mathbb{R}^2)}^{\frac{3\gamma}{2}} + C,
\end{aligned}
$$

where $\nu>0$ is small, and (5.30) has been used.

2. (estimate of I_2)

$$
\begin{aligned}
I_2 &\le \int_0^T(\|\nabla u\|_{L^2}\|\mathcal{A}_i[\rho^{\frac{\gamma}{2}}]\|_{L^2})dt \le C\int_0^T(\|\nabla u\|_{L^2}\|\rho^{\frac{\gamma}{2}}\|_{L^2})dt \\
&\le C(\int_0^T\|\nabla u\|_{L^2}^2dt)^{\frac{1}{2}}(\int_0^T\|\rho\|_{L^{\gamma}}^{\gamma}dt)^{\frac{1}{2}} \le CE_0T^{\frac{1}{2}}.
\end{aligned}
$$

3. (estimate of I_3) Using (5.30) and Hölder inequality, we know

$$
\begin{aligned}
I_3 &\leq \int_0^T (\|\nabla u\|_{L^2}\|\partial_i \mathcal{A}_i[\rho^{\frac{\gamma}{2}}]\|_{L^2})dt \leq C\int_0^T (\|\nabla u\|_{L^2}\|\rho^{\frac{\gamma}{2}}\|_{L^2})dt \\
&\leq CT^{\frac{1}{2}}\operatorname{ess\,sup}_{0\leq t\leq T}\int_{\mathbb{R}^2}\rho^{\gamma}dx(\int_0^T\int_{\mathbb{R}^2}|\nabla u|^2dxdt)^{\frac{1}{2}} \leq CE_0T^{\frac{1}{2}}.
\end{aligned}
$$

4. (estimate of I_4) Similarly as for I_2, we have

$$
\begin{aligned}
I_4 \leq \int_0^T (\|u\|_{L^2}\|\mathcal{A}_i[\rho^{\frac{\gamma}{2}}]\|_{L^2})dt &\leq C\int_0^T (\|u\|_{L^2}\|\rho^{\frac{\gamma}{2}}\|_{L^2})dt \\
&\leq CE_0.
\end{aligned}
$$

5. (estimate of I_5)

$$
\begin{aligned}
I_5 \leq \operatorname{ess\,sup}_{0\leq t\leq T}\|u\|_{L^2}\|\mathcal{A}_i[\rho^{\frac{\gamma}{2}}]\|_{L^2} &\leq C\operatorname{ess\,sup}_{0\leq t\leq T}\|u\|_{L^2}\|\rho^{\frac{\gamma}{2}}\|_{L^2} \\
&\leq CE_0.
\end{aligned}
$$

6. (estimate of I_6) Using (5.30) again, one has

$$
\begin{aligned}
I_6 \leq \int_0^T (\|u\|_{L^2}\|\mathcal{A}_i[\operatorname{div}(\rho^{\frac{\gamma}{2}}u)]\|_{L^2})dt &\leq C\int_0^T (\|u\|_{L^2}\|\rho^{\frac{\gamma}{2}}u\|_{L^2})dt \\
&\leq C(\int_0^T\int_{\mathbb{R}^2}|u|^2dxdt)^{\frac{1}{2}}(\int_0^T\int_{\mathbb{R}^2}\rho^{\gamma}u^2dxdt)^{\frac{1}{2}} \\
&\leq CT^{\frac{1}{2}}\operatorname{ess\,sup}_{0\leq t\leq T}\|u\|_{L^2}^2(\int_0^T\int_{\mathbb{R}^2}\rho^{\gamma}dxdt)^{\frac{1}{2}} \\
&\leq CTE_0^{\frac{3}{2}}.
\end{aligned}
$$

7. (estimate of I_7)

$$
\begin{aligned}
I_7 &\leq \int_0^T (\|u\|_{L^2}\|\mathcal{A}_i[\operatorname{div}u\rho^{\frac{\gamma}{2}}]\|_{L^2})dt \leq C\int_0^T (\|u\|_{L^2}\|\operatorname{div}u\rho^{\frac{\gamma}{2}}\|_{L^2})dt \\
&\leq C(\int_0^T\|u\|_{L^2}^2dt)^{\frac{1}{2}}(\int_0^T\|\operatorname{div}u\rho^{\frac{\gamma}{2}}\|_{L^2}^2dt)^{\frac{1}{2}} \\
&\leq C\operatorname{ess\,sup}_{0\leq t\leq T}T^{\frac{1}{2}}\|u\|_{L^2}(\int_0^T\int_{\mathbb{R}^2}\rho^{\gamma}(\operatorname{div}u)^2dxdt)^{\frac{1}{2}} \\
&\leq CT^{\frac{1}{2}}E_0(\int_0^T\int_{\mathbb{R}^2}(\operatorname{div}u)^2dxdt)^{\frac{1}{2}} \leq CT^{\frac{1}{2}}E_0^{\frac{3}{2}}.
\end{aligned}
$$

8. (estimate of I_8) Thanks to lemma 5.1, lemma 5.2,and (5.30), for $\gamma > 2$ we can deduce

$$
\begin{aligned}
I_8 &\le C\int_0^T (\|u\|_{L^2}\|\mathcal{A}_i[\epsilon\rho^{\frac{\gamma}{2}-1}\triangle\rho]\|_{L^2})dt\\
&\le C\int_0^T (\|u\|_{L^2}\|\mathcal{A}_i[\epsilon\triangle(\rho^{\frac{\gamma}{2}})+\epsilon|\nabla(\rho^{\frac{\gamma}{4}})|^2]\|_{L^2})dt\\
&\le C\int_0^T (\|u\|_{L^2}(\|\mathcal{A}_i[\epsilon\triangle(\rho^{\frac{\gamma}{2}})]\|_{L^2}+\|\mathcal{A}_i[\epsilon|\nabla(\rho^{\frac{\gamma}{4}})|^2]\|_{L^2})dt\\
&\le C\int_0^T \|u\|_{L^2}(\epsilon\|\nabla(\rho^{\frac{\gamma}{2}})\|_{L^2})dt + C\int_0^T \|u\|_{L^2}(\epsilon\||\nabla(\rho^{\frac{\gamma}{4}})|^2\|_{L^1})dt\\
&= C\int_0^T \|u\|_{L^2}(\epsilon\|\nabla(\rho^{\frac{\gamma}{2}})\|_{L^2})dt + C\int_0^T \|u\|_{L^2}(\epsilon\|\nabla(\rho^{\frac{\gamma}{4}})\|_{L^2}^2)dt\\
&\le C(\int_0^T \|u\|_{L^2}^2dt)^{\frac12}\epsilon(\int_0^T \|\nabla(\rho^{\frac{\gamma}{2}})\|_{L^2}^2dt)^{\frac12} + C\operatorname{ess\,sup}_{0\le t\le T}\|u\|_{L^2}(\epsilon\int_0^T \|\nabla(\rho^{\frac{\gamma}{4}})\|_{L^2}^2dt)\\
&\le C\epsilon\int_0^T \|u\|_{L^2}^2dt + C\epsilon\int_0^T \|\nabla(\rho^{\frac{\gamma}{2}})\|_{L^2}^2dt) + C\operatorname{ess\,sup}_{0\le t\le T}\|u\|_{L^2}(\epsilon\int_0^T \|\nabla(\rho^{\frac{\gamma}{4}})\|_{L^2}^2dt)\\
&\le C\epsilon E_0T + CE_0 + CE_0^{\frac12}E_1(0)\exp(\frac{\gamma-1}{2}T) < C
\end{aligned}
$$

Putting $I_1, I_2, ...,$ and I_8 into (5.33), choosing $\psi|_{(0,T)} = \phi|_K = 1$ and using Young's inequality, we can deduce our result.

For the case $\gamma = 2$, the above high regularity estimate of density still holds.

Indeed, we can choose test function of the form:

$$\varphi^i(t,x) = \psi(t)\phi(x)\mathcal{A}_i[\rho], i = 1, 2.$$

The only different term can be estimated as follows:

$$
\begin{aligned}
&I_8' \le \int_0^T (\|u\|_{L^2}\|\mathcal{A}_i[\epsilon\triangle\rho]\|_{L^2})dt \le C\int_0^T \|u\|_{L^2}(\epsilon\|\nabla\rho\|_{L^2})dt\\
&\le C\epsilon(\int_0^T \|u\|_{L^2}^2dt) + C\epsilon(\int_0^T \|\nabla\rho\|_{L^2}^2dt) \le C\epsilon E_0T + CE_0 \le CE_0, \forall\epsilon > 0 \text{ small.}
\end{aligned}
$$

In this case, the energy inequality becomes:

$$\int_{\mathbb{R}^2}[\frac12|u|^2+\rho^2](x,t)dx + \int_0^T\int_{\mathbb{R}^2}[|\nabla u|^2+4\epsilon|\nabla\rho|^2]dxdt \le E(0). \tag{5.34}$$

So similar arguments lead to the local space-time bound for ρ^3. □

Remark 5.5. *For the case $N = 3$, we can obtain the same result. Since* (5.30) *becomes*

$$
\begin{aligned}
&\|\mathcal{A}_i[h]\|_{W^{1,s}(\mathbb{R}^3)} \le C(s)\|h\|_{L^s(\mathbb{R}^3)}, 1 < s < \infty, \text{ in particular },\\
&\|\mathcal{A}_i[h]\|_{L^q(\mathbb{R}^3)} \le C(q,s)\|h\|_{L^s(\mathbb{R}^3)}, q \text{ finite, provided } \frac1q \ge \frac1s - \frac13\\
&\|\mathcal{A}_i[h]_{L^\infty(\mathbb{R}^3)} \le C(s)\|h\|_{L^s(\mathbb{R}^3)}, \text{ if } s > 3
\end{aligned} \tag{5.35}
$$

and lemma 5.1 and lemma 5.2 also hold.

holds:

$$\int_0^T \int_K \rho^{\gamma+1} dxdt \le C \tag{5.36}$$

for $\gamma \ge 2, \forall 0 < T < \infty$, and $K \subset \mathbb{R}^2$ is bounded.

Proof. Obviously, by the interpolation inequality, one can deduce from lemma 5.1 and lemma 5.3 that

$$\begin{aligned}
&\int_0^T \int_K \rho^{\gamma+1} dxdt \le \int_0^T (\|\rho\|_{L^\gamma}^{\theta} \|\rho\|_{L^{\frac{3\gamma}{2}}}^{1-\theta})^{\gamma+1} dt, \text{ for } \theta = \frac{\gamma-2}{\gamma+1}, \theta \in [0,1) \\
&\le \text{ess sup}_{0\le t\le T} \|\rho\|_{L^\gamma}^{\theta} \int_0^T \|\rho\|_{L^{\frac{3\gamma}{2}}}^{3} dt \\
&\le CT^{1-\frac{2}{\gamma}} (\int_0^T \int_K \rho^{\frac{3\gamma}{2}} dxdt)^{\frac{2}{\gamma}} \\
&\le C.
\end{aligned}$$

where C is independent of ϵ. □

Now, making use of lemma 5.4, we may suppose that

$$(\rho^\epsilon)^\gamma \to \overline{\rho^\gamma} \text{ weakly in } L^{\frac{\gamma+1}{\gamma}}(K), \tag{5.37}$$

$$(\rho^\epsilon)^{\gamma+1} \to \overline{\rho^{\gamma+1}} \text{ weakly in } \mathcal{D}'((0,T) \times \mathbb{R}^2) \tag{5.38}$$

passing to subsequence if necessary. Moreover, the uniform energy estimates implies that

$$\epsilon \triangle \rho^\epsilon \to 0, \text{ in } L^2(0,T; W^{-1,2}(\mathbb{R}^2)).$$

while lemma 5.2 yields

$\rho^\epsilon u^\epsilon$ uniformly bounded in $L^\infty(0,T; L^{\frac{2\gamma}{2+\gamma}}(\mathbb{R}^2)) \cap L^2(0,T; L^{\frac{m\gamma}{m+\gamma}}(\mathbb{R}^2)), \forall 2 < m < \infty$.

Thus using (5.4), we obtain

$\partial_t \rho^\epsilon$ uniformly bounded in $L^2(0,T; W^{-1,\frac{m\gamma}{m+\gamma}}(\mathbb{R}^2))$.

Since $L^\gamma(\mathbb{R}^2)$ is compactly embedded into $W^{-1,\frac{m\gamma}{m+\gamma}}(\mathbb{R}^2)$, we can use the Banach space version of the Arzela-Ascoli theorem to infer that

ρ^ϵ are precompact in $C([0,T], W^{-1,\frac{m\gamma}{m+\gamma}}(\mathbb{R}^2))$.

and, consequently

$$\rho^\epsilon \to \rho \text{ in } C([0,T], L^\gamma_{weak}(\mathbb{R}^2)), \text{ and weakly in } L^{\frac{3\gamma}{2}}((0,T) \times \mathbb{R}^2).$$

Combining (5.22) with (5.23), we see that (ρ, u) satisfies

$$\begin{cases} \frac{\partial \rho}{\partial t} + \text{div}(\rho u) = 0; \\ u_t - \triangle u + \nabla \overline{\rho^\gamma} = 0. \end{cases}$$

in $\mathcal{D}'((0,T) \times \mathbb{R}^2)$.

So our ultimate goal is to prove the strong convergence in (5.37) of the density.

To this end, we need the following key lemma motivated by a similar argument in[22]:

Lemma 5.5. *Let $\psi(t)$ and $\phi(x)$ be in lemma 5.3,$(\rho^\epsilon, u^\epsilon)$ be a weak solution of (5.1)-(5.3) with $\gamma \geq 2$, and (ρ, u) be a weak solution of the limit system. Then*

$$\lim_{\epsilon \to 0^+} \int_0^T \int_{\mathbb{R}^2} \psi(t)\phi(x)(\rho^\epsilon)^{\gamma+1} dxdt = \int_0^T \int_{\mathbb{R}^2} \psi(t)\phi(x)\overline{\rho^\gamma}\rho dxdt \tag{5.39}$$

Proof. As in the proof of lemma 5.3, we consider the following test function for the system (5.5):

$$\varphi^i(t,x) = \psi(t)\phi(x)\mathcal{A}_i[\rho^\epsilon], i = 1, 2$$

for the same reason as in lemma 5.3. We arrive at

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^2} \psi\phi(\rho^\epsilon)^{\gamma+1} dxdt &= \int_{\mathbb{R}^2} \psi\phi u_i^\epsilon \mathcal{A}_i[\rho^\epsilon] dx(t=0,T) \\ &- \int_0^T \int_{\mathbb{R}^2} \psi'\phi u_i^\epsilon \mathcal{A}_i[\rho^\epsilon] - \psi\phi u_i^\epsilon \mathcal{A}_i[\epsilon\triangle\rho^\epsilon] dxdt \\ &+ \int_0^T \int_{\mathbb{R}^2} \psi\phi u_i^\epsilon \mathcal{A}_i[\operatorname{div}(\rho^\epsilon u^\epsilon)] + \psi\partial_j\phi\partial_j u_i^\epsilon \mathcal{A}_i[\rho^\epsilon] dxdt \\ &+ \int_0^T \int_{\mathbb{R}^2} \psi\phi\partial_j u_i^\epsilon \partial_j \mathcal{A}_i[\rho^\epsilon] - \psi\partial_i\phi(\rho^\epsilon)^\gamma \mathcal{A}_i[\rho^\epsilon] dxdt \end{aligned} \tag{5.40}$$

Similarly, we can choose functions of the form

$$\varphi^i(t,x) = \psi(t)\phi(x)\mathcal{A}_i[\rho], i = 1, 2$$

as test functions for the limit system:

$$u_t - \triangle u + \nabla(\overline{\rho^\gamma}) = 0$$

and we obtain the following formula:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^2} \psi\phi\overline{\rho^\gamma}\rho dxdt = \int_{\mathbb{R}^2} \psi\phi u_i \mathcal{A}_i[\rho] dx(t=0,T) - \int_0^T \int_{\mathbb{R}^2} \psi'\phi u_i \mathcal{A}_i[\rho] dxdt \\ + \int_0^T \int_{\mathbb{R}^2} \psi\phi u_i \mathcal{A}_i[\operatorname{div}(\rho u)] + \psi\partial_j\phi\partial_j u_i \mathcal{A}_i[\rho] dxdt \\ + \int_0^T \int_{\mathbb{R}^2} \psi\phi\partial_j u_i \partial_j \mathcal{A}_i[\rho] - \psi\partial_i\phi\overline{\rho^\gamma}\mathcal{A}_i[\rho] dxdt \end{aligned} \tag{5.41}$$

Note that

$$\begin{aligned} &\epsilon\left|\int_0^T \int_{\mathbb{R}^2} \psi\phi u_i^\epsilon \mathcal{A}_i[\triangle\rho^\epsilon] dxdt\right| \\ &\leq \epsilon \int_0^T \|u_i^\epsilon\|_{L^2} \|\mathcal{A}_i[\triangle\rho^\epsilon]\|_{L^2} dt \\ &\leq \epsilon \int_0^T \|u_i^\epsilon\|_{L^2} \|\nabla\rho^\epsilon\|_{L^2} dt \to 0, \text{ as } \epsilon \to 0, \end{aligned} \tag{5.42}$$

by the corollary 5.1.

$$\rho^\epsilon \to \rho \text{ in } C([0,T], L^\gamma_{weak}(\mathbb{R}^2)), \text{ and weakly in } L^{\frac{3\gamma}{2}}((0,T)\times\mathbb{R}^2),$$

and $\|\mathcal{A}_i[\rho^\epsilon]\|_{W^{1,\gamma+1}(\mathbb{R}^2)} \le C(\gamma)\|\rho^\epsilon\|_{L^{\gamma+1}(\mathbb{R}^2)}$ for $\gamma \ge 2$. one deduce easily that

$$\mathcal{A}_i[\rho^\epsilon] \to \mathcal{A}_i[\rho] \text{ in } C((0,T)\times\mathbb{R}^2), \tag{5.43}$$

$$\partial_j\mathcal{A}_i[\rho^\epsilon] \to \partial_j\mathcal{A}_i[\rho] \text{ in } C([0,T]; L^\gamma_{weak}(\mathbb{R}^2)). \tag{5.44}$$

And so, the Sobolev embedding and the Arzela-Ascoli theorem imply again

$$u^\epsilon \to u \text{ strongly in } L^2(0,T;L^p(\mathbb{R}^2)), \forall 1<p<\infty,$$

$$\rho^\epsilon u^\epsilon \to \rho u, \text{ in } C\big([0,T], L^{\frac{2\gamma}{2+\gamma}}_{weak}(\mathbb{R}^2)\big);$$

$$\therefore \mathcal{A}_i[\mathrm{div}(\rho^\epsilon u^\epsilon)] \to \mathcal{A}_i[\mathrm{div}(\rho u)] \text{ in } C\big([0,T], L^{\frac{2\gamma}{2+\gamma}}_{weak}(\mathbb{R}^2)\big). \tag{5.45}$$

Comparing (5.40)(letting $\epsilon \to 0$) with (5.41),and using(5.38) and(5.42)-(5.45), we obtain our conclusion. □

To claim the strong convergence, we will make use of a(slightly modified)Minty's trick.Since the nonlinearity $P(z) = z^\gamma$ is monotone, we have

$$\int_0^T\int_{\mathbb{R}^2} \psi\phi(P(\rho^\epsilon) - P(v))(\rho^\epsilon - v)dxdt \ge 0.$$

Consequently, it follows from lemma 5.5 that

$$\int_0^T\int_{\mathbb{R}^2} \psi\phi\overline{\rho^\gamma}\rho + \psi\phi v^{\gamma+1}dxdt - \int_0^T\int_{\mathbb{R}^2} \psi\phi(\overline{\rho^\gamma}v + v^\gamma\rho)dxdt \ge 0.$$

Thus

$$\int_0^T\int_K (\overline{\rho^\gamma} - v^\gamma)(\rho - v)dxdt \ge 0.$$

We can choose $v = \rho + \eta\vartheta, \eta \to 0, \vartheta$ is arbitrary,to obtain

$$\overline{\rho^\gamma} = \rho^\gamma.$$

The above compactness argument combined with the existence Theorem 5.1, lemma 5.1 and lemma 5.2 imply the following desired result:

Theorem 5.2. *For any* $\rho_0(x) \in \dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)$ *and* $u_0(x) \in \dot{B}^{-1+\frac{2}{q}}_{q,\infty}(\mathbb{R}^2)$*,if*

$$\|\rho_0\|_{\dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}(\mathbb{R}^2)} \le \delta, \|\rho_0\|_{L^\gamma(\mathbb{R}^2)} \le C_1,$$

$$\|\rho_0\|_{L^{\frac{\gamma}{2}}(\mathbb{R}^2)} \le C_2(\text{ for } \gamma > 2 \text{ only });$$

$$\|u_0\|_{\dot{B}^{-1+\frac{2}{q}}_{q,\infty}(\mathbb{R}^2)} \le \eta, \|u_0\|_{L^2(\mathbb{R}^2)} \le C_3.$$

with $p > 2\gamma, 2 \le \gamma \le 4, q > 2, \delta$ *and* η *are small, and* C_1, C_2 *and* C_3 *are absolute constants. Then there exists a solution* $(\rho(x,t), u(x,t))$ *to the problem* (5.1)-(5.3) *in the space* $C_w([0,T]; \dot{B}^{-\frac{2}{\gamma}+\frac{2}{p}}_{p,\infty}) \times C_w([0,T]; \dot{B}^{-1+\frac{2}{q}}_{q,\infty}), \forall 0 < T < \infty.$

lution of (5.1)-(5.3), then $(\lambda^{\frac{2}{\gamma}}\rho(\lambda x, \lambda^2 t), \lambda u(\lambda x, \lambda^2 t))$ is also a solution for all $\lambda > 0$. So self-similar solutions can be obtained directly from Theorem 5.2 and Theorem 4.1 by taking ρ_0 homogeneous of degree $-\frac{2}{\gamma}$ of small $\dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}$ norm and u_0 homogeneous of degree -1 of small $\dot{B}_{q,\infty}^{-1+\frac{2}{q}}$ norm. Since $(\rho^\epsilon, u^\epsilon)$ is self-similar, so is its limit. Thus we have obtained Theorem 1.1.

Remark 5.6. *From Remark 2.2,* $(\rho_0(x), u_0(x))$ *cannot be the form of* $(C_1|x|^{-\frac{2}{\gamma}}, C_2|x|^{-1})$.

However, we can choose any function $(f(x), g(x)), f(x) \in L^\gamma(\mathbb{R}^2), g(x) \in L^2(\mathbb{R}^2)$. *Let* $\omega_k(x) = \exp(ix \cdot k), \omega_l(x) = \exp(ix \cdot l)$, *then*

$$\|\omega_k f(x)\|_{L^\gamma(\mathbb{R}^2)} = \|f(x)\|_{L^\gamma(\mathbb{R}^2)}; \|\omega_l g(x)\|_{L^2(\mathbb{R}^2)} = \|g(x)\|_{L^2(\mathbb{R}^2)}.$$

but

$$\lim_{|k|\to\infty} \|\omega_k f(x)\|_{\dot{B}_{p,\infty}^{-\frac{2}{\gamma}+\frac{2}{p}}(\mathbb{R}^2)} = 0; \lim_{|l|\to\infty} \|\omega_l g(x)\|_{\dot{B}_{q,\infty}^{-1+\frac{2}{q}}(\mathbb{R}^2)} = 0.$$

This means, the restriction of the initial date in Theorem 1.1 is reasonable.

Remark 5.7. *Our theorem can be extended to the case of three space dimension by virtue of Remark 5.3 and Remark 5.5.*

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