The Analogue of Dedekind Eta Function for CY Threefolds.

*Andrey Todorov
University of California
Department of Mathematics,
Santa Cruz, CA 95064
Bulgarian Academy of Sciences
Institute of Mathematics
Sofia, Bulgaria

November 6, 2003

Abstract

It is a well known fact that the Kronecker limit formula gives an explicit formula for regularized determinants of flat metrics on elliptic curves. It established the relation between the regularized determinant of the flat metrics on elliptic curves and their discriminants. This relation can be interpreted as follows; There exists a holomorphic section (multivalued) of the dual of the determinant line bundle such that its L^2 norm is equal to the regularized determinant of the Laplacian acting on $(0, 1)$ forms.

In this paper we prove the existence of the analogue of the Dedekind eta function for odd dimensional CY manifolds. The construction of the generalized Dedekind eta function is based on the variational formulas for the determinants of the Laplacians of a Calabi-Yau metric acting on functions and forms of type (0,q) on CY manifolds obtained in the present paper. We establish the existence of a holomorphic section of some power N of the dual of determinant line bundle on the moduli space of odd dimensional CY manifolds whose L^2 norm is the N^{th} power of the regularized determinant of the Laplacian acting on $(0, 1)$. This holomorphic section of the determinant line bundle is the analogue of the Dedekind eta function for odd dimensional CY manifolds. It is also proved that the L^2 norm on the relative dualizing sheaf is a good metric in the sense of Mumford. This implies that the Weil-Petersson volumes of the moduli spaces of CY manifolds are rational numbers. When M is a CY threefold we proved that the regularized determinant of the Laplacian acting on $(0, 1)$ forms is bounded and the section η^N vanishes on the discriminant locus.

*Partially supported by The Institute of Mathematical Sciences of The Chinese University of Hong Kong.

Contents

1 Introduction

In case of elliptic curves, it is a well known fact that the regularized determinants of the Laplacians of flat metrics are related to the discriminant locus. In this paper we established the relation between the regularized determinants of Calabi-Yau metrics with the discriminant locus. The computation of the regularized determinant in the case of the flat metric on an elliptic curve is based on the Kronecker limit formula. It states that if

$$E(s) = \sum_{n,m\in\mathbb{Z}\&(n,m)\neq(0,0)} \frac{1}{|n+m\tau|^{2s}}$$

where $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$ and ' means that the sum is over all pair of integers $(m,n) \neq (0,0)$, then $E(s)$ has a meromorphic continuation in $\mathbb{C}$ with only one pole at $s = 1$ and

$$\exp\left(-\frac{d}{ds}E(s)|_{s=0}\right) = (\operatorname{Im}\tau)^2 |\eta|^4$$

where η is the Dedekind eta function. In the case of elliptic curves

$$\{E_\tau = \mathbb{C}/(n+m\tau)|\operatorname{Im}\tau > 0\},$$

the function

$$\left(\frac{1}{2\pi}\right)^{2s} E(s)$$

is the zeta function of the Laplacians of the flat metrics on the elliptic curves E_τ. The regularized determinant of the Laplacian is

$$\exp(-\frac{d}{ds}E(s)|_{s=0}) = (\operatorname{Im}\tau)^2 |\eta|^4 \tag{1}$$

where η^{24} is equal to the discriminant of the elliptic curve E_τ. The automorphic function η^{24} vanishes at ∞ and is equal to the discriminant of the elliptic curve. The point ∞ corresponds to an elliptic curve with a node. Thus the Kronecker limit formula gives a relation between the spectrum of the Laplacian and the discriminant of elliptic curves. In the case of the elliptic curves formula (1) has the following interpretation. There exists a holomorphic section of the sixth power of the dual of the determinant line bundle whose L^2 norm is the determinant of the Laplacian of the flat metric acting on $(0,1)$ forms. Moreover the existence of such section implies the existence of a multivalued holomorphic section η^4 of the determinant line bundle whose L^2 norm is equal to the exponential of the Ray Singer analytic torsion. $(\eta^4)^6$ can be realized as a holomorphic section of the 24^{th} power of the determinant line bundle over $\mathbb{PSL}_2(\mathbb{Z})\backslash\mathfrak{h}$. Notice that

$$6 = \#\left(\mathbb{PSL}_2(\mathbb{Z})/[\mathbb{PSL}_2(\mathbb{Z}), \mathbb{PSL}_2(\mathbb{Z})]\right).$$

This implies that the zero set of the exponential of the Ray Singer analytic torsion is the same as the discriminant η^{24} in this case.

There is a simple non-formal explanation of the above mentioned interpretation of the Kronecker limit formula. As we mentioned above, the spectrum of the Laplacian of a Riemannian metric on a compact manifold is discrete. When the manifold acquires singularities then the spectrum becomes continuous and hence contains zero. This phenomenon suggests that when the metric "degenerates" together with the manifold, then the regularized determinant vanishes on the points that parametrize the singular varieties. The problem is how to relate the spectrum of the Laplacian with the discriminant locus. The relation is suggested by the theory of determinant line bundles on the moduli space, their Quillen metrics and the Ray-Singer torsion as developed recently by Quillen, Donaldson, Bismut, Gillet, Soulé and others.

The problem that we are going to study is to find the generalization of the analogue of the Dedekind eta function for three dimensional CY manifolds and to relate it to the discriminant locus in the moduli space of polarized CY manifolds. To realize this program we need to work on the moduli space of polarized CY manifolds. It is a well known fact that the moduli space $\mathcal{M}(\mathrm{M})$ of CY manifolds is obtained by factoring the Teichmüller space by an arithmetic subgroup in the mapping class group of M that preserves some polarization on the CY manifolds. According to [45] $\mathcal{M}(\mathrm{M})$ is a quasi-projective variety. From the fact that the mapping class group is an arithmetic one, we can find a subgroup of finite index in the mapping class group such that the quotient of the Teichmüller space by this group $\mathfrak{M}_L(\mathrm{M})$ is a non-singular variety. $\mathfrak{M}_L(\mathrm{M})$ is a finite covering of $\mathcal{M}(\mathrm{M})$. Over $\mathfrak{M}_L(\mathrm{M})$ there exists a family of odd dimensional CY manifolds.

In this paper we study the determinant line bundle over the moduli space $\mathfrak{M}_L(\mathrm{M})$ of CY manifolds together with the Quillen metric associated with CY metric with a fix class of cohomology of its imaginary part. We prove that the determinant line bundle $\mathcal{L}$ is trivial as a C^∞ bundle on the moduli space of odd dimensional CY manifolds and that it is isomorphic to the dualizing line bundle $\pi_*(\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})})$. Using this fact we construct a canonical C^∞ non-vanishing section $\det(\overline{\partial})$ of the determinant line bundle whose Quillen norm is exactly the analytic Ray Singer torsion. We study the zero set of $\det(\overline{\partial})$ on some compactification $\overline{\mathfrak{M}_L(\mathrm{M})}$ such that

$$\overline{\mathfrak{M}_L(\mathrm{M})} \ominus \mathfrak{M}_L(\mathrm{M}) = \mathfrak{D}$$

is a divisor with normal crossings. $\mathfrak{D}$ is called the discriminant locus. On the relative dualizing sheaf $\pi_*\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}$ we have a natural L^2 metric $\| \|^2$. One can show that the metric $\| \|^2$ is a good one in the sense that Mumford discussed in [35]. From here we deduce that the determinant line bundle $\mathcal{L}$ can be prolonged in a unique way to a line bundle $\overline{\mathcal{L}}$ over $\overline{\mathfrak{M}_L(\mathrm{M})}$ by using the metric $\| \|^2$. In order to generalize the Dedekind eta function we need to construct a section η^N of some power of the dual of the determinant line bundle whose L^2 norm is equal to the N^{th} power of the regularized determinant $\det \Delta_{\tau,1}$of a CY metric acting on $(0,1)$ forms. To construct the section η^N we need to establish the variational formulas for the regularized determinants of the Laplacians acting on $(0,q)$ forms. By the variational formulas of the determinants $\Delta_{\tau,q}$ we mean to compute the Hessian of $\log \Delta_{\tau,q}$. The method of the computation of the the Hessian of $\log \Delta_{\tau,q}$ in this paper is based on the method used in [14]. It is easy to see that in order to compute

$$\frac{\partial^2}{\partial \tau^i \overline{\partial \tau^j}} \log \Delta_{\tau,q}$$

it is enouph to compute $\frac{\partial^2}{\partial \tau^i \overline{\partial \tau^j}} \log \Delta^{"}_{\tau,q}$, where $\Delta^{"}_{\tau,q} = \Delta_{\tau,q}|_{\mathrm{Im}\,\overline{\partial}^*}$. Following the ideas in [14] it is not difficult to prove that the Hessian of the zeta function $\zeta_{\tau,q}(s)$ is given by the formula:

$$\frac{\partial^2}{\partial \tau^i \overline{\partial \tau^j}} \zeta^{"}_{\tau,q}(s) = \frac{s}{\Gamma(s)} \int_0^\infty Tr \left(\frac{1}{4\pi t^n} \exp\left(-t\Delta^{'}_{\tau,q+1}\right) \mathcal{F}\left(q+1, \phi_i \circ \overline{\phi_j}\right)\right) t^{s-1} dt,$$

where $\phi_i \in \mathbb{H}^1(\mathrm{M},\Theta)$ are Kodaira-Spencer classes viewed as bundle maps:

$$\phi_i : C^\infty\left(\mathrm{M},\Omega^{1,0}_{\mathrm{M}}\right) \to C^\infty\left(\mathrm{M},\Omega^{0,1}_{\mathrm{M}}\right).$$

$\mathcal{F}\left(q+1, \phi_i \circ \overline{\phi_j}\right)$ is the map induced by

$$\phi_i \circ \overline{\phi_j} : C^\infty\left(\mathrm{M},\Omega^{0,1}_{\mathrm{M}}\right) \to C^\infty\left(\mathrm{M},\Omega^{0,1}_{\mathrm{M}}\right)$$

on $C^\infty\left(\mathrm{M},\Omega^{0,q+1}_{\mathrm{M}}\right)$ and restricted in $\mathrm{Im}\,\overline{\partial}$. Let

$$Tr\left(\frac{1}{4\pi t^n} \exp\left(-t\Delta^{'}_{\tau,q+1}\right) \mathcal{F}\left(q+1, \phi_i \circ \overline{\phi_j}\right)\right) = \sum_{k=-n}^{1} \frac{\alpha_k}{t^k} + \alpha_0 + ..$$

be the short time asymptotic expansion. It was pointed out in [14] that

$$\frac{\partial^2}{\partial \tau^i \overline{\partial \tau^j}} \log \Delta^{"}_{\tau,q} = \alpha_0.$$

In this paper we gave an explicit formula for α_0. This can be viewed as the generalization of the results in [24] and [27].

Based on the variational formulas we construct a section of the dual of the determinant line bundle whose L^2 norm is equal $\det \Delta_{\tau,1}$. Thus we generalize the above mentioned relation between the regularized determinant of the Laplacian of a CY metric acting on $(0,1)$ forms and the Dedekind eta function from elliptic curves to odd dimensional Calabi-Yau manifolds. We show that the holomorphic section η^N vanishes on the discriminant locus when M is a CY threefold.

The results in this paper are related to the results in [2]. In [2] a relation between the section η and counting elliptic curves on CY threefold was established by using arguments from String Theory. It is a real challenge for mathematicians to establish mathematically the results obtained in [2]. Some connections between the results of this paper and those in [2] are established in [44].

The results and the conjectures stated in this paper are related to the results in [22] and [23].

We discussed the problem of finding the relations between the spectral properties of the Laplacian of CY metric on K3 surfaces with the discriminant set in [24], [25] and [27]. The results of these papers showed that the problem of relating the spectral properties of the Laplacian acting on functions of a CY metric on even dimensional CY manifolds is a very delicate one since the Ray Singer Analytic torsion is zero. The analytic torsion for Enriques surfaces is discussed in [22] from the point of view of String Theory and in [46] from a mathematical point of view. The analogue of the Dedekind eta function for Enriques surfaces was constructed in [8]. Its relations to the regularized determinants of CY metrics on Enriques surfaces is discussed in [28]. Recently H. Fang and Z. Lu obtained important results concerning the relations between BCOV analytic torsion and its relations to Hodge and Weil Petersson metric. See [16].

This article is organized as follows.

In **Section 2** we introduce some basic notions about zeta functions of Laplacians on Riemannian manifolds. We review the results from [43].

In **Section 3** We review the basic properties of the Weil-Petersson metric on the moduli of CY manifolds. See also [33].

In **Section 4** we review some facts about the Hilbert spaces of the $(0,q)$ forms and their isospectral identifications which we used in the paper. We also study traces the operators acting on the L^2 sections of some vector bundle induced by some global C^∞ section its endormorphisms composed with the heat kernel. We study the short term expansions of these operators and especially the constant term of the expansion.

In **Section 5** we establish the variational formulas for the zeta functions of the Laplacians and its regularized determinants.

In **Section 6** we review the theory of moduli of CY manifolds following [31] and also metrics with logarithmic singularities on vector bundle following Mumford's article [35]. We prove that the L^2 metric on the dualizing line bundle over the moduli space of CY manifolds is a good metric in the sense of Mumford. This implies that the volumes of the moduli spaces of CY manifolds are rational numbers.

We show that the determinant line bundle is isomorphic to the line bundle of holomorphic n-forms on the moduli space $\mathfrak{M}_L(\mathrm{M})$ and that the natural L^2 metric on that bundle has logarithmic singularities. We will use the results of the previous sections to deduce that we can prolong the determinant line bundle to a line bundle on any compactification $\overline{\mathfrak{M}_L(\mathrm{M})}$ such that

$$\overline{\mathfrak{M}_L(\mathrm{M})} \ominus \mathfrak{M}_L(\mathrm{M}) = \mathfrak{D}$$

is a divisor with normal crossings. We will show that there exists a holomorphic section η^N of some power of the prolonged determinant line bundle the support of whose zero set is contained or equal to the support of $\mathfrak{D}$. We will prove that η^N vanishes on $\mathfrak{D}_\infty$, where $\mathfrak{D}_\infty$ consists of those point in $\mathfrak{D}$, around which we can find one parameter family of polarized CY such that the monodromy operator is of infinite order in the middle cohomology.

In **Section 7** we recall the theory of determinant line bundles of Mumford, Knudsen, Bismut, Donaldson and Soulé, following the exposition of D. Freed. See [12] and [17]. We review the Quillen metric.

In **Section 8** we construct a nonvanishing section $\det(\overline{\partial})$ of the determinant line bundle $\mathcal{L}$ over the moduli space $\mathfrak{M}_L(\mathrm{M})$ of a CY manifold M of any dimension. Thus we prove that we prove that the determinant line bundle is a trivial C^∞ bundle over the moduli space $\mathfrak{M}_L(\mathrm{M})$. In the case of odd dimensional CY manifold we construct a C^∞ section of the determinant line bundle whose Quillen norm is the exponential of the Ray Singer Analytic torsion.

In **Section 9** we show that the regularized determinants of the Laplacian of CY metrics on CY threefolds acting on $(0,1)$ forms is bounded on the moduli space.

In **Section 10** we show that we can prolong the canonical C^∞ section $\det(\overline{\partial})$ constructed in **Section 9** to any compactification $\overline{\mathfrak{M}_L(\mathrm{M})}$ such that

$$\overline{\mathfrak{M}_L(\mathrm{M})} \ominus \mathfrak{M}_L(\mathrm{M}) = \mathfrak{D}$$

is a divisor with normal crossings. We prove that the zero set of $\det(\overline{\partial})$ is supported by an effective divisor contained in the discriminant locus $\mathfrak{D}$. Based on results of Kazhdan and Sullivan, we will prove that there exists a positive integer N such that the N^{th} power of the determinant holomorphic line bundle is a trivial one over the moduli space $\mathfrak{M}_L(\mathrm{M})$. Using this result we will construct a holomorphic section $\overline{\eta}^N$ of the determinant line bundle $\left(\overline{\mathcal{L}}\right)^{\otimes N}$ over $\overline{\mathfrak{M}_L(\mathrm{M})}$ whose zero set is the same as that of the exponential of the analytic Ray Singer torsion and supported by $\mathfrak{D}$.

In **Section 11** we discuss some conjectures and problems.

Acknowledgement 1 *The author wants to thank G. Moore, J. Jorgenson S. Donaldson, S.-T. Yau, G. Zuckerman, D. Kazhdan, P. Deligne, S. Lang, B. Lian, J. Li, K. Liu, Y. Eliashberg, Dai and R. Donagi for useful comments and support. I want to thank Sinan Unver for his help. Special thanks to The Institute of Mathematical Sciences at CUHK and Hangzhou for their hospitality during the preparation of this article.*

2 Preliminary Material

2.1 Basic Notions

Let M be a n-dimensional Kähler manifold with a zero canonical class. Suppose that $H^k(\mathrm{M}, \mathcal{O}_{\mathrm{M}}) = 0$ for $1 \leq k < n$. *Such manifolds are called Calabi-Yau manifolds.* A pair (M,L) will be called a polarized CY manifold if M is a CY manifold and $L \in H^2(\mathrm{M},\mathbb{Z})$[1] is a fixed class such that it represents the imaginary part of a Kähler metric on M.

Yau's celebrated theorem asserts the existence of a unique Ricci flat Kähler metric g on M such that the cohomology class $[Im(g)] = L$. (See [47].) From now on we will consider polarized CY manifolds of odd dimension. The polarization class L determines the CY metric g uniquely. We will denote by

$$\triangle_q = \overline{\partial}^* \circ \overline{\partial} + \overline{\partial} \circ \overline{\partial}^*$$

the associated Laplacians that act on smooth $(0, q)$ forms on M for $0 \leq q \leq n$. $\overline{\partial}^*$ is the adjoint operator of $\overline{\partial}$ with respect to the CY metric g.

The regularized determinants are defined as follows: Let (M,g) be an n-dimensional Riemannian manifold. Let

$$\Delta_q = dd^* + d^*d$$

be the Laplacian acting on the space of q forms on M. We recall that the spectrum of the Laplacian Δ_q is positive and discrete. Thus the non zero eigen values of Δ_q are

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...$$

We define the zeta function of Δ_q as follows:

$$\zeta_q(s) = \sum_{i=1}^{\infty} \lambda_i^{-s}.$$

It is known that $\zeta_q(s)$ is a well defined analytic function for $\mathrm{Re}(s) \gg C$, it has a meromorphic continuation in the complex plane and 0 is not a pole of $\zeta_q(s)$. Define

$$\det(\Delta_q) = \exp\left(-\frac{d}{ds}\left(\zeta_q(s)\right)|_{s=0}\right).$$

The determinant of these operators $\triangle_q$, defined through zeta function regularization, will be denoted by $\det(\triangle_q)$.

The Hodge decomposition theorem asserts that

$$\Gamma(M, \Omega^{0,q}) = \mathrm{Im}(\overline{\partial}) \oplus \mathrm{Im}(\overline{\partial}^*)$$

for $1 \leq q \leq \dim_{\mathbb{C}} M - 1$. The restriction of $\triangle_q$ on $\mathrm{Im}(\overline{\partial})$ will be denoted by $\triangle_q^{'}$ and $\triangle_q^{'} = \overline{\partial} \circ \overline{\partial}^*$ and the restriction of Δ_q on $\mathrm{Im}(\overline{\partial}^*)$ will be denoted by $\triangle_q^{"}$ and $= \overline{\partial}^* \circ \overline{\partial}$. Hence we have

$$Tr(\exp(-t\triangle_q) = Tr(\exp(-t\triangle_q^{'}) + Tr(\exp(-t\triangle_q^{"}).$$

[1]Notice that $H^{1,1}(\mathrm{M},\mathbb{R}) = H^2(\mathrm{M},\mathbb{R})$ since $H^2(\mathrm{M},\mathcal{O}_{\mathrm{M}}) = 0$ for CY manifolds.

This implies that

$$\zeta_q(s) = \sum_{k=1}^{\infty} \lambda_k^{-s} = \zeta_q^{'}(s) + \zeta_q^{"}(s),$$

where $\lambda_k > 0$ are the positive eigen values of $\triangle_q$ and $\zeta_q^{'}(s)$ & $\zeta_q^{"}(s)$ are the zeta functions of $\triangle_q^{'}$ and $\triangle_q^{"}$. From here and the definition of the regularized determinant we obtain that

$$\log\det(\triangle_q) = \log\det(\triangle_q^{'}) + \log\det(\triangle_q^{"}).$$

It is a well known fact that the action of $\triangle_q^{"}$ on $\operatorname{Im}\overline{\partial}^*$ is isospectral to the action of $\triangle_{q+1}^{'}$ on $\operatorname{Im}\overline{\partial}$, which means that the spectrum of $\triangle_q^{"}$ is equal to the spectrum of $\triangle_{q+1}^{'}$. So we have the equality

$$\det(\triangle_q^{"}) = \det(\triangle_{q+1}^{'}).$$

Let f be a map from a set A to a set B and let g be a map from the set B to the set C, then the compositions of those two maps we will denote by f$\circ$g.

2.2 Basic Notions about Complex Structures

Let M be an even dimensional C^∞ manifold. We will say that M has an almost complex structure if there exists a section

$$I \in C^\infty(\mathrm{M}, Hom(T^*, T^*))$$

such that $I^2 = -id$. T is the tangent bundle and T^* is the cotangent bundle on M. This definition is equivalent to the following one: Let M be an even dimensional C^∞ manifold. Suppose that there exists a global splitting of the complexified cotangent bundle

$$T^* \otimes \mathbf{C} = \Omega^{1,0} \oplus \Omega^{0,1},$$

where $\Omega^{0,1} = \overline{\Omega^{1,0}}$. Then we will say that M has an almost complex structure. We will say that an almost complex structure is an integrable one, if for each point $x \in$M there exists an open set $U \subset$M such that we can find local coordinates $z^1, .., z^n$, such that $dz^1, .., dz^n$ are linearly independent in each point $m \in U$ and they generate $\Omega^{1,0}|_U$.

Definition 2 *Let M be a complex manifold. Let*

$$\phi \in \Gamma(M, Hom(\Omega^{1,0}, \Omega^{0,1})),$$

then we will call ϕ a Beltrami differential.

Since

$$\Gamma(\mathrm{M}, Hom(\Omega^{1,0}, \Omega^{0,1})) \backsimeq \Gamma(\mathrm{M}, \Omega^{0,1} \otimes T^{1,0}),$$

we deduce that locally ϕ can be written as follows:

$$\phi|_U = \sum \phi^{\beta}_{\overline{\alpha}} \overline{dz}^{\alpha} \otimes \frac{\partial}{\partial z^{\beta}}.$$

From now on we will denote by A_ϕ the following linear operator:

$$A_\phi = \begin{pmatrix} id & \phi(\tau) \\ \overline{\phi(\tau)} & id \end{pmatrix}.$$

We will consider only those Beltrami differentials ϕ such that $\det(A_\phi) \neq 0$. The Beltrami differential ϕ defines an integrable complex structure on M if and only if the following equation holds:

$$\overline{\partial}\phi = \frac{1}{2}\left[\phi, \phi\right], \tag{2}$$

where

$$\left[\phi, \phi\right]|_U := \sum_{\nu=1}^{n} \sum_{1 \leqq \alpha < \beta \leqq n} \left(\sum_{\mu=1}^{n} \left(\phi^{\mu}_{\overline{\alpha}} \left(\partial_\mu \phi^{\nu}_{\overline{\beta}} \right) - \phi^{\mu}_{\overline{\beta}} \left(\partial_\nu \phi^{\nu}_{\overline{\alpha}} \right) \right) \right) \overline{dz}^{\alpha} \wedge \overline{dz}^{\beta} \otimes \frac{\partial}{dz^{\nu}} \tag{3}$$

(See [30].)

2.3 Kuranishi Space and Flat Local Coordinates

Kuranishi proved the following Theorem:

Theorem 3 *Let $\{\phi_i\}$ be a basis of harmonic $(0,1)$ forms of $\mathbb{H}^1(M, T^{1,0})$ on a Hermitian manifold M. Let G be the Green operator and let $\phi(\tau^1, .., \tau^N)$ be defined as follows:*

$$\phi(\tau^1, ..., \tau^N) = \sum_{i=1}^{N} \phi_i \tau^i + \frac{1}{2}\overline{\partial}^* G[\phi(\tau^1, ..., \tau^N), \phi(\tau^1, ..., \tau^N)]. \tag{4}$$

There exists $\varepsilon > 0$ such that if

$$\tau = (\tau^1, ..., \tau^N)$$

satisfies $|\tau_i| < \varepsilon$ then $\phi(\tau^1, ..., \tau^N)$ is a global C^∞ section of the bundle $\Omega^{(0,1)} \otimes T^{1,0}$.(See [30].)

Based on Theorem 3, we proved in [43] the following Theorem:

Theorem 4 *Let M be a CY manifold and let $\{\phi_i\}$ be a basis of harmonic $(0,1)$ forms with coefficients in $T^{1,0}$, i.e.*

$$\{\phi_i\} \in \mathbb{H}^1(M, T^{1,0}),$$

then the equation (2)*:*

$$\overline{\partial}\phi = \frac{1}{2}\left[\phi,\phi\right]$$

has a solution in the form:

$$\phi(\tau^1,...,\tau^N) = \sum_{i=1}^{N}\phi_i\tau^i + \sum_{|I_N|\geqq 2}\phi_{I_N}\tau^{I_N} =$$

$$\sum_{i=1}^{N}\phi_i\tau^i + \frac{1}{2}\overline{\partial}^* G[\phi(\tau^1,...,\tau^N),\phi(\tau^1,...,\tau^N)]$$

and

$$\overline{\partial}^*\phi(\tau^1,...,\tau^N) = 0,\ \ \phi_{I_N}\lrcorner\omega_M = \partial\psi_{I_N}$$

where $I_N = (i_1,...,i_N)$ *is a multi-index,*

$$\phi_{I_N} \in C^\infty(M,\Omega^{0,1}\otimes T^{1,0}), \tau^{I_N} = (\tau^i)^{i_1}...(\tau^N)^{i_N}$$

and if for some $\varepsilon > 0$ $|\tau^i| < \varepsilon$ *then*

$$\phi(\tau) \in C^\infty(M,\Omega^{0,1}\otimes T^{1,0})$$

where $i = 1,...,N$. (*See* [*42*] *and* [*43*]).

It is a standard fact from Kodaira-Spencer-Kuranishi deformation theory that for each

$$\tau = (\tau^1,...,\tau^N) \in \mathcal{K}$$

as in Theorem 4 the Beltrami differential $\phi(\tau^1,...,\tau^N)$ defines a new integrable complex structure on M. This means that the points of $\mathcal{K}$, where

$$\mathcal{K} : \{\tau = (\tau^1,...,\tau^N)||\tau^i| < \varepsilon\}$$

defines a family of operators $\overline{\partial}_\tau$ on the C^∞ family $\mathcal{K}\times M \to M$ and $\overline{\partial}_\tau$ are integrable in the sense of Newlander-Nirenberg. Moreover it was proved by Kodaira, Spencer and Kuranishi that we get a complex analytic family of CY manifolds

$$\pi : \mathcal{X} \to \mathcal{K},$$

where as C^∞ manifold

$$\mathcal{X} \backsimeq \mathcal{K}\times \mathrm{M}.$$

The family

$$\pi : \mathcal{X} \to \mathcal{K} \tag{5}$$

is called the Kuranishi family. The operators $\overline{\partial}_\tau$ are defined as follows:

Definition 5 *Let $\{\mathcal{U}_i\}$ be an open covering of M, with local coordinate system in $\mathcal{U}_i$ given by $\{z_i^k\}$ with $k = 1, ..., n = \dim_{\mathbb{C}} M$. Assume that $\phi(\tau^1, ..., \tau^N)|_{\mathcal{U}_i}$ is given by:*

$$\phi(\tau^1, ..., \tau^N) = \sum_{j,k=1}^{n} (\phi(\tau^1, ..., \tau^N))_{\overline{j}}^{k}\ d\overline{z}^j \otimes \frac{\partial}{\partial z^k}.$$

Then we define

$$\left(\overline{\partial}\right)_{\tau,\overline{j}} = \frac{\overline{\partial}}{\overline{\partial z^j}} - \sum_{k=1}^{n}(\phi(\tau^1, ..., \tau^N))_{\overline{j}}^{k} \frac{\partial}{\partial z^k}. \tag{6}$$

Definition 6 *The coordinates $\tau = (\tau^1, ..., \tau^N)$ defined in Theorem 4, will be fixed from now on and will be called the flat coordinate system in $\mathcal{K}$.*

2.4 Family of Holomorphic Forms

In [43] the following Theorem is proved:

Theorem 7 *There exists a family of holomorphic forms ω_τ of the Kuranishi family* (5) *such that*

$$\langle [\omega_\tau], [\omega_\tau] \rangle =$$

$$1 - \sum_{i,j} \langle \omega_0 \lrcorner \phi_i, \omega_0 \lrcorner \phi_j \rangle \tau^i \overline{\tau^j} + \sum_{i,j} \langle \omega_0 \lrcorner (\phi_i \wedge \phi_k), \omega_0 \lrcorner (\phi_j \wedge \phi_l) \rangle \tau^i \overline{\tau^j} \tau^k \overline{\tau^l} + O(\tau^5) =$$

$$1 - \sum_{i,j} \tau^i \overline{\tau^j} + \sum_{i,j} \langle \omega_0 \lrcorner (\phi_i \wedge \phi_k), \omega_0 \lrcorner (\phi_j \wedge \phi_l) \rangle \tau^i \overline{\tau^j} \tau^k \overline{\tau^l} + O(\tau^5) \text{ and}$$

$$\langle [\omega_\tau], [\omega_\tau] \rangle \leq \langle [\omega_0], [\omega_0] \rangle. \tag{7}$$

3 Weil-Petersson Metric

3.1 Basic Properties

It is a well known fact from Kodaira-Spencer-Kuranishi theory that the tangent space $T_{\tau,\mathcal{K}}$ at a point $\tau \in \mathcal{K}$ can be identified with the space of harmonic (0,1) forms with values in the holomorphic vector fields $\mathbb{H}^1(\mathrm{M}_\tau, T)$. We will view each element $\phi \in \mathbb{H}^1(\mathrm{M}_\tau, T)$ as a point wise linear map from $\Omega^{(1,0)}_{\mathrm{M}_\tau}$ to $\Omega^{(0,1)}_{\mathrm{M}_\tau}$. Given ϕ_1 and $\phi_2 \in \mathbb{H}^1(\mathrm{M}_\tau, T)$, the trace of the map

$$\phi_1 \circ \overline{\phi_2} : \Omega^{(0,1)}_{\mathrm{M}_\tau} \to \Omega^{(0,1)}_{\mathrm{M}_\tau}$$

at the point $m \in \mathrm{M}_\tau$ with respect to the metric g is simply:

$$Tr(\phi_1 \circ \overline{\phi_2})(m) = \sum_{k,l,m=1}^{n} (\phi_1)_{\overline{l}}^{k} \overline{(\phi)_{\overline{k}}^{m}} g^{\overline{l},k} g_{k,\overline{m}} \tag{8}$$

Definition 8 *We will define the Weil-Petersson metric on $\mathcal{K}$ via the scalar product:*

$$< \phi_1, \phi_2 >= \int\limits_M Tr(\phi_1 \circ \overline{\phi_2}) vol(g). \tag{9}$$

We proved in [43] that the coordinates

$$\tau = (\tau^1, ..., \tau^N)$$

as defined in Definition 6 are flat in the sense that the Weil-Petersson metric is Kähler and in these coordinates we have that the components $g_{i,\overline{j}}$ of the Weil Petersson metric are given by the following formulas:

$$g_{i,\overline{j}} = \delta_{i,\overline{j}} + R_{i,\overline{j},l,\overline{k}} \tau^l \overline{\tau^k} + O(\tau^3).$$

Very detailed treatment of the Weil-Petersson geometry of the moduli space of polarized CY manifolds can be found in [32] and [33]. In those two papers important results are obtained.

3.2 Infinitesimal Deformation of the Imaginary Part of the WP Metric

Theorem 9 *Near the point $\tau = 0$ of the Kuranishi space $\mathcal{K}$, the imaginary part $\operatorname{Im}(g)$ of the CY metric g has the following expansion in the coordinates $\tau := (\tau^1, ..., \tau^N)$:*

$$\operatorname{Im}(g)(\tau, \overline{\tau}) = \operatorname{Im}(g)(0) + O(\tau^2).$$

Proof: In [43] we proved that the forms

$$\theta^k_\tau = dz^k + \sum_{l=1} \left(\phi(\tau^1, ..., \tau^N)^k_{\overline{l}}\right) d\overline{z^l} \tag{10}$$

for

$$k = 1, ., n$$

form a basis of $(1,0)$ forms relative to the complex structure defined by $\tau \in \mathcal{K}$ in $\mathcal{U} \subset$M. Let

$$\operatorname{Im}(g_\tau) = \sqrt{-1} \left(\sum_{1 \leq k \leq l \leq n} g_{k,\overline{l}}(\tau, \overline{\tau}) \theta^k_\tau \wedge \overline{\theta^l_\tau} \right) \tag{11}$$

and

$$g_{k,\overline{l}}(\tau, \overline{\tau}) = g_{k,\overline{l}}(0) + \sum_{i=1}^N \left(\left(g_{k,\overline{l}}(1)\right)_i \tau^i + \left(g^{'}_{k,\overline{l}}(1)\right)_{\overline{i}} \overline{\tau^i} \right) + O(2). \tag{12}$$

We get the following expression for $\operatorname{Im}(g_\tau)$ in terms of dz^i and $\overline{dz^j}$, by substituting the expressions for θ^k_τ from (10) and the expressions for $g_{k,\overline{l}}(\tau, \overline{\tau})$ from formula (12) in the formula (11):

$$\operatorname{Im}(g_\tau) = \sqrt{-1}\left(\sum_{1\le k\le l\le n} g_{k,\bar{l}}(\tau,\overline{\tau})\theta_\tau^k \wedge \overline{\theta_\tau^l}\right) = \sqrt{-1}\left(\sum_{1\le k\le l\le n} g_{k,\bar{l}}(0)dz^k \wedge \overline{dz^l}\right) +$$

$$+\sqrt{-1}\left(\sum_{i=1}^N \tau^i\left(\sum_{1\le k\le l\le n}\left(\left(g_{k,\bar{l}}(1)\right)_i dz^k \wedge \overline{dz^l} + \sum_{m=1}^n (g_{k,\overline{m}}\overline{\phi^m_{i,\bar{l}}} - g_{l,\overline{m}}\overline{\phi^m_{i,\bar{k}}})dz^k \wedge dz^l\right)\right)\right)$$

$$+\frac{1}{\sqrt{-1}}\sum_{i=1}^N \overline{\tau^i}\overline{\left(\sum_{1\le k\le l\le n}\left(\left(g_{k,\bar{l}}(1)\right)_i dz^k \wedge \overline{dz^l} + (\sum_{m=1}^n (g_{k,\overline{m}}\overline{\phi^m_{i,\bar{l}}} - g_{l,\overline{m}}\overline{\phi^m_{i,\bar{k}}})dz^k \wedge dz^l\right)\right)}.$$

On page 332 of [43] the following results is proved:

Lemma 10 *Let $\phi \in \mathbb{H}^1(M,T)$ be a harmonic form with respect to the CY metric g. Let*

$$\phi|_U = \sum_{k,l=1}^n \phi^l_{\bar{k}}\,\overline{dz}^k \otimes \frac{\partial}{\partial z^l},$$

then

$$\phi_{\overline{k},\bar{l}} = \sum_{j=1}^n g_{j,\overline{k}}\ \phi^j_{\bar{l}} = \sum_{j=1}^n g_{j,\bar{l}}\ \phi^j_{\overline{k}} = \phi_{\bar{l},\overline{k}}.$$

From Lemma 10 we conclude that

$$\sum_{m=1}^n (g_{k,\overline{m}}\overline{\phi^m_{i,\bar{l}}} - g_{l,\overline{m}}\overline{\phi^m_{i,\bar{k}}}) = 0. \tag{13}$$

From (13) we get the following expression for $\operatorname{Im}(g_\tau)$:

$$\operatorname{Im}(g_\tau) = \sqrt{-1}\left(\sum_{1\le k\le l\le n} g_{k,\bar{l}}(0)dz^k \wedge \overline{dz^l}\right) +$$
$$\sqrt{-1}\left(\sum_{i=1}^N \tau^i\left(\sum_{1\le k\le l\le n}\left(g_{k,\bar{l}}(1)\right)_i dz^k \wedge \overline{dz^l}\right)\right) +$$
$$+\sqrt{-1}\left(\sum_{i=1}^N \overline{\tau^i}\overline{\sum_{1\le k\le l\le n}\left(g_{k,\bar{l}}(1)\right)_i dz^k \wedge \overline{dz^l}}\right) + O(2) \tag{14}$$

Let us define the (1,1) forms ψ_i as follows:

$$\psi_i = \sqrt{-1}\left(\sum_{1\le k\le l\le n}\left(g_{k,\bar{l}}(1)\right)_i dz^k \wedge \overline{dz^l}\right) \tag{15}$$

We derive the following formula, by substituting in the expression (14) the expression given by (15):

$$\operatorname{Im}(g_\tau) = \operatorname{Im}(g_0) + \sum_{i=1}^{N} \tau^i \psi_i + \sum_{i=1}^{N} \overline{\tau^i \psi_i} + O(\tau^2) \tag{16}$$

From the fact that the class of the cohomology of the imaginary part of the CY metric is fixed, i.e.

$$[\operatorname{Im}(g_\tau)] = [\operatorname{Im}(g_0)] = L,$$

and (16) we deduce that each ψ_i is an exact form, *i.e.*

$$\psi_i = \sqrt{-1}\partial\overline{\partial} f_i, \tag{17}$$

where f_i are globally defined functions on M. Our Theorem will follow if we prove that $\psi_i = 0$.

Lemma 11 $\psi_i = 0$.

Proof: In *[43]* we proved that

$$\det(g_\tau) = \wedge^n \operatorname{Im}(g_\tau) = \det(g_0) + O(2) \tag{18}$$

in the flat coordinates $(\tau^1, ..., \tau^N)$. We deduce from the expressions (16) and (17), by direct computations that:

$$\det(g_\tau) = \det(g_0) + \sqrt{-1}\sum_{i=1}^{N} \tau^i \left(\sum_{k,l} g^{\bar{l},k} \partial_k \overline{\partial_l}(f_i) \right) +$$

$$\frac{1}{\sqrt{-1}} \sum_{i=1}^{N} \overline{\tau^i \left(\sum_{k,l} g^{\bar{l},k} \partial_k \overline{\partial_l}(f_i) \right)} + O(2). \tag{19}$$

Combining (18) and (19) we obtain that for each i we have*:*

$$\sum_{k,l} g^{\bar{l},k} \partial_k \overline{\partial_l}(f_i) = \triangle(f_i) = 0,$$

where $\triangle$ is the Laplacian of the metric g. From the maximum principle, we deduce that all f_i are constants. Formula (17) implies that $\psi_i = 0$. Lemma 11 is proved. ■

Theorem 9 follows directly from Lemma 11. Theorem 9 is proved. ■

Corollary 12 *The imaginary part* $\operatorname{Im} g_\tau$ *of the CY metric is a constant symplectic form on the moduli space* $\mathfrak{M}_L$*(M), i.e.*

$$\frac{d}{d\tau} \operatorname{Im}(g_\tau) = 0.$$

Corollary 13 *The following formulas are true:*

$$\frac{\partial}{\partial\tau_i}\left(\overline{\partial_\tau}\right)^* = 0 \text{ and } \frac{\overline{\partial}}{\overline{\partial\tau_i}}\left(\partial_\tau\right)^* = 0. \tag{20}$$

Proof: We know from Kähler geometry that $(\overline{\partial_\tau})^* = [\Lambda_\tau, \partial_\tau]$, where Λ_τ is the contraction with *(1,1)* vector field:

$$\Lambda_\tau = \frac{\sqrt{-1}}{2}\sum_{k,l=1}^{n} g_\tau^{\overline{k},l}(\theta_\tau^l)^* \wedge (\overline{\theta_\tau^k})^* \tag{21}$$

on M_τ and $(\theta_\tau^l)^*$ is (1,0) vector field on M_τ dual to the (1,0) form

$$\theta_\tau^i = dz^i + \sum_{j=1}^{N}\tau^j(\sum_{k=1}^{n}(\phi_j)_{\overline{k}}^{i}\,\overline{dz}^k)).$$

Cor. 12 implies that

$$\frac{\partial}{\partial\tau_i}(\Lambda_\tau) = 0.$$

On the other hand, ∂_τ depends antiholomorphically on τ, i.e. it depends on

$$\overline{\tau} = (\overline{\tau_1}, ..., \overline{\tau_N}).$$

So we deduce that:

$$\frac{\partial}{\partial\tau_i}((\overline{\partial_\tau})^*) = \left([\frac{\partial}{\partial\tau_i}(\Lambda_\tau), \partial_\tau] + [\Lambda_\tau, \frac{\partial}{\partial\tau_i}(\partial_\tau)]\right) = 0.$$

Exactly in the same way we prove that

$$\frac{\overline{\partial}}{\overline{\partial\tau_i}}\left(\partial_\tau\right)^* = 0.$$

Corollary 13 is proved. ■

4 Hilbert Spaces and Trace Class Operators

4.1 Preliminary Material

Definition 14 *We will denote by $L^2_{0,q}(\mathrm{Im}(\overline{\partial}^*))$ the Hilbert subspace in $L^2(M,\Omega^{(0,q)})$ which is the L^2 completion of $\overline{\partial^*}$ exact forms in $C^\infty(M,\Omega^{(0,q)})$ for $q \geq 0$. In the same manner we will denote by $L^2_{1,q-1}(\mathrm{Im}(\partial))$ the Hilbert subspace in $L^2(M,\Omega_M^{(1,q-1)})$ which is the L^2 competition of the ∂ exact $(1,q-1)$ forms in $C^\infty(M,\Omega^{(1,q-1)})$ for $q > 0$. All the completions are with respect to the scalar product on the bundles $\Omega^{p,q}$ defined by the CY metric g.*

Let $\phi(\tau^1, .., \tau^N\ )$ be the solution of the equation (2):

$$\overline{\partial}\phi(\tau^1, ..., \tau^N) = \frac{1}{2}[\phi(\tau^1, ..., \tau^N), \phi(\tau^1, ..., \tau^N)]$$

by Theorem 4. From Definition 2 of the Beltrami differential, we know that the Beltrami differential $\phi(\tau^1, .., \tau^N\ )$ defines a linear fibrewise map

$$\phi(\tau^1, ..., \tau^N) : \Omega^{(1,0)} \to \Omega^{(0,1)}.$$

So

$$\phi(\tau^1, ..., \tau^N) \in C^\infty(\mathrm{M}, Hom(\Omega^{(1,0)}_{\mathrm{M}}, \Omega^{(0,1)}_{\mathrm{M}}). \tag{22}$$

Definition 15 *We define the following maps between vector bundles*

$$\phi \wedge id : \Omega^{(1,q-1)} \to \Omega^{(0,q)}$$

as

$$\phi(dz^i \wedge \alpha) = \phi(dz^i) \wedge \alpha$$

for each $1 \leq q \leq n$*. Clearly each fibre wise linear map* $\phi \wedge id_{q-1}$ *defines a natural linear operator*

$$F(q, \phi) : L^2(M, \Omega^{(1,q-1)}) \to L^2(M, \Omega^{(0,q)})$$

between the Hilbert spaces. The restriction of the linear operator F(q,ϕ) on the subspace

$$\mathrm{Im}(\partial) \subset L^2(M, \Omega^{(1,q-1)}_M))$$

to

$$\mathrm{Im}(\overline{\partial}) \subset L^2(M, \Omega^{(0,q)}_M)$$

will be denoted by $F'(q, \phi)$*. The restriction of the linear operator F(q,ϕ) on the subspace*

$$\mathrm{Im}(\partial^*) \subset L^2(M, \Omega^{(1,q-1)}_M))$$

to

$$\mathrm{Im}\left(\overline{\partial}^*\right) \subset L^2(M, \Omega^{(0,q)}_M)$$

will be denoted by $F"(q, \phi)$*. Let* ϕ *and* ψ *be two Kodaira Spencer classes and let*

$$\phi \circ \overline{\psi} : L^2(M, \Omega^{(0,1)}_M) \to L^2(M, \Omega^{(0,1)}_M)$$

be fibrewise linea map given by

$$\phi \circ \overline{\psi}|_U := \sum_{\alpha,\beta=1}^{n} \left(\phi \circ \overline{\psi}\right)^{\overline{\alpha}}_{\overline{\beta}} \overline{dz^\beta} \otimes \frac{\partial}{\overline{\partial z^\alpha}}. \tag{23}$$

We define the fibrewise bundle maps

$$\left(\phi \circ \overline{\psi}\right) \wedge id_{q-1} : \Omega^{0,q}_M \to \Omega^{0,q}_M, \tag{24}$$

as follows:

$$\left(\left(\phi\circ\overline{\psi}\right)\wedge id_{q-1}\right)(\omega):=\left(\sum_{\alpha,\beta=1}^{n}\left(\phi\circ\overline{\psi}\right)_{\overline{\beta}}^{\overline{\alpha}}\overline{dz^{\beta}}\otimes\overline{\frac{\partial}{\partial z^{\alpha}}}\right)\lrcorner\omega, \tag{25}$$

where $\lrcorner$ *means contraction of tensors and* ω *is some global form of type* $(0,q)$ *on M. We will define for the linear operators*

$$\mathcal{F}^{'}(q,\phi\circ\overline{\psi}):L^2_{0,q}(\mathrm{Im}(\overline{\partial}))\rightarrow L^2_{0,q}(\mathrm{Im}(\overline{\partial})) \tag{26}$$

and

$$\mathcal{F}^{"}(q,\phi\circ\overline{\psi}):L^2_{0,q}(\mathrm{Im}(\overline{\partial})^*)\rightarrow L^2_{0,q}(\mathrm{Im}(\overline{\partial})^*) \tag{27}$$

as the restriction of the operators $\left(\left(\phi\circ\overline{\psi}\right)\wedge id_{q-1}\right)$ *on* $L^2_{0,q}(\mathrm{Im}(\overline{\partial}))$ *and* $L^2_{0,q}(\mathrm{Im}(\overline{\partial})^*)$ *respectively.*

Remark 16 *It is a standard fact that we can choose globally* $\overline{\partial}$ *closed forms* $\omega_1,...,\omega_N$ *of type* $(0,q)$ *such that at each point* $z\in M$ *they span the fibre* $\Omega^{0,q}_{M,z}$*. We can deduce directly from the definitions of the operators* $F'(q,\phi),\mathcal{F}^{'}(q,\phi\circ\overline{\psi})$ *and* $F^{'}(q,\overline{\psi}\circ\phi)$ *and the existence of the forms* $\omega_1,...,\omega_N$ *that the operators* $F'(q,\phi),\mathcal{F}^{'}(q,\phi\circ\overline{\psi})$ *and* $F^{'}(q,\overline{\psi}\circ\phi)$ *pointwise will be represented by matrices of dimensions* $\binom{n}{q},\binom{n}{q}$ *and* $n\times\binom{n}{q-1}$*.*

4.2 Trace Class Operators (See [7].)

Let H be a Hilbert space with a orthonormal basis e_i. An operator A is a **Hilbert-Schmidt** operator if

$$\|A\|^2_{HS}=\sum_i\|Ae_i\|^2=\sum_{ij}|\langle Ae_i,e_j\rangle|^2<\infty$$

is finite. The number $\|A\|^2_{HS}$ is called the Hilbert-Schmidt norm of A. If A is a Hilbert-Schmid so is its adjoint A^* and

$$\|A\|^2_{HS}=\|A^*\|^2_{HS}\,.$$

If U is a bounded operator on H and A is an Hilbert-Schmidt, then $U\circ A$ and $A\circ U$ are Hilbert-Schmidt operators and

$$\|U\circ A\|_{HS}\leq\|A\circ U\|_{HS}\,.$$

In this paper we will consider the Hilbert spaces of the square integrable sections of the bundles $\Omega^{0,q}_{\mathrm{M}}\otimes|\Lambda_{\mathrm{M}}|^{1/2}$ on M, where $|\Lambda_{\mathrm{M}}|$ is the trivial density bundles generated by the volume form of the CY metric.

An operator K with square-integrable kernel

$$k(w,z)\in\Gamma_{L^2}\left(\mathrm{M}\times\mathrm{M},\left(\Omega^{0,q}_{\mathrm{M}}\otimes|\Lambda_{\mathrm{M}}|^{1/2}\boxtimes\Omega^{0,q}_{\mathrm{M}}\otimes|\Lambda_{\mathrm{M}}|^{1/2}\right)\right)$$

is Hilbert-Schmidt, and

$$\|K\|_{HS}^2 = \int\limits_{(w,z)\in \mathrm{M}\times\mathrm{M}} Tr\left(k\left(w,z\right)^{*} k\left(w,z\right)\right). \tag{28}$$

Formula (28) follows from the definition of the Hilbert-Schmidt norm

$$\|K\|_{HS}^2 = \sum_{ij} \left|\langle Ke_i, e_j\rangle\right|^2.$$

An operator K is said to be **trace class** if it has the form $A \circ B$, where A and B are Hilbert-Schmidt. For such operators, the sum

$$TrK = \sum_i \langle Ke_i, e_i\rangle$$

is absolutely summable and TrK is independent of the choice of the orthonormal basis in H and is called the trace of K.

4.3 Heat Kernels and Traces

In this subsection we study the traces of operators which are compositions of the heat kernel with operators induced by endomorphisms of some vector bundle. We will use some of the results from [14] and will adopt them to our situation.

Let h be a metric on a vector bundle E over M. Let Δ_h be the Laplacian on E. It is a well known fact that the operator $\exp(-t\triangle_h)$ can be represented by an integral kernel:

$$k_t(w,z,\tau) = \sum_j \exp\left(-t\lambda_j\right)\phi_j(\omega)\otimes\phi_j(z),$$

where λ_j and ϕ_j are the eigen values and the eigen sections of the Laplace operator Δ_h on some vector bundle E on M. $k_t(w,z,\tau)$ is an operator of trace class. We know that the following formula holds for the short term asymptotic expansion of $Tr\left(k_t(w,z,\tau)\right)$

$$Tr\left(k_t(w,z,\tau)\right) = \frac{\alpha_{-n}}{t^n} + ... + \frac{\alpha_{-1}}{t} + \alpha_0 + O(t).$$

Let E be a holomorphic vector bundle over M, let $\phi \in C^\infty(\mathrm{M},\mathrm{Hom}(E,E))$. It is easy to see that the operator $\exp(-\Delta_h)\circ\phi$ is of trace class and its trace has an asymptotic expansion

$$Tr\left(k_t(w,z,\tau)\circ\phi\right) = \frac{\beta_{-n}(\phi)}{t^n} + ... + \frac{\beta_{-1}(\phi)}{t} + \beta_0(\phi) + O(t) \tag{29}$$

according to [7]. We will study the following problem in this section:

Problem 17 *Find an explicit expression for $\beta_0(\phi)$.*

Definition 18 *We define the function $k^d_\tau(w,z,t)$ in a neighborhood of the diagonal Δ in $M\times M$ as follows: Let ρ_τ be the injectivity radius on M_τ. Let $d_\tau(w,z)$ be the distance between the points w and z on M_τ with respect to CY metric g_τ. We suppose that $|\tau|<\varepsilon$. Let δ be such that $\delta>\rho_\tau$. Then we define the function $k^d_\tau(w,z,t)$ as a C^∞ function on $M\times M$ using partition of unity by using the functions*

$$k^d_t(w,z,\tau)=\begin{cases}\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(-\frac{d^2_\tau(w,z)}{4t}\right) & if\ d_\tau(w,z)<\rho_\tau\\ 0 & if\ d_\tau(w,z)>\delta.\end{cases}\tag{30}$$

defined on the opened balls around countable points (w_k,z_k) on $M\times M$ with injectivity radius ρ_τ.

Let E be a holomorphic vector bundle on M with a Hermitian metric h on it and let $\mathcal{P}_\tau(w,z)$ be the parallel transport of the bundle E along the minimal geodesic joining the point w and z with respect to natural connection on E induced by the metric h on E. It was proved in [7] on page 87 that we can represent the operator $\exp(-t\triangle_h)$ by an integral kernel $k_t(w,z,\tau)$, where

$$k_t(w,z,\tau)=k^d_t(w,z,\tau)\left(\mathcal{P}_\tau(w,z)+O(t)\right)\tag{31}$$

and

$$\Delta_h:=\overline{\partial}^*_h\circ\overline{\partial}.$$

Definition 19 *We will define the kernel $k^\#_t(w,z,\tau)$ as the matrix operator defined as follows*

$$k^\#_t(w,z,\tau)=k^d_t(w,z,\tau)\mathcal{P}_\tau(w,z).\tag{32}$$

So we have the following formula

$$k_t(w,z,\tau)=k^\#_t(w,z,\tau)+\varepsilon_t(w,z,\tau).\tag{33}$$

Let us define

$$\Upsilon_t(\phi,\tau):=\int\limits_M Tr\left(\left(k^\#_{t/2}(w,z,\tau)\right)^*\circ\left(k^\#_{t/2}(w,z,\tau)\circ\phi\right)\right)vol(g)_w.\tag{34}$$

Proposition 20 *We have*

$$\lim_{t\to 0}\int\limits_M Tr\left(\varepsilon_t(w,z,0)\circ\phi\right)vol(g)_w=0.\tag{35}$$

Proof: The definition 19 of $k^\#_\tau(w,z,t)$ and the arguments from [14] on page 260 imply that $\varepsilon_0(w,z,t)$ is bounded and tends to zero away from the diagonal, as t tends to zero. From here we deduce that

$$\lim_{t\to 0}\int\limits_{\mathrm{M}} Tr\left(\varepsilon_t(w,z,0)\circ\phi\right)vol(g)_w=0$$

uniformly in z. Proposition 20 is proved. ■

Lemma 21 *Let E be a holomorphic vector bundle over M, let* $\phi \in C^{\infty}(M, Hom(E,E))$*, then*

$$\lim_{t\to 0}\Upsilon_t(\phi,\tau,z)$$

exists and

$$\lim_{t\to 0}\Upsilon_t(\phi,\tau,z) = Tr(\phi|_{E_z}). \tag{36}$$

Proof: We have:

$$\lim_{t\to 0}\Upsilon_t(\phi,0,z) = \lim_{t\to 0}\int\limits_{\mathrm{M}} Tr\left(\left(k^{\#}_{t/2}(w,z,\tau)\right)^{*}\circ\left(k^{\#}_{t/2}(w,z,\tau)\circ\phi\right)\right)vol(g)_w =$$

$$\lim_{t\to 0}\left(\frac{1}{(4\pi t)^{\frac{n}{2}}}\int\limits_{\mathrm{M}} Tr\left(\exp\left(-\frac{d_0^2(w,z)}{4t}\right)\circ\mathcal{P}_0(w,z)^{*}\circ\mathcal{P}_0(w,z)\circ\phi\right)vol(g)_w\right). \tag{37}$$

Using the facts that

$$\lim_{t\to 0}\exp\left(-\frac{d_\tau^2(w,z)}{4t}\right) = \delta(z-w), \tag{38}$$

$$\lim_{w\to z}\mathcal{P}_0(w,z) = id \tag{39}$$

and the explicit formula (37) for $\Upsilon_t(\phi,\tau,z)$ we obtain that

$$\lim_{t\to 0}\Upsilon_t(\phi,\tau,z) = \int\limits_{\mathrm{M}} Tr\left(\delta(z-w)\circ\phi\right)vol(g)_\omega = Tr(\phi|_{E_z}). \tag{40}$$

Lemma 21 is proved. ■

Theorem 22 *Let* $\phi \in C^{\infty}\left(M,\left(\Omega^{0,q}_M\right)^{*}\otimes\Omega^{0,q}_M\right)$ *then the operator* $\exp(-t\Delta_h)\circ\phi$ *for* $t>0$ *is of trace class and its trace is given by the formula;*

$$Tr\left(\exp(-t\Delta_h)\circ\phi\right) =$$
$$\int\limits_{M}\Upsilon_t(\phi,\tau,z)vol(g)_z + \Phi(t), \tag{41}$$

where the short term asymptotic of $\Phi(t)$ *is given by*

$$\Phi(t) = \sum_{k=1}^{N_0>0}\frac{a_{-k}}{t^k} + O(t). \tag{42}$$

Proof: The proof of Theorem 22 is based on the facts that

$$\exp(-t\Delta_h)\circ\phi = \exp\left(-\frac{t}{2}\Delta_h\right)\circ\exp\left(-\frac{t}{2}\Delta_h\circ\phi\right), \tag{43}$$

and the operators

$$\exp\left(-\frac{t}{2}\Delta_h\right) \text{ and } \exp\left(-\frac{t}{2}\Delta_h\circ\phi\right)$$

can be represented by C^∞ kernels $k_1(z,w,t)$ and $k_\phi(z,w,t)$.

As we pointed out the operators defined by the kernels $k_1(z,w,t)$ and $k_\phi(z,w,t)$ are Hilbert-Schmidt operators. Thus since the operator $\exp(-t\Delta_h)\circ\phi$ is a product of two Hilbert-Schmidt operators it is of trace class. On the other hand the definition of the trace of the operator $\exp(-t\Delta_h)\circ\phi$ implies that

$$Tr\left(\exp(-t\Delta_h)\circ\phi\right) = \left\langle \exp\left(-\frac{t}{2}\Delta_h\right)^*, \exp\left(-\frac{t}{2}\Delta_h\right)\circ\phi\right\rangle =$$

$$\int\limits_{(z,w)\in\mathrm{M}\times\mathrm{M}} Tr\left((k_1(z,w,t))^*\circ k_\phi(z,w,t)\right). \tag{44}$$

From the definitions of the function $\Upsilon_t(\phi,\tau,z)$ and the operator $\varepsilon_t(w,z,\tau)$ we deduce that

$$Tr\left(\exp(-t\Delta_h)\circ\phi\right) =$$

$$\int\limits_{\mathrm{M}}\left(\int\limits_{\mathrm{M}}\left(k_{t/2}(w,z,\tau)\right)^*\circ\left(k_{t/2}(w,z,\tau)\circ\phi\right)vol(g)_w\right)vol(g)_z =$$

$$\int\limits_{\mathrm{M}}\Upsilon_t(\phi,\tau,z)vol(g)_z +$$

$$\int\limits_{\mathrm{M}}\left(\int\limits_{\mathrm{M}} Tr\left(\left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ k^{\#}_{t/2}(w,z,\tau)\circ\phi\right)vol(g)_\omega\right)vol(g)_z +$$

$$\int\limits_{\mathrm{M}}\left(\int\limits_{\mathrm{M}} Tr\left(k^{\#}_{t/2}(w,z,\tau)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi\right)vol(g)_\omega\right)vol(g)_z. \tag{45}$$

Lemma 23 *Let*

$$\Phi_1(t) := \int\limits_M\left(\int\limits_M Tr\left(\left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ k^{\#}_{t/2}(w,z,\tau)\circ\phi\right)vol(g)_\omega\right)vol(g)_z$$

and

$$\Phi_2(t) := \int\limits_M\left(\int\limits_M Tr\left(k^{\#}_{t/2}(w,z,\tau)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi\right)vol(g)_\omega\right)vol(g)_z \tag{46}$$

then we have

$$\Phi_1(t) = \sum_{k=1}^{N_0>0} \frac{b_{-k}}{t^k} + O(t) \text{ and } \Phi_2(t) = \sum_{k=1}^{N_0>0} \frac{c_{-k}}{t^k} + O(t). \tag{47}$$

Proof: Let

$$k^{\#}_{t/2}(w,z,\tau) = \sum_{k=1}^{N_0>0} \frac{\mathcal{B}_{-k}(w,z)}{t^k} + \mathcal{B}_0(w,z) + \sum_{k=1} \mathcal{B}_k(w,z)t^k \tag{48}$$

be the short term asymptotic expansion of the operator $k^{\#}_{t/2}(w,z,\tau)$. We know that

$$\lim_{t\to 0} \varepsilon_t(w,z,\tau) = 0 \tag{49}$$

away from the diagonal $\Delta \subset \mathrm{M}\times\mathrm{M}$. Combining (48) and (49) with the definitions of operators $k^{\#}_{t/2}(w,z,\tau)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi$ and $\left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ k^{\#}_{t/2}(w,z,\tau)\circ\phi$ we obtain that

$$k^{\#}_{t/2}(w,z,\tau)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi =$$

$$\sum_{k=1}^{N_0>0} \frac{\mathcal{B}_{-k}(w,z)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi}{t^k} + \mathcal{B}_0(w,z)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi + O(t) \tag{50}$$

and

$$\left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ k^{\#}_{t/2}(w,z,\tau)\circ\phi =$$

$$\sum_{k=1}^{N_0>0} \frac{\left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ\mathcal{B}_{-k}(w,z)}{t^k} + \left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ\mathcal{B}_0(w,z) + O(t). \tag{51}$$

Combining (50), (51) with (49) we get that

$$\lim_{t\to 0}\mathcal{B}_0(w,z)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi = \lim_{t\to 0}\left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ\mathcal{B}_0(w,z) = 0$$

away from the diagonal. From here we we obtain that

$$\lim_{t\to\infty}\int_{\mathrm{M}} Tr\left(\mathcal{B}_0(w,z)\circ\varepsilon_{t/2}(w,z,\tau)\circ\phi\right) vol(g) = 0$$

and

$$\lim_{t\to\infty}\int_{\mathrm{M}} Tr\left(\left(\varepsilon_{t/2}(w,z,\tau)\right)^*\circ\mathcal{B}_0(w,z)\right) vol(g) = 0.$$

Lemma 23 is proved. ■

Theorem 22 follows directly from Lemma 23 and (45). ■

Theorem 24 *We have the following expression for* $\beta_0(\phi)$ *from* (29)*:*

$$\beta_0(\phi) = \lim_{t\to 0}\int_M \Upsilon_t(\phi,\tau,z) vol(g)_z = \int_M Tr(\phi) vol(g). \tag{52}$$

Proof: Theorem 24 follows directly from Theorem 22, Lemma 21 and the definition of $\Upsilon_t(\phi,\tau,z)$. ■

4.4 Explicit Formulas

Theorem 25 *Let $\mathcal{F}^{'}(q, \phi \circ \overline{\psi})$ be given by the formula (26). Then for $t > 0$ and $q \geq 1$ the following equality of the traces of the respective operators holds*

$$Tr\left(\exp\left(\left(-t\triangle^{"}_{q-1}\right) \circ \overline{\partial}^{-1} \circ \mathcal{F}^{'}(q, \phi \circ \overline{\psi}) \circ \overline{\partial}\right)\right) =$$

$$Tr\left(\exp\left(-t\triangle^{'}_{q}\right) \circ (\mathcal{F}^{'}(q, \phi \circ \overline{\psi}))\right). \tag{53}$$

Proof: From Proposition **2.45** on page 96 in [7] it follows directly that the operators

$$\exp(-t\Delta^{"}_{q-1}) \circ \overline{\partial}^{-1} \circ \mathcal{F}^{'}(q, \phi \circ \overline{\psi}) \circ \overline{\partial}, \;\; \exp(-t\Delta^{'}_{q}) \circ \mathcal{F}^{'}(q, \phi \circ \overline{\psi})$$

are of trace class since the operators $\exp(-t\Delta^{"}_{q-1})$ have smooth kernels for $q \geq 1$. We know from Proposition **2.45** in [7] that we have the following formula:

$$Tr(DK) = Tr(DA) \tag{54}$$

where D is a differential operator and A is an operator with a smooth kernel. By using 54 and the fact that the operators Δ_q and $\overline{\partial}$ commute we derive Theorem 25. Theorem 25 is proved. ■

Remark 26 *From Definition 15 of the operator $\mathcal{F}^{'}(q, \phi_i \circ \overline{\phi_j})$ and Remark 16 we know that it can be represented pointwise by a matrix which we will denote by $\mathcal{F}^{'}(q, (\phi_i \circ \overline{\phi_j}))$. Since $k^{\#}_t(z, w, 0)$ is also a matrix of the same dimension as the operator $\mathcal{F}^{'}(q, \phi_i \circ \overline{\phi_j})$ we get that the operator $k^{\#}_t(z, w, 0) \circ \mathcal{F}^{'}(q, \phi_i \circ \overline{\phi_j})$ will be represented pointwise by the product of finite dimensional matrices. So the integral*

$$\int\limits_M Tr\left(\left(k^{\#}_{t/2}(z, w, 0)\right)^* \circ k^{\#}_{t/2}(z, w, 0) \circ \mathcal{F}^{'}\left(q, \phi_i \circ \overline{\phi_j}\right)\right) vol(g)$$

makes sense for $t > 0$.

Theorem 27 *Let*

$$Tr\left(k_t(w, z, \tau) \circ \mathcal{F}^{'}(q, (\phi_i \circ \overline{\phi_j}))\right) =$$

$$\frac{\beta_{-n}(\phi_i \circ \overline{\phi_j})}{t^n} + ... + \frac{\beta_{-1}(\phi_i \circ \overline{\phi_j})}{t} + \beta_0(\phi_i \circ \overline{\phi_j}) + O(t) \tag{55}$$

be the short term asymptotic. Then the following limit

$$\lim_{t \to 0} \int\limits_M \left(\int\limits_M Tr\left(\left(k^{\#}_{t/2}(z, w, 0)\right)^* \circ k^{\#}_{t/2}(z, w, 0) \circ \mathcal{F}^{'}(q, \phi_i \circ \overline{\phi_j}\right) vol(g)_w\right) vol(g)$$

exists and

$$\beta_0(\phi_i \circ \overline{\phi_j}) =$$

$$\lim_{t\to 0}\int\limits_M \left(\int\limits_M Tr\left(\left(k^{\#}_{t/2}(z,w,0)\right)^* \circ k^{\#}_{t/2}(z,w,0)\circ \mathcal{F}^{'}(q,\phi_i\circ\overline{\phi_j})\right) vol(g)_w\right) vol(g) =$$

$$\int\limits_M Tr\left(\mathcal{F}^{'}\left(q,\phi_i\circ\overline{\phi_j}\right)\right) vol(g) < \infty. \tag{56}$$

Proof: Formulas (41) and (42) in Theorem 22 imply that

$$\beta_0(\phi_i \circ \overline{\phi_j}) = \lim_{t\to 0}\int\limits_{\mathrm{M}} \Upsilon_t(\mathcal{F}^{'}\left(q,\phi_i\circ\overline{\phi_j}\right), \tau, z) vol(g)_z$$

Theorem 24 imply that

$$\beta_0(\phi_i \circ \overline{\phi_j}) = \lim_{t\to 0}\int\limits_{\mathrm{M}} \Upsilon_t(\mathcal{F}^{'}\left(q,\phi_i\circ\overline{\phi_j}\right), \tau, z) vol(g)_z =$$

$$Tr\left(\mathcal{F}^{'}(q,\phi_i\circ\overline{\phi_j})\right)|_z. \tag{57}$$

Formula (57) implies formula (56). Theorem 27 is proved. ■

5 The Variational Formulas

5.1 Preliminary Formulas

Lemma 28 *The following formulas are true for* $1 \le q \le n$*:*

$$\frac{\partial}{\partial \tau^i}\left(\overline{\partial_\tau}\right)|_{\tau=0} = -F(q,\phi_i)\circ\partial \tag{58}$$

and

$$\frac{\overline{\partial}}{\overline{\partial \tau^i}}\left(\partial_\tau\right)|_{\tau=0} = -F(q,\overline{\phi_i})\circ\overline{\partial}. \tag{59}$$

Proof: From the expression of $\overline{\partial}_\tau$ given in Definition 5:

$$\overline{\partial}_\tau = \frac{\overline{\partial}}{\overline{\partial z^j}} - \sum_{m=1}^{N}\left(\sum_{k=1}^{n}(\phi_m)^k_{\overline{j}}\frac{\partial}{\partial z^k}\right)\tau^m + O(\tau^2)),$$

we conclude that

$$\frac{\partial}{\partial \tau^i}\left(\overline{\partial_\tau}\right)|_{\tau=0} = -\sum_{k=1}^{N}(\phi_i)^k_{\overline{j}}\frac{\partial}{\partial z^k}. \tag{60}$$

Formula (59) is proved in the same way as formula (60). Lemma 28 follows directly from Definition 15 of the linear operators $\mathrm{F}^{'}(q,\phi)$ and $\mathrm{F}^{"}(q,\phi)$. ■

Corollary 29 *The following formulas are true for* $1 \leq q \leq n$*:*

$$\frac{\partial}{\partial \tau^i}\left(\left(\overline{\partial_\tau}^* \circ \overline{\partial_\tau}\right)|_{\operatorname{Im}\overline{\partial_\tau}^*}\right)|_{\tau=0} = -\Delta^{"}_{0,q} \circ \overline{\partial_0}^{-1} \circ F(q,\phi_i) \circ \partial_0,$$

$$\frac{\overline{\partial}}{\overline{\partial \tau^i}}\left((\partial_\tau^* \circ \partial_\tau)|_{\operatorname{Im}\overline{\partial_\tau}^*}\right)|_{\tau=0} = -\Delta^{"}_{0,q} \circ \partial_0^{-1} \circ F(q,\overline{\phi_i}) \circ \overline{\partial}$$

and

$$\frac{\partial}{\partial \tau^j}\left(\triangle^{"}_{\tau,q}\right)|_{\tau=0} = -\triangle^{"}_{0,q} \circ \partial_0^{-1} \circ F^{'}(q+1,\overline{\phi_i}) \circ \overline{\partial_0}. \tag{61}$$

Proof: From the standard facts of Kähler geometry we obtain that on $\operatorname{Im}\overline{\partial}^*$ in $\Omega^{0,q}_{\mathrm{M}}$ we have

$$\triangle^{"}_{\tau,q}|_{\operatorname{Im}\overline{\partial}} = \partial_\tau \circ \Lambda_\tau \circ \overline{\partial_\tau} = \overline{\partial_\tau^*} \circ \overline{\partial_\tau}. \tag{62}$$

We know from (58) and (59) that

$$\frac{\partial}{\partial \tau^j}\left(\overline{\partial_\tau}\right)|_{\tau=0} = -F^{'}(q+1,\phi_j) \circ \partial,\ \frac{\partial}{\partial \tau^j}\left(\Lambda_\tau\right)|_{\tau=0} = \frac{\partial}{\partial \tau^j}\left(\partial_\tau\right)|_{\tau=0} = 0. \tag{63}$$

Combining (63), (62) and the fact that

$$\frac{\partial}{\partial \tau^j}\left(\overline{\partial_\tau}^*\right)|_{\tau=0} = 0$$

we obtain:

$$\frac{\partial}{\partial \tau^j}\left(\triangle^{"}_{\tau,q}\right)|_{\tau=0} = \frac{\partial}{\partial \tau^j}\left(\overline{\partial_\tau}^* \circ \overline{\partial_\tau}\right)|_{\tau=0} = -\overline{\partial_0}^* \circ F^{'}(q+1,\phi_j) \circ \partial_0. \tag{64}$$

Thus on $\operatorname{Im}\overline{\partial}^*$ we have

$$\partial_\tau^* = \triangle^{"}_{\tau,q} \circ \overline{\partial_\tau}^{-1}. \tag{65}$$

Substituting (65) in (64) we get

$$\frac{\partial}{\partial \tau^j}\left(\left(\triangle^{"}_{\tau,q}\right)\right)|_{\tau=0} = -\triangle^{"}_{\tau,q} \circ \overline{\partial_\tau}^{-1} \circ F^{'}(q+1,\phi_j) \circ \partial_0.$$

In the same way we prove the rest of the formulas. Corollary 29 is proved. ■

5.2 The Computation of the Antiholomorphic Derivative of $\zeta^{"}_{\tau,q-1}(s)$

First we will compute the antiholomorphic derivative of $\zeta^{"}_{\tau,q}(s)$.

Theorem 30 *The following formula is true for* $t > 0$*:*

$$\frac{\overline{\partial}}{\overline{\partial \tau^i}}\left(\zeta^{"}_{q,\tau}(s)\right)|_{\tau=0} = \frac{1}{\Gamma(s)}\int_0^\infty Tr\left(\exp(-t(\triangle^{"}_{0,q}) \circ \triangle^{"}_{0,q} \circ \partial_0^{-1} \circ \mathcal{F}^{'}(q+1,\overline{\phi_i}) \circ \overline{\partial_0}\right)t^s dt.$$

Proof: For the proof of Theorem 30 we will need the following Lemma:

Lemma 31 *The following formula is true for* $t > 0$ *and* $0 < q < n$ *:*

$$\frac{\overline{\partial}}{\overline{\partial \tau^i}}\left(Tr(\exp\left(-t\triangle^{"}_{\tau,q}\right)\right)|_{\tau=0} =$$

$$tTr\left(\exp\left(-t\triangle^{"}_{0,q}\right)\circ\Delta^{"}_{\tau,q}\circ\partial^{-1}\circ\mathcal{F}^{'}(q+1,\overline{\phi_i})\circ\overline{\partial}\right)|_{\tau=0}. \tag{66}$$

Proof: Direct computations based on Proposition **9.38.** on page 304 of the book [14] show that:

$$\frac{\overline{\partial}}{\overline{\partial \tau^i}}\left(Tr(\exp\left(-t\triangle^{"}_{\tau,q}\right)\right)|_{\tau=0} = -t\left(\exp\left(-t\triangle^{"}_{\tau,q}\right)\circ\frac{\overline{\partial}}{\overline{\partial \tau^i}}\left(\triangle^{"}_{\tau,q}\right)\right)|_{\tau=0}. \tag{67}$$

See also [7] page 98 Theorem **2.48.** Formulas (58) and (59) in Lemma 28 imply that

$$\frac{\overline{\partial}}{\overline{\partial \tau^i}}\left(\partial_\tau\right) = -F^{'}(q+1,\overline{\phi_i(\tau)})\circ\overline{\partial} \tag{68}$$

and on $\operatorname{Im}\overline{\partial}^*$ we have

$$\frac{\overline{\partial}}{\overline{\partial \tau^i}}\left(\partial^*_\tau\right) = \frac{\overline{\partial}}{\overline{\partial \tau^i}}\left(\Lambda\circ\overline{\partial_\tau}\right) = \left(\frac{\overline{\partial}}{\overline{\partial \tau^i}}\Lambda\right)\circ\overline{\partial_\tau} + \Lambda\circ\frac{\overline{\partial}}{\overline{\partial \tau^i}}\left(\overline{\partial_\tau}\right) = 0. \tag{69}$$

The last equality follows from Cor. 28 and 13. On Kähler manifolds we know that

$$\partial^*\circ\partial + \partial\circ\partial^* = \overline{\partial}^*\circ\overline{\partial} + \overline{\partial}\circ\overline{\partial}^*.$$

So we deduce that

$$\Delta^{"}_{\tau,q} = \left(\partial^*_\tau\circ\partial_\tau + \partial_\tau\circ\partial^*_\tau\right)|_{\operatorname{Im}\overline{\partial_\tau}^*} = \partial^*_\tau\circ\partial_\tau|_{\operatorname{Im}\overline{\partial_\tau}^*}.$$

Thus from formulas (68) and (69) it follows:

$$\frac{\overline{\partial}}{\overline{\partial \tau^i}}\left(\triangle^{"}_{\tau,q}\right) = \left(\partial^*_\tau\circ\frac{\overline{\partial}}{\overline{\partial \tau^i}}\left(\partial_\tau\right)\right) = -\partial^*_\tau\circ F^{'}(q+1,\overline{\phi_i(\tau)})\circ\overline{\partial}. \tag{70}$$

By substituting in (67) the expression from (70) we obtain:

$$\frac{\overline{\partial}}{\overline{\partial \tau^i}}\left(Tr(\exp\left(-t\triangle^{"}_{\tau,q}\right)\right)|_{\tau=0} =$$

$$tTr\left(\exp\left(-t\triangle^{"}_{q}\right)\circ\partial^*\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right). \tag{71}$$

The operator ∂^*_τ is well defined on the space of C^∞ (0,q) forms on M_τ. So the following formula is true on $\operatorname{Im}\overline{\partial}^*_\tau$:

$$\partial^*_\tau = \left(\triangle^{"}_{\tau,q}\right)\circ\left(\partial_\tau\right)^{-1}. \tag{72}$$

Substituting the expression for ∂_τ^* in formula (72) in (71), we deduce formula (66) :

$$\frac{\overline{\partial}}{\overline{\partial\tau^i}}\left(Tr(\exp\left(-t\triangle^{"}_{q,\tau}\right)\right)|_{\tau=0} =$$

$$tTr\left(\exp\left(-t\triangle^{"}_{0,q}\right)\circ\triangle^{"}_{0,q}\circ\partial^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial}\right).$$

Lemma 31 is proved. ■

The end of the proof of Theorem 30**:** The definition of the zeta function implies that

$$\frac{\overline{\partial}}{\overline{\partial\tau^i}}\left(\zeta_{\Delta^{"}_{\tau,q}}(s)\right)|_{\tau=0} =$$

$$\frac{\overline{\partial}}{\overline{\partial\tau^i}}\left(\frac{1}{\Gamma(s)}\int_0^\infty Tr\left(\exp(-t(\triangle^{"}_{\tau,q})\right)t^{s-1}dt\right)|_{\tau=0} =$$

$$\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{\overline{\partial}}{\overline{\partial\tau^i}}\left(Tr\exp(-t(\triangle^{"}_{\tau,q})\right)\right)|_{\tau=0}t^{s-1}dt. \qquad (73)$$

Substituting in (73) the expression for

$$\frac{\overline{\partial}}{\overline{\partial\tau^i}}\left(Tr\left(\exp(-t(\triangle^{"}_{\tau,q}))\right)\right)$$

in (66) we obtain:

$$\frac{\overline{\partial}}{\overline{\partial\tau^i}}\left(\zeta_{\Delta^{"}_{\tau,q}}(s)\right)|_{\tau=0} =$$

$$\frac{1}{\Gamma(s)}\int_0^\infty Tr\left(\left(\exp(-t(\triangle^{"}_{0,q})\right)\circ\left(\triangle^{"}_{0,q}\right)\circ\partial^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial}\right)t^s dt. \qquad (74)$$

Theorem 30 is proved. ■

5.3 The Computation of the Hessian of $\zeta^{"}_{\tau,q}(s)$

Theorem 32 *The following formula holds:*

$$\frac{\partial^2}{\partial\tau^j\overline{\partial\tau^i}}\left(\zeta^{"}_{\tau,q}(s)\right)|_{\tau=0} =$$

$$\frac{s}{\Gamma(s)}\int_0^\infty Tr\left(\left(\exp(-t(\triangle^{'}_{0,q})\right)\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\right)t^{s-1}dt+$$

$$-\frac{s^2}{\Gamma(s)}\int_0^\infty Tr\left(\left(\exp(-t(\triangle^{"}_{0,q})\right)\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\right)t^{s-1}dt. \qquad (75)$$

Proof: The facts that the operators

$$\frac{\partial}{\partial \tau^i}\left(\overline{\partial_\tau}\right) = (-\phi_i + O(\tau)) \circ \partial_0$$

depend holomorphically on τ and the operator ∂_τ^{-1} depends antiholomorphically imply that the operator

$$\partial_\tau^{-1} \circ F^{'}(q+1, \overline{\phi_i}) \circ \overline{\partial_0}$$

depends antiholomorphically on the coordinates $\tau = (\tau^1, ..., \tau^N)$. By using the explicit formula (74) for the antiholomorphic derivative of $\zeta^{"}_{\tau,q}(s)$ and

$$\frac{\partial}{\partial \tau^j}\left(\partial_\tau^{-1} \circ F^{'}(q+1, \overline{\phi_i}) \circ \overline{\partial_0}\right) = 0$$

we derive

$$\frac{\partial^2}{\overline{\partial \tau^j} \partial \tau^i}\left(\zeta^{"}_{\tau,q}(s)\right) =$$

$$\frac{1}{\Gamma(s)}\int_0^\infty Tr\left(\left(\frac{\partial}{\partial \tau^j}\exp(-t\left(\triangle^{"}_{\tau,q}\right)\right) \circ \left(\triangle^{"}_{\tau,q}\right) \circ \partial_\tau^{-1} \circ F^{'}(q+1, \overline{\phi_i}) \circ \overline{\partial_0}\right) t^s dt +$$

$$\frac{1}{\Gamma(s)}\int_0^\infty Tr\left(\left(\exp(-t(\triangle^{"}_{\tau,q})\right) \circ \frac{\partial}{\partial \tau^j}\left(\triangle^{"}_{\tau,q}\right) \circ \partial_0^{-1} \circ F^{'}(q+1, \overline{\phi_i}) \circ \overline{\partial_0}\right) t^s dt. \tag{76}$$

Lemma 33 *We have the following expression:*

$$\frac{1}{\Gamma(s)}\int_0^\infty Tr\left(\left(\frac{\partial}{\partial \tau^j}\left(\exp(-t\left(\triangle^{"}_{\tau,q}\right)\right)\right) \circ \left(\triangle^{"}_{\tau,q}\right) \circ \partial_\tau^{-1} \circ F^{'}(q+1, \overline{\phi_i}) \circ \overline{\partial_0}\right) t^s dt =$$

$$\frac{-s}{\Gamma(s)}\int_0^\infty Tr\left(\left(\exp(-t\triangle^{"}_{0,q}\right) \circ \frac{\partial}{\partial \tau^j}\left(\triangle^{"}_{\tau,q}\right) \circ \partial_\tau^{-1} \circ F^{'}(q+1, \overline{\phi_i}) \circ \overline{\partial_0}\right) t^s dt. \tag{77}$$

Proof: Direct computations show that

$$\frac{1}{\Gamma(s)}\left(\int_0^\infty Tr\left(\left(\frac{\partial}{\partial \tau^j}\left(\exp(-t\left(\triangle^{"}_{\tau,q}\right)\right)\right) \circ \left(\triangle^{"}_{\tau,q}\right) \circ \partial_\tau^{-1} \circ F^{'}(q+1, \overline{\phi_i}) \circ \overline{\partial_0}\right) t^s dt\right)|_{\tau=0} =$$

$$\frac{-1}{\Gamma(s)}\int_0^\infty \left(\frac{d}{dt}Tr\left(\left(\frac{\partial}{\partial \tau^j}\left(\exp(-t\left(\triangle^{"}_{\tau,q}\right)\right)\right)|_{\tau=0} \circ \partial_0^{-1} \circ F^{'}(q+1, \overline{\phi_i}) \circ \overline{\partial_0}\right)\right) t^s dt \tag{78}$$

By integrating by parts the right hand side of formula (78) we deduce that:

$$\frac{-1}{\Gamma(s)}\int\limits_0^\infty\left(\frac{d}{dt}Tr\left(\left(\frac{\partial}{\partial\tau^j}\left(\exp(-t\left(\triangle^{"}_{\tau,q}\right)\right)\right)|_{\tau=0}\circ\partial_0^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)\right)t^s dt=$$

$$\frac{s}{\Gamma(s)}\int\limits_0^\infty Tr\left(\left(\frac{\partial}{\partial\tau^j}\left(\exp(-t\left(\triangle^{"}_{\tau,q}\right)\right)\right)|_{\tau=0}\circ\partial_0^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^{s-1}dt. \qquad (79)$$

Direct computations of the right hand side of (79) show that:

$$\frac{s}{\Gamma(s)}\int\limits_0^\infty Tr\left(\left(\frac{\partial}{\partial\tau^j}\left(\exp(-t\triangle^{"}_{\tau,q}\right)\right)|_{\tau=0}\circ\partial_0^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^{s-1}dt=$$

$$\frac{-s}{\Gamma(s)}\int\limits_0^\infty Tr\left(\left(\exp(-t\triangle^{"}_{\tau,q}\right)\circ\frac{\partial}{\partial\tau^j}\left(\triangle^{"}_{\tau,q}\right)|_{\tau=0}\circ\partial_0^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s dt.$$

Formula (77) is proved. ∎

Substituting in (76) the expression

$$\frac{1}{\Gamma(s)}\int\limits_0^\infty Tr\left(\left(\frac{\partial}{\partial\tau^j}\left(\exp(-t\left(\triangle^{"}_{\tau,q}\right)\right)\right)\circ\left(\triangle^{"}_{\tau,q}\right)\circ\partial_\tau^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s dt$$

from (77) we get:

$$\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\left(\zeta^{"}_{q-1,\tau}(s)\right)|_{\tau=0}=$$

$$\frac{1}{\Gamma(s)}\int\limits_0^\infty Tr\left(\left(\left(\exp(-t\triangle^{"}_{\tau,q}\right)\circ\frac{\partial}{\partial\tau^j}\left(\triangle^{"}_{\tau,q}\right)\right)|_{\tau=0}\circ\partial_0^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s dt+$$

$$\frac{-s}{\Gamma(s)}\int\limits_0^\infty Tr\left(\left(\exp(-t\left(\triangle^{"}_{\tau,q}\right)\right)\circ\frac{\partial}{\partial\tau^j}\left(\triangle^{"}_{\tau,q}\right)|_{\tau=0}\circ\partial_0^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s dt. \qquad (80)$$

Lemma 34

$$\int\limits_0^\infty Tr\left(\left(\exp(-t\left(\triangle^{"}_{\tau,q}\right)\right)\circ\frac{\partial}{\partial\tau^j}\left(\triangle^{"}_{\tau,q}\right)|_{\tau=0}\circ\partial_0^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s dt=$$

$$s\int\limits_0^\infty Tr\left(\left(\exp(-t\left(\triangle^{"}_{\tau,q}\right)\right)|_{\tau=0}\circ\left(\overline{\partial_0}\right)^{-1}\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\circ\overline{\partial_0}\right)t^{s-1}dt. \qquad (81)$$

Proof: Substituting the expression of (61)

$$\frac{\partial}{\partial\tau^j}\left(\triangle^{"}_{\tau,q}\right)|_{\tau=0} = -\triangle^{"}_{0,q}\circ\left(\overline{\partial_0}\right)^{-1}\circ F^{'}(q+1,\phi_j)\circ\overline{\partial_0}$$

in the expression of

$$\int\limits_0^\infty Tr\left(\left(\exp(-t\left(\triangle^{"}_{\tau,q}\right)\right)\circ\frac{\partial}{\partial\tau^j}\left(\triangle^{"}_{\tau,q}\right)|_{\tau=0}\circ\partial_0^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s dt$$

we obtain:

$$\int\limits_0^\infty Tr\left(\left(\exp(-t\left(\triangle^{"}_{\tau,q}\right)\right)\circ\frac{\partial}{\partial\tau^j}\left(\triangle^{"}_{\tau,q}\right)|_{\tau=0}\circ\partial_0^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s dt=$$

$$-\int\limits_0^\infty Tr\left(\left(\exp(-t\left(\triangle^{"}_{0,q}\right)\right)\circ\triangle^{"}_{0,q}\circ\partial_0^{-1}\circ F^{'}(q+1,\overline{\phi_i})\circ F^{'}(q+1,\overline{\phi_i})\circ\overline{\partial_0}\right)t^s dt=$$

$$\int\limits_0^\infty Tr\left(\frac{d}{dt}\left(\exp(-t\left(\triangle^{"}_{0,q}\right)\right)\circ\left(\overline{\partial_0}\right)^{-1}\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\circ\overline{\partial_0}\right)t^s dt. \qquad (82)$$

By integrating by parts the right hand side of (82) we obtain formula (81). Lemma 34 is proved. ■

Substituting the expression of (81) in the expression (80) we get the following equality:

$$\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\left(\zeta^{"}_{q,\tau}(s)\right)|_{\tau=0}=$$

$$-\frac{s}{\Gamma(s)}\int\limits_0^\infty Tr\left(\left(\exp(-t\left(\triangle^{"}_{0,q}\right)\right)\circ\overline{\partial_0}^{-1}\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\circ\overline{\partial_0}\right)t^{s-1}dt+$$

$$\frac{1}{\Gamma(s)}\int\limits_0^\infty Tr\left(\left(\exp(-t(\triangle^{"}_{0,q})\right)\circ\overline{\partial_0}^{-1}\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\circ\overline{\partial_0}\right)t^{s-1}dt. \qquad (83)$$

Applying Theorem 25 we deduce Theorem 32. ■

5.4 The Computations of the Hessian of $\log\det\Delta_{\tau,q}$

Theorem 35 *The following formula is true:*

$$\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\left(\log\det\Delta^{"}_{\tau,q}\right)|_{\tau=0}=$$

$$-\lim_{t\to 0}\int\limits_M\left(\int\limits_M Tr\left((4\pi t)^{-\frac{n}{2}}\exp\left(-\frac{d_0^2(w,z)}{4t}\right)\circ\mathcal{P}^*\circ\mathcal{P}\circ\mathcal{F}^{'}\left(q+1,\phi_i\circ\overline{\phi_j}\right)\right)vol(g)_w\right)vol(g)_z=$$

$$-\int_M Tr\left(\mathcal{F}^{'}\left(q+1,\left(\phi_i\circ\overline{\phi_j}\right)\right)\right)vol(g). \tag{84}$$

Proof: $\zeta^{"}_{\tau,q}(s)$ is obtained from the meromorphic continuation of

$$\frac{1}{\Gamma(s)}\int_0^\infty Tr\left(\exp(-t\Delta^{"}_{\tau,q})\right)t^{s-1}dt$$

and it is a meromorphic function on $\mathbb{C}$ well defined at 0. Thus we get that

$$\zeta^{"}_{\tau,q}(s)=\mu_0(\tau)+\mu_1(\tau)s+O(s^2)$$

and

$$\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\left(\zeta^{"}_{\tau,q}(s)\right)|_{\tau=0}=\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\mu_0(\tau)|_{\tau=0}+\left(\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\mu_1(\tau)\right)|_{\tau=0}s+O(s^2)=$$

$$\alpha_0+\alpha_1s+O(s^2). \tag{85}$$

Thus from the definition of the regularized determinant

$$\log\det(\triangle^{"}_{\tau,q})=\left(\frac{d}{ds}\left(-\zeta^{"}_{\tau,q}(s)\right)\right)|_{s=0}$$

we see that

$$\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\left(\log\det\Delta^{"}_{\tau,q}\right)|_{\tau=0}=\frac{d}{ds}\left(\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\left(\left(-\zeta^{"}_{\tau,q}(s)\right)\right)|_{\tau=0}\right)|_{s=0}=-\alpha_1. \tag{86}$$

Combining formula (75)

$$\frac{\partial^2}{\partial\tau^j\overline{\partial\tau^i}}\left(\zeta^{"}_{\tau,q}(s)\right)|_{\tau=0}=$$

$$\frac{s}{\Gamma(s)}\int_0^\infty Tr\left(\left(\exp(-t(\triangle^{'}_{0,q})\right)\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\right)t^{s-1}dt+$$

$$-\frac{s^2}{\Gamma(s)}\int_0^\infty Tr\left(\left(\exp(-t(\triangle^{'}_{0,q})\right)\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\right)t^{s-1}dt$$

with the short term expansion:

$$Tr\left(\left(\exp(-t(\triangle^{'}_{0,q})\right)\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\right)=\sum_{k=-n}^{1}\frac{\nu_k}{t^k}+\nu_0+\psi(t) \tag{87}$$

where

$$\psi(t)=Tr\left(\left(\exp(-t(\triangle^{'}_{0,q})\right)\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\right)-\sum_{k=-n}^{1}\frac{\nu_k}{t^k}+\nu_0$$

we obtain:

$$\frac{\partial^2}{\partial\tau^j\overline{\partial\tau^i}}\left(\zeta^{"}_{\tau,q}(s)\right)|_{\tau=0}=$$

$$\frac{s}{\Gamma(s)}\left(\int\limits_0^1\left(\sum_{k=-n}^{1}\frac{\nu_k}{t^k}\right)t^{s-1}dt+\nu_0\int\limits_0^1 t^{s-1}dt+\int\limits_0^1\psi(t)t^{s-1}dt\right)+$$

$$\frac{s}{\Gamma(s)}\left(\int\limits_1^\infty Tr\left(\left(\exp(-t(\triangle^{'}_{0,q})\right)\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\right)t^{s-1}dt\right)-$$

$$-\frac{s^2}{\Gamma(s)}\left(\int\limits_0^1\left(\sum_{k=-n}^{1}\frac{\nu_k}{t^k}\right)t^{s-1}dt+\nu_0\int\limits_0^1 t^{s-1}dt+\int\limits_0^1\psi(t)t^{s-1}dt\right)-$$

$$-\frac{s^2}{\Gamma(s)}\left(\int\limits_1^\infty Tr\left(\left(\exp(-t(\triangle^{'}_{0,q})\right)\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\right)t^{s-1}dt\right). \tag{88}$$

By using formula (88) we will prove the following Lemma:

Lemma 36 *We have the following formula:*

$$\frac{\partial^2}{\overline{\partial\tau^j}\partial\tau^i}\left(\log\det\Delta^{"}_{\tau,q}\right)|_{\tau=0}=-\int\limits_M Tr\mathcal{F}^{'}(q+1,\phi_i\circ\overline{\phi_j})vol(g)=-\alpha_1.$$

Proof: Lemma **9.34** on page 300 of [7] or direct computations show that for $|s|<\varepsilon$ we have the following identity:

$$\frac{1}{\Gamma(s)}\int\limits_0^\infty Tr\left(\left(\exp(-t(\triangle^{'}_{0,q})\right)\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\right)t^{s-1}dt=$$

$$\frac{1}{\Gamma(s)}\left(\int\limits_0^1\left(\sum_{k=-n}^{1}\frac{\nu_k}{t^k}\right)t^{s-1}dt+\nu_0\int\limits_0^1 t^{s-1}dt+\int\limits_0^1\psi(t)t^{s-1}dt\right)+$$

$$\frac{1}{\Gamma(s)}\int\limits_1^\infty Tr\left(\left(\exp(-t(\triangle^{'}_{0,q})\right)\circ\mathcal{F}^{'}(q+1,\phi_j\circ\overline{\phi_i})\right)t^{s-1}dt=$$

$$\frac{\nu_0}{s}+\kappa+O(s). \tag{89}$$

Combining the expression in (89) with the following standard fact

$$\frac{s}{\Gamma(s)}=s^2+O(s^3)$$

we obtain from formulas (88) and (89) that for $|s| < \varepsilon$

$$\frac{\partial^2}{\partial\tau^j\overline{\partial\tau^i}}\left(\zeta^{"}_{\tau,q}(s)\right)|_{\tau=0} = \nu_0 s + O(s^2). \tag{90}$$

Thus according to (86) and (90)

$$\frac{d}{ds}\left(\frac{\partial^2}{\partial\tau^j\overline{\partial\tau^i}}\left(\zeta^{"}_{\tau,q}(s)\right)|_{\tau=0}\right)|_{s=0} = \nu_0 = \alpha_1. \tag{91}$$

Applying Theorem 24 to formula (91) we deduce that

$$\alpha_1 = \nu_0 = \int\limits_{\mathrm{M}} Tr\left(\mathcal{F}^{'}\left(q+1, \phi_i \circ \overline{\phi_j}\right)\right) vol(g).$$

Lemma 36 is proved. ■

Lemma 36 implies directly Theorem 35. ■

5.5 Some Applications of the Variational Formulas

On CY manifold we have the following duality:

$$* : \Omega^{0,q}_{\mathrm{M}} \approxeq \Omega^{0,n-q}_{\mathrm{M}}$$

induced by the Hodge star operator $*$ of a CY metric. Using this duality direct check shows that on CY manifolds we have

$$* : \operatorname{Im}\overline{\partial_q} \approxeq \operatorname{Im}\left(\overline{\partial_{n-q}}^{*}\right) \text{ and } * : \operatorname{Im}\left(\overline{\partial_q}^{*}\right) \approxeq \operatorname{Im}\left(\overline{\partial_{n-q}}\right). \tag{92}$$

Theorem 37 *The following identity holds*

$$dd^c\left(\log\det\Delta_{\tau,1}\right) = dd^c\left(\log\det\Delta^{'}_{\tau,1}\det\Delta^{"}_{\tau,1}\right) = -\operatorname{Im} W.P.$$

Proof: The proof of Theorem 37 is based on the following Lemma:

Lemma 38 *We have the following relations between the operators on a CY manifold*

$$\mathcal{F}^{'}\left(1, \phi_i \circ \overline{\phi_j}\right) = \mathcal{F}^{"}\left(n-1, \phi_i \circ \overline{\phi_j}\right) \text{ and } \mathcal{F}^{"}\left(1, \phi_i \circ \overline{\phi_j}\right) = \mathcal{F}^{'}\left(n-1, \phi_i \circ \overline{\phi_j}\right) \tag{93}$$

by identifying the Hilbert spaces $\operatorname{Im}\partial \subset L^2(M, \Omega^{0,1}_M)$ *and* $\operatorname{Im}\partial^* \subset L^2\left(M, \Omega^{0,1}_M\right)$ *with* $\operatorname{Im}\partial^* \subset L^2\left(M, \Omega^{0,n-1}_M\right)$ *and* $\operatorname{Im}\partial \subset L^2\left(M, \Omega^{0,n-1}_M\right)$ *by using* (92).

Proof: We will need the following Proposition to prove Lemma 38:

Proposition 39 *Let*

$$\phi_i \circ \overline{\phi_j} \wedge id_{n-1} : C^{\infty}(M, \Omega_M^{0,n-1}) \to C^{\infty}(M, \Omega_M^{0,n-1})$$

be the operator defined by (25) *. Then fibrewise we have the equality of the matrices*

$$\phi_i \circ \overline{\phi_j} \wedge id_{n-1} = \overline{\phi_j} \circ \phi_i \wedge id_{n-1} \tag{94}$$

of the operators $\phi_i \circ \overline{\phi_j} \wedge id_{n-1}$ *and* $\overline{\phi_j} \circ \phi_i \wedge id_{n-1}$ *given in the orthonormal basis with respect to the CY metric.*

Proof: Let $\{\omega_i\}$ be an orthonormal basis at $\Omega_x^{1,0}$. Then the operators

$$\phi_i : \Omega_x^{1,0} \to \Omega_x^{0,1}$$

in the basises $\{\omega_i\}$ and $\{\overline{\omega_i}\}$ are given by symmetric matrices by Lemma 10. From here (94) follows directly. Indeed from the relations $\phi_{\overline{\beta}}^{\alpha} = \phi_{\overline{\alpha}}^{\beta}$ we obtain

$$\left(\phi_i \circ \overline{\phi_j}\right)_{\overline{\beta}}^{\overline{\alpha}} = \sum_{\mu=1}^{n} \phi_{i,\overline{\beta}}^{\mu} \overline{\phi_{j,\overline{\mu}}^{\alpha}} = \sum_{\mu=1}^{n} \phi_{i,\overline{\mu}}^{\beta} \overline{\phi_{j,\overline{\beta}}^{\mu}} = \sum_{\mu=1}^{n} \overline{\phi_{j,\overline{\beta}}^{\mu}} \phi_{i,\overline{\mu}}^{\beta} = \left(\overline{\phi_j} \circ \phi_i\right)_{\beta}^{\alpha}. \tag{95}$$

Formula (94) follows directly from (95) and the definitions of the operators $\phi_i \circ \overline{\phi_j} \wedge id_{n-1}$ and $\overline{\phi_j} \circ \phi_i \wedge id_{n-1}$. ■

Let us define

$$\left(\phi_i \circ \overline{\phi_j} \wedge id_{n-1}\right)^* : \Omega_{\mathrm{M}}^{n-1} \to \Omega_{\mathrm{M}}^{n-1}$$

as follows:

$$\left(\phi_i \circ \overline{\phi_j} \wedge id_{n-1}\right)^* (*\overline{\omega_k}) = *\left(\left(\phi_i \circ \overline{\phi_j}\right)(\overline{\omega_k})\right). \tag{96}$$

Corollary 40 *The matrix of the operator* $\left(\phi_i \circ \overline{\phi_j} \wedge id_{n-1}\right)^*$ *in the orthonormal basis* $\overline{\omega_{i_1}} \wedge ... \wedge \overline{\omega_{i_{n-k}}}$ *can be identified withe the matrix of the operator* $\overline{\phi_j} \circ \phi_i$ *of the bundle* $\Omega_M^{1,0}$ *written in the orthonormal basis* $\{\omega_i\}$ *.*

Proof: From the definition of the operator $\left(\phi_i \circ \overline{\phi_j} \wedge id_{n-1}\right)^*$ given by (96) we get:

$$\left(\phi_i \circ \overline{\phi_j} \wedge id_{n-1}\right)^* (*\overline{\omega_k}) = *\left(\left(\phi_i \circ \overline{\phi_j}\right)(\overline{\omega_k})\right) = *\left(\sum_{k,l=1}^{n} \left(\phi_i \circ \overline{\phi_j}\right)_k^l (\overline{\omega_l})\right) =$$

$$\sum_{k,l=1}^{n} \left(\phi_i \circ \overline{\phi_j}\right)_k^l \left(\overline{\omega_1} \wedge ... \wedge \overline{\omega_{l-1}} \wedge \overline{\omega_{l+1}} \wedge ... \wedge \overline{\omega_n}\right). \tag{97}$$

Combining (95) and (97) we get

$$\left(\phi_i \circ \overline{\phi_j} \wedge id_{n-1}\right)^* (*\overline{\omega_k}) = \sum_{k,l=1}^{n} \left(\overline{\phi_j} \circ \phi_i\right)_l^k \left(\overline{\omega_1} \wedge ... \wedge \overline{\omega_{l-1}} \wedge \overline{\omega_{l+1}} \wedge ... \wedge \overline{\omega_n}\right) =$$

$$\sum_{k,l=1}^{n}\left(\overline{\phi_j}\circ\phi_i\right)_l^k(*\overline{\omega_k})=\sum_{k,l=1}^{n}\left(\overline{\phi_j}\circ\phi_i\right)_l^k\omega_k. \tag{98}$$

We can identify $\Omega_{\mathrm{M}}^{0,n-1}$ with Ω_{M}^{1} by using Hodge star operator $*$ and complex conjugation. From here and (98) we conclude Cor. 40. ■

Combining Cor. 40 with (92) and the identification $\Omega_{\mathrm{M}}^{0,q}$ with $\Omega_{\mathrm{M}}^{q,0}$ by complex conjugation we derive (93). Lemma 38 is proved. ■

From (93) we deduce that

$$\mathcal{F}^{'}\left(1,\phi_i\circ\overline{\phi_j}\right)+\mathcal{F}^{"}\left(1,\phi_i\circ\overline{\phi_j}\right)=\operatorname{Im}W.P. \tag{99}$$

Combining (99) with Theorem 35 we deduce that

$$dd^c\left(\log\det\Delta_{\tau,1}\right)=dd^c\log\left(\Delta_{\tau,1}^{'}\times\Delta_{\tau,1}^{"}\right)=-\operatorname{Im}W.P. \tag{100}$$

Theorem 37 is proved. ■

6 Moduli of CY Manifolds

6.1 Basic Construction

Definition 41 *We will define the Teichmüller space $\mathcal{T}(M)$ of a CY manifold M as follows:*

$$\mathcal{T}(M):=\mathcal{I}(M)/Diff_0(M),$$

where

$$\mathcal{I}(M):=\{\text{all integrable complex structures on } M\}$$

and Diff$_0$(M) is the group of diffeomorphisms isotopic to identity. The action of the group Diff(M$_0$) is defined as follows; Let $\phi\in$Diff$_0$(M) then ϕ acts on integrable complex structures on M by pull back, i.e. if

$$I\in C^{\infty}(M,Hom(T(M),T(M)),$$

then we define $\phi(I_\tau)=\phi^*(I_\tau)$.

We will call a pair (M; $\gamma_1,...,\gamma_{b_n}$) a marked CY manifold where M is a CY manifold and $\{\gamma_1,...,\gamma_{b_n}\}$ is a basis of H_n(M,$\mathbb{Z}$)/Tor.

Remark 42 *Let $\mathcal{K}$ be the Kuranishi space. It is easy to see that if we choose a basis of H_n(M,$\mathbb{Z}$)/Tor in one of the fibres of the Kuranishi family $\mathcal{M}\to\mathcal{K}$ then all the fibres will be marked, since as a C^∞ manifold $\mathcal{X}_{\mathcal{K}}\approxeq$M$\times\mathcal{K}$.*

In [31] the following Theorem was proved:

Theorem 43 *There exists a family of marked polarized CY manifolds*

$$\mathcal{Z}_L \rightarrow \widetilde{T}(M), \tag{101}$$

which possesses the following properties: ***a)*** *It is effectively parametrized,* ***b)*** *For any marked CY manifold M of fixed topological type for which the polarization class L defines an imbedding into a projective space* $\mathbb{CP}^N$, *there exists an isomorphism of it (as a marked CY manifold) with a fibre* M_s *of the family* $\mathcal{Z}_L$. ***c)*** *The base has dimension* $h^{n-1,1}.\gamma_{b_n}$*) a marked CY manifold where M is a CY manifold and* $\{\gamma_1, ..., \gamma_{b_n}\}$ *is a basis of* $H_n(M,\mathbb{Z})/Tor$.

Corollary 44 *Let* $\mathcal{Y} \rightarrow X$ *be any family of marked polarized CY manifolds, then there exists a unique holomorphic map*

$$\phi : X \rightarrow \widetilde{T}(M)$$

up to a biholomorphic map ψ *of M which induces the identity map on* $H_n(M,\mathbb{Z})$.

From now on we will denote by $\mathcal{T}$(M) the irreducible component of the Teichmüller space that contains our fixed CY manifold M.

Definition 45 *We will define the mapping class group* $\Gamma_1(M)$ *of any compact* C^∞ *manifold M as follows:* $\Gamma_1(M) = Diff_+(M)/Diff_0(M)$, *where* $Diff_+(M)$ *is the group of diffeomorphisms of M preserving the orientation of M and* $Diff_0(M)$ *is the group of diffeomorphisms isotopic to identity.*

Definition 46 *Let* $L \in H^2(M,\mathbb{Z})$ *be the imaginary part of a Kähler metric. Let*

$$\Gamma_2 := \{\phi \in \Gamma_1(M) | \phi(L) = L\}.$$

It is a well know fact that the moduli space of polarized algebraic manifolds $\mathcal{M}_L(\mathrm{M}) = \mathcal{T}(\mathrm{M})/\Gamma_2$. In [31] the following fact was established:

Theorem 47 *There exists a subgroup of finite index* Γ_L *of* Γ_2 *such that* Γ_L *acts freely on* $\mathcal{T}(M)$ *and* $\Gamma\backslash\mathcal{T}(M) = \mathfrak{M}_L(M)$ *is a non-singular quasi-projective variety. Over* $\mathfrak{M}_L(M)$ *there exists a family of polarized CY manifolds*

$$\mathcal{M} \rightarrow \mathfrak{M}_L(M).$$

Remark 48 *Theorem 47 implies that we constructed a family of non-singular CY manifolds*

$$\pi : \mathcal{X} \rightarrow \mathfrak{M}_L(M)$$

over a quasi-projective non-singular variety $\mathfrak{M}_L(M)$. *Moreover it is easy to see that*

$$\mathcal{X} \subset \mathbb{CP}^N \times \mathfrak{M}_L(M).$$

So $\mathcal{X}$ *is also quasi-projective. From now on we will work only with this family.*

6.2 Metrics on Vector Bundles with Logarithmic Growth

In Theorem 47 we constructed the moduli space $\mathfrak{M}_L(\mathrm{M})$ of CY manifolds. From the results in [45] and Theorem 47 we know that $\mathfrak{M}_L(\mathrm{M})$ is a quasi-projective non-singular variety. Using Hironaka's resolution theorem, we may find a compactification $\overline{\mathfrak{M}_L(\mathrm{M})}$ of $\mathfrak{M}_L(\mathrm{M})$ such that

$$\overline{\mathfrak{M}_L(\mathrm{M})} \ominus \mathfrak{M}_L(\mathrm{M}) = \mathfrak{D}$$

is a divisor with normal crossings. We need now to show how we will extend the determinant line bundle $\mathcal{L}$ to a line bundle $\overline{\mathcal{L}}$ to $\overline{\mathfrak{M}_L(\mathrm{M})}$. For this reason we are going to recall the following definitions and results from [35]. We will look at polycylinders $\mathrm{D}^N \subset \overline{\mathfrak{M}_L(\mathrm{M})}$, where D is the unit disk, $N = \dim \overline{\mathfrak{M}_L(\mathrm{M})}$ and such that

$$\mathrm{D}^N \cap \mathfrak{D}_\infty = \{\text{union of hyperplanes}; q^1 = 0, ..., q^N = 0\}.$$

Hence,

$$\mathrm{D}^N \cap \mathfrak{M}_L(M) = (\mathrm{D}^*)^k \times \mathrm{D}^{N-k},$$

where $\mathrm{D}^* = \mathrm{D} \ominus 0$. On D^* we have the Poincare metric

$$ds^2 = \frac{|dq|^2}{|q|^2 (\log|q|)^2}$$

and on D we have the simple metric $|dq|^2$, giving us a product metric on $(\mathrm{D}^*)^k \times \mathrm{D}^{N-k}$ which we call $\omega^{(P)}$.

A complex-valued C^∞ p-form η on $\mathfrak{M}_L(\mathrm{M})$ is said to have Poincare growth on $\overline{\mathfrak{M}_L(\mathrm{M})} \ominus \mathfrak{M}_L(\mathrm{M})$ if there is a set of *if polycylinders*

$$\mathcal{U}_\alpha \subset \overline{\mathfrak{M}_L(\mathrm{M})}$$

covering

$$\overline{\mathfrak{M}_L(\mathrm{M})} \ominus \mathfrak{M}_L(\mathrm{M})$$

such that in each $\mathcal{U}_\alpha$ *an estimate of the following type holds:*

$$\left|\eta\left(q^1, ..., q^N\right)\right| \leq C_\alpha \omega^{(P)}_{\mathcal{U}_\alpha}(q^1, \overline{q^1}) ..._\alpha \omega^{(P)}_{\mathcal{U}_\alpha}(q^N, \overline{q^N}).$$

This property is independent of the covering $\mathcal{U}_\alpha$ of $\overline{\mathfrak{M}_L(\mathrm{M})} \ominus \mathfrak{M}_L(\mathrm{M})$ but depends on the compactification $\overline{\mathfrak{M}_L(\mathrm{M})}$. If η_1 and η_2 both have Poincare growth on

$$\overline{\mathfrak{M}_L(\mathrm{M})} \ominus \mathfrak{M}_L(\mathrm{M}),$$

then so does $\eta_1 \wedge \eta_2$. An important property of Poincare growth is the following:

Theorem 49 *A p-form η with a Poincare growth on*

$$\overline{\mathfrak{M}_L(M)} \ominus \mathfrak{M}_L(M) = \mathfrak{D}$$

has the property that for every C^∞ $(r-p)$ form ψ on $\overline{\mathfrak{M}_L(M)}$ we have:

$$\int_{\overline{\mathfrak{M}_L(M)}\ominus\mathfrak{M}_L(M)} |\eta\wedge\psi| < \infty.$$

Hence, η defines a current $[\eta]$ on $\overline{\mathfrak{M}_L(M)}$.

Proof: For the proof see [35]. ■

Definition 50 *A complex valued C^∞ p-form η on $\overline{\mathfrak{M}_L(M)}$ will be called "good" on M if both η and $d\eta$ have Poincare growth. Let $\mathcal{E}$ be a vector bundle on $\overline{\mathfrak{M}_L(M)}$ with a Hermitian metric h. We will call h a good metric on $\overline{\mathfrak{M}_L(M)}$ if the following holds:*

1. *If for all $x\in\overline{\mathfrak{M}_L(\mathrm{M})}\ominus\mathfrak{M}_L(\mathrm{M})$, there exist sections $e_1,...,e_m$ of $\mathcal{E}$ which form a basis of $\mathcal{E}\left|_{D^r\ominus(D^r\cap\mathfrak{D}_\infty)}\right.$.*

2. *In a neighborhood D^r of $x\in\overline{\mathfrak{M}_L(\mathrm{M})}\ominus\mathfrak{M}_L(\mathrm{M})$ in which $\overline{\mathfrak{M}_L(\mathrm{M})}\ominus\mathfrak{M}_L(\mathrm{M})$ is given by $qh1\times...\times q^N=0$. The metric $\mathrm{h}_{i\overline{j}}=\mathrm{h}(e_i,e_j)$ has the following properties:* **a.**

$$\left|h_{i\overline{j}}\right|\leq C\left(\sum_{i=1}^{k}\log\left|q^i\right|\right)^{2m},\ \ (\det(h))^{-1}\leq C\left(\sum_{i=1}^{k}\log\left|q^i\right|\right)^{2m}$$

   for some $C>0$, $m\geq 0$. **b.** *The 1-forms $\left((dh)\,h^{-1}\right)$ are good forms on $\overline{\mathfrak{M}_L(\mathrm{M})}\cap D^N$.*

It is easy to prove that there exists a unique extension $\overline{\mathcal{E}}$ of $\mathcal{E}$ on $\overline{\mathfrak{M}_L(\mathrm{M})}$, i.e. $\overline{\mathcal{E}}$ is defined locally as holomorphic sections of $\mathcal{E}$ which have a finite norm in h.

Theorem 51 *Let $(\mathcal{E},h)$ be a vector bundle with a good metric on $\overline{\mathfrak{M}_L(M)}$, then the Chern classes $c_k(\mathcal{E},h)$ are good forms on $\overline{\mathfrak{M}_L(M)}$ and the currents $[c_k(\mathcal{E},h)]$ represent the cohomology classes*

$$c_k(\mathcal{E},h)\in H^{2k}(\overline{\mathfrak{M}_L(M)},\mathbb{Z}).$$

Proof: For the proof see [35]. ■

6.3 Applications of Mumford's Results to the Moduli of CY

We are going to prove the following result:

Theorem 52 *Let*

$$\pi:\mathcal{X}\rightarrow\mathfrak{M}_L(M)$$

be the flat family of non-singular CY manifolds. Then on $\pi_*\left(\omega_{\mathcal{X}/\mathfrak{M}_L(M)}\right)$ *we have a natural* L^2 *metric defined as follows:*

$$h = \|\omega_\tau\|^2 := (-1)^{\frac{n(n-1)}{2}}\left(\sqrt{-1}\right)^n \int_M \omega_\tau \wedge \overline{\omega_\tau}. \tag{102}$$

Then h *is a good metric.*

Proof: Let $q_0 \in \mathfrak{D} = \overline{\mathfrak{M}_L(\mathrm{M})} \ominus \mathfrak{M}_L(\mathrm{M})$. Let $q_0 \in (D)^N$ be a polydisk in $\overline{\mathfrak{M}_L(\mathrm{M})}$ which intersects $\mathfrak{D}$. We will assume without loss of generality that

$$(D^*)^N \subset \mathfrak{M}_L(\mathrm{M}),$$

where N is the dimension of $\mathfrak{M}_L(\mathrm{M})$. Over $(D^*)^N$ we have a family

$$\mathcal{X}_{(D^*)^N} \to (D^*)^N \tag{103}$$

of CY manifolds. The local parameters on $(D^*)^N$ will be $q^i = \exp(2\pi\sqrt{-1}\tau^i)$, $i = 1, ..., N$ and τ will be the coordinate in the upper half plane $\mathfrak{h}$. Let

$$\mathcal{X}_{\alpha_0,\alpha_1} \to D_{\alpha_0,\alpha_1} \tag{104}$$

be the restriction of the family (103) on

$$\mathrm{D}_{\alpha_0,\alpha_1} \subset (D^*)^N \subset \mathfrak{M}_L(\mathrm{M}),$$

where

$$D_{\alpha_1,\alpha_2} := \{t \in \mathbb{C} | 0 < |t| < 1 \text{ and } 0 \le \alpha_1 < \arg t < \alpha_2 < 2\pi\}$$

We will assume that the closure of $\mathrm{D}_{\alpha_0,\alpha_1}$ contains a point $\kappa_\infty \in$D and $\kappa_\infty \in \mathfrak{D} \subset \overline{\mathfrak{M}_L(\mathrm{M})}$.

Lemma 53 *There exists a global section*

$$\eta_q \in \Gamma\left(D_{\alpha_1,\alpha_2}, \omega_{\mathcal{X}_{\alpha_1,\alpha_2}/D_{\alpha_1,\alpha_2}}\right)$$

such that for the classes of cohomology $[\eta_\tau]$ *the limit*

$$\lim_{\tau \to \kappa_\infty} [\eta_q] = [\eta_{\kappa_\infty}] \tag{105}$$

exists.

Proof: Since $\mathrm{D}_{\alpha_1,\alpha_2}$ is a contractible sector in the punctured disk D^* we conclude that we can construct a section

$$\eta_q \in \Gamma\left(D_{\alpha_1,\alpha_2}, \omega_{\mathcal{X}_{\alpha_1,\alpha_2}/D_{\alpha_1,\alpha_2}}\right)$$

such that for each $q \in D_{\alpha_1,\alpha_2}$ $\eta_q^{'} \neq 0$. From local Torelli theorem we may assume that $D_{\alpha_1,\alpha_2} \subset \mathbb{P}(H^n(\mathrm{M},\mathbb{C}))$. From Lemma 53 follows directly. ■

We have two possibilities for the for the class of cohomology $[\eta_{\kappa_\infty}]$ established by Lemma 53.

Lemma 54 *Suppose that*

$$\langle [\eta_{\kappa_\infty}], [\eta_{\kappa_\infty}] \rangle > 0. \tag{106}$$

Then the metric h is good around $\kappa_\infty \in \overline{\mathfrak{M}_L(M)}$.

Proof: If (106) holds for $[\eta_{\kappa_\infty}]$ then the function

$$h(q,\overline{q}) = \langle [\eta_q], [\eta_q] \rangle$$

is bounded in $(D)^N \subset \overline{\mathfrak{M}_L\,(\mathrm{M})}$. Thus obviously we have:

$$h(q,\overline{q}) \leq C \left(\sum_{i=1}^{k} \log \left| q^i \right| \right)^{2m},$$

$$h^{-1} \leq C_1 \left(\sum_{i=1}^{k} \log \left| q^i \right| \right)^{2m}$$

and

$$\partial\left(\log(h)\right) \wedge \overline{\partial\left(\log(h)\right)} = \frac{\partial h}{h} \wedge \frac{\overline{\partial h}}{\overline{h}} < C_2 \sum_{i=1}^{k} \frac{d \log q^i \wedge \overline{d \log q^i}}{\left(\log |q^i|\right)^2}$$

in $(\mathrm{D}^*)^N \subset \mathfrak{M}_L(\mathrm{M})$. Thus we proved Lemma 54. ■

The second possibility is that

$$\langle [\eta_{\kappa_\infty}], [\eta_{\kappa_\infty}] \rangle = 0. \tag{107}$$

Assuming this let

$$\pi : (\mathrm{U})^N \to (D^*)^N \subset \mathfrak{M}_L(\mathrm{M})$$

be the universal cover of $(D^*)^N$. We will assume that U_i is the universal cover of D_i^* and it is the unit disk with a local parameter t^i. The point $0_i \in D_i = \overline{D^*}$ corresponds to a boundary point $\kappa_\infty^i \in \overline{\mathrm{U}_i}$. Let q^i be the local parameter on D_i. Let us consider

$$(D_{\alpha_1,\alpha_2})^N \subset \mathfrak{M}_L(\mathrm{M})$$

and the family

$$\mathcal{X}_{\mathrm{N},\alpha_1,\alpha_2} \to (D_{\alpha_1,\alpha_2})^N \tag{108}$$

which is the restriction of the family (103) on

$$\underbrace{D_{\alpha_1,\alpha_2} \times ... \times D_{\alpha_1,\alpha_2}}_{N} \subset \mathfrak{M}_L(\mathrm{M}).$$

Let $\tau_0 \in (D_{\alpha_1,\alpha_2})^N \subset \mathfrak{M}_L(\mathrm{M})$. Let ω_q be the family of forms constructed in Theorem 7 in a polydisk which contained the point τ_0 and is contained in$(D_{\alpha_1,\alpha_2})^N$. Then we have:

Lemma 55 *The function*

$$h(q,\overline{q}) = \langle \omega_q, \omega_q \rangle$$

is real analytic function which can be analytically continued to $(D)^N \subset \overline{\mathfrak{M}_L(M)}$ *maybe after we shrink the disks* D_i *and*

$$h(q,\overline{q})|_{\mathfrak{D}\cap \underset{N}{(D)^N}} = 0.$$

Proof: Repeating the arguments of Lemma 53 we can construct on $(D_{\alpha_1,\alpha_2})^N \subset \mathfrak{M}_L(\mathrm{M})$ a global section

$$\eta_q \in \Gamma\left((D_{\alpha_1,\alpha_2})^N, \omega_{\mathcal{X}_{N,\alpha_1,\alpha_2}/(\mathrm{D}_{\alpha_1,\alpha_2})^N}\right)$$

such that

$$\lim_{q=(q^1,\ldots,q^N)\to(\kappa^1_\infty,\ldots,\kappa^N_\infty)\in\mathfrak{D}} [\eta_q] = [\eta_{(\kappa^1_\infty,\ldots,\kappa^N_\infty)}]$$

exists and by assumption (107)

$$\left\langle [\eta_{(\kappa^1_\infty,\ldots,\kappa^N_\infty)}], [\eta_{(\kappa^1_\infty,\ldots,\kappa^N_\infty)}] \right\rangle = 0.$$

The relations between the forms $[\eta_q]$ and $[\omega_q]$ constructed in Theorem 7 are given by the formula

$$[\eta_q] = \varphi(q)[\omega_q], \tag{109}$$

where $\varphi(q)$ is a holomorphic function on the product $(D_\varepsilon)^N$ of small discs containing the point τ_0 and contained in $(D_{\alpha_1,\alpha_2})^N \subset \mathfrak{M}_L(\mathrm{M})$. According to Theorem 7

$$\langle [\omega_q], [\omega_q] \rangle \le \langle [\omega_{q_0}], [\omega_{q_0}] \rangle \tag{110}$$

Notice that the functions $\langle [\eta_q], [\eta_q] \rangle$ and $\langle [\omega_q], [\omega_q] \rangle$ are real analytic. Combining this fact with (109) , (110) and (104) we deduce that the function

$$h(q,\overline{q}) = \langle [\omega_q], [\omega_q] \rangle$$

can be analytically continued to a function on $\underbrace{D\times\ldots\times D}_{N} \subset \overline{\mathfrak{M}_L(\mathrm{M})}$ since the function $\langle [\eta_q], [\eta_q] \rangle$ is well defined on $\underbrace{D\times\ldots\times D}_{N}$ and so

$$\lim_{q=(q^1,\ldots,q^N)\to(\kappa^1_\infty,\ldots,\kappa^N_\infty)\in\mathfrak{D}} [\omega_q] = [\omega_{(\kappa^1_\infty,\ldots,\kappa^N_\infty)}]$$

exists and

$$\langle [\omega_{\kappa_\infty}], [\omega_{\kappa_\infty}] \rangle|_{\mathfrak{D}\cap(D)^N} = h(q,\overline{q})|_{\mathfrak{D}\cap(D)^N} = 0.$$

Lemma 55 is proved. ■

The local coordinate t^i in the disk U_i which is the universal cover of D_i^* satisfies

$$t^i = \frac{\tau^i + \kappa^i_\infty}{\tau^i - \kappa^i_\infty}.$$

Lemma 56 *On each of the components* U_i *of the universal cover*

$$\pi : (\mathrm{U})^N \to (D^*)^N \subset \mathfrak{M}_L(M)$$

of $(D^*)^N$ *we have the following expression of the restriction of* $\langle \omega_{t^i}, \omega_{t^i} \rangle$

$$\langle \omega_{t^i}, \omega_{t^i} \rangle = 1 - \left| t^i \right|^2 + \psi(t^i) \ \textit{and} \lim_{t^i \to \kappa^i_\infty} \psi(t^i) = 0. \tag{111}$$

Proof: (111) follows directly from the expression for

$$h(t, \overline{t}) = \langle [\omega_t], [\omega_t] \rangle =$$

$$1 - \sum_{i,j} \langle \omega_0 \lrcorner \phi_i, \omega_0 \lrcorner \phi_j \rangle \, t^i \overline{t^j} + \sum_{i,j} \langle \omega_0 \lrcorner (\phi_i \wedge \phi_k), \omega_0 \lrcorner (\phi_j \wedge \phi_l) \rangle \, t^i \overline{t^j} t^k \overline{t^l} + O(t^5)$$

given by formula (7) of Theorem 7 when restricted to the unit disc U_i. Lemma 56 is proved. ■

Lemma 57 *On*

$$(D)^N \subset \overline{\mathfrak{M}_L(M)}$$

the function $h(q, \overline{q}) := \langle \omega_q, \omega_q \rangle$ *satisfies the inequality*

$$h(\omega_q, \omega_q) \le C \left(\sum_{i=1}^{k} \log |q_i| \right)^{2m}.$$

Proof: We know from the theory of Hodge structures that if $\{\gamma_1, ..., \gamma_{b_n}\}$ is a basis of H^n(M,$\mathbb{Z}$)/Tor, then the functions:

$$\left(..., \int_{\gamma_i} \omega_q, ... \right)$$

for $0 < |q| < 1$ and $0 < \arg(q) < 2\pi$ are solutions of a differential equation with regular singularities. From the fact that the solutions of any differential equation with regular singularities have logarithmic growth and

$$h(\omega_q, \omega_q) = \left(..., \int_{\gamma_i} \omega_q, ... \right) (\langle \gamma_i, \gamma_j \rangle) \left(..., \int_{\gamma_i} \overline{\omega_q}, ... \right)^t,$$

we deduce that

$$h(\omega_q, \omega_q) \le C \left(\sum_{i=1}^{k} \log |q_i| \right)^{2m}.$$

Lemma 57 is proved. ■

Next we will prove that on $(D^*)^N \subset \mathfrak{M}_L$(M) we have

$$h^{-1} := \frac{1}{\langle \omega_\tau, \omega_\tau \rangle} \le C \left(\sum_{i=1}^{k} \log |q_i| \right)^{2m} \tag{112}$$

and

$$\partial\left(\log(h)\right)\wedge\overline{\partial\left(\log(h)\right)}\leq C\sum_{i=1}^{k}\frac{dq_i\wedge\overline{dq_i}}{|q_i|^2\left(\log|q_i|\right)^2}=C\sum_{i=1}^{k}\frac{d\log q_i\wedge\overline{d\log q_i}}{\left(\log|q_i|\right)^2}. \quad (113)$$

We will work on each of the components U_i of the universal cover $\pi:(\mathrm{U})^N\to(D^*)^N\subset\mathfrak{M}_L(\mathrm{M})$ of $(D^*)^N$.

Proof of (112) : From the relations between the coordinates q,τ and t

$$t=\frac{\tau+\kappa_\infty}{\tau-\kappa_\infty}\text{ and }q=\exp\left(2\pi\sqrt{-1}t\right)$$

we deduce that on

$$\left(D^*\right)^N\subset\mathfrak{M}_L(\mathrm{M})$$

the function $h(q,\overline{q})$ on each D_i will be given by

$$h(q,\overline{q})|_{\mathrm{D}_i}=|q_i|^2\left(\log|q_i|^2\right)+\phi(q_i),$$

where

$$\lim_{q\to0}\phi(q)=0. \quad (114)$$

Formula (112) follows directly from formula (114). Thus (112) is proved. ■

Proof of (113) : From the explicit formula (7) for the family $[\omega_t]$ and the definition of the function h we obtain:

$$h(t,\overline{t})|_{\mathrm{U}_i}=\langle[\omega_{t^i}],[\omega_{t^i}]\rangle=1-\left|\frac{t^i}{\kappa_\infty^i}\right|^2+\psi(t^i)$$

where $\psi(t^i)$ is C^∞ and

$$\lim_{t\to\kappa_\infty}\psi(t^i)=0.$$

Thus we obtain that

$$\left(\frac{\partial h}{h}\wedge\overline{\frac{\partial h}{h}}\right)|_{\mathrm{D}_i}=\frac{\left|\left(1-\frac{\partial}{\partial t^i}h(t^i)\right)\right|^2dt^i\wedge\overline{dt^i}}{h^2}. \quad (115)$$

Since the function

$$\frac{\partial}{\partial t^i}h(t^i)=-\frac{\overline{t^i}}{|\kappa_\infty^i|^2}+\psi(t^i)$$

is bounded and the formula (111) we obtain from (115):

$$\frac{\partial h}{h}\wedge\overline{\frac{\partial h}{h}}|_{\mathrm{U}_i}\leq C\frac{\sqrt{-1}}{2}\frac{dt^i\wedge\overline{dt^i}}{\left(1-|t^i|^2\right)^2}. \quad (116)$$

Notice that (116) is equivalent to (111) since the upper half plane is conformally equivalent to the unit disk and by the conformal transformation the Poincare metric

$$\frac{-\sqrt{-1}}{2}\frac{dq_i\wedge\overline{dq_i}}{|q_i|^2\left(\log|q_i|\right)^2}$$

on the upper half plane transforms to

$$\frac{-\sqrt{-1}}{2}\frac{dt^i\wedge\overline{dt^i}}{(1-|t^i|^2)^2}.$$

Thus relation (111) is proved. ■

Theorem 52 is proved. ■

Theorem 58 *The Weil-Petersson volume of the moduli space of polarized CY manifolds is finite and it is a rational number.*

Proof: Theorem 52 implies that the metric on the relative dualizing sheaf $\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}$ defined by (102) is a good metric. This implies that the Chern form of any good metric defines a class of cohomology in $H^2(\overline{\mathfrak{M}_L(\mathrm{M})},\mathbb{Z})\cap H^{1,1}(\overline{\mathfrak{M}_L(\mathrm{M})},\mathbb{Z})$. See Theorem 51. We know from [43] that the Chern form of the metric h is equal to minus the imaginary part of the Weil-Petersson metric. So the imaginary part of the Weil-Petersson metric is a good form in the sense of Mumford. This implies that

$$\int\limits_{\overline{\mathfrak{M}_L(\mathrm{M})}}\wedge^{\dim_{\mathbb{C}}\overline{\mathfrak{M}_L(\mathrm{M})}}c_1(h)\in\mathbb{Z}$$

since $\overline{\mathfrak{M}_L(\mathrm{M})}$ is a smooth manifold. Since $\mathfrak{M}_L(\mathrm{M})$ is a finite over the moduli space $\mathcal{M}_L(\mathrm{M})$ then the Weil-Petersson volume of $\mathcal{M}_L(\mathrm{M})$ will be a rational number. Theorem 58 is proved. ■

In the paper [33] the authors proved that the Weil-Petersson volumes of the moduli space of CY manifolds are finite.

7 The Theory of Determinant Line Bundles and the Quillen Metric

7.1 Geometric Data

In order to construct the determinant line bundle, we need the following data:

1. A smooth fibration of manifolds $\pi:\mathcal{X}\rightarrow\mathfrak{M}_L(\mathrm{M})$. In our case it will be the smooth fibration of the family of CY manifolds over the moduli space as defined in Theorem 47. Let n=$\dim_{\mathbb{C}}$M.
2. A metric along the fibres, that is, a metric g(τ) on the relative tangent bundle $\mathcal{T}(\mathcal{X}/\mathfrak{M}_L(\mathrm{M}))$. In this paper the metric will be the families of CY metrics g(τ) such that the class of cohomology $[\mathrm{Im}(g(\tau))]=L$ is fixed.

From this data we will construct the determinant line bundle $\mathcal{L}$ over the moduli space of CY manifolds $\mathfrak{M}_L(\mathrm{M})$. We will consider the relative $\overline{\partial}_{\mathcal{X}/\mathfrak{M}_L(M)}$ complex:

$$0\rightarrow ker\overline{\partial}_{\mathcal{X}/\mathfrak{M}_L(M)}\rightarrow C^\infty(\mathcal{X})\overset{\overline{\partial}_{0,\mathcal{X}/\mathfrak{M}_L(M)}}{\rightarrow}\Omega^{0,1}_{\mathcal{X}/\mathfrak{M}_L(M)}\overset{\overline{\partial}_{1,\mathcal{X}/\mathfrak{M}_L(M)}}{\rightarrow}$$

$$\stackrel{\overline{\partial}_{1,\mathcal{X}/\mathfrak{M}_L(M)}}{\rightarrow} \Omega^{0.n-1}_{\mathcal{X}/\mathfrak{M}_L(M)} \stackrel{\overline{\partial}_{n-1,\mathcal{X}/\mathfrak{M}_L(M)}}{\rightarrow} \Omega^{0.n}_{\mathcal{X}/\mathfrak{M}_L(M)} \rightarrow 0.$$

For each $\tau \in \mathfrak{M}_L(\mathrm{M})$ and k, we will define D to be:

$$D_k := \overline{\partial}_{k,\mathcal{X}/\mathfrak{M}_L(M)} + \left(\overline{\partial}_{k,\mathcal{X}/\mathfrak{M}_L(M)}\right)^*$$

and

$$D_{k,\tau} := \mathrm{D}_k\,|_{\mathrm{M}_\tau} = \overline{\partial}_{k,\tau} + \left(\overline{\partial}_{k,\tau}\right)^*.$$

Definition 59 *We will call the above complex the relative Dolbault complex.*

Let us define

$$\left(\mathcal{H}^k\right)_\tau := L^2(M_\tau, \Omega^{0,k}_\tau).$$

Furthermore, as τ varies over $\mathfrak{M}_L(\mathrm{M})$, the spaces $\left(\mathcal{H}^k_\tau\right)$ fit together to form continuous Hilbert bundles $\mathcal{H}^k$ over $\mathfrak{M}_L(\mathrm{M})$.[2]. Thus we can view $\overline{\partial}_{k,\mathcal{X}/\mathfrak{M}_L(M)}$ as bundle maps:

$$\overline{\partial}_{k,\mathcal{X}/\mathfrak{M}_L(M)} : \mathcal{H}^k \rightarrow \mathcal{H}^{k+1}.$$

The Hilbert bundles $\mathcal{H}^k$ carry L^2 metrics by definition.

Now we are ready to construct the Determinant line bundle $\mathcal{L}$ of the operator $\overline{\partial}_{\mathcal{X}/\mathfrak{M}_L(M)}$. We will recall some basic consequences of the ellipticity of D_τ. Each fibre $\mathcal{H}^k_\tau$ of the Hilbert bundles $\mathcal{H}^k$ decomposes into direct sums of eigen spaces of non-negative Laplacians $D^*_k D_k$ and $D^*_k D_k$. The spectrums of these operators are discrete, and the nonzero eigen values $\{\lambda_{k,i}\}$ of $D^*_k D_k$ and $D_k D^*_k$ agree and D_k defines a canonical isomorphisms between the corresponding eigen spaces.

Definition 60 ***1.*** *Let*

$$\mathcal{U}_a : \{\tau \in \mathfrak{M}_L(M) | a \notin Spec(D_k D^*_k)$$

for $0 \leq k \leq n$ and any $a > 0\}$. (U_a are open sets in $\mathfrak{M}_L(M)$ and they form an open covering of $\mathfrak{M}_L(M)$ since the spectrum of $D^*_\tau D_\tau$ is discrete.) ***2.*** *Let the fibres of $\mathcal{H}^k_a$ be the vector subspaces in $\mathcal{H}^k_{\tau,a}$ spanned by eigen vectors with eigen values less than a over $\mathcal{U}_a$. Then we can define the complex:*

$$0 \rightarrow \Gamma(\mathcal{U}_a, \mathcal{O}_{\mathcal{U}_a}) \rightarrow \mathcal{H}^0_a \stackrel{\overline{\partial}_{0,\mathcal{X}/\mathfrak{M}_L(M)}}{\rightarrow} \ldots$$

$$\ldots \rightarrow \mathcal{H}^{n-1}_a \stackrel{\overline{\partial}_{n-1,\mathcal{X}/\mathfrak{M}_L(M)}}{\rightarrow} \mathcal{H}^n_a \rightarrow \ker(D_{n-1} \circ D^*_n) \rightarrow 0.$$

If $b > a$ we set $\mathcal{H}^k_{a,b} := \mathcal{H}^k_b/\mathcal{H}^k_a$. The spaces $\mathcal{H}^k_a$ form smooth finite dimensional C^∞ bundles over an open set $\mathcal{U}^a \subset \mathfrak{M}_L(M)$. For the proof of this fact see [1].

[2] These bundles are not smooth since the composition $L^2 \times C^\infty \rightarrow L^2$ is not a differentiable map.

7.2 Construction of the Generating Sections $\det(D_a)$ over $\mathcal{U}_a$

Definition 61 *Let*

$$\omega_1^k, .., \omega_{m_k}^k, \psi_1^k, .., \psi_{N_k}^k, \phi_1^k, .., \phi_{M_k}^k$$

be an orthonormal basis in the trivial vector bundle $\mathcal{H}_a^k$ over $\mathcal{U}_a$, where

$$D_k\omega_i^k = 0,\ \overline{\partial}_k^*(\overline{\partial}_k\psi_j^k) = \lambda_j^k\psi_j^k,\ \overline{\partial}_k(\overline{\partial}_k^*\phi_j^k) = \lambda_j^k\phi_j^k, \phi_j^k \in \operatorname{Im}\overline{\partial}_{k-1,\mathcal{X}/\mathfrak{M}_L(M)}$$

and

$$\psi_j^k \in \operatorname{Im}(\overline{\partial}_{k,\mathcal{X}/\mathfrak{M}_L(M)}^*)$$

for 1$\leq i \leq k$ and $0 < \lambda_j < a$ for 1$\leq j \leq N$. Let

$$\det(\overline{\partial}_{k,a}) =$$

$$\omega_1^k\wedge...\wedge\omega_{m_k}^k\wedge(\overline{\partial}_{k-1,\mathcal{X}/\mathfrak{M}_L(M)}\psi_1^{k-1})\wedge...\wedge(\overline{\partial}_{k-1,\mathcal{X}/\mathfrak{M}_L(M)}\psi_{N_k}^{k-1})\wedge\left(\phi_1^k\wedge...\wedge\phi_{M_k}^k\right)^{(-1)^k}.$$

We will define the line bundle $\mathcal{L}$ restricted on $\mathcal{U}_a$ as follows:

$$\mathcal{L}^a := \mathcal{L}|_{\mathcal{U}_a} = \otimes_{k=0}^n\left(\wedge^{\dim\mathcal{H}_a^k}\mathcal{H}_a^k\right)^{(-1)^k}.$$

Definition 62 *The generator $\det(\overline{\partial}_a)$ of $\mathcal{L}^a := \mathcal{L}|_{\mathcal{U}^a}$ is defined as follows:*

$$\det(\overline{\partial}_a) := \underset{k}{\otimes}\det(\overline{\partial}_{k,a}).$$

We will define how we patch together $\mathcal{L}^a$ and $\mathcal{L}^b$ over $\mathcal{U}^a\cap\mathcal{U}^b$[3]. On that intersection we have:

$$\mathcal{L}^b = \mathcal{L}^a\otimes\mathcal{L}^{a,b},$$

where

$$\mathcal{L}^{a,b} := \otimes_k(\det\mathcal{H}_{a,b}^k)^{(-1)^k}$$

on $\mathcal{U}^a\cap\mathcal{U}^b$. We can identify $\mathcal{L}^a, b$ over $\mathcal{U}^a\cap\mathcal{U}^b$ with the line bundle spanned by the section

$$\det(\overline{\partial}_{a,b}) = \otimes_{k=0}^n\det(\overline{\partial}_{k,a,b})^{(-1)^k},$$

where

$$\det(\overline{\partial}_{k,a,b}) :=$$

$$= \left(\overline{\partial}_{k-1,\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}\psi_1^{k-1}\right)\wedge...\wedge\left(\overline{\partial}_{k-1,\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}\psi_{N_k}^{k-1}\right)\wedge\phi_1^k\wedge...\wedge\phi_{M_k}^k,$$

$$\phi_j^k\in\operatorname{Im}\overline{\partial}_{k-1,\mathcal{X}/\mathfrak{M}_L(\mathrm{M})},\ \psi_j^k\in\operatorname{Im}(\overline{\partial}_{0,\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}^*),$$

$$\Delta_k\phi_j^k = \lambda_j^k\phi_j^k,\ \Delta_k(\overline{\partial}(\psi_i^{k-1})) = \lambda_i^k(\overline{\partial}(\psi_i^{k-1}))$$

and $a < \lambda_i^k < b$.

[3]We may suppose that $b > a$.

Remark 63 *We can view* $\det(\overline{\partial}_{a,b})$ *as a section of the line bundle* $\mathcal{L}^{a,b}$ *over* $\mathcal{U}^a \cap \mathcal{U}^b$. *It defines canonical smooth isomorphisms over* $\mathcal{U}^a \cap \mathcal{U}^b$:

$$\mathcal{L}^a \to \mathcal{L}^a \otimes \mathcal{L}^{a,b} = \mathcal{L}^b (s \to s \otimes \det(\overline{\partial}_{a,b})$$

for all $0 < a < b$.

We define the determinant line bundle $\mathcal{L}$ by patching the trivial line bundles $\mathcal{L}^a$ over $\mathcal{U}^a$ by using the canonical isomorphism defined in Remark 63.

7.3 The Description of the Quillen Metric on the Determinant Line Bundle

We now proceed to describe the Quillen metric on $\mathcal{L}$. Fix $a > 0$. Then the subbundles $\mathcal{H}_a^k$ of the Hilbert bundles $\mathcal{H}^k$ on $\mathcal{U}_a$ inherit metrics from $\mathcal{H}^k$. According to standard facts from linear algebra, metrics are induced on determinants, duals, and tensor products. So the $\mathcal{L}^a$ inherits a natural metric. We will denote by g^a the L^2 norm of the section $\det(\overline{\partial}_a)$. Clearly,

$$g^a = \prod_{k=0}^{n} \left(\lambda_1^k ... \lambda_{n_k}^k\right)^{(-1)^k},$$

and λ_i^k are all nonzero eigen values of the operators $\overline{\partial}_k^* \overline{\partial}_{k-1}$which are less than a.

If $b > a$, then under the isomorphism defined in Remark 63, we have two metrics on $\mathcal{L}^b$ and their ratio is a real number equal to the L^2 norm of the section$\left\|\det(\overline{\partial}_{a,b})\right\|^2$. The definition of the section $\det(\overline{\partial}_{a,b})$ implies that we have the following formula:

$$\left\|\det(\overline{\partial}_{a,b})\right\|^2 = \prod_{k=0}^{n} \prod_{i=1} \left\|\phi_i^k\right\|^2 \prod_{j=1}^{n} \left(\left\|\overline{\partial}\psi_j^k\right\|^2\right)^{(-1)^k} =$$

$$= \prod_{i=1} \left\|\phi_i^k\right\|^2 \prod_{j=1} \left\langle \overline{\partial}_k^* \overline{\partial}_{k-1} \psi_j^k, \psi_j^k \right\rangle^{(-1)^k} = \prod_{k=1}^{n} \left(\lambda_i^k\right)^{(-1)^k}$$

where λ_i^k are all the non-zero eigen values of the operators $\overline{\partial}_k^* \overline{\partial}_{k-1}$ such that $a < \lambda_i^k < b$. In other words, on $\mathcal{U}^a \cap \mathcal{U}^b$

$$g^b = g^a \prod_{a < \lambda_i^k < b} \left(\lambda_i^k\right)^{(-1)^k}.$$

To correct this discrepancy, we define

$$\overline{g}^a = g^a \det(\overline{\partial}^* \overline{\partial}) \mid_{\lambda > a},$$

where

$$\det(\overline{\partial}^* \overline{\partial}) \mid_{\lambda > a} = \prod_{k=1}^{n} \left(\det(\overline{\partial}_k^* \overline{\partial}_{k-1} \mid_{\lambda > a})\right)^{(-1)^k},$$

$$\det(\overline{\partial}_k^*\overline{\partial}_{k-1}\mid_{\lambda>a}) = -\exp(-\frac{d}{ds}\zeta_k^a(s)|_{s=0}),$$

and

$$\zeta_k^a(s) = \sum_{\lambda_i>a}^{\infty} \left(\lambda_i^k\right)^s.$$

The crucial property of this regularization is that it behaves properly with respect to the finite number of eigen values, i.e.

$$\det(\overline{\partial}_k^*\overline{\partial}_{k-1}\mid_{\lambda>b}) = \det(\overline{\partial}_k^*\overline{\partial}_{k-1}\mid_{\lambda>a})\prod_{a<i<b}^{N}\lambda_i^k$$

on the intersection $\mathcal{U}^a\cap\mathcal{U}^b$. From the last remark we deduce that $\overline{g}^a$ and $\overline{g}^b$ agree on $\mathcal{U}^a\cap\mathcal{U}^b$. Thus $\overline{g}^a$ and $\overline{g}^b$ patch together to a Hermitian metric $\mathrm{g}^{\mathcal{L}}$ on $\mathcal{L}$. The metric $\mathrm{g}^{\mathcal{L}}$ will be called the Quillen metric on $\mathcal{L}$.

Definition 64 *We will define the holomorphic Ray Singer analytic torsion I(M) for CY manifold M as follows:*

$$I(M) := \log\left(\prod_{q=1}^{n}(\det(\triangle_q^{'})^{(-1)^q}\right).$$

See [39].

Remark 65 *It is easy to see that if M is a CY manifold and* $\dim_{\mathbb{C}}M=2n$*, then* log*I(M)=0. We know that for odd dimensional CY manifolds I(M)* $\neq 0$*. So from now on we will consider only odd dimensional CY manifolds.*

We will need the following result from [5] on p. 55:

Theorem 66 *The Quillen norm of the* C^∞ *section* $\det(\overline{\partial}_a)$ *on* $\mathcal{U}_a$ *of* $\mathcal{L}$ *is equal to* exp(*I(M)*)*.*

Proof: It follows from Definition 61 of the section $\det(\overline{\partial})\mid_{\mathcal{U}^a}$ of $\mathcal{L}$ and the definition of the Quillen metric that at each point $\tau\in\mathfrak{M}_L(\mathrm{M})$ the following formula is true:

$$\left\|\det(\overline{\partial})_\tau\mid_{\mathcal{U}^a}\right\|_Q^2 = \exp(\mathrm{I}(\mathrm{M}_\tau))|_{\mathcal{U}^a},$$

where $\left\|\det(\overline{\partial})_\tau\mid_{\mathcal{U}^a}\right\|_Q^2$ means the Quillen norm of the section $\det(\overline{\partial})_\tau\mid_{\mathcal{U}^a}$. Theorem 66 is proved. ■

8 Construction of a C^∞ Non-Vanishing Section of the Determinant Line Bundle $\mathcal{L}$ for Odd Dimensional CY Manifolds

8.1 Some Preliminary Results

Let us denote by

$$\pi_*\left(\omega_{\mathcal{X}/\mathfrak{M}_L(M)}\right) := \pi_*\left(\Omega^{n,0}_{\mathcal{X}/\mathfrak{M}_L(M)}\right)$$

the relative dualizing sheaf. The local sections of $\pi_*\left(\omega_{\mathcal{X}/\mathfrak{M}_L(M)}\right)$ are families of holomorphic n-forms ω_τ on M_τ.

Theorem 67 *If the dimension of the CY manifold is even, then $\mathcal{L}$ is isomorphic to the dual of the line bundle $\pi_*\left(\omega_{\mathcal{X}/\mathfrak{M}_L(M)}\right)$. If the dimension of the CY manifold is odd, then $\mathcal{L}$ is isomorphic to the line bundle $\pi_*\left(\omega_{\mathcal{X}/\mathfrak{M}_L(M)}\right)$ over $\mathfrak{M}_L(M)$.*

Proof: The definition of CY states that:

$$\dim_{\mathbb{C}} H^q(\mathrm{M},\mathcal{O}_{\mathrm{M}}) = \begin{cases} 1 & q=0 \text{ or } q=n \\ 0 & \text{for } q\neq 0 \text{ or } n \end{cases}. \tag{117}$$

(117) and the definition of CY manifolds imply that

$$R^q\pi_*\mathcal{O}_{\mathrm{M}} = \begin{cases} \left(\pi_*\omega_{\mathcal{X}/\mathfrak{M}_L(M)}\right)^* & q=n \\ \mathcal{O}_{\mathfrak{M}_L(M)} & \text{for } q\neq\ n \end{cases}. \tag{118}$$

From the definition of $\mathcal{L}$ it follows that

$$\mathcal{L}\backsimeq\prod_{q=0}^{n}(-1)^q\det\left(R^q\pi_*\mathcal{O}_{\mathrm{M}}\right).$$

Combining (117) and (118) we directly deduce Theorem 67. ■

Corollary 68 *Let M be a CY manifold of odd dimension $n = 2m+1$. Then the index of the operator $\overline{\partial}$ on the complex defined in Definition 59 is zero.*

8.2 Construction of a C^∞ Section of the Determinant Line Bundle $\mathcal{L}$ with Quillen Norm Ray-Singer Analytic Torsion

Definition 69 *Let*

$$\mathcal{H}_+ = \underset{k}{\oplus} L^2\left(M,\Omega_M^{0,2k}\right),$$

$$\mathcal{H}_- = \underset{k}{\oplus} L^2\left(M,\Omega_M^{0,2k+1}\right)$$

and

$$D = \overline{\partial}_{\mathcal{X}/\mathfrak{M}_L(M)} + \overline{\partial}^*_{\mathcal{X}/\mathfrak{M}_L(M)}.$$

Theorem 70 *Let M be an odd dimensional CY manifold. Then the $(1,1)$ form $dd^c I(M_\tau)$ is a good form on $\overline{\mathfrak{M}_L(M)}$ in the sense of Mumford.*

Proof: The definition of the Ray Singer analytic torsion together with Theorem 35 imply that

$$dd^c\mathrm{I}(\mathrm{M}_\tau) = \sum_{q=0}^{2n}(-1)^{q+1}dd^c\log\det\Delta'_{\tau,q} =$$

$$\sum_{q=0}^{2n}(-1)^{q+1}\left(\frac{\sqrt{-1}}{2}\sum_{i,j}\left(\int\limits_{\mathrm{M}} Tr\left(\mathcal{F}^{'}\left(q+1,\phi_i\circ\overline{\phi_j}\right)\right)vol(g)\right)d\tau^i\wedge\overline{d\tau^j}\right). \tag{119}$$

Let us define the $(1,1)$ forms $\beta_q(1,1)$ on $\mathfrak{M}_L(\mathrm{M})$ as follows:

$$\beta_q(1,1)=\frac{\sqrt{-1}}{2}\sum_{i,j}\left(\frac{\partial^2}{\partial\tau^j\overline{\partial\tau^i}}\left(-\log(\det(\triangle^{"}_{\tau,q}))\right)\right)d\tau^j\wedge\overline{d\tau^i}. \tag{120}$$

From Theorem 35 and (120) we deduce that we have the following equality:

$$\beta_q(1,1)=\frac{\sqrt{-1}}{2}\sum_{i,j}Tr\left(\mathcal{F}^{'}_{\tau}\left(q+1,\phi_i\circ\overline{\phi_j}\right)\right)d\tau^i\wedge\overline{d\tau^j}. \tag{121}$$

Lemma 71 *Pointwise on $\mathfrak{M}_L(M)$ we have*

$$\beta_q(1,1)\leq\binom{2n}{q-1}\operatorname{Im}W.P.\ . \tag{122}$$

Proof: We need the following Proposition:

Proposition 72 *Let F be a linear map of a vector space V of dimension n. Then the linear operator $F\wedge id$ as defined by* (23) *acting on $\wedge^q V$ has a trace given by the formula:*

$$Tr(F\wedge id)=\binom{n-1}{q-1}Tr(F).$$

Proof: The proof of Proposition 72 is an exercise in linear algebra. ■

From Proposition 72, the definition $Tr\left(\mathcal{F}^{'}_{\tau}\left(q,\phi_i\circ\overline{\phi_j}\right)\right)$ and the fact that

$$Tr\left(\mathcal{F}^{'}_{\tau}\left(q,\phi_i\circ\overline{\phi_j}\right)\right)+Tr\left(\mathcal{F}^{"}_{\tau}\left(q,\phi_i\circ\overline{\phi_j}\right)\right)=\int\limits_{\mathrm{M}}Tr((\phi_i\circ\overline{\phi_j})\wedge id_{q-1})vol(g)=$$

$$=\binom{2n}{q-1}\int\limits_{\mathrm{M}}Tr\left(\phi_i\circ\overline{\phi_j}\wedge id_{q-1}\right)vol(g).$$

we can conclude that pointwise on $\mathfrak{M}_L(\mathrm{M})$

$$Tr\left(\mathcal{F}^{'}\left(q,\phi_i\circ\overline{\phi_j}\right)\right)\leq Tr\left(\mathcal{F}\left(q,\phi_i\circ\overline{\phi_j}\right)\right)=\binom{2n}{q-1}\operatorname{Im}W.P.. \tag{123}$$

From (123) we deduce (122). Lemma 71 is proved. ■

According to Theorem 52 $\operatorname{Im}W.P.$ is a good form and so Lemma 71 implies that $\beta_q(1,1)$ will be good forms. From here we deduce that $dd^c\mathrm{I}(\mathrm{M}_\tau)$ will be a good form. Theorem 70 is proved. ■

Corollary 73 *Let $\overline{\mathcal{L}}$ be the extended line bundle of the relative dualizing sheaf on $\overline{\mathfrak{M}_L(M)}$ with respect to the good metric h as defined by 102. Then $dd^c I(M_\tau)$ represents the Chern class of the extended line bundle $\overline{\mathcal{L}}$. (For the fact that h is a good metric see Theorem 52.)*

Corollary 74 *The determinant line bundle $\mathcal{L}$ is trivial as a C^∞ bundle.*

Proof: We know that $\exp(I(\mathrm{M}_\tau))$ is a globally defined function on $\mathfrak{M}_L(\mathrm{M})$ which is non zero. It defines a metric on the determinant line bundle which we proved that is isomorphic to $\mathcal{L} := \pi_*\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}$. Thus $dd^c(I(\mathrm{M}))$ defines a good form by Theorem 70 and is the first Chern class of $\pi_*\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}$.Thus the first Chern class of the line bundle $\pi_*\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}$ is zero and Cor. 74 is proved. ■

Theorem 75 *There exists a global C^∞ section* $\det(\overline{\partial})$ *of the determinant line bundle*

$$\mathcal{L} \rightarrow \mathfrak{M}_L(M)$$

which has no zeroes on $\mathfrak{M}_L(M)$ and whose Quillen norm is the exponential of the Ray Singer Analytic Torsion when M is an odd dimensional.

Proof: The proof of Theorem 75 is based on the following Lemma:

Lemma 76 *There exists a nonvanishing global section* $\det(\overline{\partial})$ *of the determinant line bundle $\mathcal{L}$ such that the Quillen norm of* $\det(\overline{\partial})$ *is* $\exp(I(M))$.

Proof: From Corollary 74 we can conclude the existence of a global C^∞ section ω_τ of the line bundle

$$\mathcal{L} \rightarrow \mathfrak{M}_L(\mathrm{M})$$

which has no zeroes on $\mathfrak{M}_L(\mathrm{M})$ and which for each $\tau \in \mathfrak{M}_L(\mathrm{M})$ has L^2 norms 1, i.e. we have $\|\omega_\tau\|^2 = 1$. Since M_τ is an odd dimensional CY manifold we know from Theorem 67 that $\mathcal{L}$ is isomorphic to $\pi_*(\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})})$. The nonvanishing section ω_τ of the determinant line bundle $\mathcal{L}$ can be interpreted as a family of (2n+1,0) forms ω_τ which generate the kernel of

$$D^* : \mathcal{H}_- \rightarrow \mathcal{H}_+.$$

The kernel of

$$D : \mathcal{H}_+ \rightarrow \mathcal{H}_-$$

is generated by the constant 1. This follows directly from the definition of the CY manifold.

From Definition 61 of the section $\det(\overline{\partial}_a)$ on the open set $\mathcal{U}_a$ in $\mathfrak{M}_L(\mathrm{M})$, the existence of a global C^∞ family of antiholomorphic forms ω_τ with L^2 norm 1, which trivializes $R^{2n+1}\pi_*(\mathcal{O}_{\mathcal{X}}\ )$ over $\mathfrak{M}_L(\mathrm{M})$, and the definition of the transition functions $\{\sigma_{a,b}\}$ of $\mathcal{L}$ with respect to the covering $\{\mathcal{U}_a\}$, we deduce that for $b > a$ we have on $\mathcal{U}_a \cap \mathcal{U}_b$

$$\det(\overline{\partial}_b) = \det(\overline{\partial}_b)\,(\sigma_{a,b})\,.$$

This fact and Theorem 66 imply that we have constructed a global C^{∞} section $\det(\overline{\partial})$ of $\mathcal{L}$ whose Quillen norm is exp(I(M)). So the determinant line bundle $\mathcal{L}$ is a trivial C^{∞} line bundle. Theorem 75 is proved. ■

In [5] a canonical smooth isomorphism is constructed between the holomorphic determinant of the Grothendieck-Knudsen-Mumford and Quillen determinant bundle. More precisely, the following theorem is proved:

Theorem 77 *Let*

$$\pi : \mathcal{X} \rightarrow \mathfrak{M}_L(M)$$

be a holomorphic fibration with smooth fibres. Suppose $\mathcal{X}$ admits a closed (1,1) form ψ which restricts to a Kähler metric on each fibre. Let $\mathcal{E} \rightarrow \mathcal{X}$ be a holomorphic Hermitian bundle with its Hermitian connection. Then the determinant line bundle $\mathcal{L} \rightarrow \mathfrak{M}_L(M)$ of the relative $\overline{\partial}$ complex (coupled to $\mathcal{E}$) admits a holomorphic structure. The canonical connection (constructed in [3]) on $\mathcal{L}$ is the Hermitian connection for the Quillen metric. Finally, if the index of $\overline{\partial}$ is zero, the section $\det(\overline{\partial}_E)$ of $\mathcal{L}$ is holomorphic.

From now on we will consider the family of CY manifolds $\mathcal{X} \rightarrow \mathfrak{M}_L$(M) as defined in Remark 48 together with the trivial line bundle $\mathcal{L}$ over $\mathfrak{M}_L$(M). It is easy to see that the family $\mathcal{X} \rightarrow \mathfrak{M}_L$(M) fulfills the conditions of Theorem 77. So we get the following Corollary:

Corollary 78 *The determinant line bundle as a holomorphic bundle is flat over $\mathfrak{M}_L(M)$.*

9 The Regularized Determinant $\det \Delta_{\tau,1}$ are Bounded

9.1 Invariants of the Short Term Asymptotic Expansion of the Heat Kernel

Theorem 79 *Suppose that M is a three dimensional CY manifold and g is a CY metric. Then the coefficients a_{2k} for $k = 3, 2, 1$ and 0 in the expression (124) for the short term assymptotic expansion of $Tr(\exp(-t\Delta_{\tau,1}))$ are constants which depend only on the CY manifolds and the fixed class of cohomology of the CY metric.*

Proof: We know that the Heat kernel has the following asymptotic expansion:

$$Tr(\exp(-t\triangle_{\tau,1})) = \frac{a_{-n}(g)}{t^n} + \frac{a_{n-1}(g)}{t^{n-1}} + \frac{a_{n-2}(g)}{t^{n-2}} + ... + a_0(g) + h(t,\tau,\overline{\tau}). \quad (124)$$

(See [38].) We will apply (124) for three dimensional CY manifolds. In [18] on page 118 one can find the following formulas for $a_{-3}(g), a_{-2}(g)$ and $a_{-1}(g)$:

$$\alpha_{-3}(g) = \frac{vol(g)}{4\pi},\ a_{-2}(g) = \frac{-\int\limits_{\text{M}} k(g)vol(g)}{24\pi}$$

and

$$a_{-1}(g) = \frac{-12\left(\int\limits_{\mathrm{M}} \Delta_g(k(g))vol(g)\right) + 5\left\|Ric(g)\right\|^2 - 2\left\|R(g)\right\|^2}{1440\pi} \tag{125}$$

where $k(g)$ is the scalar curvature of the metric g, $\|Ric(g)\|^2$ is the L^2 norm of the Ricci tensor of g and $\|R(g)\|^2$ is the L^2 norm of the curvature of the metric g. Using the fact that g is a Calabi-Yau metric, i.e. $Ric(g) = k(g) = 0$, we obtain:

$$\alpha_{-3}(g) = \frac{vol(g)}{4\pi},\ a_{-2}(g) = 0 \text{ and } a_0(g) = \frac{-\left\|R(g)\right\|^2}{720\pi}. \tag{126}$$

In [11] Calabi proved on page 264 the following Proposition:

Proposition 80 *The following formula holds on a complex Kähler manifold M with a fixed cohomology class L of the imaginary part of a Kähler metric:*

$$2\left\|Ric(g)\right\|^2 - \left\|R(g)\right\|^2 - \int\limits_M k(g)^2 vol(g) = -\int\limits_M c_2(M)\wedge\omega_g^{n-2}, \tag{127}$$

where $c_2(M)$ is the second Chern class of M.

Applying formula (127) to a CY metric, we obtain that on a three dimensional CY manifold with a Calabi Yau metric g we have:

$$a_{-3}(g) = \frac{1}{4\pi}\int\limits_{\mathrm{M}} L^n,\ a_{-2}(g) = 0 \text{ and } a_{-1}(g) = -\frac{1}{720\pi}\int\limits_{\mathrm{M}} c_2(\mathrm{M})\wedge L^{n-2}. \tag{128}$$

Theorem 79 follows directly from (128) since (128) implies that $a_{-3}(g), a_{-2}(g)$ and $a_{-1}(g)$ are topological invariants. We need to prove that $a_0(g)$ is a constant in order to deduce Theorem 79.

Lemma 81 *Let*

$$Tr(\exp(-t\Delta_{\tau,1}) = \frac{a_{-n}}{t^n} + \frac{a_{-n-1}}{t} + ... + a_0 + O(t)$$

be the asymptotic expansion of $Tr(\exp(-t\Delta_{\tau,1})$ with respect to a CY metric with a fixed class of cohomology of its imaginary part. Then the coefficient a_0 is a constant, i.e.

$$\frac{\partial}{\partial\tau}a_0(\tau,\overline{\tau}) = 0.$$

Proof: According to [7] the following equality is true:

$$\zeta_{\tau,1}(0) = a_0(\tau), \tag{129}$$

where a_0 is a real valued function on the moduli space of polarized CY manifolds. Since

$$\zeta_{\tau,q}(s) = \zeta''_{\tau,q}(s) + \zeta''_{\tau,q+1}(s)$$

it will be enouph to prove

$$\frac{\partial}{\partial \tau^i}\left(\zeta^{"}_{q,\tau}(0)\right)=0 \tag{130}$$

then Lemma 81 will follow directly from (130).

From Lemma 30 we know that

$$\frac{\partial}{\partial \tau^i}\left(\zeta^{"}_{q,\tau}(s)\right)=$$

$$\frac{1}{\Gamma(s)}\int\limits_0^\infty Tr\left(\exp(-t\triangle^{"}_{\tau,q})\circ\triangle^{"}_{\tau}\circ\left(\overline{\partial}_\tau\right)^{-1}\circ F^{'}(q,\frac{\partial}{\partial\tau^i}\phi(\tau))\circ\partial_0\right)t^s dt=$$

$$\frac{1}{\Gamma(s)}\int\limits_0^\infty Tr\left(\frac{d}{dt}\left(\exp(-t\triangle^{"}_{\tau,q})\right)\circ\left(\overline{\partial}_\tau\right)^{-1}\circ F^{'}(q,\frac{\partial}{\partial\tau^i}\phi(\tau))\circ\partial_0\right)t^s dt. \tag{131}$$

By integrating by parts the expressions in (131) , we obtain:

$$\frac{\partial}{\partial \tau^i}\left(\zeta^{"}_{\tau,q}(s)\right)=$$

$$\frac{s}{\Gamma(s)}\int\limits_0^\infty Tr\left(\exp(-t\triangle^{"}_{\tau,q})\circ\left(\overline{\partial}_\tau\right)^{-1}\circ \mathcal{F}^{'}(q,\frac{\partial}{\partial\tau^i}\phi(\tau))\circ\partial_0\right)t^{s-1} dt. \tag{132}$$

We can rewrite the integral in the right hand side of (132) as follows:

$$\frac{s}{\Gamma(s)}\int\limits_0^\infty Tr\left(\exp(-t\triangle^{"}_{\tau,q})\circ\left(\overline{\partial}_\tau\right)^{-1}\circ F^{'}(q,\frac{\partial}{\partial\tau^i}\phi(\tau))\circ\partial_0\right)t^{s-1} dt=$$

$$\frac{s}{\Gamma(s)}\int\limits_0^1 Tr\left(\exp(-t\triangle^{"}_{\tau,q})\circ\left(\overline{\partial}_\tau\right)^{-1}\circ F^{'}(q,\frac{\partial}{\partial\tau^i}\phi(\tau))\circ\partial_0\right)t^{s-1} dt+$$

$$\frac{s}{\Gamma(s)}\int\limits_1^\infty Tr\left(\exp(-t\triangle^{"}_{\tau,q})T\circ\left(\overline{\partial}_\tau\right)^{-1}\circ F^{'}(q,\frac{\partial}{\partial\tau^i}\phi(\tau))\circ\partial_0\right)t^{s-1} dt. \tag{133}$$

From the short term asymptotic expansion of the heat kernel

$$Tr\left(\exp(-t\triangle^{"}_{\tau,q})T\circ\left(\overline{\partial}_\tau\right)^{-1}\circ F^{'}(q,\frac{\partial}{\partial\tau^i}\phi(\tau))\circ\partial_0\right)=$$

$$\frac{c_n(\tau)}{t^n}+...+\frac{c_1(\tau)}{t}+c_0(\tau)+\psi(t) \tag{134}$$

we deduce on the basis of Lemma **9.34** on page 300 of [7] that

$$\frac{s}{\Gamma(s)}\int\limits_0^\infty Tr\left(\exp(-t\triangle^{"}_{\tau,q})\circ\left(\overline{\partial}_\tau\right)^{-1}\circ F^{'}(q,\frac{\partial}{\partial\tau^i}\phi(\tau))\circ\partial_\tau\right)t^{s-1} dt=$$

$$\frac{s}{\Gamma(s)}\left(\int\limits_0^1\left(\frac{c_n(\tau)}{t^n}+...+\frac{c_1(\tau)}{t}+c_0(\tau)\right)t^{s-1}dt+\int\limits_1^\infty\psi(t)t^{s-1}dt\right)=$$
$$\frac{s}{\Gamma(s)}\left(\frac{c_0(\tau)}{s}+d_0(\tau)+O(s)\right), \tag{135}$$

where

$$\psi(t)=Tr\left(\exp(-t\triangle^{"}_{\tau,q})\circ\left(\overline{\partial}_\tau\right)^{-1}\circ F^{'}(q,\frac{\partial}{\partial\tau^i}\phi(\tau))\circ\partial_\tau\right)-\frac{c_n(\tau)}{t^n}+...+\frac{c_1(\tau)}{t}+c_0(\tau)$$

From (135) and the fact that

$$\frac{s}{\Gamma(s)}=s^2+O(s^3)$$

we get that

$$\frac{\partial}{\partial\tau^i}\left(\zeta^{"}_{0,\tau}(s)\right)=s^2\left(\frac{c_0(\tau)}{s}+\gamma_0(\tau)+O(s)\right)=c_0(\tau)s+s^2\gamma_0(\tau)+....$$

From the last formula we obtain that

$$\frac{\partial}{\partial\tau^i}\left(\left(\zeta^{"}_{\tau,q}\right)(s)\right)|_{s=0}=0.$$

Lemma 81 is proved. ■

Lemma 81 implies Theorem 79. ■

9.2 The Regularized Deeterminant is Bounded

Theorem 82 *For CY threefolds the regularized determinants of the Laplacians $\Delta_{\tau,0}$ of the Calabi Yau metrics $g(\tau,\overline{\tau})$ with a fixed cohomology class L are bounded as function on the moduli space, i.e. we have:*

$$0\leq\det\left(\triangle_{\tau,1}\right)\leq C.$$

Proof: The bound of $\det(\triangle_{\tau,1})$ is based on the following expression for the zeta function of the Laplacian acting on functions:

$$\zeta_1(s)=\frac{1}{\Gamma(s)}\int\limits_0^\infty Tr(\exp(-t\triangle_{\tau,1})t^{s-1}dt=b_0+b_1s+O(s^2).$$

From the definition of $\det(\triangle_{\tau,1})$ it follows that

$$\det(\triangle_{\tau,1})=\exp(-b_1(\tau)). \tag{136}$$

So if the function $b_1(\tau)$ is bounded from bellow, i.e.

$$c_1\leq b_1(\tau) \tag{137}$$

then Theorem 82 will be proved. The bound of b_1 is based on several facts.

The first fact that we will use is the following explicit formula for b_1 as stated in [1]:

$$b_1 = \gamma a_0 + \sum_{k=1}^{3} \frac{a_{2k}}{k} + \psi_1 + \psi_2, \tag{138}$$

where γ is the Euler constant, ψ_1 is given by the formula

$$\psi_1(t, \tau, \overline{\tau}) = \int_0^1 \left(Tr(\exp(-t\triangle_{\tau,1})) - \sum_{k=0}^{3} \frac{a_{2k}}{t^k} \right) \frac{dt}{t}$$

and ψ_2 by

$$\psi_2(t, \tau, \overline{\tau}) = \int_1^\infty Tr(\exp(-t\triangle_{\tau,1})) \frac{dt}{t}.$$

The second fact is Theorem 79 which implies that the expression:

$$\gamma a_0 + \sum_{k=1}^{3} \frac{a_{2k}}{k}$$

in (138) is a constant. Clearly

$$\psi_2(t, \tau, \overline{\tau}) > 0.$$

The third fact is that

$$\psi_1(t, \tau, \overline{\tau}) \geq c_0,$$

where c_0 is a constant. Combining all these facts we get (137). Combining (137) with the explicit formula (138) for the $\det(\Delta_{\tau,1})$we will obtain that

$$0 \leq \det(\Delta_{\tau,1}) \leq C < \infty.$$

So we need to prove the following Lemma:

Lemma 83 *The following inequality holds:*

$$\psi_1(t, \tau, \overline{\tau}) \geq c_0.$$

Proof: Let

$$h(t, \tau, \overline{\tau}) = Tr(\exp(-t\triangle_{\tau,1})) - \sum_{k=0}^{3} \frac{a_{2k}}{t^k} \tag{139}$$

We also know that $h(t, \tau, \overline{\tau}) = th_1(t, \tau, \overline{\tau})$. According to Theorem 79, the expression

$$\sum_{k=0}^{3} \frac{a_{2k}}{t^k}$$

is a function which does not depend on τ and $\overline{\tau}$. We also know that $Tr(\exp(-t\triangle_{\tau,0}))$ is a strictly positive function for $t>0$ which depends on t,τ and $\overline{\tau}$. These two facts imply that for each $t>0$, $\inf_{\tau} h(t,\tau,\overline{\tau})$ exists. Let $\phi(t) := \inf_{\tau} h(t,\tau,\overline{\tau})$ for fixed t. On the other hand, we know that at $t=0$,

$$h(0,\tau,\overline{\tau}) = 0.$$

So in the interval $0 \leq t \leq 1$, $\phi(t)$ is a well defined function. It is an easy exercise to prove that $\phi(t)$ is a continuous function in $[0,1]$. Let $c_0 = \min_{0\leq t\leq 1} \phi(t)$. So we have

$$\int_0^1 (h(t,\tau,\overline{\tau}) - c_0)dt \geq 0.$$

On the other hand, we have

$$\int_0^1 (h(t,\tau,\overline{\tau}) - c_0)dt = \int_0^1 h(t,\tau,\overline{\tau})dt - c_0 = \psi_1(t,\tau,\overline{\tau}) - c_0 \geq 0.$$

So $\psi_1(t,\tau,\overline{\tau}) \geq c_0$. Lemma 83 is proved. ■

Theorem 82 is proved. ■

Theorem 84 *Let M be an odd dimensional CY manifold, then*

$$\det \Delta^{'}_{\tau,1} \det \Delta^{''}_{\tau,1} = \det \Delta_{\tau,1}$$

defines a good metric in the sense of Mumford. For CY manifold M there exists a global holomorphic section σ of the dual of the extended determinant line bundle $\overline{\mathcal{L}}$ on $\overline{\mathfrak{M}_L(M)}$ such that $\|\sigma\|^2_{L^2} = \det \Delta_{\tau,1}$. If M is three dimensional then

$$\|\sigma\|^2_{L^2} = \det \Delta_{\tau,1} < C_1. \tag{140}$$

The extension $\overline{\mathcal{L}}$ of the determinant line bundle $\mathcal{L}$ on $\overline{\mathfrak{M}_L(M)}$ with respect to the good metric $\det \Delta_{\tau,1}$ *is the same as the extension to the good metric h defined by* (102)*.*

Proof: The proof of Theorem 84 is based on Theorem 37. In [43] we proved that

$$dd^c \log \langle \omega_\tau, \omega_\tau \rangle = -\operatorname{Im} W.P. \tag{141}$$

From Theorem 37 and formula (141) we deduce that on any open set $U \subset \overline{\mathfrak{M}_L(\mathrm{M})}$ we have

$$\det \Delta_{\tau,1}|_U = \left(\det \Delta^{'}_{\tau,1} \times \det \Delta^{''}_{\tau,1}\right)|_U = \langle \omega_\tau, \omega_\tau \rangle |f_U|^2, \tag{142}$$

where f_U is a holomorphic function on U. Thus from 142 we can conclude that the holomorphic functions f_U define a global section σ of the dual line bundle of the determinant line bundle on $\mathfrak{M}_L(\mathrm{M})$. Theorems 82 and 70 imply that the function $\det \Delta_{\tau,1} = \det \Delta^{'}_{\tau,1} \times \det \Delta^{"}_{\tau,1}$ is bounded on $\mathfrak{M}_L(\mathrm{M})$ and it defines a good metric. So we can conclude that the holomorphic section σ is globally defined and has a L^2 norms equal to $\det \Delta_{\tau,1}$. From Theorem 37 and formula (141) we deduce that the Chern classes of the extended line bundles obtained from the determinant line bundle, with respect to the to the good metrics $\det \Delta_{\tau,1}$ and h defined by (102) have the same Chern classes on $\overline{\mathfrak{M}_L(\mathrm{M})}$. So from here we conclude that the extension $\overline{\mathcal{L}}$ of the determinant line bundle $\mathcal{L}$ on $\overline{\mathfrak{M}_L(\mathrm{M})}$ with respect to the good metric $\det \Delta_{\tau,1}$ is the same as the extension to the good metric h. Theorem 84 is proved. ■

10 The Dedekind Eta Function for CY Manifolds

10.1 Construction of the Dedekind η Function

In this paragraph we will construct a holomorphic section η^N of some power of the dual of the determinant line bundle for any odd dimensional CY manifold. The construction of the Dedekind eta function is based on the following Theorem of Kazhdan:

Theorem 85 *For any arithmetic groups Γ of rank ≥ 2 the abelian group $\Gamma/[\Gamma,\Gamma]$ is finite. See [10].*

According to Sullivan the subgroup Γ of the mapping class group $\Gamma(\mathrm{M})$ defined in **Section 7.1** is an arithmetic group of rank ≥ 2

Theorem 86 *Let M be an odd dimensional CY manifold. Let $N = \#\Gamma_L/[\Gamma_L,\Gamma_L]$. Then $\mathcal{L}^{\otimes N}$ is a trivial complex analytic line bundle over $\mathfrak{M}_L(M)$.*

Proof: According to Theorem 67

$$\mathcal{L} \approxeq R^0\pi_*(\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}),$$

where $\dim_{\mathbb{C}} M = 2n+1$. Therefore, $\mathcal{L}$ is a subbundle of the flat vector bundle $R^{2n+1}\pi_*\mathbb{C}$, where $\mathbb{C}$ is the locally constant sheaf on $\mathcal{X}$, and

$$\pi : \mathcal{X} \to \mathfrak{M}_L(\mathrm{M})$$

is the versal family of CY manifolds over $\mathfrak{M}_L(\mathrm{M})$. We know from Theorem 47 that

$$\mathfrak{M}_L(\mathrm{M}) = \mathcal{T}(\mathrm{M})/\Gamma_L,$$

where $\mathcal{T}(\mathrm{M})$ is the Teichmüller space and Γ_L is a subgroup of finite index in the subgroup the mapping class group of M that preserve the polarization class. According to [41], Γ_L is an arithmetic group.

If we lift the flat bundle $R^n\pi_*\mathbb{C}$ on $\mathcal{T}(\mathrm{M})$, then $R^{2n+1}\pi_*\mathbb{C}$ will be the trivial bundle, i.e.

$$R^{2n+1}\pi_*\mathbb{C} \approxeq \mathcal{T}(\mathrm{M}) \times H^{2n+1}(\mathrm{M}_0, \mathbb{C}).$$

Let us denote by

$$\sigma : \mathcal{T}(\mathrm{M}) \to \mathfrak{M}_L(\mathrm{M}) = \mathcal{T}(\mathrm{M})/\Gamma_L$$

the natural projection map. Clearly $\sigma^*(\mathcal{L})$ will be a flat complex analytic subbundle of the trivial bundle $\mathcal{T}(\mathrm{M}) \times H^{2n+1}(\mathrm{M}_0, \mathbb{C})$.

Proposition 87 *Let N be a quasi-projective variety, $\mathcal{E} \approxeq \mathbb{C}^n \times N$ be a trivial bundle and $\mathcal{L}$ be a flat line bundle over N such that the dual $\mathcal{L}^*$ of $\mathcal{L}$ satisfies*

$$\mathcal{L}^* \subset \mathcal{E},$$

then $\mathcal{L}$ is also trivial.

Proof: The proof of Proposition 87 is obvious. ■

Proposition 87 implies we that $\sigma^*(\mathcal{L})$ will be a trivial line bundle. So we get that

$$\mathcal{L} \approxeq \mathbb{C} \times \mathcal{T}(\mathrm{M})/\Gamma_L,$$

where Γ_L acts in a natural way on the Teichmüller space and it acts by a character

$$\chi \in Hom(\Gamma_L, \mathbb{C}_1^*) \approxeq Hom(\Gamma_L/[\Gamma_L, \Gamma_L], \mathbb{C}_1^*)$$

of the group Γ_L on the fibre $\mathbb{C}$. For CY manifolds Γ_L is an arithmetic group of rank ≥ 2 according to [41]. From here and Theorem 85 we deduce that $\mathcal{L}^N$ will be a trivial bundle on $\mathfrak{M}_L(\mathrm{M})$, where

$$N = \#\Gamma_L/[\Gamma_L, \Gamma_L]. \tag{143}$$

Theorem 86 is proved. ■

Definition 88 *We will define $\mathfrak{D}_\infty$ as follows; Let*

$$\mathfrak{D} := \overline{\mathfrak{M}_L(M)} \ominus \mathfrak{M}_L(M)$$

be the discriminant locus then a point τ_∞ is in $\mathfrak{D}_\infty$ if around τ_∞ we can find a disk $\mathcal{D}$ such that

$$\tau_\infty \in \mathcal{D},\ \mathcal{D} \ominus \tau_\infty \subset \mathfrak{M}_L(M)$$

and over $\mathcal{D} \ominus \tau_\infty$ the family of polarized CY manifolds has a monodromy group of infinite order in $H_n(M_\tau, \mathbb{Q})$.

Theorem 89 *Let M be a three dimensional CY. There exists a holomorphic section η^N of $(\mathcal{L}^*)^{\otimes N}$ such that it can be prolonged to a holomorphic section $\overline{\eta^N}$ of the line bundle $\left(\overline{\mathcal{L}^*}\right)^{\otimes N}$ such that for each point $m \in \mathfrak{M}_L(M)$ $\eta^N(m) \neq 0$, i.e. the support of the zero set of $\overline{\eta^N}$ is contained in the support of the divisor $\mathfrak{D}$ or it is equal to it.*

Proof: Let $\mathfrak{D} = \bigcup_i D_i$ be the decomposition of the divisor $\mathfrak{D}$ on irreducible components. Theorem 86 implies that the line bundle $(\mathcal{L}^*)^{\otimes N}$ is holomorphic trivial bundle on $\mathfrak{M}_L(\mathrm{M})$. N is defined as in (143) . So we can conclude that

$$\left(\overline{\mathcal{L}^*}\right)^{\otimes N} \approxeq \mathcal{O}_{\overline{\mathfrak{M}_L(\mathrm{M})}}(\sum_j k_j D_j), \tag{144}$$

where D_j are the components of $\mathfrak{D}$. We will prove that the multiplicities k_i are non negative integers. Indeed we know from Theorem 52 that the metric defined by formula (102) on the line bundle $\mathcal{L}$ is a good one in the sense of Mumford. So the Chern form $c_1(\mathcal{L}^*,\mathrm{h})$ of the good metric h defined by (102) is a positive current on $\overline{\mathfrak{M}_L(\mathrm{M})}$. The Poincare dual of the cohomology of the current

$$[c_1(\mathcal{L}^*, \mathrm{h})] \in H^2(\overline{\mathfrak{M}_L(\mathrm{M})}, \mathbb{Z})$$

is

$$\mathcal{P}\left([c_1(\mathcal{L}^*, \mathrm{h})]\right) = \sum_j k_j[D_j] \in H_{2n-2}(\overline{\mathfrak{M}_L(\mathrm{M})}, \mathbb{Z}). \tag{145}$$

where the coefficients k_i are defined as in (144). The positivity of the current $c_1(\mathcal{L}^*, h)$ implies that its Poincare dual current

$$\sum_j k_j[D_j]$$

is positive. From here we can conclude that the coefficients k_i are positive integers. Indeed, let $[\omega_{D_i}] \in H^{2n-2}(\mathrm{M},\mathbb{Z})$ be such classes of cohomology that:

$$\int_{D_j} [\omega_{D_i}] = \delta_{ij}. \tag{146}$$

Since the current $\sum_j k_j[D_j]$ is positive (146) implies

$$\left\langle \sum_j k_j[D_j], [\omega_{D_i}] \right\rangle = k_j \geq 0. \tag{147}$$

From (147) Theorem 89 follows. ■

Remark 90 *It is not difficult to prove that in the case of odd dimensional CY manifolds the points* τ *in* $\overline{\mathfrak{M}_L(M)}$ *around which we can find one parameter family of polarized CY manifolds whose monodromy operator acting on the middle homology is of finite order form a complex analytic submanifold of codimension greater or equal to 2 and are contained in* $\mathfrak{M}_L(M)$.

Theorem 91 *Let M be a three dimensional CY manifold and let*

$$N = \#\Gamma_L/[\Gamma_L, \Gamma_L].$$

Then the holomorphic section η^N *of the line bundle* $\overline{(\mathcal{L}^*)^{\otimes N}}$ *constructed in Theorem 89 is the same up to a non zero constant as the section* σ^N *constructed in Theorem 84. The zero set of* η^N *is a non zero effective divisor whose support contains or is equal to the support of* $\mathfrak{D}_\infty$*, where* $\mathfrak{D}_\infty$ *is defined in Definition 88 and*

$$\left\|\eta^N\right\|^2_{L^2} = (\det \Delta_{\tau,1})^N.$$

.

Proof: According to Theorem 84 the extensions of $\mathcal{L}$ with respect to the good metrics $\det \Delta_{\tau,1}$ and h defined by (102) are the same on $\overline{\mathfrak{M}_L(\mathrm{M})}$. Let η^N be the holomorphic section constructed in Theorem 89. The sections σ^N and η^N of $\overline{\mathcal{L}}$ are defined on $\overline{\mathfrak{M}_L(\mathrm{M})}$ and do not vanish on $\mathfrak{M}_L(\mathrm{M})$. From here we conclude that

$$\sigma^N = \eta^N$$

after multiplying η^N by a suitable constant. Thus we can conclude that

$$\left\|\eta^N\right\|^2_{L^2} = (\det \Delta_{\tau,1})^N.$$

From here and Theorem 82 we deduce that for each point $\tau \in \mathfrak{M}_L(\mathrm{M})$ and an open set $\mathcal{U}_\tau$ such that $\tau \in \mathcal{U}_\tau$ we have

$$\det \Delta_{\tau,1} = \left\|\eta^N\right\|^2_{L^2} |_U = \langle\omega_\tau, \omega_\tau\rangle \left|f_{\mathcal{U}_\tau}(\tau)\right|^2 \leq C_1, \tag{148}$$

where $f_{\mathcal{U}_\tau}(\tau)$ is a holomorphic function in $\mathcal{U}_\tau$.

Proposition 92 *Let us choose* $\mathcal{U}$ *such that* $\mathcal{U} \cap \mathfrak{D}_\infty \neq \emptyset$. *Let*

$$\tau_\infty \in \mathfrak{D}_\infty \cap \mathcal{U}.$$

Let $f_{\mathcal{U}}(\tau)$ *be the holomorphic function defined by* (148) *in* $\mathcal{U} \ominus \mathcal{U} \cap \mathfrak{D}_\infty$. *Then* $f_{\mathcal{U}}$ *is well defined at the point*

$$\tau_\infty \in \mathfrak{D}_\infty = \overline{\mathfrak{M}_L(M)} \ominus \mathfrak{M}_L(M)$$

and $f_{\mathcal{U}}(\tau_\infty) = 0$.

Proof: Indeed, we proved in Theorem 52 that $\langle\omega_\tau, \omega_\tau\rangle$ have a logarithmic growth around $\tau_\infty \in \mathfrak{D}_\infty$. Combining this fact with (148) we conclude that

$$\lim_{\tau \to \tau_0} f_{\mathcal{U}}(\tau_\infty) = 0 \tag{149}$$

From (149) we deduce that $f_{\mathcal{U}}$ can be continued analytically around any point

$$\tau_\infty \in \mathfrak{D} = \overline{\mathfrak{M}_L(\mathrm{M})} \ominus \mathfrak{M}_L(\mathrm{M}).$$

Proposition 92 is proved. ■

Theorem 91 follows directly from Proposition 92. ■

11 Open Problems

It is natural to ask if the generic point of the discriminant locus of an odd dimensional CY manifold corresponds to a manifold with conic singularity. This is not true. Let us take double covers of $\mathbb{CP}^3$ ramified over eight planes in a general position. After the resolution of the singularities we will get a CY threefold. The discriminant locus corresponds to a double covering which is ramified over eight planes three of them meeting in one point. So the generic point of the discriminant locus does not correspond to a threefolds with conic singularities since the monodromy group around these points is finite. The monodromy group of a conic singularity is infinite.

Problem 1. Show that the moduli space of CY threefolds that are double cover of $\mathbb{CP}^3$ ramified over eight planes in a general position is a locally symmetric space associated with $\mathbb{SU}(3,3)/\mathbb{S}(\mathbb{U}(3)\times\mathbb{U}(3))$. If the moduli space of CY manifolds that are double covers of $\mathbb{CP}^n$ ramified over $2n+2$ planes is a locally symmetric space then it should be $\mathbb{SU}(n,n)/\mathbb{S}(\mathbb{U}(n)\times\mathbb{U}(n))$.

One can show that the moduli space of a CY threefold is a locally symmetric space of rank greater or equal to two if and only if the Yukawa coupling has no quantum corrections. It is a well known fact that any moduli space of CY manifolds that is one dimensional is a locally symmetric space and the famous example of Candelas and coauthors shows that there exists a CY manifolds whose moduli space is one dimensional locally symmetric space and there are quantum corrections. In the Candelas example the action of the mapping class group on the upper half plane is not arithmetic. It will be interesting to construct CY manifolds whose moduli space is one dimensional and action of the mapping class group is arithmetic on the upper half plane. I do not know how arithmeticity is related to the existence of quantum corrections to Yukawa coupling.

Problem 1 was also discussed in [8]. It was stated that I. Dolgachev conjectured that the moduli space of CY manifolds that are double covers of $\mathbb{CP}^n$ ramified over $2n+2$ planes is the tube domain $S_n(\mathbb{C})+\sqrt{-1}S_n^+(\mathbb{C})$, where $S_n(\mathbb{C})$ is the space of $n\times n$ Hermitian matrices and $S_n^+(\mathbb{C})$ is the space of positive Hermitian matrices.

The basis of proposing Problem 1 is the following Lemma:

Lemma 93 *Let $\mathbb{C}^{2n}$ be equipped with a Hermitian metric $\langle u,u\rangle$ with signature (n,n). Let*

$$\mathbb{C}^{2n}=V\oplus\overline{V},$$

where $\langle u,u\rangle$ when restricted to V is positive and on $\overline{V}$ is negative. Then $\wedge^n(V\oplus\overline{V})$ is a variation of Hodge Structures of weight n with $\dim_{\mathbb{C}}H^{n,0}=1$. *This variation of Hodge structures is parametrized by* $\mathbb{SU}(n,n)/\mathbb{S}(\mathbb{U}(n)\times\mathbb{U}(n))$.

Problem 2. Suppose that M is a CY manifold whose moduli space is not a locally symmetric space. Is it true in this case that the generic point of the discriminant locus corresponds to a CY manifold with a conic singularity? Characterize all CY three folds whose moduli spaces are locally symmetric spaces.

B. Gross in [21] classified all the symmetric domains that are also tube domains and over them one can cocntsruct a variations of Hodge structure of weight three with $\dim H^{3,0} = 1$. It is an open problem posed by B. Gross can one find a geometric realization of the variations of the Hodge structure described in [21]?

Problem 2 is closely related to Miles Ried's conjecture that the moduli spaces of all CY threefolds are connected. So one can ask the following question:

Problem 3. Is it true that a CY threefold such that its moduli space is a locally symmetric space and the moduli space is contained in the discriminant locus of the moduli space a CY manifold and the generic point of the discriminant locus corresponds to a manifold with a conic singularity?

Let $\tau \in \mathfrak{M}_L(\mathrm{M})$. Then we know that τ correponds to a CY threefold M_τ. Let us denote by ω_τ a non zero holomorphic three form on M_τ. Let $\beta \in H_3(\mathrm{M},\mathbb{Z})$, then we will denote by

$$\langle \tau, \beta \rangle := \int_\beta \omega_\tau .$$

Problem 4. Can one find a product formula for the analogue of the Dedekind eta function of CY threefolds

$$\eta^N = \exp\left(2\pi\sqrt{-1}\,\langle \gamma, \tau \rangle\right) \times \prod_i \left(1 - \exp 2\pi\sqrt{-1}\,\langle \tau, \beta_i \rangle\right),$$

around points of maximal degenerations, which would mean that around such points the monodromy operator has an index of unipotency $n+1$, β_i are the vanishing invariant cycles of the monodromy operators of infinite order and $\tau = (\tau^1, ..., \tau^N)$ are the flat local coordinates? For a discussion of the product formulas for automorphic forms see [9].

Problem 4 is closely related to paper [2] and more precisely to the counting problem of elliptic curves on the CY threefold. For discussion of the

Problem 5. Prove that $\det \Delta_{\tau,1}$ is bounded on the moduli space $\mathfrak{M}_L(\mathrm{M})$ of any CY manifold M.

This problem will follow directly if one can prove that the coefficients a_k for $k = -n, ..., 1$ of the short term asymptotic expansion

$$Tr\left(\exp\left(-t\Delta_{\tau,1}\right)\right) = \sum_{k=-n}^{1} \frac{a_k}{t^k} + a_0 + ...$$

are constants. We prove that a_0 is a constant if M is a CY. The solution of Problem 4 will show that the analogue of the Dedekind eta function η^N vanishes on the discriminant locus. This will imply that the section $\overline{\eta}^N$ constructed in this paper will be related to the algebraic discriminant as defined by Gelfand, Kapranov and Zelevinsky.

References

[1] D. Abramovich, J. -F. Burnol, J. Kramer and C. Soulé, *"Lectures on Arakelov Geometry"*, Cambridge Studies In Advanced Mathematics Volume **33**, Cambridge University Press, 1992.

[2] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *"Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitude"*, Comm. Math. Phys. **165** (1994), 311-428.

[3] J.-M. Bismut and D. Freed, *"The Analysis of Elliptic Families I. Metrics and Connections on Determinant Line Bundles",* Comm. Math. Phys. **106,** 159-176(1986).

[4] E. Bishop, *"Conditions for the Analyticity of Certain Sets."* Mich. Math. J. **11**(1964) 289-304.

[5] J.-M. Bismut, H. Gillet and C. Soulè, *"Analytic Torsion and Holomorphic Determinant Bundles I. Bott-Chern Forms and Analytic Torsion",* Comm. Math. Phys.**115,** 49-78(1988).

[6] J.-M. Bismut, H. Gillet and C. Soulè, *"Analytic Torsion and Holomorphic Determinant Bundles II. Direct Images and Bott-Chern Forms ",* Comm. Math. Phys.**115,** 79-126(1988).

[7] N. Berline, E. Getzler and M. Vergne, *"Heat Kernels and Dirac Operators",* Springer Verlag, 1991.

[8] R. Borcherds, *"The Moduli of Enriques Surfaces and the Fake Monster Lie Superalgebra",* Topology **35,** (1996), 699-710.

[9] R. Borcherds, *"Automorphic Forms with Singularities on Grassmanians",* Invent. Math. **132** (1998), No 3, 491-562.

[10] A. A. Kirillov and Cl. Delaroche, *"Sur les Relations entre l'espace dual d'un group et la struture de ses sous-groupes fermés" (d'après D. A. Kazhdan),* Bourbaki No. 343, 1967/68.

[11] E. Calabi, *"Extremal Kähler Metrics",* Seminar on Differential Geometry, ed. S.-T. Yau, Annals of Mathematical Studies, vol. **102**, Princeton University Press.

[12] S. Donaldson, *"Infinite Determinants, Stable Bundles and Curvature",* Duke Mathematical Journal, vol. **54, Number 1**(1987), 231-248.

[13] J.-P. Demailly, *"L^2 Vanishing Theorems for Positive Line Bundles and Adjunction Theory",* Lecture Notes in Mathematics **vol.** 1646, p. 1-97.

[14] S. Donaldson and P. Kronheimer, *"The Geometry of Four Manifolds",* Oxford Mat. Monog., Oxford Science Publications, Oxford University Press, New York 1990.

[15] A. Grothemdieck, "*Éléments de Géométrie Algébrique",* IHES, Publications Mathématiques, **vol.** 8.

[16] H. Fang and Z. Lu, "*Generalized Hodge Metric and BCOV Torsion on Calabi-Yau Moduli", **DG/0310007.***

[17] D. Freed, "*On Determinant Line Bundles",* Mathematical Aspects of String Theory, ed. S.-T. Yau, World Scientific, Singapore, New Jersey, Hong Kong, p. 189-238(1998).

[18] P. Gilkey, "*Invariance Theory, The Heat Equation, And the Atiyah-Singer Index Theorem",* Mathematics Lecture Series vol. **11**, Publish or Parish, Inc. Wilmington, Delaware (USA) 1984.

[19] Ph. Griffiths, *"Periods of Integrals on Algebraic Manifolds" I and II,* Amer. Jour. Math. **90** (1968), 568-626 and 805-865.

[20] Ph. Griffiths and J. Harris, *"Principles of Algebraic Geometry".*

[21] B. Gross, *"A Remark on Tube Domains",* Math. Research Letters **1,** 1-9(1994).

[22] J. Harvey and G. Moore, "*Algebras, BPS States, and Strings",* Nucl. Phys. **B 463**(1996), 315-368, hep-th/9510182 .

[23] J. Harvey and G. Moore, "*On the Algebras of BPS States",* hep-th/9609017.

[24] J. Jorgenson and A. Todorov, "*A Conjectural Analogue of Dedekind Eta Function for K3 Surfaces",* Math. Research Lett. **2**(1995) 359-360.

[25] J. Jorgenson and A. Todorov, "*Analytic Discriminant for Manifolds with Zero Canonical Class",* Manifolds and Geometry, ed. P. de Bartolomeis, F. Tircerri and E. Vesantini, Symposia Mathematica **36,** (1996) 223-260.

[26] J. Jorgenson and A. Todorov, "*Ample Divisors, Automorphic Forms and Shafarevich's Conjecture"*, Mirror Symmetry IV, ed. S-T. Yau and Phong, AMS.

[27] J. Jorgenson and A. Todorov, "*Analytic Discriminant for Polarized Algebraic K3 Surfaces",* Mirror Symmetry III, ed. S-T. Yau and Phong, AMS, p. 211-261.

[28] J. Jorgenson and A. Todorov*, "Enriques Surfaces, Analytic Discriminants, and Borcherd's Φ Function",* Com. Math. Phys. **191,** (1998), 249-264.

[29] J. Jorgenson and A. Todorov *"Correction to "Enriques Surfaces Analytic Discriminants, and Borcherd's* Φ Function", Com. Math. Phys. (1999).

[30] K. Kodaira and Morrow, "*Complex Manifolds".*

[31] K. Liu, A. Todorov, Shing-Tung Yau and K. Zuo, "*The Analogue of Shafarevich Conjecture for CY Manifolds I.",* preprint.

[32] Z. Lu, "*On the Curvature Thensor of the Hodge Metric of Moduli of Polarized Calabi-Yau Threefolds",* J. Geom. Analysis, **11**(2001), No 4, 633-645.

[33] Z. Lu and X. Sun, "*Weil-Petersson Geometry on Moduli space of Polarized Calabi-Yau Manifolds."* preprint.

[34] T. Matsusaka and D. Mumford, "*Two Fundamental Theorems on Deformations of Polarized Varieties",* Amer. J. Math. **86**(1964).

[35] D. Mumford, "*Hirzebruch's Proportionality Principle in the Non-Compact Case",* Inv. Math. **42**(1977), 239-272.

[36] R. Palais, "*On the Existance of Slices for Actions of Non-Compact Lie Groups",* Ann. Math. (2) **73** (1961), 295-323.

[37] M. S. Raghunathan, "*Discrete Subgroups of Lie Groups",* Ergebnisse Der Mathematik und Ihrer Grenzgebiete, Band **68,** Springer-Verlag 1972.

[38] J.Roe, "*Elliptic Operators, Topology and Asymptotic Methods"* Pitman Research Notes in Mathematics Series **179,** Longman Scientific & Technical, 1988.

[39] D. Ray and I. Singer, "*Analytic Torsion for Complex Manifolds",* Ann. Math. **98** (1973) 154-177.

[40] A. Grothendieck, *"Séminaire de Géométrie Algébrique",* IHES, Paris, 1960-1961.

[41] D. Sullivan, "*Infinitesimal Computations in Topology", Publ. Math. IHES, No **47** (1977), 269-331.*

[42] G. Tian, "*Smoothness of the Universal Deformation Space of Calabi-Yau Manifolds and its Petersson-Weil Metric",* Math. Aspects of String Theory, ed. S.-T.Yau, World Scientific (1998), 629-346.

[43] A. Todorov, *"The Weil-Petersson Geometry of Moduli Spaces of SU(n≥ 3) (Calabi-Yau Manifolds) I",* Comm. Math. Phys. **126** (1989), 325-346.

[44] A. Todorov, *"Witten's Geometric Quantization of Moduli of CY manifolds",* preprint 1999.

[45] E. Viehweg, *"Quasi-Projective Moduli for Polarized Manifolds",* Ergebnisse der Mathematik und iher Grenzgebiete 3. Folge, Band **30**, Springer-Ver;ag, 1991.

[46] Yoshikawa, "*Generalized Enriques Surfaces and Analytic Torsion",* preprint (1998).

[47] S. T. Yau, "*On the Ricci Curvature of Compact Kähler Manifolds and Complex Monge-Amper Equation I*", Comm. Pure and App. Math. **31,** 339-411(1979).