TRANSONIC SHOCK IN A NOZZLE I, 2-D CASE

Zhouping Xin

(Department of Mathematics and The IMS, CUHK, Shatin, N.T., Hong Kong)

Huicheng Yin

(Department of Mathematics, Nanjing University, Nanjing 210093, P.R.China)

(The Institute of Mathematical Sciences, CUHK, Shatin, N.T., Hong Kong)

Abstract

In this paper we establish the existence and uniqueness of a transonic shock for the steady flow through a general two-dimensional nozzle with variable sections. The flow is governed by the inviscid potential equation, and is supersonic upstream, has no-flow boundary conditions on the nozzle walls, and a given pressure at the exit of the exhaust section. The transonic shock is a free boundary dividing two regions of $C^{1,1-\delta_0}$ flow in the nozzle. The potential equation is hyperbolic upstream where the flow is supersonic, and elliptic in the downstream subsonic region. In particular, our results show that there exists a solution to the corresponding free boundary problem such that the equation is always subsonic in the downstream region of the nozzle when the pressure in the exit of the exhaustion section is appropriately larger than that in the entry. This confirms exactly the conjecture of Courant and Friedriches on the transonic phenomena in a nozzle [10]. Furthermore, the stability of the transonic shock is also proved when the upstream supersonic flow is a small steady perturbation for the uniform supersonic flow or the pressure at the exit has a small perturbation. The main ingredients of our analysis are a generalized hodograph transformation and multiplier methods for elliptic equation with mixed boundary conditions and corner singularities.

Keywords: Steady potential equation, transonic flow, shock wave, nozzle

Mathematical Subject Classification: 35L70, 35L65, 35L67, 76N15

§1. Introduction and the main results

In this paper we study the problem of the existence and uniqueness of a solution with a transonic shock to the steady flow through a general 2-dimensional nozzle with variable sections. Phenomena involving transonic flows and transonic flows with shocks is a fundamental subject in fluid dynamics, especially in gas dynamics, and has been studied extensively in the literature ([10], [4], [24-25], [12], [15],

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[23], [27-30] and the references therein). Profound understanding has been achieved both physically and mathematically by Morawetz [27, 30] and others ([10], [4]) on smooth transonic flows. While for transonic flows with shocks, most of previous studies involve either experimental and numerically simulations or analysis of special wave patterns ([10], [4], [16], [11]), except the rigorous results on the existence and stability of the quasi-one dimensional transonic shocks (see [23] and [15]). Recently, some important wave patterns involving truly multi-space-dimensional transonic shock wave have been established for various models and geometries (see [7], [6], [5], [31], [33] and the references therein). In this paper, we will study the problem on the existence and uniqueness of a transonic shock to the steady flow through a general 2-dimensional nozzle with variable sections. As conjectured by Courant-Friedrichs in [10], the following structure of transonic flows with shocks in a nozzle is expected: Given the appropriately large receiver pressure p_r, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle a shock front intervenes and the gas is compressed and slowed down to subsonic speed. The position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes p_r. One of the major goals of this paper is to establish the existence and structural stability of the above-mentioned structure of the flow field in two-dimensional nozzles with slowly-varying sections.

The Euler system for steady compressible flow in two dimensional spaces is

$$\begin{cases} \partial_1(\rho u_1) + \partial_2(\rho u_2) = 0 \\ \partial_1(P + \rho u_1^2) + \partial_2(\rho u_1 u_2) = 0 \\ \partial_1(\rho u_1 u_2) + \partial_2(P + \rho u_2^2) = 0 \\ \partial_1((\rho e + \frac{1}{2}\rho|u|^2)u_1) + \partial_2((\rho e + \frac{1}{2}\rho|u|^2)u_2) = 0 \end{cases} \tag{1.1}$$

where $u = (u_1, u_2)$, P, ρ and e represent the velocity, pressure, density and the inner energy respectively. Moreover, $P = P(\rho, e)$ is a smooth function of ρ, e and $c^2(\rho, e) = \partial_\rho P(\rho, e) > 0$ for $\rho > 0$.

Suppose that there is a uniform supersonic flow $(u_1, u_2) = (q_0, 0)$ with constant density $\rho_0 > 0$ which comes from negative infinity, and the flow enters the $2-D$ nozzle from the entrance. In this paper, we always assume that the two nozzle walls are a small perturbation of two straight line segments $x_2 = -1$ and $x_2 = 1$ with $-1 \le x_1 \le 1$ respectively. Therefore, the flow in the nozzle can be approximately considered to be irrotational. Besides, for weak shocks, the changes in the entropy is of third order and so may be ignored [10]. Thus one may also assume the flow to be isentropic. In this case, one can introduce the velocity potential to simplify the system (1.1) and obtain a second order quasilinear equation which is called the potential equation. This equation and its variants have been extensively studied in the literature, and are regarded as good physical models which are susceptible of rigorous analysis (see [4], [10], [6], [30], [9]). Various interesting wave-phenomena involving this model had been pointed out and investigated by Courant-Friedriches in [9] and numerous rigorous mathematical rigorous theory (mainly on 2-dimensional smooth flows) had been surveyed by Bers in [4]. Particularly, Morawetz has used the potential equation and its variants to treat many important problems in transonic flows, see [27, 28, 29, 30] and the references therein.

Let $\varphi(x)$ be the potential of velocity, i.e. $(\partial_1\varphi, \partial_2\varphi) = (u_1, u_2)$, then Bernoulli's law implies that

$$\frac{1}{2}|\nabla\varphi|^2 + h(\rho) \equiv C_0 = \frac{1}{2}q_0^2 + h(\rho_0) \tag{1.2}$$

where $h(\rho)$ is the specific enthalpy. For the given state equation $P = P(\rho)$ with $P'(\rho) = c^2(\rho) > 0$ and $P''(\rho) \geq 0$ for $\rho > 0$, then $h'(\rho) = \frac{c^2(\rho)}{\rho}$.

Since $h'(\rho) > 0$, one then can define the inverse function of $h(\rho)$ as $H(s)$, namely,

$$\rho = H(C_0 - \frac{1}{2}|\nabla\varphi|^2) \tag{1.3}$$

Substituting (1.3) into the first equation of (1.1) leads to

$$\partial_1(\partial_1\varphi H) + \partial_2(\partial_2\varphi H) = 0 \tag{1.4}$$

The equation (1.4) can be rewritten as

$$((\partial_1\varphi)^2 - c^2)\partial_1^2\varphi + 2\partial_1\varphi\partial_2\varphi\partial_{12}^2\varphi + ((\partial_2\varphi)^2 - c^2)\partial_2^2\varphi = 0 \tag{1.5}$$

It is easy to verify that (1.5) is strictly hyperbolic for $|\nabla\varphi| > c(\rho)$ and uniformly elliptic for $|\nabla\varphi| < c(\rho)$.

Suppose that the two walls of the nozzle are given respectively by

$$x_2 = f_2(x_1) \qquad \textit{and} \qquad x_2 = f_1(x_1) \tag{1.6}$$

satisfying

$$|\frac{d^k}{dx_1^k}(f_2(x_1) - 1)| \leq \varepsilon, \qquad |\frac{d^k}{dx_1^k}(f_1(x_1) + 1)| \leq \varepsilon, \qquad \textit{for} \qquad -1 \leq x_1 \leq 1, k \leq 4, k \in \mathbb{N} \cup \{0\} \tag{1.7}$$

where ε is a suitable positive constant.

Without loss of generality and for the simplicity of presentation, it will be assumed that

$$f_1(-1) = f_1(1) = -1, \qquad f_2(-1) = f_2(1) = 1, \qquad f_i^{(k)}(-1) = 0, \qquad \textit{for} \qquad i = 1, 2; \qquad 1 \leq k \leq 4 \tag{1.8}$$

When the uniform supersonic flow $(q_0, 0)$ enters the entry of the nozzle, the potential $\varphi_-(x)$ in the nozzle will be determined by the following equation and the boundary conditions

$$\begin{cases} ((\partial_1\varphi_-)^2 - c_-^2)\partial_1^2\varphi_- + 2\partial_1\varphi_-\partial_2\varphi_-\partial_{12}^2\varphi_- + ((\partial_2\varphi_-)^2 - c_-^2)\partial_2^2\varphi_- = 0 \\ \varphi_-|_{x_1=-1} = -q_0 \\ \partial_1\varphi_-|_{x_1=-1} = q_0 \\ \partial_2\varphi_- = f_i'(x_1)\partial_1\varphi_-, \qquad \textit{on} \qquad x_2 = f_i(x_1), \qquad i = 1, 2 \end{cases} \tag{1.9}$$

where $c_- = c(H_-)$ and $H_- = H(C_0 - \frac{1}{2}|\nabla\varphi_-|^2)$.

It will follow from Lemma 2.1 in §2 that (1.9) has a C^4 solution $\varphi_-(x)$ in the nozzle $\{(x_1, x_2) : -1 \leq x_1 \leq 1, f_1(x_1) \leq x_2 \leq f_2(x_1)\}$, moreover $|\nabla_x^\alpha(\varphi_-(x) - q_0x_1)| \leq C\varepsilon$ holds for $|\alpha| \leq 4$ with a uniform positive constant C.

Let an appropriate pressure $P_+ = P(\rho_+)$ be prescribed at the exit (right end) of the nozzle such that the density ρ_+ and the velocity $|\nabla\varphi| = q_+$ on $x_1 = 1$ satisfy the following relations

$$\frac{1}{2}q_+^2 + h(\rho_+) = C_0, \qquad \rho_+ q_+ = \rho_0 q_0, \qquad \textit{and} \qquad q_+ < c(\rho_+) \tag{1.10}$$

As stated in the first paragraph of the paper, it is expected that there appears a transonic shock $x_1 = \xi(x_2)$ in the nozzle. To assure the uniqueness, we also require that the shock $x_1 = \xi(x_2)$ goes through a specified point, say, $(0,0)$, namely,

$$\xi(0) = 0 \tag{1.11}$$

Indeed, the position of the shock can be roughly determined up to a shift by the mass conservation law (for details, see [10]). Under the assumptions (1.10) and (1.11), it will be shown that the transonic steady flow exists uniquely. As indicated in [10] (pages 372), it is a question of great importance to know under what circumstances a steady flow involving shocks is uniquely determined by the boundary conditions and by the conditions at the entrance, and when further conditions at the exit are appropriate.

Denote by $\varphi_+(x)$ the potential across the shock $x_1 = \xi(x_2)$. By the properties of the shock for the second order equations [4,10], one may require that the potential $\varphi(x)$ is continuous across the shock front, i.e.,

$$\varphi_+(x) = \varphi_-(x), \qquad \textit{on} \qquad x_1 = \xi(x_2) \tag{1.12}$$

and the derivative of $\varphi_+(x)$ must satisfy the Rankine-Hugoniot condition

$$[\partial_1\varphi_+ H] - \xi'(x_2)[\partial_2\varphi_+ H] = 0, \qquad \textit{on} \qquad x_1 = \xi(x_2) \tag{1.13}$$

In addition, φ_+ should satisfy the physical entropy condition(see [9]):

$$H(C_0 - \frac{1}{2}|\nabla\varphi_-|^2) < H(C_0 - \frac{1}{2}|\nabla\varphi_+|^2), \qquad \textit{on} \qquad x_1 = \xi(x_2) \tag{1.14}$$

The boundary condition at the exit of the nozzle is

$$H(C_0 - \frac{1}{2}|\nabla\varphi_+|^2) = \rho_+, \qquad \textit{on} \qquad x_1 = 1 \tag{1.15}$$

Finally, the velocity of the flow is tangent to the nozzle walls so that

$$\partial_2\varphi_+ = f_i'(x_1)\partial_1\varphi_+, \qquad \textit{on} \qquad x_2 = f_i(x_1), \qquad i = 1,2 \tag{1.16}$$

The main purpose of this paper is to establish the existence and the uniqueness of the solution to the equation (1.5) with the boundary conditions (1.11)-(1.16). Our main result is

Theorem 1.1. Assume that (1.7), (1.8) and (1.10) hold. Then for suitably small $\varepsilon > 0$, there exists a unique pair $(\varphi(x), \xi(x_2))$ with the property that $\varphi(x)$ is piecewise smooth and

$$\varphi(x) = \begin{cases} \varphi_-(x), & for \quad x_1 < \xi(x_2) \\ \varphi_+(x), & for \quad x_1 > \xi(x_2) \end{cases}$$

such that $(\varphi(x), \xi(x_2))$ solves the problems (1.5), (1.9) and (1.11) - (1.16).

Furthermore, for a fixed constant $\delta_0 \in (0, \frac{1}{8})$, there exists a constant C independent of ε with the following properties

(i). (Regularity of the supersonic flow) $\varphi_-(x) \in C^4(\bar{\Omega})$ is a solution of the equation (1.9), here $\Omega = \{(x_1, x_2) : -1 < x_1 < 1, f_1(x_1) < x_2 < f_2(x_1)\}$. Moreover

$$\|\varphi_-(x) - q_0 x_1\|_{C^4(\bar{\Omega})} \leq C\varepsilon$$

(ii). (Regularity of the shock surface) Denote by $\tilde{P}_i = (x_1^i, x_2^i)(i = 1, 2)$ the intersection point of $x_1 = \xi(x_2)$ with $x_2 = f_i(x_1)$ and by $|d_{x_2}| = min\{dist(x, \tilde{P}_1), dist(x, \tilde{P}_2)\}$ with $x = (\xi(x_2), x_2)$. Then $\xi(x_2) \in C^{1,1-\delta_0}[x_2^1, x_2^2] \cap C^3(x_2^1, x_2^2)$, and

$$\|\xi(x_2)\|_{C^{1,1-\delta_0}} \leq C\varepsilon, \qquad |\frac{d^k \xi(x_2)}{dx_2^k}| \leq \frac{C\varepsilon}{|d_{x_2}|^{k-2+\delta_0}} \qquad for \qquad k = 2, 3; \qquad x_2 \in (x_2^1, x_2^2)$$

(iii). (Regularity of the subsonic flow) Denote by $\Omega_+ = \{(x_1, x_2) : \xi(x_2) < x_1 < 1, f_1(x_1) < x_2 < f_2(x_1)\}$. For $x \in \Omega_+$, write $|d_x| = \min_{1 \leq i \leq 4} \{dist(x, \tilde{P}_i)\}$ with $\tilde{P}_3 = (1, 1)$ and $\tilde{P}_4 = (1, -1)$. Then $\varphi_+(x) \in C^{1,1-\delta_0}(\bar{\Omega}_+) \cap C^3(\bar{\Omega}_+ \setminus \cup_{i=1}^4 \tilde{P}_i)$ and satisfies

$$\|\varphi_+(x) - q_+ x_1\|_{C^{1,1-\delta_0}} \leq C\varepsilon, \qquad |\nabla_x^k(\varphi_+(x) - q_+ x_1)| \leq \frac{C\varepsilon}{|d_x|^{k-2+\delta_0}} \qquad for \qquad k = 2, 3; \qquad x \in \Omega_+$$

(iv). The physical entropy condition (1.14) is satisfied on $x_1 = \xi(x_2)$.

Remark 1.1. It should be noted that the transonic shock in the theorem is perpendicular to the walls of the nozzle. This fact follows easily from the boundary conditions (1.12), (1.16), and the ones on the nozzle walls for $\varphi_-(x)$ in (1.9).

Remark 1.2. Following the proof of Theorem 1.1, one can also obtain the stability of the solution $\varphi(x)$ and the shock $x_1 = \xi(x_2)$ with respect to small perturbations of the initial state at the entrance or the pressure at the exit of the nozzle in the sense that if

$|\frac{d^k}{dx_2^k}(\hat{\varphi}_-(-1, x_2) + q_0)| \leq \varepsilon$ and $|\frac{d^k}{dx_2^k}(\partial_1 \hat{\varphi}_-(-1, x_2) - q_0)| \leq \varepsilon$ or $|\frac{d^k}{dx_2^k}(\hat{\rho}_+(1, x_2) - \rho_+)| \leq \varepsilon$ hold for $0 \leq k \leq 4$, then the corresponding solution pair $(\hat{\varphi}_+(x), \hat{\xi}(x_2))$ still satisfies

$$\|\hat{\varphi}_+(x) - q_+ x_1\|_{C^{1,1-\delta_0}(\overline{\hat{\Omega}_+})} \leq C\varepsilon, \qquad \|\hat{\xi}(x)\|_{C^{1,1-\delta_0}[\hat{x_2^1}, \hat{x_2^2}]} \leq C\varepsilon$$

where $\hat{\Omega}_+ = \{(x_1, x_2) : \hat{\xi}(x_2) < x_1 < 1, f_1(x_1) < x_2 < f_2(x_1)\}$, $(\hat{x_1^i}, \hat{x_2^i})(i = 1, 2)$ are the intersection points of $x_1 = \hat{\xi}(x_2)$ with $x_2 = f_i(x_1)$, and the constant C is independent of ε.

Remark 1.3. The condition (1.10) is needed to fix the shift of the shock location as shown in the case of a nozzle with constant sections where the location of the shock is not uniquely determined by (1.5), (1.9), (1.12) - (1.15).

Remark 1.4. It should be noted that the main assumption in theorem 1.1 is that the walls of the nozzle vary slowly, i.e., ε is suitably small. This is in general necessary for the existence of such a transonic shock wave pattern as described in theorem 1.1. Since for general nozzle, there might be supersonic shocks in the supersonic region, and furthermore, there may be supersonic bubbles surrounded by subsonic flow, see [4, 10].

Remark. 1.5. The method in this paper can be used to show the stability of a transonic shock for the uniform supersonic flow past a curved sharp wedge when the appropriately large downstream pressure is given. As described in [10] (pages 317-318), when a uniform supersonic flow (P_0, ρ_0, q_0), which comes from negative infinity, hits a sharp wedge with the angle θ_K, and if θ_K is less than a critical value determined by P_0, ρ_0 and q_0, then two oblique shock fronts are possible through which the flow turned through the angle θ_K, a weak and a strong one. The questions arise which of the two actually occurs. It has frequently been stated that the strong one is unstable and that, therefore, only the weak one could occur. A convincing proof of this instability has apparently never been given. Quite aside from the question of stability, the problem of determining which of the possible shocks occurs cannot be formulated and answered without taking the boundary conditions at infinity into account. The flow may be considered as the limiting case of the flow in a duct as the duct becomes infinitely wide and the inclined section infinitely long. If the pressure prescribed there is below an appropriate limit, the weak shock occurs in the corner. If, however, the pressure at the downstream end is sufficiently high, a strong shock may be needed for adjustment. Under appropriate circumstances the strong shock may begin just in the corner. Indeed, by applying the ideas and method developed here, one can easily treat this problem to prove the existence and structural stability of a steady flow pattern with a transonic shock through the vertex of the wedge. Since the analysis is similar to the nozzle problem, we will not present the details.

Remark 1.6. Many of the ideas developed here are also useful for the treatment of the problems of transonic shocks in a multi-dimensional nozzle with variable sections. This will be reported in a forthcoming publication [32].

We now comment on the proof of Theorem 1.1. One of the main difficulties is the treatment of the free boundary - the transonic shock. A typical method is to perform a transformation to fix the unknown shock. One useful approach to fix a shock is the partial hodograph transformation introduced in [26] and [24]. The partial hodograph transformation corresponding to our problem is to take the unknown function $\psi(x) = \varphi_-(x) - \varphi_+(x)$ as the X_1-variable in the new coordinates $X = (X_1, X_2)$. In this case, the shock $x_1 = \xi(x_2)$ becomes the fixed boundary $X_1 = 0$. However, the fixed boundary $x_1 = 1$ will become unknown. To overcome this difficulty, as in [8] and [9], we need to introduce a generalized hodograph transformation which will fix the shock while at the same time change the fixed boundary $x_1 = 1$ into a new fixed boundary $X_1 = 1$. In addition, to avoid the undesirable effects due to the possible appearance of negative eigenvalues for the linearization of the nonlinear boundary value problem of the second order nonlinear elliptic equation derived from (1.5) - (1.16) by the partial hodograph transformation, one needs to have a better choice of the partial hodograph transformation than that used in [8, 9]. This will be given explicitly in §2. With this transformation, the quasilinear equation (1.5), whose coefficients contain only the first order derivatives of $\varphi(x)$, will become a new second order nonlinear equation with its coefficients and source term containing the unknown function $V(X)$ and its first order derivatives $\nabla_X V(X)$. Correspondingly, the boundary conditions (1.13), (1.14) and (1.16) are also transformed into the new nonlinear boundary conditions which contain $V(X)$ and $\nabla_X V(X)$. Thanks to the appropriate choice of the hodograph transformation, we can avoid discussing the eigenvalue problems of the resulting linearized problem since the coefficients of $V(X)$ and $\nabla_X V(X)$ in the second order elliptic equation and the coefficients of $V(X)$ in the boundary conditions are all suitably small in appropriately weighted spaces. This fact plays a key role in all our analysis.

To solve the new nonlinear elliptic equations with the corresponding nonlinear mixed boundary conditions, we intend to use the Schauder fixed point theorem to show the existence of the solution. It is here that one has to deal with another main difficulty, the corner singularities. Indeed, since the domain under consideration has four corners, then one cannot expect that the solution has the $C^{2,\alpha}$ or even C^2 regularity up to the boundaries (for some fixed $\alpha > 0$). It should be noted that the issue of singularities due to the corners at the boundaries is more pronounced here for the potential equation in that the coefficients of the principal terms in (1.5) depend on both the unknown and its first order derivatives. We will employ some weighted Hölder spaces to deal with the corner singularities as motivated by the works in [13,19-21]. In order to apply the Schauder theorem, one needs an L^∞ estimate on the solution of the corresponding linear problem. This is achieved by getting the a priori H^2 estimate by looking for suitable multipliers. It seems difficult to use the maximum principle to obtain the L^∞ estimate as in [20] since our linear equation and the boundary conditions do not satisfy the requirements in Lemma 1.1 or Corollary 2.4 in [20]. After we establish the existence of the solution, the uniqueness can be proved by the energy estimates based on new choices of multipliers.

Next we would like to discuss some of the recent notable studies on multi-dimensional transonic shocks for various models (see [2], [5-7], [31,34] and the references therein). In particular, we comment on two recent interesting works which are closely related to this paper. The first one is the study of the two-dimensional transonic small disturbance equation (TSDE) by Canic-Keyfitz-Lieberman in [6]. The TSDE can be derived by taking the first term in the geometric optics expansion to (1.5) near a certain physical boundary, and can be formulated into a second order quasilinear equation of mixed type with coefficients depending only on the unknown function itself. In [6], the authors establish the existence and the stability of a uniform planar transonic shock. As already pointed in [6], the property that the coefficients of the TSDE are independent of the gradient of the unknown function plays a key role in the analysis of [6]. However, as commented in [6], it seems difficult to transform the potential equation (1.5) into such a from since the coefficients of (1.5) depend on the gradient of the unknown function. Thus we need a different approach from that in [6] to treat the transonic shock problem for (1.5).

The second one is the study of the multidimensional transonic shocks for the nonlinear potential equation of mixed type in [7], where Chen and Feldman prove the existence and stability of a steady multidimensional transonic shock when the flow in the flat channel $\tilde{\Omega} = (0,1)^{n-1} \times (-1,1)$ with a Dirichlet boundary condition for the potential at the exit of the channel, which gave an impetus for us to investigate the transonic shock flow in a general nozzle. Note that the boundary of $\tilde{\Omega}$ in [6] is straight, and so the domain $\tilde{\Omega}$ can be extended periodically and the solution may be thought to be periodic. As a result, the influences of the corners of $\tilde{\Omega}$ are avoided. Furthermore, due to the Dirichlet boundary condition for the potential imposed at the exit in [7], Chen and Feldman can apply the maximum principle to establish some key a priori estimates for the existence and use the technique of shifting the free boundary to achieve the uniqueness as in [2]. These seem to be crucial ingredients in the analysis in [7]. However, it seems to be difficult to modify the approach in [7] to treat our problem due to the curved section of the nozzle and boundary condition prescribingat the sure at exit of the nozzle. Thus, our approach in this paper is quite different from those in [7]. Furthermore, by use of the method introduced in this paper, one can also deduce the corresponding results in [7], see Remark 7.1 in §7.

The rest of the paper is organized as follows. In Sect.2, we reformulate the problem (1.5) with the boundary conditions (1.11)-(1.16) by introducing a generalized hodograph transformation and state the main results on the reformulated problem. Although the computations in this section are very tedious, the concrete expressions of the resulting equation and boundary conditions are important to obtain the uniform estimates on the solution of the corresponding linear problem so that one can apply the Schauder fixed point theorem to show the existence of solution to the nonlinear problem. Besides, these expressions are also very useful in order to show the uniqueness in §6. In Sect.3, some preliminaries results on the linearization of the reformulated problem will be given. The aim in this part is to find a second order linear elliptic equation with linear boundary conditions related to the nonlinear problem in §2. Furthermore, the detailed analysis on the coefficients of the linear equation and its boundary conditions is also carried out. In Sect.4 and Sect.5, the L^∞-norm estimates and the higher order estimates for the solution to the linear problem in §3 are derived respectively. Besides, we will complete the proof on the existence in Theorem 2.2. In Sect.6, we prove the uniqueness of the solution in Theorem 2.2. The proof of Theorem 1.1 and some remarks are given in §7. Finally, some of elementary but important formulas used in §2 and §3 are computed in the appendix (§8).

We will use the following convention in this paper:

$O(M\varepsilon)$ means that there exists a generic constant C such that $|O(M\varepsilon)| \le CM\varepsilon$, where C is independent of M and ε.

§2. The reformulation of the problem and the generalized hodograph transformation

As outlined in the introduction, with the help of the solution to the initial-boundary value problem, (1.9), for a nonlinear second order hyperbolic equation, the original problem of transonic shocks in a nozzle (1.5), (1.9), (1.11) - (1.16), which is an initial-boundary value problem for a nonlinear mixed-type equation, can be reduced to a boundary-value problem for a second order quasi-linear elliptic equation with a free boundary. In this section, we will introduce a generalized hodograph transformation and a coordinate transformation to reduce the free-boundary value problem, (1.5), (1.11) - (1.16), to a boundary value problem for a quasi-linear elliptic equation with nonlinear mixed boundary conditions on a rectangular domain $Q = [0,1] \times [-1,1]$. The structure of the new reformulated problem will provide some keys to our later on analysis. We start with an estimate on the potential for the hyperbolic flow, $\varphi_-(x)$ in (1.9). Recall that ε is the measurement of the variation of the nozzle sections, see (1.7).

Lemma 2.1. Under the assumptions (1.7) and (1.8), the problem (1.9) has a $C^4(\bar{\Omega})$ solution φ_-. Moreover, for small $\varepsilon > 0$, there exists a constant C independent of ε such that

$$\|\varphi_-(x) - q_0 x_1\|_{C^4(\bar{\Omega})} \le C\varepsilon$$

Proof. Set $\tilde{\varphi}(x) = \varphi_-(x) - q_0 x_1$. Then $\tilde{\varphi}(x)$ solves the following problem

$$\begin{cases} ((q_0 + \partial_1\tilde{\varphi})^2 - c_-^2)\partial_1^2\tilde{\varphi} + 2(q_0 + \partial_1\tilde{\varphi})\partial_2\tilde{\varphi}\partial_{12}^2\tilde{\varphi} + ((\partial_2\tilde{\varphi})^2 - c_-^2)\partial_2^2\tilde{\varphi} = 0 \\ \tilde{\varphi}(x)|_{x_1=-1} = 0 \\ \partial_1\tilde{\varphi}(x)|_{x_1=-1} = 0 \\ \partial_2\tilde{\varphi} = f_i'(x_1)\partial_1\tilde{\varphi} + q_0 f_i'(x_1), \qquad on \qquad x_2 = f_i(x_1), \qquad i = 1,2 \end{cases} \tag{2.1}$$

where $c_- = c(H(C_0 - \frac{1}{2}(|q_0 + \partial_1\tilde{\varphi}|^2 + |\partial_2\tilde{\varphi}|^2)))$.

It follows from (1.8) that the initial-boundary values in (2.1) satisfy the compatibility conditions up to the third order.

Since $q_0 > c(\rho_0)$, then (2.1) is strictly hyperbolic with respect to x_1-direction for the small perturbation of solution. By the standard Picard iteration and the characteristics method (for example, see [18]), for small ε we know that (2.1) has a $C^4(\bar{\Omega})$ solution and there exists a constant C independent of ε such that

$$\|\tilde{\varphi}(x)\|_{C^4(\bar{\Omega})} \leq C\varepsilon$$

Hence Lemma 2.1 is proved.

With $\varphi_-(x)$ at hand, the original problem is reduced to find a solution to the free-boundary-value problem, (1.5), (1.11) - (1.16). Rescaling and shifting if necessary, without loss of generality and for the sake of convenience in presentation, we may assume that $f_1(0) = -f_2(0)$ and $q_0 - q_+ = 1$ from now on. To fix the free boundary and meanwhile keep the fixed boundary fixed, we introduce the following hodograph type transformation:

$$\begin{cases} X_1 = 1 - \frac{1-x_1}{1-x_1+(\varphi_-(x)-\varphi_+(x))} \\ X_2 = \frac{2x_2-(f_1+f_2)(x_1)}{f_2(x_1)-f_1(x_1)} \end{cases} \tag{2.2}$$

As it will be shown that $|\partial_x^\beta(\varphi_+(x) - q_+x_1)| \leq C\varepsilon$ for $\beta = 0, 1$, it follows from Lemma 2.1 that $\partial_1(\varphi_-(x) - \varphi_+(x)) = \partial_1(\varphi_-(x) - q_0x_1) - \partial_1(\varphi_+(x) - q_+x_1) + q_0 - q_+ > \frac{q_0-q_+}{2} > 0$ for small ε. This implies $\varphi_-(x) > \varphi_+(x)$ when $x_1 > \xi(x_2)$. Thus, (2.2) is an invertible transformation from the domain $\bar{\Omega}_+$ to $\bar{Q}_+ = \{(X_1, X_2) : 0 \leq X_1 \leq 1, -1 \leq X_2 \leq 1\}$. Correspondingly, the boundaries $x_1 = \xi(x_2)$, $x_1 = 1$, $x_2 = f_1(x_1)$ and $x_2 = f_2(x_1)$ are changed into $X_1 = 0$, $X_1 = 1$, $X_2 = -1$ and $X_2 = 1$ respectively. In addition, the origin $(x_1, x_2) = (0, 0)$ becomes $P_0 = (0, \frac{f_1(0)+f_2(0)}{f_1(0)-f_2(0)}) = (0, 0)$.

Set

$$V(X) = 1 - x_1 + (\varphi_-(x) - \varphi_+(x)).$$

Later on, $V(X)$ will be chosen to be the new unknown function to study the equation (1.5) with the boundary conditions (1.11)-(1.16). It follows from the previous discussion that $V(X) = 1 + O(\varepsilon)$ and $\nabla_X V(X) = O(\varepsilon)$. These properties will be important in our analysis.

Note that

$$\begin{cases} x_1 = 1 + (X_1 - 1)V(X) \\ x_2 = \frac{1}{2}\Big(X_2(f_2 - f_1)(x_1) + (f_1 + f_2)(x_1)\Big)\Big|_{x_1=1+(X_1-1)V(X)} \end{cases} \tag{2.3}$$

In terms of the new variables, one can rewrite equation (1.5) in the domain Q_+ as follows:

$$a_{11}(X, V, \nabla_X V)\partial^2_{X_1}V + 2a_{12}(X, V, \nabla_X V)\partial^2_{X_1X_2}V + a_{22}(X, V, \nabla_X V)\partial^2_{X_2}V + F_0(X, V, \nabla_X V) = 0, \tag{2.4}$$

where $a_{ij}(X, V, \nabla_X V)(i, j = 1, 2)$ and $F_0(X, V, \nabla_X V)$ are smooth functions and can be explicitly computed as given in the appendix. It is important to note that the quasilinear equation (2.4) is uniformly elliptic in the regime we are interested provided that ε is suitably small. This and other important properties of $a_{ij}(X,V,\nabla_X V)$ and $F_0(X,v,\nabla_X V)$ are listed in Lemma 3.5 in the next section.

Next, we turn to the forms of the boundary conditions (1.11) - (1.13), (1.15) and (1.16) in the new coordinates. First, it follows from (1.12) that the boundary condition (1.13) is equivalent to the following equality:

$$[\partial_1\varphi H]\partial_1(\varphi_+-\varphi_-)+[\partial_2\varphi H]\partial_2(\varphi_+-\varphi_-)=0 \qquad on \qquad x_1=\xi(x_2).$$

which has been transformed into

$$G(X,V,\nabla_X V)=0 \qquad on \qquad X_1=0 \tag{2.5}$$

where

$$G(X,V,\nabla_X V)=H(C_0-\frac{1}{2}((1+\partial_{x_1}V-\partial_1\varphi_-)^2+(\partial_{x_2}V-\partial_2\varphi_-)^2))\Big((\partial_1\varphi_- -\partial_{x_1}V-1)(1+\partial_{x_1}V)$$
$$+(\partial_2\varphi_- -\partial_{x_2}V)\partial_{x_2}V\Big)-(\partial_1\varphi_-(1+\partial_{x_1}V)+\partial_2\varphi_-\partial_{x_2}V)H(C_0-\frac{1}{2}|\nabla\varphi_-|^2)$$

Similarly, (1.15) and (1.16) become respectively

$$H(C_0-\frac{1}{2}((1+\partial_{x_1}V-\partial_1\varphi_-)^2+(\partial_{x_2}V-\partial_2\varphi_-)^2))=\rho_+, \qquad on \qquad X_1=1 \tag{2.6}$$

$$(-f_1',1)\frac{\partial X}{\partial x}\nabla_X V=f_1'(1-\partial_1\varphi_-)+\partial_2\varphi_-, \qquad on \qquad X_2=-1 \tag{2.7}$$

$$(-f_2',1)\frac{\partial X}{\partial x}\nabla_X V=f_2'(1-\partial_1\varphi_-)+\partial_2\varphi_-, \qquad on \qquad X_2=1 \tag{2.8}$$

where the variable $x=(x_1,x_2)$ is a function of $X=(X_1,X_2)$ and $V(X)$. It will be clear from next section that boundary conditions (2.5) and (2.6) represent the nonlinear uniform oblique derivative boundary conditions for (2.4). Finally, it follows from (1.11) and the assumption on $P_0=(0,0)$ that

$$V(0,0)=1. \tag{2.9}$$

Thus, our problem is reduced to solve the quasilinear equation (2.4) on the domain Q_+ with the nonlinear boundary condition (2.5) - (2.9). Then the main results on this reformulated problem are the following theorem:

Theorem 2.2. Let $\delta_0\in(0,\frac{1}{8})$ be a given fixed constant. Assume that conditions (1.7), (1.8), and (1.10) hold. Then there exist positive constants ε_0 and C depending only on ρ_+,q_+ and δ_0 such that, for $\varepsilon\in(0,\varepsilon_0)$, the equation (2.4) with boundary conditions (2.5)-(2.9) has a unique solution $V(X)\in C^{1,1-\delta_0}(\bar{Q}_+)\cap C^{3,\delta_0}(\bar{Q}_+\setminus\cup_{i=1}^4 P_i)$ satisfying

$$\|V(X)-1\|_{C^{1,1-\delta_0}}\le C\varepsilon, \qquad |\nabla_X^k V(X)|\le\frac{C\varepsilon}{|R_X|^{k-2+\delta_0}}, \qquad for \qquad k=2,3, \tag{2.10}$$

$$\sup_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^4 P_i}\sum_{k=3}|d_{X,Y}|^{1+2\delta_0}\frac{|\nabla^k V(X)-\nabla^k V(Y)|}{|X-Y|^{\delta_0}}\le C\varepsilon, \tag{2.11}$$

where $P_1=(0,-1),P_2=(0,1),P_3=(1,1),P_4=(1,-1)$,$R_X=X_1(1-X_1)+(1-X_2)(1+X_2)$ and $d_{X,Y}=min\{R_X,R_Y\}$.

Once Theorem 2.2 is established, Theorem 1.1 can be verified directly based on Theorem 2.2 and the generalized hodograph transformation. It thus remains to prove Theorem 2.2, which will be done in the rest of this paper.

§3. **Some preliminaries for the reformulated problem**

In this section, we state some preliminary results and estimates on the reformulated problem, which are basics for the proof of Theorem 2.2.

To prove Theorem 2.2, we will use a Schauder fixed point theorem on a weighted Hölder space. The following version of the Schauder fixed point theorem will be used.

Theorem 3.1. (Theorem 11.1 in [14]) Let $\mathbb{K}$ be a compact and convex subset of a Banach space $\mathbb{B}$, and let J be a continuous mapping from $\mathbb{K}$ into itself. Then J has a fixed point in $\mathbb{K}$.

The suitable Banach space will be a weighted Hölder space defined as

$$\mathbb{B} = \{W(X) \in C^{1,1-\tilde{\delta}_0}(\bar{Q}_+) \cap C^{3,\tilde{\delta}_0}(\bar{Q}_+ \setminus \cup_{i=1}^4 P_i) : \|W\|_{C^{1,1-\tilde{\delta}_0}} \leq C, \qquad \sup_X |R_X|^{\tilde{\delta}_0}|\nabla_X^2 W| \leq C,$$

$$\sup_X |R_X|^{1+\tilde{\delta}_0}|\nabla_X^3 W| \leq C, \quad \sup_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^4 P_i} \sum_{k=3} |d_{X,Y}|^{1+2\tilde{\delta}_0} \frac{|\nabla^k W(X) - \nabla^k W(Y)|}{|X-Y|^{\frac{\delta_0}{2}}} \leq C, \delta_0 < \tilde{\delta}_0 < \frac{1}{8}\} \tag{3.1}$$

The norm in $\mathbb{B}$ is defined to be

$$\|W\|_{\mathbb{B}} = \|W\|_{C^{1,1-\tilde{\delta}_0}} + \sum_{k=2}^{3} \sup_X |R_X|^{k-2+\tilde{\delta}_0}|\nabla_X^k W| +$$
$$\sup_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^4 P_i} \sum_{k=3} |d_{X,Y}|^{1+2\tilde{\delta}_0} \frac{|\nabla^k W(X) - \nabla^k W(Y)|}{|X-Y|^{\frac{\delta_0}{2}}} \qquad for \qquad W \in \mathbb{B} \tag{3.2}$$

It is easy to verify that $\mathbb{B}$ is a Banach space with this norm (or see the reference [13]). The role of R_X in $\mathbb{B}$ is to measure the loss of regularity of $W(X)$ near the corners $P_i (i = 1, 2, 3, 4)$. Sometimes we neglect the subscript X in R_X for convenience.

Next, we define a compact and convex set $\mathbb{K}$ in $\mathbb{B}$ as

$$\mathbb{K} = \{W(X) \in C^{1,1-\delta_0}(\bar{Q}_+) \cap C^{3,\delta_0}(\bar{Q}_+ \setminus \cup_{i=1}^4 P_i) | \|W - 1\|_{C^{1,1-\delta_0}} \leq M\varepsilon,$$
$$\sup_X |R|^{\delta_0}|\nabla_X^2 W| \leq M\varepsilon, \qquad \sup_X |R|^{1+\delta_0}|\nabla_X^3 W| \leq M\varepsilon,$$
$$\sup_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^4 P_i} \sum_{k=3} |d_{X,Y}|^{1+2\delta_0} \frac{|\nabla^k W(X) - \nabla^k W(Y)|}{|X-Y|^{\delta_0}} \leq M\varepsilon, W(0) = 1\}, \tag{3.3}$$

where $M \geq 1$ is to be determined.

It should be clear that $\mathbb{K}$ is a convex subset of $\mathbb{B}$. Furthermore, it can be shown that $\mathbb{K}$ is also compact in $\mathbb{B}$ by the reference [13].

In order to prove Theorem 2.2, we will define a continuous map J, which maps $\mathbb{K}$ into itself, by solving an appropriate boundary value problem for some second order linear elliptic equation on a fixed domain with linear boundary conditions, which is an appropriate linearzation of the nonlinear problem (2.4)-(2.9). More precisely, for any $W \in \mathbb{K}$, we define

$$JW = \tilde{V} + 1 \tag{3.4}$$

where $\tilde{V}$ is required to solve

$$\begin{aligned} a_{11}(X,W,\nabla_X W)\partial^2_{X_1}\tilde{V} + 2a_{12}(X,W,\nabla_X W)\partial^2_{X_1X_2}\tilde{V} + a_{22}(X,W,\nabla_X W)\partial^2_{X_2}\tilde{V} \\ + F_0(X,W,\nabla_X W) = 0 \qquad on \qquad Q_+ \end{aligned} \tag{3.5}$$

with boundary conditions on ∂Q_+, which are suitable linearization of the nonlinear boundary conditions (2.5) - (2.8). Since the forms of these boundary conditions will be important for our analysis, here we discuss them in some details.

We start with the boundary condition for $\tilde{V}$ on $X_1 = 0$. Note that the nonlinear boundary condition (2.5) can be rewritten as

$$\sum_{i=1}^{2} B_{1i}(X,V,\nabla_X V)\partial_{X_i}V + B_1(X,V,\nabla_X V)(V-1) = G(X,1,0,0), \qquad on \qquad X_1 = 0 \tag{3.6}$$

where

$$\begin{aligned} B_{1i}(X,V,\nabla_X V) &= \int_0^1 \partial_{\partial_{X_i}V} G(X,\theta(V-1)+1,\theta\nabla_X V)d\theta, \qquad i=1,2, \qquad and \\ B_1(X,V,\nabla_X V) &= \int_0^1 \partial_V G(X,\theta(V-1)+1,\theta\nabla_X V)d\theta. \end{aligned}$$

It can be shown that the coefficients in (3.6) have the following estimates:

Lemma 3.2. Assume that $W \in \mathbb{K}$. If ε is small such that $M\varepsilon \le \frac{1}{2}$ and $M^2\varepsilon \le \frac{1}{2}$, then it holds that

$$\begin{aligned} &\sum_{k=0}^{3} |\nabla^k_X G(X,1,0,0)| \le \frac{1}{2}, \\ &B_{11}(X,W,\nabla_X W) = \frac{\rho_+(q_0-q_+)(c^2(\rho_+)-q_+^2)}{c^2(\rho_+)}(1+O(M\varepsilon)) \\ &B_{12}(X,W,\nabla_X W) = O(M\varepsilon), \qquad B_1(X,W,\nabla_X W) = O(M\varepsilon), \\ &\nabla^k_X B_{1i}(X,W,\nabla_X W) = O\left(\frac{M\varepsilon}{R^{k-1+\delta_0}}\right), \qquad k=1,2; \quad i=1,2, \\ &\sup_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^4 P_i} \sum_{k=2} |d_{X,Y}|^{1+2\delta_0}\frac{|\nabla^k B_{1i}(X)-\nabla^k B_{1i}(Y)|}{|X-Y|^{\delta_0}} = O(M\varepsilon), \qquad i=1,2, \\ &\sup_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^4 P_i} \sum_{k=2} |d_{X,Y}|^{1+2\delta_0}\frac{|\nabla^k B_1(X)-\nabla^k B_1(Y)|}{|X-Y|^{\delta_0}} = O(M\varepsilon), \end{aligned}$$

where C depends only on ρ_+ and q_+.

Proof. We will only prove the first inequality here. The rest of the Lemma 3.2 is proved in the appendix. It follows from the definition of $G(X,V,\nabla_X V)$ that

$$G(X,1,0,0) = \left(H(C_0 - \frac{1}{2}((1-\partial_1\varphi_-)^2 + (\partial_2\varphi_-)^2)(\partial_1\varphi_- - 1) - H(C_0 - \frac{1}{2}|\nabla\varphi_-|^2)\partial_1\varphi_-)\right)(\bar{x})$$

with $\bar{x} = (\bar{x}_1, \bar{x}_2)$ given by

$$\begin{cases} \bar{x}_1 = X_1, \\ \bar{x}_2 = \frac{1}{2}(X_2(f_2(X_1) - f_1(X_1)) + (f_1(X_1) + f_2(X_1))). \end{cases}$$

Making use of Lemma 2.1, one can compute that

$$G(X,1,0,0) = \Big(H(C_0 - \frac{1}{2}(q_0-1)^2 + O(\varepsilon))(q_0 - 1 + O(\varepsilon)) - H(C_0 - \frac{1}{2}q_0^2 + O(\varepsilon))(q_0 + O(\varepsilon))\Big)(\bar{x})$$
$$= \Big(H(C_0 - \frac{1}{2}(q_0-1)^2)(q_0-1) - H(C_0 - \frac{1}{2}q_0^2)q_0\Big)(\bar{x}) + O(\varepsilon).$$

It now follows from (1.10) and the assumption, $q_0 - q_+ = 1$, that

$$|G(X,1,0,0)| \le C\varepsilon$$

with C a uniform constant depending only on ρ_+ and q_+. Similarly, one can show that

$$\sum_{k=1}^{3} |\nabla_X^k G(X,1,0,0)| \le C\varepsilon,$$

where we have also used the assumption (1.7). Thus, we have shown the first inequality in Lemma 3.2. The rest of the Lemma 3.2 can be proved by lengthy computations and the details are given in the appendix.

We can now require that $\tilde{V}$ satisfies the following linear boundary condition on $X_1 = 0$:

$$\sum_{i=1}^{2} B_{1i}(X,W,\nabla_X W)\partial_{X_i}\tilde{V} + B_1(X,W,\nabla_X W)(W-1) = G(X,1,0,0), \qquad on \qquad X_1 = 0. \tag{3.7}$$

Since $c(\rho_+) < q_+ < q_0$, so Lemma 3.2 implies that $B_{11}(X,W,\nabla_X W) \neq 0$ for suitably small ε. Thus one can rewrite the boundary condition (3.7) as

$$\partial_{X_1}\tilde{V} + \tilde{B}_{11}(X,W,\nabla_X W)\partial_{X_2}\tilde{V} + \tilde{B}_1(X,W,\nabla_X W) = 0, \qquad on \qquad X_1 = 0. \tag{3.8}$$

It follows from Lemma 3.2 that the coefficients in (3.8) have the following estimates:

Lemma 3.3. Assume that $W \in \mathbb{K}$. If $M\varepsilon \le \frac{1}{2}$ and $M^2\varepsilon \le \frac{1}{2}$, then

$$\begin{aligned}
&\tilde{B}_{11}(X,W,\nabla_X W) = O(M\varepsilon),\\
&\tilde{B}_1(X,W,\nabla_X W) = O(\varepsilon),\\
&\nabla_X^k \tilde{B}_{11}(X,W,\nabla_X W) = O(\frac{M\varepsilon}{R^{k-1+\delta_0}}), \qquad k = 1,2,\\
&\nabla_X^k \tilde{B}_1(X,W,\nabla_X W) = O(\frac{\varepsilon}{R^{k-1+\delta_0}}), \qquad k = 1,2,\\
&\sup_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^4 P_i} \sum_{k=2} |d_{X,Y}|^{1+2\delta_0} \frac{|\nabla^k \tilde{B}_{11}(X) - \nabla^k \tilde{B}_{11}(Y)|}{|X-Y|^{\delta_0}} = O(M\varepsilon),\\
&\sup_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^4 P_i} \sum_{k=2} |d_{X,Y}|^{1+2\delta_0} \frac{|\nabla^k \tilde{B}_{1}(X) - \nabla^k \tilde{B}_{1}(Y)|}{|X-Y|^{\delta_0}} = O(\varepsilon).
\end{aligned}$$

Here we emphasize that the fact, $\tilde{B}_1(X,W,\nabla_X W) = O(\varepsilon)$, is critical to determine the constant M in $\mathbb{K}$.

Next, we determine the appropriate boundary condition for $\tilde{V}(X)$ on $X_1 = 1$.

Set

$$\tilde{G}(X,V,\nabla_X V)=H(C_0-\frac{1}{2}((1+\partial_{x_1}V-\partial_1\varphi_-)^2+(\partial_{x_2}V-\partial_2\varphi_-)^2))-H(C_0-\frac{1}{2}q_+^2)$$

It follows from $H(C_0-\frac{1}{2}q_+^2)=\rho_+$ that the boundary condition (2.6) becomes

$$\tilde{G}(X,V,\nabla_X V)=0 \qquad on \qquad X_1=1,$$

which may be rewritten as

$$\sum_{i=1}^{2}B_{2i}(X,V,\nabla_X V)\partial_{X_i}V+B_2(X,V,\nabla_X V)(V-1)=\tilde{G}(X,1,0,0), \qquad on \qquad X_1=1$$

where

$$\begin{aligned}
B_{2i}(X,V,\nabla_X V)&=\int_0^1\partial_{\partial_{X_i}V}\tilde{G}(X,\theta(V-1)+1,\theta\nabla_X V)d\theta,\\
B_2(X,V,\nabla_X V)&=\int_0^1\partial_V\tilde{G}(X,\theta(V-1)+1,\theta\nabla_X V)d\theta,\\
\tilde{G}(X,1,0,0)&=\Big(H(C_0-\frac{1}{2}((1-\partial_1\varphi_-)^2+(\partial_2\varphi_-)^2))-H(C_0-\frac{1}{2}q_+^2)\Big)(\bar{x}).
\end{aligned}$$

Direct computations as in Lemma 3.2 yield

$$\begin{aligned}
&B_{21}(X,W,\nabla_X W)=-\frac{\rho_+q_+}{c^2(\rho_+)}(1+O(M\varepsilon)),\\
&B_{22}(X,W,\nabla_X W)=O(M\varepsilon),\\
&B_2(X,W,\nabla_X W)=O(M\varepsilon),\\
&\tilde{G}(X,1,0,0)=O(\varepsilon).
\end{aligned}$$

Naturally, we pose the following boundary condition for $\tilde{V}$ on $X_1=1$:

$$\sum_{i=1}^{2}B_{2i}(X,W,\nabla_X W)\partial_{X_i}\tilde{V}+B_2(X,W,\nabla_X W)(W-1)=\tilde{G}(X,1,0,0), \qquad on \qquad X_1=1$$

For convenience, one can rewrite this as

$$\partial_{X_1}\tilde{V}+\tilde{B}_{22}(X,W,\nabla_X W)\partial_{X_2}\tilde{V}+\tilde{B}_2(X,W,\nabla_X W)=0, \qquad on \qquad X_1=1 \tag{3.9}$$

where $\tilde{B}_{22}$ and $\tilde{B}_2$ have the same properties as $\tilde{B}_{11}$ and $\tilde{B}_1$ do in Lemma 3.3 respectively.

In a similar way, one can derive from (2.7) and (2.8) the desirable boundary conditions for $\tilde{V}$ on $X_2=-1$ and $X_2=1$ respectively as follows:

$$\partial_{X_2}\tilde{V}+\tilde{B}_{33}(X,W,\nabla_X W)\partial_{X_1}\tilde{V}+\tilde{B}_3(X,W)=0, \qquad on \qquad X_2=-1 \tag{3.10}$$

$$\partial_{X_2}\tilde{V}+\tilde{B}_{44}(X,W,\nabla_X W)\partial_{X_1}\tilde{V}+\tilde{B}_4(X,W)=0, \qquad on \qquad X_2=1 \tag{3.11}$$

where

$$\begin{aligned}
&\nabla_X^k\tilde{B}_i(X,W)=O(\varepsilon), \qquad k=0,1; \qquad i=3,4 \quad ,\\
&\nabla_X^k\tilde{B}_i(X,W)=O(\frac{\varepsilon}{R^{k-2+\delta_0}}), \qquad k=2,3; \qquad i=3,4 \quad ,\\
&\sup_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^4P_i}\sum_{k=3}|d_{X,Y}|^{1+2\delta_0}\frac{|\nabla^k\tilde{B}_i(X)-\nabla^k\tilde{B}_i(Y)|}{|X-Y|^{\delta_0}}=O(\varepsilon), \qquad i=3,4 \quad ,
\end{aligned}$$

and $\tilde{B}_{ii}(i=3,4)$ possess the same properties as $\tilde{B}_{11}$ does in Lemma 3.3.

Thus we have derived the boundary conditions (3.8) - (3.11) for $\tilde{V}$. Finally, in light of (2.9), one may require $\tilde{V}$ to satisfy

$$\tilde{V}(0,0)=0 \tag{3.12}$$

It will be shown that the problem, (3.5) and (3.8)-(3.12), has a unique solution $\tilde{V}$ so that the mapping J in (3.4) is well-defined, continuous on $\mathbb{K}$, and $\tilde{V}+1\in\mathbb{K}$ for an appropriate constant M. To this end, more information on $a_{ij}(X,W,\nabla_X W)$ and $F_0(X,W,\nabla_X W)$ in (3.5) is desired. We list some of the important properties in the following lemma without proofs since they involve mainly tedious computations, see the appendix.

Lemma 3.4. Assume that $W\in\mathbb{K}$. If $M\varepsilon\le\frac{1}{2}$ and $M^2\varepsilon\le\frac{1}{2}$, then

$$\begin{aligned}
&a_{11}(X,W,\nabla_X W)=(c^2(\rho_+)-q_+^2)(1+O(M\varepsilon))\\
&a_{12}(X,W,\nabla_X W)=O(\varepsilon+M\varepsilon)\\
&a_{22}(X,W,\nabla_X W)=c^2(\rho_+)(1+O(M\varepsilon))\\
&F_0(X,W,\nabla_X W)=O(\varepsilon)\\
&\nabla_X^k a_{ij}(X,W,\nabla_X W)=O(\frac{M\varepsilon}{R^{k-1+\delta_0}}),\qquad k=1,2\\
&\nabla_X^k F_0(X,W,\nabla_X W)=O(\frac{\varepsilon}{R^{k-1+\delta_0}}),\qquad k=1,2\\
&\sup_{X,Y\in\bar{Q}_+\setminus\cup_{i=1}^4 P_i}\sum_{k=2}|d_{X,Y}|^{1+2\delta_0}\frac{|\nabla^k a_{ij}(X)-\nabla^k a_{ij}(Y)|}{|X-Y|^{\delta_0}}=O(M\varepsilon)\qquad i,j=1,2
\end{aligned}$$

So far we have outlined the construction of the mapping J, and derived some preliminary estimates on the coefficients of the equation (3.5) and the boundary conditions (3.8)-(3.11). In the subsequent sections, we will focus on solving (3.5) with (3.8)-(3.12) and deriving the necessary estimates on the solutions.

§4. The basic L^∞-estimates

With the preparations in §3 at hand, we can proceed to derive a priori L^∞ estimate on $\tilde{V}(X)$. It should be noted that it seems to be quite difficult to bound the $\|\tilde{V}\|_{L^\infty(Q_+)}$ by using the maximum principle for the following reasons: First, the Hopf lemma cannot be applied directly here since the domain Q_+ does not satisfy the interior ball condition. Second, it is known that the maximum principle is available for solutions to the following uniform oblique derivative problem for linear second elliptic equation

$$\begin{cases}
-\sum_{i,j=1}^n a_{ij}(x)\partial_{ij}^2 u+\sum_{i=1}^n b_i\partial_i u+c(x)u=f(x), & in \quad \Omega\\
\sum_{i=1}^n \beta_i\partial_i u+\mu(x)u=g(x), & on \quad \partial\Omega
\end{cases}$$

where $(a_{ij}(x))$ is a positive definite matrix, without requirement that the domain Ω has interior ball property provided that $c(x)\ge 0$ and $\mu(x)\ge 0$ with $c(x)+\mu(x)>0$ (for details, see [20-22] and the references therein). However, this theory does not apply directly to the problem (3.5) with (3.8)-(3.12) since $c(x)\equiv\mu(x)\equiv 0$ in this case. Our approach is to estimate the H^2-energy $\int\int_{Q_+}(|\tilde{V}(X)|^2+$

$|\nabla_X \tilde{V}(X)|^2 + |\nabla_X^2 \tilde{V}(X)|^2)dX$. Then $\|\tilde{V}\|_{L^\infty(\bar{Q}_+)}$ is bounded by the Sobolev's imbedding theorem. Our main estimate in this section is the following proposition.

Proposition 4.1. Assume that $W \in \mathbb{K}$. If $\tilde{V}(X) \in C^{1,1-\delta_0}(\bar{Q}_+) \cap C^{3,\delta_0}(\bar{Q}_+ \setminus \cup_{i=1}^4 P_i)$ is a solution of (3.5) with the boundary conditions (3.8)-(3.12), then for small ε, there exists a constant C_0 independent of M and ε such that

$$|\tilde{V}(X)| \le C_0 \varepsilon \qquad for \qquad X \in \bar{Q}_+ \tag{4.1}$$

Proof. First, we show that the solution to (3.5) and (3.8)-(3.12) is unique.

Indeed, if there are two solutions $\tilde{V}_1(X)$ and $\tilde{V}_2(X)$ to equation (3.5) with boundary conditions (3.8)-(3.12), then $U(X) = \tilde{V}_1(X) - \tilde{V}_2(X)$ solves

$$\begin{cases} a_{11}(X, W, \nabla_X W)\partial_{X_1}^2 U + 2a_{12}(X, W, \nabla_X W)\partial_{X_1X_2}^2 U + a_{22}(X, W, \nabla_X W)\partial_{X_2}^2 U = 0 \\ \partial_{X_1} U + \tilde{B}_{11}(X, W, \nabla_X W)\partial_{X_2} U = 0, \qquad on \qquad X_1 = 0 \\ \partial_{X_1} U + \tilde{B}_{22}(X, W, \nabla_X W)\partial_{X_2} U = 0, \qquad on \qquad X_1 = 1 \\ \partial_{X_2} U + \tilde{B}_{33}(X, W, \nabla_X W)\partial_{X_1} U = 0, \qquad on \qquad X_2 = -1 \\ \partial_{X_2} U + \tilde{B}_{44}(X, W, \nabla_X W)\partial_{X_1} U = 0, \qquad on \qquad X_2 = 1 \\ U(0,0) = 0 \end{cases}$$

By the Corollary 2.4 in [20], one deduces that $U \equiv C$ with a constant C. Since $U(0,0) = 0$, then $U \equiv 0$. Namely, the solution $\tilde{V}$ is unique.

Next, we establish the a priori L^∞-estimate (4.1). We start with the L^2-estimates on the derivatives of $\tilde{V}$. To overcome the difficulties caused by the uniform oblique derivative boundary condition on $\tilde{V}$ ([21]), we will introduce two vector-fields D_1 and D_2 so that $D_i\tilde{V}$ solves a new second order elliptic equation with some mixed boundary conditions, which can be estimated by energy method.

Set

$$U_1(X) = \partial_{X_1}\tilde{V} + \gamma_1(X)\partial_{X_2}\tilde{V} \equiv D_1\tilde{V},$$
$$U_2(X) = \partial_{X_2}\tilde{V} + \gamma_2(X)\partial_{X_1}\tilde{V} \equiv D_2\tilde{V},$$

where

$$\gamma_1(X) = (1 - X_1)\tilde{B}_{11} + X_1\tilde{B}_{22},$$
$$\gamma_2(X) = \frac{1}{2}((1 - X_2)\tilde{B}_{33} + (X_2 + 1)\tilde{B}_{44}).$$

Then the boundary conditions (3.8)-(3.11) imply that

$$\begin{cases} U_1(X) = -\tilde{B}_1(X, W, \nabla_X W), \qquad on \qquad X_1 = 0 \\ U_1(X) = -\tilde{B}_2(X, W, \nabla_X W), \qquad on \qquad X_1 = 1 \\ U_2(X) = -\tilde{B}_3(X, W), \qquad on \qquad X_2 = -1 \\ U_2(X) = -\tilde{B}_4(X, W), \qquad on \qquad X_2 = 1 \end{cases} \tag{4.2}$$

Moreover $U_1, U_2 \in C^{1-\delta_0}(\bar{Q}_+) \cap C^{2,\delta_0}(\bar{Q}_+ \setminus \cup_{i=1}^4 P_i)$ by the assumption on $\tilde{V}$. Next, we determine the boundary conditions for U_1 on $X_2 = -1, X_2 = 1$ and U_2 on $X_1 = 0, X_1 = 1$ respectively,

Let $N_1 = \partial_{X_2} + T_1(X)\partial_{X_1}$, here $T_1(X)$ will be determined later. Then

$$N_1U_1 = (1 + T_1\gamma_1)\partial_{X_1X_2}^2\tilde{V} + T_1\partial_{X_1}^2\tilde{V} + \gamma_1\partial_{X_2}^2\tilde{V} + (\partial_{X_2}\gamma_1 + T_1\partial_{X_1}\gamma_1)\partial_{X_2}\tilde{V}.$$

On the other hand, (4.2) implies that

$$\partial^2_{X_1X_2}\tilde V = -\gamma_2\partial^2_{X_1}\tilde V - \partial_{X_1}\gamma_2\partial_{X_1}\tilde V - \partial_{X_1}\tilde B_3(X,W), \qquad on \qquad X_2=-1$$

This, together with the equation (3.5), yields that

$$\partial^2_{X_2}\tilde V = \frac{2\gamma_2a_{12}-a_{11}}{a_{22}}\partial^2_{X_1}\tilde V + \frac{2a_{12}\partial_{X_1}\gamma_2}{a_{22}}\partial_{X_1}\tilde V + \frac{2a_{12}\partial_{X_1}\tilde B_3 - F_0}{a_{22}}, \qquad on \qquad X_2=-1$$

Hence

$$N_1U_1 = (T_1-(1+T_1\gamma_1)\gamma_2 + \frac{(2\gamma_2a_{12}-a_{11})\gamma_1}{a_{22}})\partial^2_{X_1}\tilde V + (\frac{2a_{12}\gamma_1\partial_{X_1}\gamma_2}{a_{22}} - (1+T_1\gamma_1)\partial_{X_1}\gamma_2)\partial_{X_1}\tilde V$$
$$+ (\partial_{X_2}\gamma_1 + T_1\partial_{X_1}\gamma_1)\partial_{X_2}\tilde V - (1+T_1\gamma_1)\partial_{X_1}\tilde B_3 + \frac{\gamma_1(2a_{12}\partial_{X_1}\tilde B_3 - F_0)}{a_{22}}, \qquad on \qquad X_2=-1$$

Setting

$$T_1 = \frac{1}{1-\gamma_1\gamma_2}\{\gamma_2 - \frac{(2\gamma_2a_{12}-a_{11})\gamma_1}{a_{22}}\} \in C^{1-\delta_0}(\bar Q_+)\cap C^{2,\delta_0}(\bar Q_+\setminus\cup_{i=1}^4P_i)$$

and noting that

$$\begin{cases} \partial_{X_1}\tilde V = \frac{U_1-\gamma_1U_2}{1-\gamma_1\gamma_2}, \\ \partial_{X_2}\tilde V = \frac{U_2-\gamma_2U_1}{1-\gamma_1\gamma_2}, \end{cases} \tag{4.3}$$

one then arrives at

$$N_1U_1 + d_1(X)U_1 + d_2(X)U_2 = g(X), \qquad on \qquad X_2=-1, \tag{4.4}$$

where $d_1(X)$, $d_2(X)$ and $g(X)$ can be written explicitly. However, their explicit forms are not important here. One needs only to know that $d_1(X), d_2(X), g(X) \in C^{1,\delta_0}(\bar Q_+\setminus\cup_{i=1}^4P_i)$ and

$$|d_1(X)| + |d_2(X)| \le \frac{CM\varepsilon}{R^{\delta_0}}, \qquad |g(X)| \le \frac{C\varepsilon}{R^{\delta_0}},$$

which follow from the explicit forms and the estimates in §3, as can be checked easily. Here and below C represents a generic constant independent of M and ε.

Similarly, one has

$$N_1U_1 + \bar d_1(X)U_1 + \bar d_2(X)U_2 = \bar g(X), \qquad on \qquad X_2=1, \tag{4.5}$$

where $\bar d_1(X)$, $\bar d_2(X)$, and $\bar g(X)$ have the same properties as $d_1(X)$, $d_2(X)$, and $g(X)$ do respectively.

In a similar way, one can choose $N_2 = \partial_{X_1} + T_2\partial_{X_2}$ with

$$T_2 = \frac{1}{1-\gamma_1\gamma_2}\left\{\gamma_1 - \frac{(2\gamma_1a_{12}-a_{22})\gamma_1}{a_{11}}\right\} \in C^{1-\delta_0}(\bar Q_+)\cap C^{2,\delta_0}(\bar Q_+\setminus\cup_{i=1}^4P_i)$$

such that

$$N_2U_2 + e_1(X)U_1 + e_2(X)U_2 = h(X), \qquad on \qquad X_1=0 \tag{4.6}$$
$$N_2U_2 + \bar e_1(X)U_1 + \bar e_2(X)U_2 = \bar h(X), \qquad on \qquad X_1=1 \tag{4.7}$$

where $e_1(X), e_2(X), h(X), \bar e_1(X), \bar e_2(X), \bar h(X) \in C^{1,\delta_0}(\bar Q_+\setminus\cup_{i=1}^4P_i)$ and

$$|e_1(X)| + |e_2(X)| + |\bar e_1(X)| + |\bar e_2(X)| \le \frac{CM\varepsilon}{R^{\delta_0}}, \qquad |h(X)| + |\bar h(X)| \le \frac{C\varepsilon}{R^{\delta_0}}$$

Next we derive the equations satisfied by U_1 and U_2.

Note that

$$
\begin{aligned}
&a_{11}\partial^2_{X_1}(\partial_{X_i}\tilde V)+2a_{12}\partial^2_{X_1X_2}(\partial_{X_i}\tilde V)+a_{22}\partial^2_{X_2}(\partial_{x_i}\tilde V)+\partial_{X_i}a_{11}\partial^2_{X_1}\tilde V+2\partial_{X_i}a_{12}\partial^2_{X_1X_2}\tilde V\\
&\quad+\partial_{X_i}a_{22}\partial^2_{X_2}\tilde V+\partial_{X_i}F_0=0,\qquad i=1,2
\end{aligned}
$$

By using the relation (4.3), Lemma 3.3 and Lemma 3.4, one can derive

$$
\begin{aligned}
&A_{11}(X)\partial^2_{X_1}U_1+2A_{12}(X)\partial^2_{X_1X_2}U_1+A_{22}(X)\partial^2_{X_2}U_1+A_1(X)\partial_{X_1}U_1+A_2(X)\partial_{X_2}U_1+A_3(X)\partial_{X_1}U_2\\
&\quad+A_4(X)\partial_{X_2}U_2+C_1(X)U_1+C_2(X)U_2+F(X)=0, \qquad (4.8)\\
&\bar A_{11}(X)\partial^2_{X_1}U_2+2\bar A_{12}(X)\partial^2_{X_1X_2}U_2+\bar A_{22}(X)\partial^2_{X_2}U_2+\bar A_1(X)\partial_{X_1}U_1+\bar A_2(X)\partial_{X_2}U_1+\bar A_3(X)\partial_{X_1}U_2\\
&\quad+\bar A_4(X)\partial_{X_2}U_2+\bar C_1(X)U_1+\bar C_2(X)U_2+\bar F(X)=0, \qquad (4.9)
\end{aligned}
$$

where $A_{ij}=\bar A_{ij}=a_{ij}\in C^{1-\delta_0}(\bar Q_+)\cap C^{2,\delta_0}(\bar Q_+\setminus\cup_{i=1}^4P_i)(i,j=1,2)$; $A_i(X),\bar A_i(X)\in C^{1,\delta_0}(\bar Q_+\setminus\cup_{i=1}^4P_i)(i=1,2,3,4)$; $C_i(X)$, $\bar C_i(X)\in C^{\delta_0}(\bar Q_+\setminus\cup_{i=1}^4P_i)(i=1,2)$; $F(X),\bar F(X)\in C^1(\bar Q_+\setminus\cup_{i=1}^4P_i)$. Furthermore,

$$
\begin{aligned}
&A_{11}(X)=(c_+^2-q_+^2)(1+O(M\varepsilon))\\
&A_{12}(X)=O(M\varepsilon)\\
&A_{22}(X)=c_+^2(1+O(M\varepsilon))\\
&\nabla_XA_{ij}(X)=O(\frac{M\varepsilon}{R^{\delta_0}}),\qquad i,j=1,2\\
&A_i(X)=O(\frac{M\varepsilon}{R^{\delta_0}}),\qquad i=1,2,3,4\\
&C_i(X)=O(\frac{M\varepsilon}{R^{1+\delta_0}}),\qquad i=1,2\\
&F(X)=O(\frac{\varepsilon}{R^{\delta_0}})
\end{aligned}
$$

and $\bar A_{ij}(X),\bar A_i(X),\bar C_i(X)$ and $\bar F(X)$ have the same properties as those of $A_{ij}(X),A_i(X),C_i(X)$ and $F(X)$ do respectively.

Now we can derive the basic energy estimates on U_1 and U_2. Indeed, multiplying (4.8) and (4.9) on both sides by U_1 and U_2 respectively and integrating by parts over the domain Q_+, one gets

$$
\int\int_{Q_+}(|\nabla U_1|^2+|\nabla U_2|^2)dX\le C(M\varepsilon\int\int_{Q_+}\frac{|U_1|^2+|U_2|^2}{R^{1+\delta_0}}dX+\varepsilon\int\int_{Q_+}\frac{|U_1|+|U_2|}{R^{\delta_0}}dX+\varepsilon^2+|Q|+|\bar Q|), \qquad (4.10)
$$

here one has used the estimates on A_{ij}, $\bar A_{ij}$, A_i, $\bar A_i$, C_i, $\bar C_i$, F and $\bar F$. Besides, Q and $\bar Q$ represent the boundary integrals as follows:

$$
\begin{aligned}
Q=&\int_{-1}^1\{((A_{11}\partial_1U_1+2A_{12}\partial_2U_1)U_1)(1,X_2)-((A_{11}\partial_1U_1+2A_{12}\partial_2U_1)U_1)(0,X_2)\}dX_2\\
&+\int_0^1\{(A_{22}\partial_2U_1U_1)(X_1,1)-(A_{22}\partial_2U_1U_1)(X_1,-1)\}dX_1\\
\bar Q=&\int_{-1}^1\{((\bar A_{11}\partial_1U_2+2\bar A_{12}\partial_2U_2)U_2)(1,X_2)-((\bar A_{11}\partial_1U_2+2\bar A_{12}\partial_2U_2)U_2)(0,X_2)\}dX_2\\
&+\int_0^1\{(\bar A_{22}\partial_2U_2U_2)(X_1,1)-(\bar A_{22}\partial_2U_2U_2)(X_1,-1)\}dX_1
\end{aligned}
$$

The terms in the right hand side of (4.10) will be treated separately as follows.

(i). The estimate on $\int\int_{Q_+} \frac{|U_1|^2+|U_2|^2}{R^{1+\delta_0}} dX$.

By (4.2), Lemma 3.3 and the Poincare's inequality, one has

$$\begin{aligned}\int\int_{Q_+} |U_1|^2 dX &\le \int\int_{Q_+} |U_1+\tilde{B}_1|^2 dX + \int\int_{Q_+} |\tilde{B}_1|^2 dX \\ &\le C(\int\int_{Q_+} |\nabla(U_1+\tilde{B}_1)|^2 dX + \varepsilon^2) \\ &\le C(\int\int_{Q_+} |\nabla U_1|^2 dX + \varepsilon^2). \end{aligned} \tag{4.11}$$

In addition, Sobolev's imbedding theorem implies

$$\Big(\int\int_{Q_+} |U_1|^p dX\Big)^{\frac{1}{p}} \le C_p \Big(\int\int_{Q_+} (|U_1|^2 + |\nabla U_1|^2) dX\Big)^{\frac{1}{2}}, \qquad for \qquad p \in [1,+\infty) \tag{4.12}$$

Hence, combining (4.11) with (4.12), one shows that

$$\Big(\int\int_{Q_+} |U_1|^p dX\Big)^{\frac{1}{p}} \le C_p(\int\int_{Q_+} |\nabla U_1|^2 dX + \varepsilon^2)^{\frac{1}{2}}, \qquad for \qquad p \in [1,+\infty). \tag{4.13}$$

Then

$$\begin{aligned}\int\int_{Q_+} \frac{|U_1|^2}{R^{1+2\delta_0}} dX &\le \Big(\int\int_{Q_+} \frac{1}{R^{(1+2\delta_0)p_1}} dX\Big)^{\frac{1}{p_1}} \Big(\int\int_{Q_+} |U_1|^{2p_2} dX\Big)^{\frac{1}{p_2}} \\ &\le C\Big(\int\int_{Q_+} |U_1|^{2p_2} dX\Big)^{\frac{1}{p_2}} \end{aligned} \tag{4.14}$$

where $p_1 = \frac{3}{2(1+2\delta_0)} > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$.

Similar estimates hold for $\int\int_{Q_+} \frac{|U_2|^2}{R^{1+2\delta_0}} dX$.

It follows from (4.13) and (4.14) that

$$\int\int_{Q_+} \frac{|U_1|^2 + |U_2|^2}{R^{1+\delta_0}} dX \le C(\int\int_{Q_+} (|\nabla U_1|^2 + |\nabla U_2|^2) dX + \varepsilon^2). \tag{4.15}$$

(ii). The estimate on $\varepsilon \int\int_{Q_+} \frac{|U_1|+|U_2|}{R^{\delta_0}} dX$.

Since for small constant $\delta > 0$, there exists a constant $C_\delta > 0$ such that $\frac{\varepsilon|U_1|}{R^{\delta_0}} \le C_\delta \frac{\varepsilon^2}{R^{2\delta_0}} + \delta|U_1|^2$, then it follows from (4.11) and (4.12) that

$$\varepsilon \int\int_{Q_+} \frac{|U_1| + |U_2|}{R^{\delta_0}} dX \le C(C_\delta \varepsilon^2 + \delta \int\int_{Q_+} (|\nabla U_1|^2 + |\nabla U_2|^2) dX) \tag{4.16}$$

(iii). The estimate on $|Q|$.

First, we treat the term $I_1 = \int_{-1}^1 ((A_{11}\partial_1 U_1 + 2A_{12}\partial_2 U_1)U_1)(1, X_2) dX_2$ in Q.

Since

$$\partial_1 U_1 = \partial_{X_1}^2 \tilde{V} + \gamma_1 \partial_{X_1X_2}^2 \tilde{V} + \partial_{X_1}\gamma_1 \partial_{X_2} \tilde{V},$$

and

$$\begin{cases} \partial^2_{X_2}\tilde V + \gamma_2\partial^2_{X_1X_2}\tilde V + \partial_{X_2}\gamma_2\partial_{X_1}\tilde V = \partial_2 U_2, \\ \partial^2_{X_1X_2}\tilde V + \gamma_1\partial^2_{X_2}\tilde V + \partial_{X_2}\gamma_1\partial_{X_2}\tilde V = \partial_2 U_1, \\ a_{11}\partial^2_{X_1}\tilde V + 2a_{12}\partial^2_{X_1X_2}\tilde V + a_{22}\partial^2_{X_2}\tilde V + F_0 = 0, \end{cases}$$

then one has the following expression for $\partial_1 U_1$,

$$\partial_1 U_1 = P_1(X)\partial_2 U_1 + P_2(X)\partial_2 U_2 + E_1(X)U_1 + E_2(X)U_2 + P_3(X) \tag{4.17}$$

where $P_1(X), P_2(X), P_3(X) \in C^{1-\delta_0}(\bar Q_+) \cap C^2(\bar Q_+ \setminus \cup_{i=1}^4 P_i)$ and $E_1(X), E_2(X) \in C^{1,\delta_0}(\bar Q_+ \setminus \cup_{i=1}^4 P_i)$. Moreover

$$\sum_{i=1}^2 |P_i(X)| \le C, \qquad |P_3(X)| \le C\varepsilon,$$
$$\sum_{i=1}^2 |\nabla_X P_i(X)| \le \frac{CM\varepsilon}{R^{\delta_0}}, \qquad \sum_{i=1}^2 |E_i(X)| \le \frac{CM\varepsilon}{R^{\delta_0}}.$$

Substituting (4.17) into I_1, integrating by parts, and using the boundary conditions (4.2), we obtain

$$|I_1| \le C(\varepsilon^2 + M\varepsilon\int_{-1}^1 \frac{U_2^2(1,X_2)}{R^{\delta_0}}dX_2 + \int_{-1}^1 \frac{\varepsilon|U_2(1,X_2)|}{R^{\delta_0}}dX_2) \tag{4.18}$$

In addition, by the trace theorem and the argument as for (4.11) one has

$$\begin{aligned} \left(\int_{-1}^1 |U_2(1,X_2)|^p dX_2\right)^{\frac1p} &\le C_p\left(\int\int_{Q_+} (|U_2|^2 + |\nabla U_2|^2)dX\right)^{\frac12} \\ &\le C_p\left(\int\int_{Q_+} |\nabla U_2|^2 dX + \varepsilon^2\right)^{\frac12}, \qquad for \qquad p \in [1,+\infty) \end{aligned} \tag{4.19}$$

So it follows from the Holder inequality and (4.19) that

$$|I_1| \le C(C_\delta\varepsilon^2 + (M\varepsilon + \delta)\int\int_{Q_+} |\nabla U_2|^2 dX) \tag{4.20}$$

Similarly, one can obtain

$$|I_2| = |\int_{-1}^1 ((A_{11}\partial_1 U_1 + 2A_{12}\partial_2 U_1)U_1)(0,X_2)dX_2| \le C(C_\delta\varepsilon^2 + (M\varepsilon+\delta)\int\int_{Q_+} |\nabla U_2|^2 dX) \tag{4.21}$$

Next, we estimate $I_3 = \int_0^1((A_{22}\partial_2 U_1 U_1)(X_1,1))dX_1$ in Q.

Note here that the term $I_4 = \int_0^1((A_{22}\partial_2 U_1 U_1)(X_1,-1))dX_1$ in Q can be treated similarly.

It follows from the boundary condition (4.5) that

$$I_3 = \int_0^1 (A_{22}(\bar g - \bar d_1 U_1 - \bar d_2 U_2 - T_1\partial_1 U_1)U_1)(X_1,1)dX_1.$$

By the properties of $A_{22}, \bar g, \bar d_i$, and T_1, one obtains by using the Holder inequality and integration by parts that

$$|I_3| \le C\left(\varepsilon^2 + M\varepsilon(\int_0^1 |U_1(X_1,1)|^4 dX_1)^{\frac12} + \varepsilon\int_0^1 \frac{|U_1(X_1,1)|}{R^{\delta_0}}dX_1\right)$$

Similar argument as for (4.20) leads to

$$|I_3| \le C(C_\delta \varepsilon^2 + (M\varepsilon + \delta) \int\int_{Q_+} |\nabla U_1|^2 dX).$$

Collecting all the estimates on I_1, I_2, I_3 and I_4, we arrive at

$$|Q| \le C(C_\delta \varepsilon^2 + (M\varepsilon + \delta) \int\int_{Q_+} (|\nabla U_1|^2 + |\nabla U_2|^2) dX). \tag{4.22}$$

(iv). The estimate on $|\bar{Q}|$.

By a similar treatment as for Q, one can show that

$$|\bar{Q}| \le C(C_\delta \varepsilon^2 + (M\varepsilon + \delta) \int\int_{Q_+} (|\nabla U_1|^2 + |\nabla U_2|^2) dX). \tag{4.23}$$

Therefore, substituting (4.15), (4.16), (4.22) and (4.23) into (4.10) and choosing $M\varepsilon$ and δ small enough, then one gets

$$\int\int_{Q_+} (|U_1|^2 + |U_2|^2 + |\nabla U_1|^2 + |\nabla U_2|^2) dX \le C\varepsilon^2, \tag{4.24}$$

(4.24), together with (4.3) and Lemma 3.3, yields the desirable estimate on $\iint_{Q_+} (|\nabla \tilde{V}|^2 + |\nabla^2 \tilde{V}|^2) dX$. Finally, we estimate $\iint_{Q_+} |\tilde{V}|^2 dX$.
Since $\tilde{V}(0,0) = 0$, then

$$\tilde{V}(X) = \tilde{V}(X) - \tilde{V}(X_1, 0) + \tilde{V}(X_1, 0) - \tilde{V}(0,0) = \int_0^{X_2} \partial_{X_2} \tilde{V}(X_1, X_2) dX_2 + \int_0^{X_1} \partial_{X_1} \tilde{V}(X_1, 0) dX_1$$

Hence, one obtain by the trace theorem that

$$\begin{aligned} \int\int_{Q_+} |\tilde{V}|^2 dX &\le C(\int\int_{Q_+} |\partial_{X_2} \tilde{V}|^2 dX + \int_0^1 |\partial_{X_1} \tilde{V}(X_1, 0)|^2 dX_1) \\ &\le C \int\int_{Q_+} (|\nabla \tilde{V}|^2 + |\nabla^2 \tilde{V}|^2) dX \qquad . \end{aligned} \tag{4.25}$$

Noting (4.13), (4.14) and (4.24), then one has from (4.25) that

$$\begin{aligned} \int\int_{Q_+} |\tilde{V}|^2 dX &\le C(\int\int_{Q_+} (|U_1|^2 + |U_2|^2 + |\nabla U_1|^2 + |\nabla U_2|^2) dX + \varepsilon^2 \int\int_{Q_+} \frac{|U_1|^2 + |U_2|^2}{R^{2\delta_0}} dX) \\ &\le C\Big(\int\int_{Q_+} (|U_1|^2 + |U_2|^2 + |\nabla U_1|^2 + |\nabla U_2|^2) dX + \varepsilon^2\Big) \\ &\le C\varepsilon^2 \end{aligned} \tag{4.26}$$

Combining (4.24) with (4.26), we obtain

$$\int\int_{Q_+} (|\tilde{V}|^2 + |\nabla \tilde{V}|^2 + |\nabla^2 \tilde{V}|^2) dX \le C\varepsilon^2,$$

which implies (4.1) by Sobolev's imbedding theorem. Thus Proposition 4.1 is proved.

Remark 4.1. It follows easily from the proof of Proposition 4.1 that we actually obtain the uniform $C^\alpha(\bar{Q}_+)$ estimate on $\tilde{V}(X)$ since $H^2(\bar{Q}_+) \subset C^\alpha(\bar{Q}_+)$ for any fixed $0 < \alpha < 1$.

§5. The higher order estimates and the proof of the existence in Theorem 2.2

With the L^∞ estimate on $\tilde V$ at hand, we can now give the higher order estimates for $\tilde V$.

Proposition 5.1. Assume that $W \in \mathbb{K}$. If $\tilde V(X) \in C^{1,1-\delta_0}(\bar Q_+) \cap C^{3,\delta_0}(\bar Q_+ \setminus \cup_{i=1}^4 P_i)$ is a solution of (3.5) with the boundary conditions (3.8)-(3.9), then for small ε, there exists a constant C_0 independent of M and ε such that

$$\begin{aligned}
&\|\tilde V\|_{C^{1,1-\delta_0}} \le C_0\varepsilon,\\
&\sup_X R^{k-2+\delta_0}|\nabla_X^k \tilde V| \le C_0\varepsilon, \qquad k=2,3,\\
&\sup_{X,Y\in \bar Q_+\setminus \cup_{i=1}^4 P_i} \sum_{k=3} |d_{X,Y}|^{1+2\delta_0} \frac{|\nabla^k \tilde V(X) - \nabla^k \tilde V(Y)|}{|X-Y|^{\delta_0}} \le C_0\varepsilon.
\end{aligned}$$

Proof. By Lemma 3.4, without loss of generality and for simplicity, we may assume that

$$a_{11} = 1 + O(M\varepsilon), \qquad a_{22} = 1 + O(M\varepsilon)$$

Denote by $\Gamma_1 = \{X_1 = 0, -1 < X_2 < 1\}$, $\Gamma_2 = \{0 < X_1 < 1, X_2 = 1\}$, $\Gamma_3 = \{X_1 = 1, -1 < X_2 < 1\}$, and $\Gamma_4 = \{0 < X_1 < 1, X_2 = -1\}$. Then $\partial Q_+ = \overline{\cup_{i=1}^4 \Gamma_i}$. Consider a subdomain Q_1 of Q_+ with the property that $\partial Q_1 \cap \partial Q_+$ lies in the interior of ∂Q_+. Since the solution $\tilde V$ is unique, then by the classical Schauder estimates on the second order elliptic equation with the uniform oblique derivative boundary conditions(see [13] or [19]), there exists a constant $C(\|\tilde B_{11}\|_{C^{1,1-\delta_0}(\bar Q_1)}, ..., \|\tilde B_{44}\|_{C^{1,1-\delta_0}(\bar Q_1)})$ which depends on $\|\tilde B_{ii}\|_{C^{1,1-\delta_0}(\bar Q_1)} (i = 1,2,3,4)$ such that

$$\begin{aligned}
\|\tilde V\|_{C^{2,1-\delta_0}(\bar Q_1)} \le\; & C(\|\tilde B_{11}\|_{C^{1,1-\delta_0}(\bar Q_1)}, ..., \|\tilde B_{44}\|_{C^{1,1-\delta_0}(\bar Q_1)})(\|\tilde V\|_{L^\infty(Q_+)} + \|F_0\|_{C^{1-\delta_0}(Q_+)}\\
&+ \sum_{i=1}^4 \|\tilde B_i\|_{C^{1,1-\delta_0}(Q_+)})
\end{aligned} \tag{5.1}$$

Thus, the main task is to estimate $\tilde V$ and its derivative near the corners.

Let $G_i(r_0)(i = 1,2,3,4)$ represent the part of the disk centered at P_i with radius $0 < r_0 < \frac{1}{16}$ which is included between Γ_i and Γ_{i-1}(here we define $\Gamma_0 = \Gamma_4$). Define a C^∞ function $\chi_i(X)$ such that

$$\chi_i(X) = \begin{cases} 1, & X \in G_i(\frac{r_0}{2}) \\ 0, & X \in Q_+ \backslash G_i(\frac{2}{3}r_0) \end{cases}$$

We divide the proof of Proposition 5.1 into four steps.

Step 1. There exist constants C_1^i and C_2^i $(1 \le i \le 4)$ such that

$$\left|\sum_{i=1}^4 \chi_i(X)(\tilde V(X) - (\tilde V(P_i) + C_1^i(X_1 - X_1^i) + C_2^i(X_2 - X_2^i)))\right| \le C\varepsilon R^{2-\delta_0},$$

here $P_i = (X_1^i, X_2^i)$.

Set $V_1(X) = \chi_1(X)(\tilde{V} - \tilde{V}(P_1))$. Then it follows from (3.5), (3.8) and (3.10) that $V_1(X)$ satisfies the following elliptic equation and the boundary conditions

$$\begin{cases} a_{11}\partial^2_{X_1}V_1 + 2a_{12}\partial^2_{X_1X_2}V_1 + a_{22}\partial^2_{X_2}V_1 = F_1(X) \\ \partial_{X_1}V_1 + \tilde{B}_{11}(X)\partial_{X_2}V_1 = q_1(X), \qquad on \qquad X_1 = 0 \\ \partial_{X_2}V_1 + \tilde{B}_{33}(X)\partial_{X_1}V_1 = q_2(X), \qquad on \qquad X_2 = -1 \\ V_1 = 0 \qquad on \qquad |X_1|^2 + |X_2+1|^2 = r_0^2 \end{cases} \tag{5.2}$$

where

$$\begin{aligned} F_1(X) &= 2\Big(a_{11}\partial_1\chi_1\partial_1\tilde{V} + a_{12}(\partial_1\chi_1\partial_2\tilde{V} + \partial_2\chi_1\partial_1\tilde{V}) + a_{22}\partial_2\chi_1\partial_2\tilde{V}\Big) + (a_{11}\partial_1^2\chi_1 + 2a_{12}\partial_{12}^2\chi_1 \\ &\quad + a_{22}\partial_2\chi_1)\tilde{V} - \chi_1 F_0 \\ q_1(X) &= (\tilde{B}_{11}\partial_2\chi_1 + \partial_1\chi_1)\tilde{V} - \chi_1\tilde{B}_1 \\ q_2(X) &= (\tilde{B}_{33}\partial_1\chi_1 + \partial_2\chi_1)\tilde{V} - \chi_1\tilde{B}_3 \end{aligned}$$

Without loss of generality, we may assume that P_1 is the origin (0,0) from now on (otherwise, one can take a transformation: $\tilde{X}_1 = X_1, \tilde{X}_2 = X_2 + 1$ to achieve this).

Set $\tilde{V}_1 = V_1 - \chi_1(X)(C_1^1X_1 + C_2^1X_2)$, here

$$C_1^1 = \frac{\tilde{B}_3(0)\tilde{B}_{11}(0) - \tilde{B}_1(0)}{1 - \tilde{B}_{11}(0)\tilde{B}_{33}(0)},$$

$$C_2^1 = \frac{\tilde{B}_1(0)\tilde{B}_{33}(0) - \tilde{B}_3(0)}{1 - \tilde{B}_{11}(0)\tilde{B}_{33}(0)}.$$

Then $\tilde{V}_1$ solves the following problem

$$\begin{cases} a_{11}\partial^2_{X_1}\tilde{V}_1 + 2a_{12}\partial^2_{X_1X_2}\tilde{V}_1 + a_{22}\partial^2_{X_2}\tilde{V}_1 = \tilde{F}_1(X), \\ \partial_{X_1}\tilde{V}_1 + \tilde{B}_{11}(X)\partial_{X_2}\tilde{V}_1 = \tilde{q}_1(X), \qquad on \qquad X_1 = 0 \\ \partial_{X_2}\tilde{V}_1 + \tilde{B}_{33}(X)\partial_{X_1}\tilde{V}_1 = \tilde{q}_2(X), \qquad on \qquad X_2 = 0 \\ \tilde{V}_1 = 0 \qquad on \qquad |X_1|^2 + |X_2|^2 = r_0^2 \end{cases} \tag{5.3}$$

where

$$\begin{aligned} \tilde{F}_1(X) &= F_1(X) - (C_1^1X_1 + C_2^1X_2)(a_{11}\partial_1^2\chi_1 + 2a_{11}\partial_{12}^2\chi_1 + a_{22}\partial_2^2\chi_1) \\ &\qquad - (2a_{11}\partial_1\chi_1C_1^1 + 2a_{12}(\partial_2\chi_1C_1^1 + \partial_1\chi_1C_2^1) + 2a_{22}\partial_2\chi_1C_2^1) \\ \tilde{q}_1(X) &= q_1(X) - \chi_1(X)(C_1^1 + C_2^1\tilde{B}_{11}(X)) - (C_1^1X_1 + C_2^1X_2)(\partial_1\chi_1 + \tilde{B}_{11}(X)\partial_2\chi_1) \\ \tilde{q}_2(X) &= q_2(X) - \chi_1(X)(C_2^1 + C_1^1\tilde{B}_{33}(X)) - (C_1^1X_1 + C_2^1X_2)(\partial_2\chi_1 + \tilde{B}_{33}(X)\partial_1\chi_1) \end{aligned}$$

with $\tilde{q}_1(0) = \tilde{q}_2(0) = 0$. In addition, it follows from Proposition 4.1, Remark 4.10, and Lemmas 3.3 - 3.4 that

$$\|\tilde{q}_1(X)\|_{C^{1-\delta_0}(\overline{G_1(r_0)})} + \|\tilde{q}_2(X)\|_{C^{1-\delta_0}(\overline{G_1(r_0)})} \le C\varepsilon.$$

Noting $\tilde{V}_1(0) = \nabla_X\tilde{V}_1(0) = 0$, hence we need only to show $|\nabla_X\tilde{V}_1(X)| \le C\varepsilon r^{1-\delta_0}$ to conclude Step 1, here $r = \sqrt{X_1^2 + X_2^2}$. First, note that the boundary of $G_1(r_0)$ satisfies the uniform exterior ball property. Then by the standard method of barrier functions (for example, see [17]) one can show that

$$|\nabla_X\tilde{V}_1| \le C\varepsilon, \quad on \quad G_1(r_0). \tag{5.4}$$

To obtain the Holder continuity of $\nabla_X \tilde{V}_1$ at P_1, further estimates are needed. As in the proof of Proposition 4.1, we set

$$U_1 = \partial_{X_1}\tilde{V}_1 + \tilde{B}_{11}\partial_{X_2}\tilde{V}_1$$
$$U_2 = \partial_{X_2}\tilde{V}_1 + \tilde{B}_{33}\partial_{X_1}\tilde{V}_1$$

Then by similar simplifications and an analogous formula as (4.17) in the proof of Proposition 4.1, one has

$$\begin{cases} A_{11}(X)\partial^2_{X_1}U_1 + 2A_{12}(X)\partial^2_{X_1X_2}U_1 + A_{22}(X)\partial^2_{X_2}U_1 + A_1(X)\partial_{X_1}U_1 + A_2(X)\partial_{X_2}U_1 \\ \qquad +C_1(X)U_1 + C_2(X)U_2 = F_{11}(X) \\ U_1 = \tilde{q}_1(X), \qquad on \qquad X_1 = 0 \\ \partial_{X_2}U_1 + T(X)\partial_{X_1}U_1 + d_1(X)U_1 + d_2(X)U_2 = g(X), \qquad on \qquad X_2 = 0 \\ U_1 = 0, \qquad on \qquad X_1^2 + X_2^2 = r_0^2 \end{cases} \tag{5.5}$$

and

$$\begin{cases} \bar{A}_{11}(X)\partial^2_{X_1}U_2 + 2\bar{A}_{12}(X)\partial^2_{X_1X_2}U_2 + \bar{A}_{22}(X)\partial^2_{X_2}U_2 + \bar{A}_1(X)\partial_{X_1}U_2 + \bar{A}_2(X)\partial_{X_2}U_2 \\ \qquad +\bar{C}_1(X)U_1 + \bar{C}_2(X)U_2 = \bar{F}_{11}(X) \\ \partial_{X_1}U_2 + \bar{T}(X)\partial_{X_2}U_2 + \bar{d}_1(X)U_1 + \bar{d}_2(X)U_2 = \bar{g}(X), \qquad on \qquad X_1 = 0 \\ U_2 = \tilde{q}_2(X), \qquad on \qquad X_2 = 0 \\ U_2 = 0, \qquad on \qquad X_1^2 + X_2^2 = r_0^2 \end{cases} \tag{5.6}$$

where $A_{ij}, \bar{A}_{ij}, A_i, \bar{A}_i, C_i, \bar{C}_i, T, \bar{T}, d_i, \bar{d}_i, g_1, \bar{g}_1, F_{11}$ and $\bar{F}_{11}$ have the same properties as those of the corresponding terms in (4.8), (4.9), (4.4) and (4.6) respectively.

One can rewrite (5.5) as

$$\begin{cases} A_{11}(X)\partial^2_{X_1}U_1 + 2A_{12}(X)\partial^2_{X_1X_2}U_1 + A_{22}(X)\partial^2_{X_2}U_1 + A_1(X)\partial_{X_1}U_1 + A_2(X)\partial_{X_2}U_1 = H_1(X) \\ U_1 = \tilde{q}_1(X), \qquad on \qquad X_1 = 0 \\ \partial_{X_2}U_1 + T(X)\partial_{X_1}U_1 = g_1(X), \qquad on \qquad X_2 = 0 \\ U_1 = 0, \qquad on \qquad X_1^2 + X_2^2 = r_0^2 \end{cases} \tag{5.7}$$

where

$$H_1(X) = O(\frac{\varepsilon}{r^{1+\delta_0}})$$
$$g_1(X) = O(\frac{\varepsilon}{r^{\delta_0}})$$

The strategy is to find some subsolution and supersolution of (5.7) so that one can get a control on U_1. Set $W_{sub}(X) = -C_0\varepsilon r^{1-\delta_0}\ cos((1-\frac{\delta_0}{2})\theta + \frac{\delta_0}{2})$, here $\theta = arctg\frac{X_2}{X_1}$. Noting $\nabla\chi_1 = \nabla^2\chi_1 \equiv 0$ for $|X| \le \frac{r_0}{2}$ and using the estimate (5.4), one shows by director computations that for large C_0 (independent of M and ε) and $0 \le \theta \le \frac{\pi}{2}$, there holds

$$\begin{cases} (A_{11}\partial^2_{X_1} + 2A_{12}\partial^2_{X_1X_2} + A_{22}\partial^2_{X_2} + A_1\partial_{X_1} + A_2\partial_{X_2})W_{sub} \\ \qquad = C_0\varepsilon r^{-1-\delta_0}\delta_0(1-\frac{3}{4}\delta_0)(cos((1-\frac{\delta_0}{2})\theta + \frac{\delta_0}{2}) + O(M\varepsilon)) \ge -H_1(X) \\ W_{sub} \le \tilde{q}_1(X), \qquad on \qquad X_1 = 0 \\ \partial_{X_2}W_{sub} + T\partial_{X_1}W_{sub} = C_0(1-\frac{\delta_0}{2})\varepsilon r^{-\delta_0}(sin((1-\frac{\delta_0}{2})\theta + \frac{\delta_0}{2}) + O(M\varepsilon))) \\ \qquad \ge -g_1(X), \qquad on \qquad X_2 = 0 \\ W_{sub} < 0, \qquad on \qquad X_1^2 + X_2^2 = r_0^2 \end{cases}$$

By the comparison principle (see Corollary 2.4. in [20]), one knows that

$$U_1 \geq W_{sub}$$

Similarly, one can prove

$$U_1 \leq W_{sup} = -W_{sub}$$

Hence

$$|U_1| \leq C_0 \varepsilon r^{1-\delta_0}.$$

Setting $\tilde{W}_{sub} = -C_0 \varepsilon r^{1-\delta_0} sin((1-\frac{\delta_0}{2})\theta + \frac{\delta_0}{2})$ and $\tilde{W}_{sup} = -\tilde{W}_{sub}$, then we also have

$$|U_2| \leq C_0 \varepsilon r^{1-\delta_0}.$$

Namely, $|\nabla_X \tilde{V}_1(X)| \leq C\varepsilon r^{1-\delta_0}$ holds.

Similarly, we can treat $\tilde{V}(X)$ near the other points P_2, P_3 and P_4. In addition, away from the domain $\cup_{i=1}^4 G_i(r_0)$, then (5.1) and Proposition 4.1 imply that

$$\|\tilde{V}(X)\|_{C^{2,1-\delta_0}(\overline{Q_+ \backslash \cup_{i=1}^4 G_i(r_0)})} \leq C\varepsilon$$

Hence Step 1 is proved.

Step 2. We will show that $|\nabla_X^2 \tilde{V}| \leq C\varepsilon R^{-\delta_0}$.

It will suffice to prove that $|\nabla_X^2 \tilde{V}_1(X)| \leq C\varepsilon R^{-\delta_0}$ for $X \in G_{\frac{1}{2}}(r_0)$.

Motivated by the analysis in [3], we will consider the mixed boundary value problem (5.3) in the following domains in $G_{\frac{1}{2}}(r_0)$:

$$\begin{cases} D_m = \left\{X \in G(r_0), \frac{r_0}{2^{m+4}} \leq r \leq \frac{r_0}{2^{m+3}}\right\}, \qquad m = -2, -1, 0, 1, \cdots, N, \\ \tilde{D}_l = D_{l-1} \cup D_l \cup D_{l+1}, \qquad l = 0, 1, \cdots, N. \\ \tilde{\tilde{D}}_l = D_{l-2} \cup \tilde{D}_l \cup D_{l+2} \quad . \end{cases} \tag{5.8}$$

We then rescale the independent variables by the transformation

$$X = \frac{\tilde{X}}{2^l} \quad . \tag{5.9}$$

This transformation changes D_l, $\tilde{D}_l$, and $\tilde{\tilde{D}}_l$ onto D_0, $\tilde{D}_0$, and $\tilde{\tilde{D}}_0$ respectively. Furthermore, set

$$\bar{V}_1(\tilde{X}) = \tilde{V}_1(\frac{\tilde{X}}{2^l}) \qquad for \qquad \tilde{X} \in \tilde{\tilde{D}}_0 \quad .$$

Then it follows from (5.3) that on $\tilde{\tilde{D}}_0$, $\bar{V}_1(\tilde{X})$ solves the following elliptic problem:

$$\begin{cases} \tilde{a}_{11}(\tilde{X})\partial^2_{\tilde{X}_1}\bar{V}_1 + 2\tilde{a}_{12}(\tilde{X})\partial^2_{\tilde{X}_1\tilde{X}_2}\bar{V}_1 + \tilde{a}_{22}(\tilde{X})\partial^2_{\tilde{X}_2}\bar{V}_1 = \frac{1}{2^{2l}}\bar{F}_1(\tilde{X}), \\ \partial_{\tilde{X}_1}\bar{V}_1 + \bar{B}_{11}(\tilde{X})\partial_{\tilde{X}_2}\bar{V}_1 = \frac{1}{2^l}\bar{q}_1(\tilde{X}) \qquad on \qquad \tilde{X}_1 = 0, \\ \partial_{\tilde{X}_2}\bar{V}_1 + \bar{B}_{22}(\tilde{X})\partial_{\tilde{X}_1}\bar{V}_1 = \frac{1}{2^l}\bar{q}_2(\tilde{X}) \qquad on \qquad \tilde{X}_2 = 0, \end{cases} \tag{5.10}$$

where $\tilde{a}_{ij}(\tilde{X}) = a_{ij}(\frac{\tilde{X}}{2^l})$, $\bar{F}_1(\tilde{X}) = \tilde{F}_1(\frac{\tilde{X}}{2^l})$, $\bar{B}_{ii}(\tilde{X}) = \tilde{B}_{ii}(\frac{\tilde{X}}{2^l})$, and $\bar{q}_i(\tilde{X}) = \tilde{q}_i(\frac{\tilde{X}}{2^l})$, $i, j = 1, 2$.

Using the structures of those coefficients in (5.10), Lemma 3.3-3.4, Proposition 4.1, and the facts in the proof of Step 1, one can show by lengthy computations that

$$\begin{cases} \|\tilde{a}_{ij}\|_{C^{1-\delta_0}(\tilde{\tilde{D}}_0)} \leq C, \\ \|\bar{B}_{ii}\|_{C^{1,1-\delta_0}(\tilde{\tilde{D}}_0)} \leq CM\varepsilon, \\ \|\bar{q}_i\|_{C^{1,1-\delta_0}(\tilde{\tilde{D}}_0)} \leq C\varepsilon 2^{-l(1-\delta_0)}, \\ \|\bar{F}_1\|_{C^{1-\delta_0}(\tilde{\tilde{D}}_0)} \leq C\varepsilon. \end{cases} \tag{5.11}$$

It follows from the classical Schauder estimates on the second order elliptic equations with the uniform oblique derivative boundary conditions (see Lemma 1 in [19]) and (5.10)-(5.11) that there exists a uniform positive constant C such that

$$\begin{aligned} \|\bar{V}_1\|_{C^{2,1-\delta_0}(\tilde{D}_0)} \leq C\Big\{ & \|\bar{V}_1\|_{L^\infty(\tilde{\tilde{D}}_0)} + \frac{1}{2^l}\|\bar{q}_1\|_{C^{1,1-\delta_0}(\tilde{\tilde{D}}_0)} \\ & + \frac{1}{2^l}\|\bar{q}_2\|_{C^{1,1-\delta_0}(\tilde{\tilde{D}}_0)} + \frac{1}{2^{2l}}\|\bar{F}_1\|_{C^{1-\delta_0}}(\tilde{\tilde{D}}_0)\Big\} \\ \leq & C(\|\bar{V}_1\|_{L^\infty}(\tilde{\tilde{D}}_0) + \varepsilon 2^{-l(2-\delta_0)} + \varepsilon 2^{-2l}). \end{aligned}$$

On the other hand, it follows from $\tilde{V}_1(0) = 0$ and the estimate in Step 1 that

$$\|\tilde{V}_1\|_{L^\infty(\tilde{\tilde{D}}_0)} \leq C\varepsilon 2^{-l(2-\delta_0)} \quad .$$

Consequently, we arrive at

$$\|\bar{V}_1\|_{C^{2,1-\delta_0}(\tilde{D}_0)} \leq C_0\varepsilon 2^{-l(2-\delta_0)} \quad . \tag{5.12}$$

Since $\nabla^k_{\tilde{X}}\tilde{V}_1(\tilde{X}) = \frac{1}{2^{lk}}\nabla^k_X\tilde{V}_1(X)$, one can deduce easily from (5.12) that

$$|\nabla^k_X\tilde{V}_1(X)| \leq C\varepsilon 2^{-l(2-k-\delta_0)} \qquad for \qquad X \in \tilde{D}_l, \qquad k = 1, 2 \quad .$$

In particular, one has shown

$$|\nabla^2_X\tilde{V}_1(X)| \leq C\varepsilon R^{-\delta_0}, \qquad X \in G_{\frac{1}{2}}(r_0) \quad . \tag{5.13}$$

A similar analysis shows that $|\nabla^2_X\tilde{V}_i| \leq C\varepsilon R^{-\delta}$ holds for $i = 2, 3, 4$. Therefore the claim in Step 2 is proved.

Step 3. It holds that $|\nabla^3_X\tilde{V}(X)| \leq C\varepsilon R^{-1-\delta_0}$ for an uniform positive constant C.

As in Step 2, it suffices to show that

$$|\nabla^3_X\tilde{V}_1(X)| \leq C\varepsilon R^{-1-\delta_0} \qquad for \qquad X \in G_{\frac{1}{2}}(r_0) \quad .$$

To this end, we consider U_1 and U_2 introduced in Step 1. As in Step 2, we note that in $\tilde{D}_0$, the function $\tilde{U}_1(\tilde{X}) = U_1(\frac{\tilde{X}}{2^l})$ solves the following problem:

$$\begin{cases} \tilde{A}_{11}(\tilde{X})\partial^2_{\tilde{X}_1}\tilde{U}_1 + 2\tilde{A}_{12}(\tilde{X})\partial^2_{\tilde{X}_1\tilde{X}_2}\tilde{U}_1 + \tilde{A}_{22}(\tilde{X})\partial^2_{\tilde{X}_2}\tilde{U}_1 + \frac{1}{2^l}\tilde{A}_1(\tilde{X})\partial_{\tilde{X}_1}\tilde{U}_1 \\ \qquad + \frac{1}{2^l}\tilde{A}_2(\tilde{X})\partial_{\tilde{X}_2}\tilde{U}_1 = \frac{1}{2^{2l}}\tilde{H}_1(\tilde{X}), \\ \tilde{U}_1(\tilde{X}) = \tilde{q}_1(\tilde{X}), \qquad on \qquad \tilde{X}_1 = 0, \\ \partial_{\tilde{X}_2}\tilde{U}_1 + \tilde{T}(\tilde{X})\partial_{\tilde{X}_1}\tilde{U}_1 + \frac{1}{2^l}\tilde{d}_1(\tilde{X})\tilde{U}_1 = \frac{1}{2^l}\tilde{g}_1(\tilde{X}) \qquad on \qquad \tilde{X}_2 = 0, \end{cases} \tag{5.14}$$

where $\tilde{g}_{ij}(\tilde{X}) = g(\frac{\tilde{X}}{2^l}) - d_2(\frac{\tilde{X}}{2^l})\tilde{q}_2(\frac{\tilde{X}}{2^l})$, $\bar{q}_1(\tilde{X}) = \tilde{q}_1(\frac{\tilde{X}}{2^l})$, and

$$(\tilde{A}_{ij}, \tilde{A}_i, \tilde{H}_1, \tilde{T}, \tilde{d}_1)(\tilde{X}) = (A_{ij}, A_i, H_1, T, d_1)(\frac{\tilde{X}}{2^l}).$$

Due to the structures of these coefficients as given in Step 1, one can show by using Lemmas 3.3-3.4, (5.12), and (5.13) that

$$\begin{cases} \|\tilde{A}_{ij}\|_{C^{1-\delta_0}}(\tilde{D}_0) \le C, \|\tilde{A}_i\|_{C^{1-\delta_0}}(\tilde{D}_0) \le CM\varepsilon 2^{2\delta_0 l}, & i,j=1,2 \\ \|\tilde{T}\|_{C^{1,1-\delta_0}}(\tilde{D}_0) \le CM\varepsilon, \|\tilde{d}_i\|_{C^{1,1-\delta_0}}(\tilde{D}_0) \le CM\varepsilon 2^{2\delta_0 l}, & i=1,2 \\ \|\tilde{g}_1\|_{C^{1,1-\delta_0}}(\tilde{D}_0) \le C\varepsilon 2^{l\delta_0}, \\ \|\bar{q}_1\|_{C^{2,\delta_0}}(\tilde{D}_0) \le C\varepsilon 2^{-l(1-\delta_0)}, \\ \|\tilde{H}_1\|_{C^{1-\delta_0}}(\tilde{D}_0) \le C\varepsilon 2^{l\delta_0} \end{cases} \tag{5.15}$$

Now, by the classical Schauder estimates (see Lemma 1 in [19] and [1,3]) applying to (5.14) on domains D_0 and $\tilde{D}_0$, one gets

$$\begin{aligned} \|\tilde{U}_1\|_{C^{2,\delta_0}(D_0)} &\le C(\|\tilde{U}_1\|_{L^\infty(\tilde{D}_0)} + \frac{1}{2^{2l}}\|\tilde{H}_1\|_{C^{\delta_0}(\tilde{D}_0)} + \frac{1}{2^l}\|\tilde{g}_1\|_{C^{1,\delta_0}(\tilde{D}_0)} + \|\bar{q}_1\|_{C^{2,\delta_0}}(\tilde{D}_0)) \\ &\le C\varepsilon(\frac{1}{2^{l(1-\delta_0)}} + \frac{1}{2^{2l}}2^{\delta_0 l} + \frac{1}{2^l}2^{\delta_0 l} + 2^{-l(1-\delta_0)}) \\ &= C\varepsilon 2^{-l(1-\delta_0)}, \end{aligned}$$

where we have used (5.15), Step 1, (5.12) in Step 2, and the choice of δ_0 such that $\delta_0 < 1-\delta_0$. Thus we have obtained that

$$\|\tilde{U}_1\|_{C^{2,\delta_0}}(D_0) \le C\varepsilon 2^{-l(1-\delta_0)}. \tag{5.16}$$

As a consequence of (5.16), one gets

$$|\nabla_X^2 U_1(X)| \le C\varepsilon R^{-1-\delta_0} \qquad for \qquad X \in G_{\frac{1}{2}}(r_0) \tag{5.17}$$

In exactly same way, one can show that

$$|\nabla_X^2 U_2(X)| \le C\varepsilon R^{-1-\delta_0} \qquad for \qquad X \in G_{\frac{1}{2}}(r_0). \tag{5.18}$$

It follows from (5.17), (5.18), and the relations between (U_1, U_2) and $\nabla_X \tilde{V}_1$ that

$$|\nabla_X^3 \tilde{V}_1(X)| \le C\varepsilon R^{-1-\delta_0} \tag{5.19}$$

where we have also used Lemma 3.4.

Similar analysis yields that $|\nabla_X^3 \tilde{V}_i| \le C\varepsilon R^{-1-\delta_0}$ for $i = 2,3,4$. Consequently, we may conduct that

$$|\nabla_X^3 V| \le C\varepsilon R^{-1-\delta_0}. \tag{5.20}$$

Step 4. We claim that $\|\tilde{V}\|_{C^{1,1-\delta_0}(\bar{Q}_+)} \le C\varepsilon$.

To prove the claim, it follow from Proposition 4.1 and (5.4) that one needs only to prove

$$\sup_{X,Y\in Q_+}\frac{|\nabla\tilde V(X)-\nabla\tilde V(Y)|}{|X-Y|^{1-\delta_0}}\le C\varepsilon.$$

As in the previous steps, it suffices to show

$$\sup_{X,Y\in G_1(r_0)}\frac{|\nabla\tilde V_1(X)-\nabla\tilde V_1(Y)|}{|X-Y|^{1-\delta_0}}\le C\varepsilon. \tag{5.21}$$

Without loss of generality, we may assume $|Y|\le|X|$.

If $|Y|\le\frac12|X|$, then $|X-Y|\ge\frac12|X|$. Then it follows from Step 1 that

$$|\nabla\tilde V_1(X)-\nabla\tilde V_1(Y)|\le C\varepsilon(|X|^{1-\delta_0}+|Y|^{1-\delta_0})\le C|X|^{1-\delta_0},$$

hence (5.21) holds for $|Y|\le\frac12|X|$.

If $|Y|\ge\frac12|X|$, as in Step 2, one can consider the domain

$$\tilde D=\{\tilde X\in G_1(r_0):\frac{|X|}{2}\le|\tilde X|\le|X|\}.$$

Set

$$\tilde X_i=\frac{2|X|X_i'}{r_0},\qquad i=1,2.$$

Then the domain $\tilde D$ is changed into $D'=\{X':\frac{r_0}{4}\le|X'|\le\frac{r_0}{2}\}$.

Let $W_1(X')=\tilde V_1(\frac{2|X|}{r_0}X')$. Then a similar argument as in the derivation of (5.12) in Step 2, shows that

$$\|W_1(X')\|_{C^{2,1-\delta_0}(D')}\le C\varepsilon|X|^{2-\delta_0}. \tag{5.22}$$

Obviously, (5.22) implies the following fact

$$\sup_{X,Y\in\tilde D}\frac{|\nabla\tilde V_1(X)-\nabla\tilde V_1(Y)|}{|X-Y|^{1-\delta_0}}\le C\varepsilon.$$

This leads to (5.21). Therefore, $\|\tilde V_1\|_{C^{1,1-\delta_0}(\overline{G_1(r_0)})}\le C\varepsilon$. Namely, Step 4 is completed.

Step 5. It holds that

$$\sup_{X,Y\in\bar Q_+\setminus\cup_{i=1}^4P_i}\sum_{k=3}|d_{X,Y}|^{1+2\delta_0}\frac{|\nabla^k\tilde V(X)-\nabla^k\tilde V(Y)|}{|X-Y|^{\delta_0}}\le C\varepsilon.$$

As in Step 4, it suffices to show

$$\sup_{X,Y\in G_1(r_0)}\sum_{k=3}|d_{X,Y}|^{1+2\delta_0}\frac{|\nabla^k\tilde V_1(X)-\nabla^k\tilde V_1(Y)|}{|X-Y|^{\delta_0}}\le C\varepsilon$$

with $d_{X,Y}=min\{|X|,|Y|\}$.

Assume $|Y|\le|X|$. If $|Y|\le\frac12|X|$, then it follows from Step 3 that

$$\sum_{k=3}|d_{X,Y}|^{1+2\delta_0}\frac{|\nabla^k\tilde V_1(X)-\nabla^k\tilde V_1(Y)|}{|X-Y|^{\delta_0}}\le C\varepsilon|Y|^{1+\delta_0}(\frac{1}{|X|^{1+\delta_0}}+\frac{1}{|Y|^{1+\delta_0}})\le C\varepsilon$$

If $|Y| \geq \frac{|X|}{2}$, as in Step 3 and Step 4, set $W_{11}(X') = U_1(\frac{2|X|}{r_0}X')$ and $W_{12}(X') = U_2(\frac{2|X|}{r_0}X')$, one can obtain as for (5.16) in Step 3 that

$$\|W_{11}(X')\|_{C^{2,\delta_0}(D')} + \|W_{12}(X')\|_{C^{2,\delta_0}(D')} \leq C\varepsilon |X|^{1-\delta_0}. \tag{5.23}$$

This yields

$$|d_{X,Y}|^{1+2\delta_0} \sum_{|l|=2} \sum_{i=1}^{2} \frac{|\nabla^l U_i(X) - \nabla^l U_i(Y)|}{|X-Y|^{\delta_0}} \leq C|X|^{1+2\delta_0} \frac{1}{|X|^{2+\delta_0}} \sum_{i=1}^{2} \|W_{1i}(X')\|_{C^{2,\delta_0}(D')} \leq C\varepsilon.$$

Since

$$\begin{cases} \partial_{X_1}\tilde{V}_1 = \frac{U_1 - \tilde{B}_{11}U_2}{1-\tilde{B}_{11}\tilde{B}_{33}} \\ \partial_{X_2}\tilde{V}_1 = \frac{U_2 - \tilde{B}_{33}U_2}{1-\tilde{B}_{11}\tilde{B}_{33}} \end{cases}$$

then

$$\begin{aligned} \sum_{|k|=3} \nabla_X^k \tilde{V}_1 = &\sum_{|l|=2} (C_l^1(X)\nabla^l U_1 + C_l^2(X)\nabla^l U_2) + \sum_{|m|=1} (C_m^3(X)\nabla^m U_1 + C_m^4(X)\nabla^m U_2) \\ &+ C_5(X)U_1 + C_6(X)U_2 \end{aligned}$$

where

$$\begin{aligned} &\sum_{|l|=2} (|C_l^1(X)| + |C_l^2(X)|) \leq C, \qquad \sum_{|l|=2} (|\nabla C_l^1(X)| + |\nabla C_l^2(X)|) \leq \frac{CM\varepsilon}{|X|^{\delta_0}}, \\ &\sum_{|m|=1} (|\nabla^k C_m^3(X)| + |\nabla^k C_m^4(X)|) \leq \frac{CM\varepsilon}{|X|^{k+\delta_0}}, \qquad k = 0, 1, \\ &|\nabla^k C_5(X)| + |\nabla^k C_6(X)|) \leq \frac{CM\varepsilon}{|X|^{k+1+\delta_0}}, \qquad k = 0, 1. \end{aligned}$$

It then follows from Step 2 that

$$\begin{aligned} &|d_{X,Y}|^{1+2\delta_0} \sum_{k=3} \frac{|\nabla^k \tilde{V}_1(X) - \nabla^k \tilde{V}_1(Y)|}{|X-Y|^{\delta_0}} \\ &\leq C|X|^{1+2\delta_0} \left(\sum_{|l|=2} \sum_{i=1}^{2} \frac{|\nabla^l U_i(X) - \nabla^l U_i(Y)|}{|X-Y|^{\delta_0}} + \frac{CM\varepsilon^2}{|X|^{3\delta_0}} \right) \\ &\leq C\varepsilon \end{aligned}$$

Thus the claim in Step 5 is proved.

Combining Step 1 - Step 5, we have completed the proof of Proposition 5.1.

Based on Proposition 4.1 and Proposition 5.1, it follows from the standard continuity method as given in [14] (or see Lemma 2.3 in [21]) that the linear equation (3.5) with the standard boundary conditions (3.8)-(3.12) is solvable in the set $\mathbb{K}$. Furthermore, Proposition 4.1 and Proposition 5.1 imply that one can choose the constant C_0 as the constant M in $\mathbb{K}$. Hence the mapping J in (3.4) is well-defined and maps from $\mathbb{K}$ into $\mathbb{K}$. Furthermore, one has

Lemma 5.2. *J is a continuous mapping from $\mathbb{K} \to \mathbb{K}$.*

Proof. To prove Lemma 5.2, we need to verify the assertion:

If $W_l(X), W_0(X) \in \mathbb{K}$ and $W_l(X) \to W_0(X)$ in $\mathbb{K}$ as $l \to \infty$, then the corresponding solutions $V_l(X) \to V_0(X)$ in $\mathbb{B}$. (5.24)

First, it follows from Proposition 5.1 that $\{V_l(X)\}_{l=1}^{\infty}$ and $V_0(X)$ are uniformly bounded in $\mathbb{K}$. To prove (5.24), it suffices to show $\|V_l(X) - V_0(X)\|_{C(\bar{Q}_+)} \to 0$ by the interpolation inequality on the weighted Holder space(see [12]).

Set $\bar{V}_l = V_l(X) - V_0(X)$. Then $\bar{V}_l$ solves the following problem

$$\begin{cases} a_{11}(X, W_l, \nabla_X W_l)\partial^2_{X_1}\bar{V}_l + 2a_{12}(X, W_l, \nabla_X W_l)\partial^2_{X_1X_2}\bar{V}_l + a_{22}(X, W_l, \nabla_X W_l)\partial^2_{X_2}\bar{V}_l \\ \quad +F^1_l(X)\partial_{X_1}(W_l - W_0) + F^2_l(X)\partial_{X_2}(W_l - W_0) + F^0_l(X)(W_l - W_0) = 0 \\ \partial_{X_1}\bar{V}_l + \tilde{B}_{11}(X, W_l, \nabla_X W_l)\partial_{X_2}\bar{V}_l + B^1_{l1}(X)\partial_{X_1}(W_l - W_0) + B^2_{l1}(X)\partial_{X_2}(W_l - W_0) \\ \quad +B^0_{l1}(X)(W_l - W_0) = 0 \qquad on \quad X_1 = 0 \\ \partial_{X_1}\bar{V}_l + \tilde{B}_{22}(X, W_l, \nabla_X W_l)\partial_{X_2}\bar{V}_l + B^1_{l2}(X)\partial_{X_1}(W_l - W_0) + B^2_{l2}(X)\partial_{X_2}(W_l - W_0) \\ \quad +B^0_{l2}(X)(W_l - W_0) = 0 \qquad on \quad X_1 = 1 \\ \partial_{X_2}\bar{V}_l + \tilde{B}_{33}(X, W_l, \nabla_X W_l)\partial_{X_1}\bar{V}_l + B^1_{l3}(X)\partial_{X_1}(W_l - W_0) + B^2_{l3}(X)\partial_{X_2}(W_l - W_0) \\ \quad +B^0_{l3}(X)(W_l - W_0) = 0 \qquad on \quad X_1 = -1 \\ \partial_{X_2}\bar{V}_l + \tilde{B}_{44}(X, W_l, \nabla_X W_l)\partial_{X_1}\bar{V}_l + B^1_{l4}(X)\partial_{X_1}(W_l - W_0) + B^2_{l4}(X)\partial_{X_2}(W_l - W_0) \\ \quad +B^0_{l4}(X)(W_l - W_0) = 0 \qquad on \quad X_1 = 1 \\ \bar{V}_l(0) = 0 \end{cases} \tag{5.25}$$

where

$$\begin{aligned} F^i_l(X) &= \sum_{j=1}^{2} \partial^2_{X_j} V_0 \int_0^1 (\partial_{W_i} a_{jj})(X, \theta W_l + (1-\theta)W_0, \theta\nabla W_l + (1-\theta)\nabla W_0)d\theta \\ &\quad + 2\partial^2_{X_1X_2} V_0 \int_0^1 (\partial_{W_i} a_{12})(X, \theta W_l + (1-\theta)W_0, \theta\nabla W_l + (1-\theta)\nabla W_0)d\theta \\ &\quad + \int_0^1 (\partial_{W_i} F_0)(X, \theta W_l + (1-\theta)W_0, \theta\nabla W_l + (1-\theta)\nabla W_0)d\theta, \qquad i = 0, 1, 2 \\ B^i_{lj}(X) &= \partial_{X_2} V_0 \int_0^1 (\partial_{W_i} \tilde{B}_{jj})(X, \theta W_l + (1-\theta)W_0, \theta\nabla W_l + (1-\theta)\nabla W_0)d\theta \\ &\quad + \int_0^1 (\partial_{W_i} \tilde{B}_j)(X, \theta W_l + (1-\theta)W_0, \theta\nabla W_l + (1-\theta)\nabla W_0)d\theta, \qquad i = 0, 1, 2; j = 1, 2 \\ B^i_{lj}(X) &= \partial_{X_1} V_0 \int_0^1 (\partial_{W_i} \tilde{B}_{jj})(X, \theta W_l + (1-\theta)W_0, \theta\nabla W_l + (1-\theta)\nabla W_0)d\theta \\ &\quad + \int_0^1 (\partial_{W_i} \tilde{B}_j)(X, \theta W_l + (1-\theta)W_0, \theta\nabla W_l + (1-\theta)\nabla W_0)d\theta, \qquad i = 0, 1, 2; j = 3, 4 \end{aligned}$$

here we have used the notations $(W_0, W_1, W_2) = (W(X), \partial_{X_1} W, \partial_{X_2} W)$ for convenience.

It follows from the expressions of $F^i_l(X)$ and $B^i_{lj}(X)$ and Lemmas 3.2 - 3.4 that

$$\begin{aligned} &|F^i_l(X)| \le \frac{CM\varepsilon}{R^{\delta_0}}, \\ &|\nabla_X F^i_l(X)| \le \frac{CM\varepsilon}{R^{1+\delta_0}}, \\ &|B^i_{lj}(X)| \le CM\varepsilon, \\ &|\nabla_X B^i_{lj}(X)| \le \frac{CM\varepsilon}{R^{\delta_0}}. \end{aligned}$$

Therefore, analyzing in a similar way as in the proof of Proposition 4.1 (in fact, it is even simpler), we can obtain

$$\int\int_{Q_+}(|\bar{V}_l|^2+|\nabla_X\bar{V}_l|^2+|\nabla_X^2\bar{V}_l|^2)dX\le C(\|W_l-W_0\|^2_{C^{1,1-\delta_0}(\bar{Q}_+)}+\sum_{k=2}\|R^{\delta_0}\nabla_X^k(W_l-W_0)\|^2_{L^\infty(Q_+)})$$

This, together with the Sobolev's imbedding theorem, implies

$$\|V_l(X)-V_0(X)\|_{C(\bar{Q}_+)}\to 0 \qquad as \qquad l\to\infty$$

Hence we complete the proof of Lemma 5.2.

Proof of existence in Theorem 2.2.

It follows from Proposition 4.1, Proposition 5.1 and Lemma 5.2 that the mapping J satisfies all the requirements of Theorem 3.1. By the choice of J, one can obtain the existence of solution in Theorem 2.2.

It remains to prove the uniqueness of solution in Theorem 2.2. This will be given in the next section.

§6. The proof of uniqueness for Theorem 2.2.

This section is devoted to prove the uniqueness of the solution to the nonlinear elliptic equation (2.4) with the boundary conditions (2.5)-(2.9).

Proposition 6.1. (Uniqueness) If ε is a sufficiently small constant depending on M,ρ_+,q_+ and δ_0, then there exists at most one solution $V(X)$ to equation (2.4) with the boundary conditions (2.5)-(2.9) such that

$$\|V(X)-1\|_{C^{1,1-\delta_0}(\bar{Q}_+)}\le M\varepsilon,\qquad |\nabla_X^2V(X)|\le\frac{M\varepsilon}{R^{\delta_0}},\qquad |\nabla_X^3V(X)|\le\frac{M\varepsilon}{R^{1+\delta_0}}. \tag{6.1}$$

Proof. If there exist two solutions $V_1(X)$ and $V_2(X)$, to the problem (2.4) - (2.9), both satisfy the estimtes (6.1), one then can check through the expressions in (2.3) and the appendix that $v(X)=V_1(X)-V_2(X)$ solves the following problem

$$\begin{cases} a_{11}(X,V_1,\nabla_XV_1)\partial^2_{X_1}v+2a_{12}(X,V_1,\nabla_XV_1)\partial^2_{X_1X_2}v+a_{22}(X,V_1,\nabla_XV_1)\partial^2_{X_2}v+b_1(X)\partial_{X_1}v \\ \qquad +b_2(X)\partial_{X_2}v+c(X)v=0, \qquad\qquad X\in Q_+ \\ \partial_{X_1}v+\gamma_1(X)\partial_{X_2}v+d_1(X)v=0, \qquad on \qquad X_1=0, \\ \partial_{X_1}v+\gamma_2(X)\partial_{X_2}v+d_2(X)v=0, \qquad on \qquad X_1=1, \\ \partial_{X_2}v+\bar{\gamma}_1(X)\partial_{X_1}v+\bar{d}_1(X)v=0, \qquad on \qquad X_2=-1, \\ \partial_{X_2}v+\bar{\gamma}_2(X)\partial_{X_1}v+\bar{d}_2(X)v=0, \qquad on \qquad X_2=1, \\ v(0)=0, \end{cases} \tag{6.2}$$

with $b_i(X),c(X)\in C^1(\bar{Q}_+\setminus\cup_{i=1}^4P_i)$ and $\gamma_i(X),d_i(X),\bar{\gamma}_i(X),\bar{d}_i(X)\in C^2(\bar{Q}_+\setminus\cup_{i=1}^4P_i)\cap C^{1-\delta_0}(\bar{Q}_+)$ such that

$$\begin{cases} \sum_{i=1}^{2}|\nabla_X^k b_i(X)|+|\nabla_X^k c(X)|\le \frac{C\varepsilon}{R^{k+\delta_0}}, \qquad k=0,1\\ \sum_{i=1}^{2}(\|\gamma_i\|_{C^{1-\delta_0}}+\|\bar\gamma_i\|_{C^{1-\delta_0}}+\|d_i\|_{C^{1-\delta_0}}+\|\bar d_i\|_{C^{1-\delta_0}})\le C\varepsilon\\ \sum_{i=1}^{2}(|\nabla_X^k\gamma_i|+|\nabla_X^k\bar\gamma_i|+|\nabla_X^k d_i|+|\nabla_X^k\bar d_i|)\le \frac{C\varepsilon}{R^{k-1+\delta_0}}, \qquad k=1,2. \end{cases} \tag{6.3}$$

Here and below the generic constant C can depend on M.

Although the problem (6.2) is similar to the equation (3.5) with the boundary conditions (3.8)-(3.12), there are some differences because (6.2) contains the low regularity coefficients in the first order derivative $\nabla_X v(X)$ and the function $v(X)$ itself. Hence additional care is required to treat the problem (6.2). In addition, it seems to be difficult to use the maximum principle to prove the uniqueness of solution to (6.2) since $c(X), d_i(X)$ and $\bar d_i(X)$ can change their signs. Here we would like to emphasize that it is critical to choose the "good" transformation (2.2) to study the uniqueness since we can obtain the "smallness" estimates on $c(X), d_i(X)$ and $\bar d_i(X)$ in some appropriate sense, which are the keys to show $v(X)\equiv 0$ in (6.2).

Set

$$\begin{cases} v_1=\partial_{X_1}v+((1-X_1)\gamma_1(X)+X_1\gamma_2(X))\partial_{X_2}v+((1-X_1)d_1(X)+X_1d_2(X))v,\\ v_2=\partial_{X_2}v+\frac12((1-X_2)\bar\gamma_1(X)+(1+X_2)\bar\gamma_2(X))\partial_{X_1}v+\frac12((1-X_2)\bar d_1(X)+(1+X_2)\bar d_2(X))v. \end{cases} \tag{6.4}$$

As in §4, one can show that v_1 and v_2 solve the following problems respectively:

$$\begin{cases} A_{11}(X)\partial^2_{X_1}v_1+2A_{12}(X)\partial^2_{X_1X_2}v_1+A_{22}(X)\partial^2_{X_2}v_1+A_1(X)\partial_{X_1}v_1+A_2(X)\partial_{X_2}v_1+A_3(X)\partial_{X_1}v_2\\ \qquad +A_4(X)\partial_{X_2}v_2+c_1(X)v_1+c_2(X)v_2+c_0(X)v=0,\\ v_1=0, \qquad \text{on} \qquad X_1=0 \qquad \text{and} \qquad X_1=1,\\ \partial_{X_2}v_1+T(X)\partial_{X_1}v_1+e_1(X)v_1+e_2(X)v_2+e_0(X)v=0 \qquad \text{on} \qquad X_2=-1 \qquad \text{and} \qquad X_2=1, \end{cases} \tag{6.5}$$

and

$$\begin{cases} \bar A_{11}(X)\partial^2_{X_1}v_2+2\bar A_{12}(X)\partial^2_{X_1X_2}v_2+\bar A_{22}(X)\partial^2_{X_2}v_2+\bar A_1(X)\partial_{X_1}v_1+\bar A_2(X)\partial_{X_2}v_1+\bar A_3(X)\partial_{X_1}v_2\\ \qquad +\bar A_4(X)\partial_{X_2}v_2+\bar c_1(X)v_1+\bar c_2(X)v_2+\bar c_0(X)v=0,\\ \partial_{X_1}v_2+\bar T(X)\partial_{X_2}v_2+\bar e_1(X)v_1+\bar e_2(X)v_2+\bar e_0(X)v=0 \qquad \text{on} \qquad X_1=0 \qquad \text{and} \qquad X_1=1\\ v_2=0, \qquad \text{on} \qquad X_2=-1 \qquad \text{and} \qquad X_2=1, \end{cases} \tag{6.6}$$

where $A_{ij}(X), \bar A_{ij}(X)\in C^{1-\delta_0}(\bar Q_+)\cap C^2(\bar Q_+\setminus\cup_{i=1}^4P_i)$; $A_i(X), \bar A_i(X)\in C^1(\bar Q_+\setminus\cup_{i=1}^4P_i)$; $c_i(X), \bar c_i(X)\in C(\bar Q_+\setminus\cup_{i=1}^4P_i)$; $T(X)\in C^{1-\delta_0}(\bar Q_+)\cap C^2(\bar Q_+\setminus\cup_{i=1}^4P_i)$ and $e_i(X)\in C^1(\bar Q_+\setminus\cup_{i=1}^4P_i)$. Moreover,

$$\begin{cases} A_{11}(X) = (c_+^2 - q_+^2)(1 + O(M\varepsilon)), \\ A_{12}(X) = O(M\varepsilon), \\ A_{22}(X) = c_+^2(1 + O(M\varepsilon)), \\ |\nabla_X A_{ij}(X)| \le \frac{C\varepsilon}{R^{\delta_0}}, \\ \displaystyle\sum_{i=1}^{4} |A_i(X)| \le \frac{C\varepsilon}{R^{\delta_0}}, \\ \displaystyle\sum_{i=0}^{2} |c_i(X)| \le \frac{C\varepsilon}{R^{1+\delta_0}}, \\ |T(X)| \le C\varepsilon, \\ |\nabla_X T(X)| \le \frac{C\varepsilon}{R^{\delta_0}}, \\ \displaystyle\sum_{i=0}^{2} |e_i(X)| \le \frac{C\varepsilon}{R^{\delta_0}}, \end{cases} \tag{6.7}$$

while $\bar{A}_{ij}(X)$, $\bar{A}_i(X)$, $\bar{c}_i(X)$, $\bar{T}(X)$, and $\bar{e}_i(X)$ have the same properties as $A_{ij}(X)$, $A_i(X)$, $c_i(X)$, $T(X)$, and $e_i(X)$ do respectively.

Multiplying the two equations in (6.5) and (6.6) on both sides by v_1 and v_2 respectively, integrating by parts over the domain Q_+, and using the properties in (6.7), one can derive in a similar way as in §4 that

$$\int\int_{Q_+} (|\nabla_X v_1|^2 + |\nabla_X v_2|^2) dX \le C\Big\{\varepsilon\Big(\int\int_{Q_+} \frac{|v_1|^2 + |v_2|^2}{R^{1+\delta_0}} dX + \int\int_{Q_+} \frac{|v|^2}{R^{1+\delta_0}} dX\Big) + |Q_1| + |\bar{Q}_1|\Big\} \tag{6.8}$$

with

$$Q_1 = \int_0^1 \{(A_{22}\partial_{X_2} v_1 v_1)(X_1, 1) - (A_{22}\partial_{X_2} v_1 v_1)(X_1, -1)\} dX_1,$$
$$\bar{Q}_1 = \int_{-1}^1 \{(\bar{A}_{11}\partial_{X_1} v_2 v_2)(1, X_2) - (\bar{A}_{11}\partial_{X_1} v_2 v_2)(0, X_2)\} dX_2.$$

Since $v_1 = 0$ on $X_1 = 0$ and $v_2 = 0$ on $X_2 = -1$, then Poincare's inequality implies

$$\Big(\int\int_{Q_+} (|v_1|^p + |v_2|^p) dX\Big)^{\frac{1}{p}} \le C_p \Big(\int\int_{Q_+} (|\nabla_X v_1|^2 + |\nabla_X v_2|^2) dX\Big)^{\frac{1}{2}} \qquad for \qquad p \in [1, +\infty) \tag{6.9}$$

In addition, as in §4, in light of $v(0) = 0$, one shows that

$$\int\int_{Q_+} |v|^2 dX \le C \int\int_{Q_+} (|\nabla_X v|^2 + |\nabla_X^2 v|^2) dX. \tag{6.10}$$

From the definitions of v_1 and v_2, one has

$$\begin{cases} \partial_{X_1} v = r_1(X) v_1 + r_2(X) v_2 + r_3(X) v \\ \partial_{X_2} v = \bar{r}_1(X) v_2 + \bar{r}_2(X) v_1 + \bar{r}_3(X) v \end{cases} \tag{6.11}$$

where $r_i(X), \bar{r}_i(X) \in C^2(\bar{Q}_+ \setminus \cup_{i=1}^4 P_i)) \cap C^{1-\delta_0}(\bar{Q}_+)$ such that

$$|r_1| + |\bar{r}_1| \le C, \qquad \sum_{i=2}^{3} (|r_i| + |\bar{r}_i|) \le C\varepsilon, \qquad \sum_{i=1}^{3} (|\nabla_X r_i| + |\nabla_X \bar{r}_i|) \le \frac{C\varepsilon}{R^{\delta_0}}.$$

Substituting (6.11) into (6.10) yields

$$\int\int_{Q_+} |v|^2 dX \le C(\int\int_{Q_+} (|v_1|^2 + |v_2|^2 + |\nabla_X v_1|^2 + |\nabla_X v_2|^2) dX + \varepsilon^2 \int\int_{Q_+} \frac{|v_1|^2 + |v_2|^2}{R^{2\delta_0}} dX$$
$$+ \varepsilon^2 \int\int_{Q_+} \frac{|v|^2}{R^{2\delta_0}} dX). \tag{6.12}$$

It follows from the Sobolev's imbedding theorem and (6.11) that

$$\Big(\int\int_{Q_+} |v|^p dX\Big)^{\frac{1}{p}} \le C_p \Big(\int\int_{Q_+} (|v|^2 + |\nabla_X v|^2) dX\Big)^{\frac{1}{2}}$$
$$\le C_p \Big(\int\int_{Q_+} (|v|^2 + |v_1|^2 + |v_2|^2) dX\Big)^{\frac{1}{2}}, \qquad p \in (1, +\infty). \tag{6.13}$$

Combining the Holder inequality with (6.13), one has from (6.12) that

$$\int\int_{Q_+} |v|^2 dX \le C(\int\int_{Q_+} (|v_1|^2 + |v_2|^2 + |\nabla_X v_1|^2 + |\nabla_X v_2|^2) dX. \tag{6.14}$$

Taking into account of (6.9), (6.13) and (6.4), we conclude from (6.8) that

$$\int\int_{Q_+} (|\nabla_X v_1|^2 + |\nabla_X v_2|^2) dX \le C(|Q_1| + |\bar{Q}_1|). \tag{6.15}$$

Next we analyze Q_1 and $\bar{Q}_1$.

It follows from the boundary condition on v_1 and integration by parts that

$$|\int_0^1 (A_{22} \partial_{X_2} v_1 v_1)(X_1, 1) dX_1| \le C\varepsilon \int_0^1 \Big(\frac{|v|^2 + |v_2|^2}{R^{\delta_0}}\Big)(X_1, 1) dX_1. \tag{6.16}$$

By the Holder inequality and the trace theorem, one gets

$$\int_0^1 \Big(\frac{|v_1|^2}{R^{\delta_0}}\Big)(X_1, 1) dX_1 \le (\int_0^1 \frac{1}{R^{2\delta_0}})^{\frac{1}{2}} (\int_0^1 |v_1|^4 (X_1, 1) dX_1)^{\frac{1}{2}}$$
$$\le C \int\int_{Q_+} (|v_1|^2 + |\nabla_X v_1|^2) dX. \tag{6.17}$$

Similarly, it follows from the trace theorem, (6.13) and (6.14) that

$$\int_0^1 \Big(\frac{|v|^2}{R^{\delta_0}}\Big)(X_1, 1) dX_1 \le C \int\int_{Q_+} (|v|^2 + |\nabla v|^2) dX$$
$$\le C \int\int_{Q_+} (|v_1|^2 + |v_2|^2 + |\nabla v_1|^2 + |\nabla v_2|^2) dX. \tag{6.18}$$

The other terms in Q_1 and $\bar{Q}_1$ can be treated similarly. Hence we can derive from (6.15)-(6.18) that

$$\int\int_{Q_+} (|\nabla_X v_1|^2 + |\nabla_X v_2|^2) dX = 0$$

It follows from (6.9) that

$$v_1 = v_2 \equiv 0$$

Finally, (6.14) implies

$$v \equiv 0$$

Namely, Lemma 6.1 is proved.

Now, the uniqueness in Theorem 2.2 follows from Lemma 6.1.

§7. The proof of Theorem 1.1 and remarks

In this section, we will give the proof of Theorem 1.1.

Proof of Theorem 1.1.

First, the regularity and estimates on the supersonic flow, (i) in Theorem 1.1, follow from Lemma 2.1 directly.

Next, by the regularity and uniqueness of $V(X)$ in Theorem 2.2, we conclude that the inverse transformation, (2.3), has the following properties:

$$x_1(X), x_2(X) \in C^{1,1-\delta_0}(\bar{Q}_+) \cap C^{3,\delta_0}(Q_+) \tag{7.1}$$

Since the shock $x_1 = \xi(x_2)$ corresponds to $X_1 = 0$ in $\bar{Q}_+$, then $\xi(x_2) \in C^{1,1-\delta_0}[x_2^1, x_2^2] \cap C^{3,\delta_0}(x_2^1, x_2^2)$, where $(x_1^1, x_2^1) = (x_1(P_1), x_2(P_1))$ and $(x_1^2, x_2^2) = (x_1(P_2), x_2(P_2))$ with $P_1 = (0, -1)$ and $P_2 = (0, 1)$. Now, the other conclusions in Theorem 1.1.(ii), (iii) and (iv) follow easily from the properties of $V(X)$ in Theorem 2.2. We omit these simple proofs. Thus the proof of Theorem 1.1 is completed.

We now conclude this section by giving a remark.

Remark. 7.1. If instead of the boundary condition (1.11), one imposes a Dirichlet condition on the potential $\varphi_+(x)$ at the exit of the nozzle as in [7], then the similar conclusions as in Theorem 1.1 still hold. Furthermore, the general approach introduced in the previous sections still works except the analysis is much simpler in this case. This is due to the following reasons:

(i). If the potential $\varphi_+(x)$ is given at the exit of the nozzle, $x_1 = 1$, then, instead of the generalized hodograph transformation (2.2), we can use the following partial hodograph transformation

$$\begin{cases} X_1 = \varphi_-(x) - \varphi_+(x), \\ X_2 = \frac{2x_2 - (f_1 + f_2)(x_1)}{f_2(x_1) - f_1(x_1)}, \end{cases} \tag{7.2}$$

and it also suffices to take $x_1 = x_1(X)$ as the new unknown function for solving the induced nonlinear problem. In the present case, the resulting equation after the transformation (7.2) is much simpler than (2.4), and the boundary conditions, (2.5) - (2.9), are replaced by the analogues of (2.5), (2.7), (2.8), and $x_1(X) = 1$ on the exit of the nozzle.

(ii). Thus, in this case, the linearized problem, corresponding to (3.5), (3.8) - (3.12), becomes the following second linear order elliptic equation with the mixed boundary conditions

$$\begin{cases} -\sum_{i,j=1}^{2} a_{ij}(X)\partial_{ij}^2 u + \sum_{i=1}^{2} b_i \partial_i u + c(X)u = f(X), \qquad in \qquad \Omega \\ \sum_{i=1}^{2} \beta_i \partial_i u + \mu(X) u = g(X), \qquad on \qquad \partial\Omega_1 \\ u = g_2(X), \qquad on \qquad \partial\Omega_2 \end{cases} \tag{7.3}$$

where $c(X) \geq 0$, $\mu(X) \geq 0$, $\partial\Omega = \overline{\partial\Omega_1 \cup \partial\Omega_2}$, $\partial\Omega_1$ and $\partial\Omega_2$ are piecewise C^1, and $|\partial\Omega_1| \neq 0$, $|\partial\Omega_2| \neq 0$. The major advantage of (7.3) over (3.5) is the Dirichlet boundary condition on $\partial\Omega_2$. Furthermore, the boundary condition on $\partial\Omega_1$ is uniform oblique. For appropriate singularity assumptions on the coefficients of (7.3), the maximum principle is available for (7.3), so many difficulties encountered in the previous sections can be avoided.

However, as described in [10], it is more natural and physical to prescribe the pressure at the exit of the nozzle than prescribe the value of the potential function.

§8. **Appendix**

In this appendix, we will provide some details on the structures of the coefficients of the boundary-value problem, (1.5), (1.9) and (1.11) - (1.16) in the new variables after the generalized hodograph transformation (2.2). First, we will study some properties of the hodograph transformation. Second, we give the precise expressions of the coefficients in the equation (2.4). Finally, we will sketch the verifications of Lemma 3.2 - 3.4.

First, by direct computations using (2.2), (2.3), and the definition of the new dependent variable V, one gets

$$\begin{cases} \frac{\partial X_2}{\partial x_1} = \frac{2(f_1f_2'-f_2f_1')(x_1)-2x_2(f_2'-f_1')(x_1)}{(f_2-f_1)(x_1)} = O(\varepsilon), \\ \frac{\partial X_2}{\partial x_2} = \frac{2}{f_2(x_1)-f_1(x_1)} = 1+O(\varepsilon), \\ \frac{\partial X_1}{\partial x_1} = \frac{1}{V+(X_1-1)\partial_{X_1}V}(1-(X_1-1)\partial_{X_2}V\frac{\partial X_2}{\partial x_1}), \\ \frac{\partial X_1}{\partial x_2} = \frac{(1-X_1)\partial_{X_2}V}{V+(X_1-1)\partial_{X_1}V}\frac{\partial X_2}{\partial x_2}. \end{cases} \tag{8.1}$$

To compute the second order derivatives of the hodograph transformation, we use the following notations:

$$\begin{cases} D(X,V,\nabla_X V) = \frac{1}{V+(X_1-1)\partial_{X_1}V} \\ b_{lm}^{jk}(X,V,\nabla_X V) = \frac{1}{2}(1-X_1)D(X,V,\nabla_X V)(\frac{\partial X_l}{\partial x_j}\frac{\partial X_m}{\partial x_k}+\frac{\partial X_l}{\partial x_k}\frac{\partial X_m}{\partial x_j}), \qquad j,k,l,m=1,2 \\ b_0^{jk}(X,V,\nabla_X V) = -D(X,V,\nabla_X V)(2\frac{\partial X_1}{\partial x_j}\frac{\partial X_1}{\partial x_k}, \frac{\partial X_1}{\partial x_j}\frac{\partial X_2}{\partial x_k} \\ \qquad +\frac{\partial X_1}{\partial x_k}\frac{\partial X_2}{\partial x_j}+(X_1-1)\frac{\partial^2 X_2}{\partial x_j\partial x_k})\nabla_X V \end{cases} \tag{8.2}$$

If follows from (8.1), (8.2), and direct lengthy computations that

$$\begin{cases} \frac{\partial^2 X_2}{\partial x_2^2} = 0, \\ \frac{\partial^2 X_2}{\partial x_1 \partial x_2} = \frac{2(f_1'-f_2')(x_1)}{(f_2(x_1)-f_1(x_1))^2}|_{x_1=1+(X_1-1)V(X)} = O(\varepsilon) \\ \frac{\partial^2 X_2}{\partial x_1^2} = \frac{d}{dx_1}(\frac{2(f_1f_2'-f_2f_1')(x_1)-2x_2(f_2'-f_1')(x_1)}{(f_2(x_1)-f_1(x_1))^2}|_{x_1=1+(X_1-1)V(X)} = O(\varepsilon) \\ \frac{\partial^2 X_1}{\partial x_k \partial x_l} = \sum_{i=1}^{2} b_{ii}^{kl}(X,V,\nabla_X V)\partial_{X_i}^2 V + b_{12}^{kl}(X,V,\nabla_X V)\partial_{X_1X_2}^2 V \\ \qquad +b_0^{kl}(X,V,\nabla_X V), \qquad k,l=1,2 \end{cases} \tag{8.3}$$

Using (8.3) and (2.1), one can derive the new equation (2.4) from equation (1.5) by direct calculations. Furthermore, the coefficients in (2.4) are given by the following formula:

$$
\begin{aligned}
a_{ij}(X,V,\nabla_X V) = &-\sum_{k=1}^{2}((\partial_{x_k}\varphi_+)^2 - c_+^2)(\frac{\partial X_i}{\partial x_k}\frac{\partial X_j}{\partial x_k} + b_{ij}^{kk}\partial_1 V)\\
&+ 2\partial_{x_1}\phi_+\partial_{x_2}\phi_+(b_{ij}^{12}\partial_1 V + \frac{1}{2}(\frac{\partial X_i}{\partial x_1}\frac{\partial X_j}{\partial x_2} + \frac{\partial X_j}{\partial x_1}\frac{\partial X_i}{\partial x_2})), \qquad i,j = 1,2
\end{aligned}
\tag{8.4}
$$

and

$$
\begin{aligned}
F_0(X,V,\nabla_X V) = &-\nabla_X V(\sum_{i=1}^{2}((\partial_{x_i}\varphi_+)^2 - c_+^2)b_0^{ii} + 2\partial_{x_1}\varphi_+\partial_{x_2}\varphi_+ b_0^{12},\\
&\sum_{i=1}^{2}((\partial_{x_i}\varphi_+)^2 - c_+^2)\frac{\partial^2 X_2}{\partial x_i^2} + 2\partial_{x_1}\phi_+\partial_{x_2}\varphi_+\frac{\partial^2 X_2}{\partial x_1\partial x_2})\\
&-\sum_{i=1}^{2}((\partial_{x_i}\varphi_-)^2 - (\partial_{x_i}\varphi_+)^2 + c_+^2 - c_)^2 - 2(\partial_{x_1}\varphi_-\partial_{x_2}\varphi_- - \partial_{x_1}\varphi_+\partial_{x_2}\varphi_+)\partial^2_{x_1x_2}\varphi_- \quad ,
\end{aligned}
\tag{8.5}
$$

where $c_+ = c(H_+)$ and $H_+ = H(c_0 - \frac{1}{2}|\nabla\varphi_+|^2)$.

We are now ready to complete the proof of Lemma 3.2.

Proof of Lemma 3.2. First, one notes that (2.3) and (8.1) imply that

$$
\begin{cases}
\frac{\partial x_1}{\partial V} = X_1 - 1,\\
\frac{\partial x_2}{\partial V} = \frac{X_1-1}{2}\{X_2(f_2'(x_1) - f_2'(x_1)) + (f_1'(x_1) + f_2'(x_1))\}|_{x_1=1+(X_1-1)V}\\
\frac{\partial(\partial_{x_i}V)}{\partial(\partial_{X_j}V)} = \frac{V}{V+(X_1-1)\partial_{X_1}V}\frac{\partial X_j}{\partial x_i}, \qquad i,j=2\\
\frac{\partial(\partial_i\varphi_-)}{\partial V} = \sum_{j=1}^{2}\partial_{ij}^2\varphi_-(x)\frac{\partial x_j}{\partial V}, \qquad i=1,2\\
\frac{\partial(\partial_i\varphi_+)}{\partial V} = \sum_{j=1}^{2}\left(\partial_{ij}^2\varphi_-(x)\frac{\partial x_j}{\partial V} - \partial_{X_j}V\frac{\partial}{\partial V}(\frac{\partial X_j}{\partial x_i})\right), \qquad i=1,2.
\end{cases}
\tag{8.6}
$$

It follows from $G(X,V,\nabla_X V) \equiv \bar{G}(\nabla\varphi_+,\nabla\varphi_-) = \sum_{i=1}^{2}[\partial_{x_i}\varphi H]\partial_i(\varphi_+ - \varphi_-)$ that

$$
\begin{aligned}
&\partial_{\partial_{X_i}V}G = -\frac{V}{V+(X_1-1)\partial_{X_1}V}(\partial_{\partial_1\varphi_+}\bar{G}\frac{\partial X_i}{\partial x_1} + \partial_{\partial_2\varphi_+}\bar{G}\frac{\partial X_i}{\partial x_2}), \qquad i=1,2\\
&\partial_V G = \partial_{\partial_1\varphi_+}\bar{G}\frac{\partial(\partial_1\varphi_+)}{\partial V} + \partial_{\partial_2\varphi_+}\bar{G}\frac{\partial(\partial_2\varphi_+)}{\partial V} + \partial_{\partial_1\varphi_-}\bar{G}\frac{\partial(\partial_1\varphi_-)}{\partial V} + \partial_{\partial_2\varphi_-}\bar{G}\frac{\partial(\partial_2\varphi_-)}{\partial V},\\
&\partial_{\partial_i\varphi_+}\bar{G} = \sum_{j=1}^{2}\left([\partial_j\varphi H]\delta_{ij} + (H_+\delta_{ij} + \partial_j\phi_+\partial_i\varphi_+H_+')(\partial_j\varphi_+ - \partial_j\varphi_-)\right) \qquad i=1,2,\\
&\partial_{\partial_i\varphi_-}\bar{G} = -\sum_{j=1}^{2}\left([\partial_j\varphi H]\delta_{ij} + (H_-\delta_{ij} + \partial_j\varphi_-H_-'\partial_i\varphi_-)\partial_j(\varphi_+ - \varphi_-)\right) \qquad i=1,2,
\end{aligned}
\tag{8.7}
$$

Replacing $V(X)$ by $W(X) \in \mathbb{K}$ in the transformation (2.2) and (2.3), one obtains from (8.1) and the assumptions on W in Lemma 3.2 that

$$
\begin{aligned}
&(\frac{\partial X_1}{\partial x_j})(X,W,\nabla_X W) = \delta_{1j} + O(M\varepsilon), \qquad j=1,2\\
&(\frac{\partial X_2}{\partial x_j})(X,W,\nabla_X W) = \delta_{2j} + O(\varepsilon)
\end{aligned}
\tag{8.8}
$$

Similarly, it follows from (8.6), (8.8), and Lemma 2.1 that

$$\begin{cases} \frac{\partial(\partial_i\varphi_-)}{\partial V}(X,W) = O(\varepsilon) \\ \frac{\partial(\partial_i\varphi_+)}{\partial V}(X,W) = O(\varepsilon + M\varepsilon) \\ \frac{\partial(\partial_{x_i}V)}{\partial(\partial_{X_j}V)}(X,W,\nabla_X W) = \delta_{ij} + O(M\varepsilon) \end{cases} \tag{8.9}$$

Next, we claim that

$$\begin{cases} \partial_{\partial_1\varphi_+}\bar{G} = \frac{\rho_+(q_+-q_0)(c^2(\rho_+)-q_+^2)}{c^2(\rho_+)}(1+O(M\varepsilon)), \\ \partial_{\partial_1\varphi_-}\bar{G} = \frac{\rho_0(q_+-q_0)(c^2(\rho_0)-q_0^2)}{c^2(\rho_0)}(1+O(M\varepsilon)), \\ \partial_{\partial_2\varphi_\pm}\bar{G} = O(\varepsilon + M\varepsilon). \end{cases} \tag{8.10}$$

Indeed, note that we have normalized so that $q_+ - q_0 = 1$. Then it follows from $W(X) = 1 - x_1 + \varphi_-(x) - \varphi_+(x)$ that $\partial_i\varphi_+ = \partial_i\varphi_- - \delta_{1i} - \partial_{x_i}W = \partial_i\varphi_- - \delta_{1i} - \nabla_X W\frac{\partial X}{\partial x_i}$. Hence Lemma 2.1 and (8.8) imply that

$$\begin{cases} \partial_1\varphi_- = q_0 + O(\varepsilon) \\ \partial_1\varphi_+ = q_+ + O(\varepsilon + M\varepsilon) \\ \partial_2\varphi_\pm = O(\varepsilon + M\varepsilon). \end{cases} \tag{8.11}$$

On the other hand, the Bernoull's law, (1.2) and (1.3), yield that $c^2(H) = \frac{H}{H'}$. Thus, (8.7) yields,

$$\begin{aligned} \partial_{\partial_1\varphi_+}\bar{G} &= [\partial_1\varphi H] + (H_+ - (\partial_1\varphi_+)^2 H'_+)(\partial_1\varphi_+ - \partial_1\varphi_-) + \partial_1\varphi_+\partial_2\varphi_+(\partial_2\varphi_+ - \partial_2\varphi_-)H'_+ \\ &= (\partial_1\varphi_+H_+ - \partial_1\varphi_-H_-) + H_+(\partial_1\varphi_+ - \partial_1\varphi_-)(1 - (\partial_1\varphi_+)^2\frac{H'_+}{H_+}) + O(M\varepsilon) \\ &= (q_+\rho_+ - q_0\rho_0) + \rho_+(q_+ - q_0)(1 - q_+^2\frac{1}{c_+^2}) + O(M\varepsilon) \\ &= \frac{\rho_+(q_+-q_0)(c^2(\rho_+)-q_+^2)}{c^2(\rho_+)}(1+O(M\varepsilon)) \end{aligned}$$

The others in (8.10) can be verified similarly.

It now follows from (8.6) - (8.10) and the formulas for $B_{1i}(i=1,2)$ and B_1 that

$$\begin{aligned} &B_{11}(X,W,\nabla_X W) = -\frac{\rho_+(q_+-q_0)(c^2(\rho_+)-q_+^2)}{c^2(\rho_+)}(1+O(M\varepsilon)), \\ &B_{12}(X,W,\nabla_X W) = O(M\varepsilon), \\ &B_1(X,W,\nabla_X W) = O(M\varepsilon). \end{aligned}$$

The other properties on $B_{1i}(X,W,\nabla_X W)$ and $B_1(X,W,\nabla_X W)$ can be computed directly. Hence Lemma 3.2 is proved.

Finally, we sketch the proof of Lemma 3.4. By (8.4), we have

$$\begin{aligned} a_{11}(X,W,\nabla_X W) = &-\sum_{k=1}^{2}((\partial_{x_k}\varphi_+)^2 - c_+^2)((\frac{\partial X_1}{\partial x_k})^2 + b_{11}^{kk}\partial_{X_1}W) \\ &- 2\partial_{x_1}\varphi_+\partial_{x_2}\varphi_+(b_{11}^{12}\partial_{X_1}W + \frac{\partial X_1}{\partial x_1}\frac{\partial X_1}{\partial x_2}). \end{aligned}$$

It follows from (8.8), (8.11), (8.2), and the assumptions on W that

$$a_{11}(X,W,\nabla_X W) = -(q_+^2 - c^2(\rho_+))(1+O(M\varepsilon)) \tag{8.12}$$

Similarly,

$$\begin{aligned} a_{22}(X,W,\nabla_X W) &= -\sum_{k=1}^{2}((\partial_{x_k}\varphi_+)^2 - c_+^2)((\frac{\partial X_2}{\partial x_k})^2 + b_{22}^{kk}\partial_{X_1}W) \\ &\quad - 2\partial_{x_1}\varphi_+\partial_{x_2}\varphi_+(b_{22}^{12}\partial_{X_1}W + \frac{\partial X_2}{\partial x_1}\frac{\partial X_2}{\partial x_2}) \\ &= c_+^2(\frac{\partial X_2}{\partial x_2})^2 + O(M\varepsilon) \\ &= c^2(\rho_+)(1+O(M\varepsilon)), \end{aligned} \tag{8.13}$$

and

$$\begin{aligned} a_{12}(X,W,\nabla_X W) &= -\sum_{k=1}^{2}((\partial_{x_k}\varphi_+)^2 - c_+^2)(\frac{\partial X_1}{\partial x_k}\frac{\partial X_2}{\partial x_k} + b_{12}^{kk}\partial_{X_1}W) \\ &\quad - 2\partial_{x_1}\varphi_+\partial_{x_2}\varphi_+(b_{12}^{12}\partial_{X_1}W + \frac{1}{2}(\frac{\partial X_1}{\partial x_1}\frac{\partial X_2}{\partial x_2} + \frac{\partial X_1}{\partial x_2}\frac{\partial X_2}{\partial x_1})) \\ &= O(\varepsilon + M\varepsilon). \end{aligned} \tag{8.14}$$

Next, one can obtain from (8.2), (8.3), (8.8), and the properties of W that

$$|b_0^{jk}(X,W,\nabla_X W)| = O(\varepsilon + M\varepsilon).$$

This, together with (8.5) and Lemma 2.1, leads to

$$|F_0(X,W,\nabla_X W)| \le O(\varepsilon) + O(M\varepsilon)(\varepsilon + M\varepsilon) = O(\varepsilon) \tag{8.15}$$

by the assumption $M^2\varepsilon \le \frac{1}{2}$. The rest of the proof of Lemma 3.4 follows from similar line and direct computations. Thus the proof of Lemma 3.4 is completed.

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