GRADIENT KÄHLER-RICCI SOLITONS AND A UNIFORMIZATION CONJECTURE

ALBERT CHAU[1] AND LUEN-FAI TAM[2]

Abstract. In this article we study the limiting behavior of the Kähler-Ricci flow on complete non-compact Kähler manifolds. We provide sufficient conditions under which a complete non-compact gradient Kähler-Ricci soliton is biholomorphic to $\mathbb{C}^n$. We also discuss the uniformization conjecture by Yau [15] for complete non-compact Kähler manifolds with positive holomorphic bisectional curvature.

1. introduction

In this paper, we show when a complete non-compact gradient Kähler-Ricci soliton is biholomorphic to $\mathbb{C}^n$. We will also discuss when a general solution to the Kähler-Ricci flow on a non-compact Kähler manifold converges after rescaling to a complete flat Kähler limit metric.

Canonical examples of such solitons on $\mathbb{C}^n$ were first provided by Cao[1, 2]. These examples are all rotationally symmetric with positive holomorphic bisectional curvature. It would be interesting to know how many other complete gradient Kähler-Ricci soliton metrics there are on $\mathbb{C}^n$. Our results may be of use here. Another reason for our interest in gradient Kähler Ricci solitons is that they may serve as models for the uniformization conjecture by Greene-Wu [6], Siu [14] and in the most general form by Yau [15] which states that any complete non-compact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{C}^n$. Using our techniques and ideas we shed light on recent approaches to proving this conjecture using the Kähler-Ricci flow [11, 12, 10].

A gradient Ricci soliton is defined as follows. Let $g_{ij}(x,t)$ be a family of metrics on a Riemannian manifold M satisfying the Ricci flow equation:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} - 2\rho g_{ij}. \tag{1.1}$$

for $0 \le t < \infty$, where R_{ij} denotes the Ricci tensor at time t and ρ is a constant. $g_{ij}(x,t)$ is said to be a *gradient Ricci soliton of steady type*, if $\rho = 0$ and if there is a potential function f and a family of diffeomorphisms φ_t

Date: October 2003.

2000 *Mathematics Subject Classification.* Primary 53C44; Secondary 58J37, 35B35.

[1]Research partially supported by The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, Hong Kong, China.

[2]Research partially supported by Earmarked Grant of Hong Kong #CUHK4032/02P.

generated by the gradient of $-f$ with respect to $g_{ij}(x,0)$ such that $g_{ij}(x,t) = \varphi_t^*(g_{ij}(x,0))$. If $\rho > 0$ (respectively $\rho < 0$), then it is said to be of *expanding type* (respectively *shrinking type*). If $g_{ij}(x,t)$ is a gradient Ricci soliton with potential function f then one has

$$f_{ij} = 2R_{ij}(x,0) + 2\rho g_{ij}(x,0) \tag{1.2}$$

where f_{ij} is the Hessian of f with respect to $g_{ij}(x,0)$.

If $(M, g_{\alpha\bar\beta}(x,0))$ is a Kähler manifold, (1.1) is referred to as the Kähler-Ricci flow and is written as

$$\frac{\partial}{\partial t} g_{\alpha\bar\beta} = -R_{\alpha\bar\beta} - 2\rho g_{\alpha\bar\beta}. \tag{1.3}$$

A gradient Ricci-soliton solution to (1.3) is referred to as a gradient Kähler-Ricci soliton . In this case (1.2) takes the form

$$\begin{aligned} f_{\alpha\bar\beta} &= R_{\alpha\bar\beta} + 2\rho g_{\alpha\bar\beta} \\ f_{\alpha\beta} &= 0. \end{aligned} \tag{1.4}$$

Hence the gradient of f is a holomorphic vector field and the diffeomorphism φ_t is a biholomorphism. At times, we may refer to a Riemannian manifold (M, g_{ij}) as a Ricci-soliton if the corresponding solution to (1.1) is a Ricci-soliton . We do likewise in the Kähler case.

We consider gradient Kähler-Ricci solitons which are either (i) steady with positive Ricci curvature so that the scalar curvature attains maximum at some point; or (ii) expanding with nonnegative Ricci curvature. Under either of these conditions, it is not hard to prove that there is a unique equilibrium point p where the gradient of the potential function f is zero. Our main result for gradient Kähler Ricci solitons is:

Theorem 1.1. *Let $(M, g_{\alpha\bar\beta})$ be a complete non-compact gradient Kähler-Ricci soliton with potential f satisfying either of the conditions mentioned above, and let $g_{\alpha\bar\beta}(x,t)$ be the corresponding solution to (1.3). Let p be the equilibrium point and let $\mathbf{v}_p \in T_p^{1,0}(M)$ be a fixed nonzero vector with $|\mathbf{v}_p|_0 = 1$. Then for any sequence of times $t_k \to \infty$, the sequence of complete Kähler metrics $\frac{1}{|\mathbf{v}_p|^2_{t_k}} g_{\alpha\bar\beta}(x,t_k)$ subconverges on compact sets of M to a complete flat Kähler metric $h_{\alpha\bar\beta}$ on M if and only if $R_{\alpha\bar\beta}(p) = \beta g_{\alpha\bar\beta}(p)$ at $t = 0$ for some constant β. In particular, if the condition is satisfied then M is biholomorphic to $\mathbb{C}^n$.*

Here for a tangent vector $\mathbf{v}$ on M, $|\mathbf{v}|_t$ denotes the length of $\mathbf{v}$ in the metric $g(t)$.

Next, we consider general complete non-compact Kähler manifolds with nonnegative holomorphic bisectional curvature. In [11, 12] (see also [10]), W.-X. Shi proved that on a complete noncompact Kähler manifold $(M, g_{\alpha\bar\beta})$ with bounded nonnegative holomorphic bisectional curvature such that

$$\frac{1}{V_x(r)} \int_{B_x(r)} R \le \frac{C}{1+r^2} \tag{1.5}$$

for some constant C for all $x \in M$ and for all r, the Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}}(x,t) = -R_{\alpha\bar{\beta}}(x,t)$$

with initial condition $g_{\alpha\bar{\beta}}(x,0) = g_{\alpha\bar{\beta}}(x)$ has a long time solution. Moreover, useful estimates were obtained. In [11], an approach by Shi to prove the uniformization conjecture of Greene-Wu-Siu-Yau for manifolds satisfying (1.5) is to use the Kähler-Ricci flow to produce a complete flat Kähler metric h the Kähler manifold M. More precisely, one considers the rescaled metrics $\frac{1}{|\mathbf{v}_p|_t^2} g_{\alpha\bar{\beta}}(x,t)$ and shows that a subsequence will converge to a flat complete Kähler metric h. Here $\mathbf{v}_p$ is a fixed vector in $T_p^{1,0}(M)$ and $|\mathbf{v}_p|_t$ is its length in $g(t)$. However, the proof in [11] is not quite satisfactory. First, as noted in [3] the completeness of h is unclear from [11] and has yet to be verified. On the existence of h, the authors would like to point out that the proof in [11] depends critically on a bound for a quantity Q (see (4.3) for more details) and that Shi's proof of this bound appears to be incorrect. More specifically, the formula on [11, p.156] for $\frac{\partial}{\partial t} Q$ seems to be incorrect. In this paper we partially rectify these issues by providing a proof for the completeness of h assuming we have an a priori bound for Q. We do this in section 4 (Theorem4.2). In general, in the absence of such a bound, we prove that completeness is in many cases a natural condition that follows from the existence of h alone. In this direction our main result is:

Theorem 1.2. *There exists a constant $C(n)$ depending only on n such that if M^n is a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature satisfying:*

(i)
$$\frac{1}{V_x(r)} \int_{B_x(r)} R \le \frac{C(n)}{1+r^2}$$
for all $x \in M$ and for all $r > 0$; and

(ii) *there exist a point $p \in M$ and a sequence $t_k \to \infty$ such that at p $\frac{1}{|\mathbf{v}_p|_{t_k}^2} g(p,t_k)$ are uniformly equivalent to $g(p,0)$, where $\mathbf{v}_p$ is a fixed vector in $T_p^{1,0}(M)$ with $|\mathbf{v}_p|_0 = 1$.*

Then the metrics $\frac{1}{|\mathbf{v}_p|_{t_k}^2} g(x,t_k)$ subconverge uniformly in the C^∞ topology in compact sets to a complete Kähler flat metric on M. In particular, the universal covering space of M is biholomorphic to $\mathbb{C}^n$.

The authors would like to thank Prof. S.T.Yau for helpful discussions and support. The first Author would also like to thank Prof. Richard Hamilton for helpful discussions.

2. A necessary condition for convergence

In this section, we prove the necessary part of Theorem 1.1. In fact, we have the following:

Theorem 2.1. *Let $g_{ij}(x,t)$ be a gradient Ricci soliton with function f and diffeomorphisms φ_t generated by $\nabla_0(-f)$, where ∇_0 is the covariant derivative respect to $g_{ij}(x,0)$. Suppose the flow φ_t has an equilibrium point p and suppose there exist a subsequence $t_k \to \infty$ and positive numbers $\sigma(t_k)$ such that $\sigma(t_k)g_{ij}(x,t_k)$ converges uniformly in a neighborhood of p to a Riemannian metric h_{ij}. Then at $t=0$, $R_{ij}(p)=\beta g_{ij}(p)$ for some constant β.*

Proof. In the following $g_{ij}(x,0)$ will simply be denoted by g and the metric at time t will be denoted explicitly by $g(t)$.

Choose a coordinate neighborhood V of p with coordinates $\mathbf{x}=(x^1,\ldots,x^n)$ such that $\mathbf{x}(p)=0$, $g_{ij}(0)=\delta_{ij}$, $\frac{\partial}{\partial x^k}g_{ij}(0)=0$, $\frac{\partial^2 f}{\partial x^i\partial x^j}(0)=\lambda_i\delta_{ij}$. By (1.2), it is sufficient to prove that $\lambda_i=\lambda_j$ for all $i,\ j$.

Let $\mathbf{v}_0\in T_p(M)$ such that

$$\mathbf{v}_0=\sum_k v_0^k\frac{\partial}{\partial x^k}. \tag{2.1}$$

Let $F^i(\mathbf{x})=g^{ij}(\mathbf{x})\frac{\partial f}{\partial x^j}(\mathbf{x})$. Since p is an equilibrium point, $\varphi_t(0)=0$ for all t, $\nabla_0 f(p)=0$. Hence $F^i(0)=0$ and $\frac{\partial F^i}{\partial x^j}(0)=\lambda_i\delta_{ij}$.

We may assume that there is a constant C_1 such that $|F(\mathbf{x})|\le C_1|\mathbf{x}|$ on V, where $|\mathbf{x}|^2=\sum_i(x^i)^2$. Hence for any $T>0$, there exists a constant $a>0$ such that the equation

$$\begin{cases}\frac{d\mathbf{x}}{dt}= & F(\mathbf{x}(t))\\ \mathbf{x}(0)= & \mathbf{x}_0\end{cases} \tag{2.2}$$

has a unique solution on $[0,T]$ with image inside V whenever $|\mathbf{x}_0|^2=\sum_i\left(x_0^i\right)^2\le a^2$.

Consider the curve $\alpha(s)=(sv_0^1,\ldots,sv_0^n)$ so that $\alpha'(0)=\mathbf{v}_0$. There exists $s_0>0$ such that $|\alpha(s)|\le a$ for all $0\le s\le s_0$. Hence for all $0\le s\le s_0$, the solution $\mathbf{x}(t;\alpha(s))$ of (2.2) with initial value $\alpha(s)$ is defined on $0\le t\le T$ with image inside V. Since $\varphi_t(0)=0$ for all t, $(\varphi_t)_*(\mathbf{v}_0)=\frac{\partial}{\partial s}\varphi_t(\alpha(s))\Big|_{s=0}\in T_p(M)$. Denote $(\varphi_t)_*(\mathbf{v}_0)$ by

$$\sum_k v^k(t)\frac{\partial}{\partial x^k}. \tag{2.3}$$

In local coordinates $\varphi_t(\alpha(s))=\mathbf{x}(t;\alpha(s))=\left(x^1(t;\alpha(s)),\ldots,x^n(t;\alpha(s))\right)$ and

$$\frac{\partial}{\partial s}\varphi_t(\alpha(s))\Big|_{s=0}=\frac{\partial x^k}{\partial s}(t;\alpha(s))\Big|_{s=0}\frac{\partial}{\partial x^k}. \tag{2.4}$$

Hence $v^k(t)$ is given by

$$v^k(t)=\frac{\partial x^k}{\partial s}(t;\alpha(s))\Big|_{s=0}. \tag{2.5}$$

Now for $0 \le t \le T$,

$$
\begin{aligned}
\frac{d}{dt} v^k(t) &= \frac{\partial}{\partial t}\frac{\partial x^k}{\partial s}\left(t;\alpha(s)\right)\Big|_{s=0} \\
&= \frac{\partial}{\partial s}\frac{\partial x^k}{\partial t}\left(t;\alpha(s)\right)\Big|_{s=0} \\
&= \frac{\partial}{\partial s} F^k\left(\mathbf{x}(t;\alpha(s))\right)\Big|_{s=0} \\
&= \left[\frac{\partial}{\partial x^i} F^k\left(\mathbf{x}(t;\alpha(s))\right)\frac{\partial x^i}{\partial s}\left(\mathbf{x}\left(t;\alpha(s)\right)\right)\right]_{s=0} \\
&= \lambda_k \delta_{ki} v^i(t) \\
&= \lambda_k v^k(t),
\end{aligned} \tag{2.6}
$$

where we have used (2.5), the fact that $\mathbf{x}(t;\alpha(0)) = \mathbf{x}(t;0) = 0$ because $F(0) = 0$, and the fact that $\frac{\partial F^i}{\partial x^k} = \lambda_i \delta_{ik}$ at 0. Using the initial condition, we conclude that

$$v^k(T) = \exp(\lambda_k T) v_0^k \tag{2.7}$$

and

$$(\varphi_T)_* \left(\sum_k v_0^k \frac{\partial}{\partial x^k}\right) = \sum_k \exp(\lambda_k T) v_0^k \frac{\partial}{\partial x^k}. \tag{2.8}$$

Hence

$$|\mathbf{v}_0|^2_{g(T)} = |\mathbf{v}_0|^2_{\varphi_T^*(g_0)} = |\left(\varphi_T\right)_* \mathbf{v}_0|^2_g = \sum_i \exp(2\lambda_i T)\left(v_0^i\right)^2 \tag{2.9}$$

for all $T > 0$. Suppose there exist $t_k \to \infty$, $\sigma(t_k)g(t_k)$ converges in C^∞ topology to a Riemannian metric h on a neighborhood of p. Then there exists a constant $C_2 > 0$ such that for any $\mathbf{v},\ \mathbf{w} \in T_p(M)$ with $|\mathbf{v}|_g = |\mathbf{w}|_g$, and for all k, we have

$$C_2^{-1} \le \frac{|\mathbf{v}|^2_{\sigma(t_k)g(t_k)}}{|\mathbf{w}|^2_{\sigma(t_k)g(t_k)}} \le C_2. \tag{2.10}$$

In the coordinates $(x^1, \cdots, x^n)$, by (2.9), we have

$$C_2^{-1} \le \frac{\sum_i \exp(2\lambda_i t_k)\left(v_0^i\right)^2}{\sum_i \exp(2\lambda_i t_k)\left(w_0^i\right)^2} \le C_2 \tag{2.11}$$

for all k, whenever $\sum_i \left(v_0^i\right)^2 = \sum_i \left(w_0^i\right)^2 = 1$. Since $t_k \to \infty$, $\lambda_i = \lambda_j$ for all i and j. □

3. A sufficient condition for convergence

In this section we prove the sufficient part of Theorem 1.1. First, we have the following on the existence of an equilibrium point.

Lemma 3.1. *Let $(M^n, g_{\alpha\bar{\beta}})$ be a complete non-compact gradient Kähler-Ricci soliton with potential f satisfying either of the following two conditions:*

1. *At $t = 0$, $f_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}}$ and $R_{\alpha\bar{\beta}} > 0$ so that the scalar curvature R attains maximum at some point in M.*
2. *At $t = 0$, $f_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}} + g_{\alpha\bar{\beta}}$ and $R_{\alpha\bar{\beta}} \geq 0$.*

Then there is a unique point $p \in M$ at which $\nabla_0 f(p) = 0$, where ∇_0 is the covariant derivative with respect to $g(0)$. Also, M is diffeomorphic to $\mathbb{R}^{2n}$.

Proof. It will suffice to show that f is a strictly convex exhaustion function, see [5, Theorem 3].

In case (1), this follows from the proof of [9, Theorem 20.1], see also[4].

In case (2), we begin by noting that (2) together with (1.4) imply that the Hessian of f with respect to $g(0)$ satisfies $D^2 f \geq g(0)$, thus f is indeed strictly convex. Next, let q be a fixed point and consider an arbitrary geodesic $\gamma(s)$ originating at q parametrized by arc length in $g(0)$. Then along $\gamma(s)$ we have $\frac{d^2 f}{ds^2}(\gamma(s)) = D^2 f(\gamma'(s), \gamma'(s)) \geq 1$. Integrating this we get

$$
\begin{aligned}
f(\gamma(s)) - f(q) =& f(\gamma(s)) - f(\gamma(0)) \\
=& \int_0^s \left[\int_0^\tau \frac{d^2 f}{ds^2}(\gamma(\rho)) d\rho - \frac{df}{ds}(\gamma(0)) \right] d\tau \\
\geq& \int_0^s \left(\int_0^\tau d\rho - |\nabla_0 f|(q) \right) d\tau \\
\geq& \int_0^s (\tau - |\nabla_0 f|(q)) d\tau \\
\geq& \frac{s^2}{2} - |\nabla_0 f|(q) s
\end{aligned}
\tag{3.1}
$$

where ∇_0 is the covariant derivative with respect to $g(0)$. It is now clear that f is an exhaustion function on M. This completes the proof of the lemma. □

The sufficient part of Theorem 1.1 will follow from Lemma 3.1 and the following lemmas. In the following, when we say case (1) (respectively case (2)), we mean that the potential f in Theorem 1.1 satisfies condition (1) (respectively condition (2)) in Lemma 3.1.

Let p be the equilibrium point in Theorem 1.1, whose existence is implied by Lemma 3.1. $B_t(R)$ will denote the geodesic ball of radius R with respect to the metric $g(t)$ with center p. In particular $B_0(R)$ is the geodesic ball of radius R with respect to the initial metric $g(0)$.

Lemma 3.2. *With the same assumptions and notations as in Lemma 3.1, for any $R > 0$, the following are true:*

(i) *$B_{t_1}(R) \subset B_{t_2}(R)$ for all $t_1 \leq t_2$;*
(ii) *for any $T \geq 0$, $q \in B_T(R)$, $\mathbf{w}_q \in T^{1,0}(M)$,*

$$|\mathbf{w}_q|_t \leq \exp\left(-C_R(t - T)\right) |\mathbf{w}_q|_T$$

for all $t \geq T$, where $C_R > 0$ is a constant depending only on R and $g(0)$; and

(iii) *for any integer $k \geq 0$, for any $t \geq 0$,*

$$||\nabla_t^k Rm(t)||_{g(t)} \leq C(R,k)$$

on $B_0(R)$ for some constant $C(R,k)$ depending only on R, k and $g(0)$, where ∇_t is the covariant derivative with respect to $g(t)$ and $Rm(t)$ is the curvature tensor of $g(t)$.

Proof. Let φ_t be the biholomorphism of M generated by the gradient of $-f$ so that $g(t) = \varphi_t^*(g(0))$. Then $\varphi_t(p) = p$ by the definition of p. Since $R_{\alpha\bar\beta} \geq 0$ in both cases in the assumptions of Lemma 3.1, $g_{\alpha\bar\beta}(t_2) \leq g_{\alpha\bar\beta}(t_1)$ if $t_1 \leq t_2$. From these, it is easy to see that (i) is true.

Since $\varphi_t : (M, g(t)) \to (M, g(0))$ is an isometry and $\varphi_t(p) = p$, φ_t will map $(B_t(R), g(t))$ isometrically onto $(B_0(R), g(0))$. Hence by (i) if $t \geq T$, the greatest lower bound of the Ricci curvature of $g(t)$ in $B_T(R)$ is no less than the greatest lower bound of the Ricci curvature of $g(T)$ in $B_T(R)$, which is the same as the greatest lower bound of the Ricci curvature of $g(0)$ on $B_0(R)$.

Now let $q \in B_T(R)$ and if $\mathbf{w} = \mathbf{w}_q \in T^{1,0}(M)$,

$$\begin{aligned} \frac{\partial}{\partial t}\left(g_{\alpha\bar\beta}(q,t)w^\alpha w^{\bar\beta}\right) &= \left(-R_{\alpha\bar\beta}(q,t) - 2\rho g_{\alpha\bar\beta}(q,t)\right) w^\alpha w^{\bar\beta} \\ &\leq -C_1 g_{\alpha\bar\beta}(q,t) w^a w^{\bar\beta} \end{aligned} \tag{3.2}$$

for some constant $C_1 > 0$ depending only on R and $g(0)$. In fact, if case (2) is assumed so that $\rho = 1/2$, then C_1 can be taken to be 1. If case (1) is assumed so that $\rho = 0$ then C_1 can be taken to be twice the greatest lower bound of the Ricci curvature of $g(0)$ in $B_0(R)$, which is positive. Dividing both sides of the above inequality by $g_{\alpha\bar\beta}(q,t)w^a w^{\bar\beta}$ and integrating from T to t, (ii) follows.

Since $B_0(R) \subset B_t(R)$ for $t \geq 0$ and since $(B_t(R), g(t))$ is isometric to $(B_0(R), g(0))$, it is easy to see that (iii) is true. □

Lemma 3.3. *With the same assumptions and notations as in Lemma 3.1, let $R > 0$ and $T \geq 0$. Then there exists a constant $C_R > 0$ which depends only on R and $g(0)$ with the following property: For any $q \in B_T(R)$, $\mathbf{u}_p \in T^{1,0}(M)$, $\mathbf{w}_q \in T^{1,0}(M)$ with $|\mathbf{u}_p|_T = |\mathbf{w}_q|_T$,*

$$C_R^{-1} \leq \frac{|\mathbf{u}_p|_t}{|\mathbf{w}_q|_t} \leq C_R \tag{3.3}$$

for all $t \geq T$.

Proof. For any $t \geq T \geq 0$, let q, $\mathbf{w}_q$ and $\mathbf{u}_p$ as in the assumptions. Let $\gamma_t(s)$ be a minimal geodesic from p to q in the metric $g(t)$. Let $\mathbf{w}(s)$ be a parallel vector field with respect to $g(t)$ along $\gamma_t(s)$ such that $\mathbf{w}(\gamma_t(d)) = \mathbf{w}_q$, where

$d = d_t(p,q)$ is the distance between p and q in $g(t)$. Then in both case (1) and case (2)

(3.4)
$$\begin{aligned}\frac{\partial}{\partial t}\log\left[\frac{g_{\alpha\bar\beta}(p,t)u_p^\alpha u_p^{\bar\beta}}{g_{\alpha\bar\beta}(q,t)w_q^\alpha w_q^{\bar\beta}}\right] &= \frac{R_{\alpha\bar\beta}(q,t)w^\alpha(d)w^{\bar\beta}(d)}{g_{\alpha\bar\beta}(q,t)w^\alpha(d)w^{\bar\beta}(d)} - \frac{R_{\alpha\bar\beta}(p,t)u_p^\alpha u_p^{\bar\beta}}{g_{\alpha\bar\beta}(p,t)u_p^\alpha v_p^{\bar\beta}}\\ &= \frac{R_{\alpha\bar\beta}(q,t)w^\alpha(d)w^{\bar\beta}(d)}{g_{\alpha\bar\beta}(q,t)w^\alpha(d)w^{\bar\beta}(d)} - \frac{R_{\alpha\bar\beta}(p,t)w^\alpha(0)w^{\bar\beta}(0)}{g_{\alpha\bar\beta}(p,t)w^\alpha(0)w^{\bar\beta}(0)}\\ &= \int_0^d \frac{\partial}{\partial s}\left(\frac{R_{\alpha\bar\beta}(\gamma_t(s),t)w^\alpha(s)w^{\bar\beta}(s)}{g_{\alpha\bar\beta}(\gamma_t(s),t)w^\alpha(s)w^{\bar\beta}(s)}\right)ds\\ &\le d\max_{0\le s\le d}||\nabla_t R_{\alpha\bar\beta}||_{g(t)}(\gamma_t(s),t).\end{aligned}$$

Here we have used the assumption that $R_{\alpha\bar\beta}(p,0) = \beta g_{\alpha\bar\beta}(p,0)$ for some constant β and hence $R_{\alpha\bar\beta}(p,t) = \beta g_{\alpha\bar\beta}(p,t)$ for all t because $\varphi_t(p) = p$. Since $q \in B_T(R)$, by Lemma 3.2(ii) for all $t \ge T$, we have

$$d \le R\exp(-C_1(t-T))$$

for some positive constant C_1 depending only on R and $g(0)$. By Lemma 3.2(iii), we have

$$\left|\frac{\partial}{\partial t}\log\left[\frac{g_{\alpha\bar\beta}(p,t)u_p^\alpha u_p^{\bar\beta}}{g_{\alpha\bar\beta}(q,t)w_q^\alpha w_q^{\bar\beta}}\right]\right| \le C_2 R\exp\left(-C_1(t-T)\right) \tag{3.5}$$

where C_2 is a constant depending only on R and $g(0)$. Integrating (3.5) from T to t, using that fact that $|\mathbf{u}_p|_T = |\mathbf{w}_q|_T$, the resut follows. □

Lemma 3.4. *With the same assumptions as in Lemma 3.1, for any sequence of times $t_k \to \infty$, the sequence of complete Kähler metrics $h(k) = \frac{1}{|\mathbf{v}_p|^2_{t_k}}g(t_k)$ has a subsequence converging in C^∞ on compact sets of M to a flat Kähler metric H on M.*

Proof. For any $t \ge 0$, let $h(t) = \frac{1}{\sigma(t)}g(t)$, where $\sigma(t) = |\mathbf{v}_p|^2_t$. In the following $\widehat{Rm}(t)$ and $\widehat{\nabla}$ will denote the curvature tensor and the covariant derivative of $h(t)$, and $Rm(t)$ and ∇ will denote the curvature tensor and the covariant derivative of $g(t)$.

By Lemma 3.2, for any interger $m \ge 0$ and $R > 0$, there is a constant C_1 depending only on m, R and $g(0)$ such that

$$||\widehat{\nabla}^m\widehat{Rm}(t)||^2_{h(t)} = \sigma^{m+2}(t)||\nabla^m Rm||_{g(t)} \le C_1\sigma^{m+2}(t) \tag{3.6}$$

on $B_0(R)$ with

$$\sigma(t) \le \exp(-C_2 t) \tag{3.7}$$

for some constant $C_2 > 0$ depending only on $g(0)$. By Lemma 3.3 and the definition of $h(t)$, there is a constant $C_3 > 0$ depending only on R and $g(0)$

such that

$$C_3^{-1}g(0) \le h(t) \le C_3 g(0) \tag{3.8}$$

for all $t \ge 0$.

Let $(z^1, \ldots, z^m)$ be a fixed local coordinates in an coordinates neighborhood $U \subset B_0(R)$. We want to prove that

$$\left|\frac{\partial}{\partial z^\xi} h_{\alpha\bar\beta}\right|(x,t) \le C_4 \tag{3.9}$$

for some constant C_4 for all $x \in U$ and for all t.

Let $\Gamma^\gamma_{\alpha\beta}$ be the Christoffel symbols of $h(t)$ which is also the Christoffel symbols of $g(t)$ in the coordinates z^α and let $\widetilde\Gamma^\tau_{\alpha\xi}$ be the Christoffel symbols of $g(0)$. Let $A^\tau_{\alpha\xi} = \Gamma^\tau_{\alpha\xi} - \widetilde\Gamma^\tau_{\alpha\xi}$, then $A^\tau_{\alpha\xi}$ is a tensor and

$$\widetilde\nabla_\xi g_{\alpha\bar\beta} = A^\tau_{\alpha\xi} g_{\tau\bar\beta}$$

where $\widetilde\nabla$ is the covariant derivative with respect to $g(0)$. Then the norm of A with respect to $g(0)$ is given by

$$||A||_0^2 = \widetilde g_{\gamma\bar\delta}\widetilde g^{\alpha\bar\beta}\widetilde g^{\xi\bar\zeta} A^\gamma_{\alpha\xi}\overline{A^\delta_{\beta\zeta}}.$$

By (1.3), we have

$$\frac{\partial}{\partial t}||A||_0^2 = -\widetilde g_{\gamma\bar\delta}\widetilde g^{\alpha\bar\beta}\widetilde g^{\xi\bar\zeta}\left[g^{\gamma\bar\sigma}\nabla_\alpha R_{\xi\bar\sigma}\overline{A^\delta_{\beta\zeta}} + A^\gamma_{\alpha\xi}\overline{g^{\delta\bar\sigma}\nabla_\beta R_{\zeta\bar\sigma}}\right] \tag{3.10}$$

Since the equality does not depends on coordinates, we choose holomorphic coordinates $(u^1, \ldots, u^n)$ in U such that $\widetilde g_{\alpha\bar\beta} = \delta_{\alpha\beta}$, $g_{\alpha\bar\beta} = \lambda_\alpha\delta_{\alpha\beta}$ at a point. Then

(3.11)

$$\begin{aligned}\left|\widetilde g_{\gamma\bar\delta}\widetilde g^{\alpha\bar\beta}\widetilde g^{\xi\bar\zeta} g^{\gamma\bar\sigma}\nabla_\alpha R_{\xi\bar\sigma}\overline{A^\delta_{\beta\zeta}}\right| &\le C_5\left(\sum_{\alpha,\xi,\lambda}\lambda_\gamma^{-1}\left|\nabla_\alpha R_{\xi\bar\gamma}\right|\right)||A||_0 \\ &\le C_6\left(\sum_{\alpha,\xi,\lambda}\lambda_\gamma^{\frac12}\lambda_\alpha^{-\frac12}\lambda_\xi^{-\frac12}\lambda_\gamma^{-\frac12}\left|\nabla_\alpha R_{\xi\bar\gamma}\right|\right)||A||_0 \\ &\le C_7\exp(-C_7 t)||\nabla_\alpha R_{\xi\bar\gamma}||\,||A||_0 \\ &\le C_8\exp(-C_7 t)||A||_0\end{aligned}$$

for some constants $C_5 - C_8$ depending only on R and $g(0)$ where we have used Lemma 3.2 and Lemma 3.3. Combining this with (3.10), we have

$$\frac{\partial}{\partial t}||A||_0^2 \le C_9\exp(-C_7 t)||A||_0$$

for some constant C_9 depending only on R and $g(0)$. Since $A = 0$ at $t = 0$, we conclude that

$$||A||_0^2(x,t) \le C_{10}$$

for some constant C_{10} for all $x \in U$ and for all t. From this and (3.8), it is easy to see that (3.9) is true.

Now

$$\widehat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\frac{\partial^2}{\partial z^\gamma \partial \bar{z}^\delta} h_{\alpha\bar{\beta}} + h^{\sigma\bar{\tau}} \left(\frac{\partial}{\partial \bar{z}^\delta} h_{\sigma\bar{\beta}} \right) \left(\frac{\partial}{\partial z^\gamma} h_{\alpha\bar{\tau}} \right). \tag{3.12}$$

By (3.6)–(3.8), there is a constant C_{11} independent of t such that

$$\left|\Delta_0 h_{\alpha\bar{\beta}}\right| = 4 \left| \sum_\gamma \frac{\partial^2}{\partial z^\gamma \partial \bar{z}^\gamma} h_{\alpha\bar{\beta}} \right| \leq C_{11}$$

in U. By [7, Theorem 8.32], for any open set $U' \subset\subset U$ there are constants $C_{12} > 0$ and $1 > \alpha > 0$ independent of t such that the $C^{1,\alpha}$ normed of $h_{\alpha\bar{\beta}}$ satisfies

$$|h_{\alpha\bar{\beta}}|_{1,\alpha,U'} \leq C_{12}. \tag{3.13}$$

Also, by (3.6)–(3.8), we conclude that

$$\left| \frac{\partial}{\partial z^\sigma} \widehat{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} \right| \leq C_{13}$$

in U for some constant C_{13} independent of t. Hence we can conclude from (3.12) and (3.13) that the C^α norm of $\Delta_0 h_{\alpha\bar{\beta}}$ in U' is also bounded by a constant independent of t. Therefore the $C^{2,\alpha}$ norm of $h_{\alpha\bar{\beta}}$ in any $U' \subset\subset U$ can be bounded by the constant independent of t. Similarly, one can prove that the $C^{k,\alpha}$ norm of $h_{\alpha\bar{\beta}}$ is bounded by a constant independent of t. From this, (3.6), (3.8) and (3.7) it is easy to see the lemma is true. □

Lemma 3.5. *H is complete.*

Proof. We may assume $h(k)$ converge to H. Suppose H is not complete. Then there is a divergent path $\gamma(\tau) : [0, \infty) \to M$ from p such that $\ell_H(\gamma) = L < \infty$, here ℓ_H is the length with respect to the metric H. Given $0 < \epsilon < L$. Let $a > 0$ be such that $\ell_H(\gamma|_{[0,a]}) = L - \epsilon/2$. Since $\gamma|_{[0,a]}$ is compact, there exists k_0 such that for all $k \geq k_0$,

$$L + \epsilon \geq \ell_k(\gamma|_{[0,a]}) \geq L - \epsilon \tag{3.14}$$

where ℓ_k is the length with respect to $h(t_k)$. By Lemma 3.3 and by the fact that $h(k) \geq g(t_k)$ because $|\mathbf{v}_p|_{t_k} \leq 1$ by Lemma 3.2, there is a constant which is independent of k and k_0, such that for any $k \geq k_0$, $q \in \widetilde{B}_{k_0}(3L)$, $\mathbf{w}_q \in T^{1,0}(M)$, $\mathbf{w}_p \in T^{1,0}(M)$ such that if $|\mathbf{w}_q|_{h(k_0)} = |\mathbf{w}_p|_{h(k_0)}$, then

$$C^{-1} \leq \frac{|\mathbf{w}_p|_{h(k)}}{|\mathbf{w}_q|_{h(k)}} \leq C \tag{3.15}$$

for some constant $C > 0$ depending only on L and $g(0)$. Here $\widetilde{B}_{k_0}(3L)$ is the geodesic ball of radius $3L$ in the metric $h(k_0)$ with center at p. Now reparametrized γ by arc length s with respect to $h(t_{k_0})$. Let $\gamma|_{(0\leq\tau\leq a)} = \gamma|_{(0\leq s\leq b)}$ where b is the length of $\gamma|_{(0\leq\tau\leq a)}$ with respect to $h(k_0)$. By (3.14), we have $L-\epsilon \le b \le L+\epsilon$. In particular, $2b \le 3L$ and so $\gamma([0,2b]) \subset \widetilde{B}_{k_0}(3L)$. By (3.15), for $k \ge k_0$ and for any $b \le s \le 2b$, we have

$$|\gamma'(s)|_{h(k)} \ge C^{-1}|\gamma'(b-s)|_{h(k)} \tag{3.16}$$

where we have used the fact that $|\gamma'(s)|_{h(k_0)} = |\gamma'(b-s)|_{h(k_0)} = 1$. Hence

$$\ell_k(\gamma|_{(b\le s\le 2b)}) \ge C^{-1}\ell_k(\gamma|_{(0\le s\le b)}) \tag{3.17}$$

and

$$\begin{aligned} \ell_k(\gamma|_{(0\le s\le 2b)}) &\ge \left(1+C^{-1}\right)\ell_k(\gamma|_{(0\le s\le b)}) \\ &\ge \left(1+C^{-1}\right) b \\ &\ge \left(1+C^{-1}\right)(L-\epsilon). \end{aligned} \tag{3.18}$$

Let $k\to\infty$, we have

$$\ell_H(\gamma|_{(0\le s\le 2b)}) \ge \left(1+C^{-1}\right)(L-\epsilon). \tag{3.19}$$

Since C does not depend on ϵ, if we let $\epsilon \to 0$, we have

$$\ell_H(\gamma|_{(0\le s\le 2b)}) \ge \left(1+C^{-1}\right) L > L. \tag{3.20}$$

This contradicts the definition of L. □

Proof. (Sufficient part of Theorem 1.1): The first part of the conclusion follows from Lemma 3.4 and Lemma 3.5. In particular, H is a complete flat Kähler metric on M and thus M is biholomorphic to a quotient of $\mathbb{C}^n$ by a group of biholomorphic isometries. But by Lemma 3.1 we know that M is diffeomorphic to $\mathbb{R}^{2n}$. Thus we must have M biholomorphic to $\mathbb{C}^n$. □

4. Convergence of Kähler-Ricci flows

In this section we study a general solution to the Kähler-Ricci flow focusing on Shi's program [11] for the uniformization conjecture of Greene-Wu-Siu-Yau. We will study the Kähler-Ricci flow equation

$$\frac{\partial}{\partial t} g_{\alpha\bar\beta}(x,t) = -R_{\alpha\bar\beta}(x,t). \tag{4.1}$$

More precisely, we are interested in the following situation. Let $(M^n, g_{\alpha\bar\beta})$ be a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature such that the scalar curvature R satisfies:

$$\frac{1}{V_x(r)}\int_{B_x(r)} R \le \frac{C}{1+r^2} \tag{4.2}$$

for some constant C for all $x \in M$ and for all r. By [11, 12, 10], we have the following:

Theorem 4.1. *Let (M^n, g) be as above. Then the Kähler-Ricci flow (4.1) has long time solution with initial value $g_{\alpha\bar\beta}(x,0) = g_{\alpha\bar\beta}(x)$. Moreover, the following are true:*

(1) *for any* $t \geq 0$, $g(x,t)$ *is Kähler with nonnegative holomorphic bisectional curvature;*

(2) *for any* $T > 0$, *there exists a constant* $C_1 > 0$ *such that*

$$C_1^{-1} g(x,0) \leq g(x,t) \leq C_1 g(x,0)$$

for all $x \in M$ *and for all* $0 \leq t \leq T$*;*

(3) *for any integer* $m \geq 0$, *there is a constant* C_2 *depending only on* m *and the initial metric such that*

$$||\nabla^m Rm||^2(x,t) \leq \frac{C_2}{(1+t)^{2+m}},$$

for all $x \in M$ *and for all* t *if* $m = 0$ *and for all* $t \geq 1$ *if* $m \geq 1$, *where* ∇ *is the covariant derivative with respect to* $g(t)$ *and the norm is also taken in* $g(t)$.

For the rest of the paper, we will always assume the conditions of Theorem 4.1. For any $T \geq 0$, define

$$Q(x,t;T) = \Big(1 + g^{\alpha\bar{\delta}}(x,t) g^{\gamma\bar{\beta}}(x,t) g^{\xi\bar{\zeta}}(x,T) \widetilde{\nabla}_\xi g_{\alpha\bar{\beta}}(x,t) \widetilde{\nabla}_{\bar{\zeta}} g_{\gamma\bar{\delta}}(x,t)\Big)^{\frac{1}{2}},$$

where $t \geq T$, and $\widetilde{\nabla}$ is the derivatives with respect to $g(T)$. In [11], a bound on Q was derived in order to prove the existence of a rescaled limit metric h on M. However, the derivation of this bound seems to be incorrect. In particular the formula of $\frac{\partial Q}{\partial t}$ on p. 156 in [11] is not correct. Moreover, the proof of the completeness of h is absent in [11]. In the first part of this section, we will prove that the limit metric is complete under the assumption that a bound on Q exists.

Let $p \in M$ be a fixed point and let $B_t(R)$ denote the geodesic ball of radius R in $g(t)$ with center at p. Let $\mathbf{v}_p \in T^{1,0}(M)$ be a fixed vector with length 1 in $g(0)$. As before, the norm of a vector in $g(t)$ is denoted by $|\mathbf{v}|_t$. We want to prove that:

Theorem 4.2. *Same assumptions as in Theorem 4.1. Moreover, suppose* M *has positive holomorphic bisectional curvature and suppose for any* $R > 0$, *there is a constant* C *such that*

$$Q(x,t;T) \leq C \tag{4.3}$$

for all $T \geq 0$, *for all* $x \in B_T(R)$ *and for all* $t \geq T$. *Then there exists a sequence* $t_k \to \infty$ *such that the metrics* $\frac{1}{|\mathbf{v}_p|^2_{t_k}} g(t_k)$ *converge uniformly in* C^∞ *topology to a complete Kähler flat metric on* M. *In particular, the universal covering space of* M *is biholomorphic to* $\mathbb{C}^n$.

The crucial point is Lemma 3.3, which is also true under the assumptions of the theorem.

Lemma 4.1. *With the same assumptions as in Theorem 4.2, let* $R > 0$ *and* $T \geq 0$. *Then there exists a constant* $C_R > 0$ *which is independent of* T *with* *the following property: For any* $q \in B_T(R)$, $\mathbf{w}_p \in T^{1,0}(M)$, $\mathbf{w}_q \in T^{1,0}(M)$ *with* $|\mathbf{w}_p|_T = |\mathbf{w}_q|_T$,

$$C_R^{-1} \le \frac{|\mathbf{w}_p|_t}{|\mathbf{w}_q|_t} \le C_R$$

for all $t \ge T$. *The constant* C_R *is also independent of* q, $\mathbf{w}_p$ q *and* $\mathbf{w}_q$.

Proof. This was basically proved in [11]. Let q, $\mathbf{w}_p$, and $\mathbf{w}_q$ as in the lemma. Let γ be a minimal geodesic with respect to $g(T)$ from q to p parametrized by arc length and with length $\ell \le R$. Parallel translate $\mathbf{w}_q$ along γ with respect to $g(T)$ to obtain a vector field $\mathbf{w}(s)$ on γ such that $\mathbf{w}(0) = \mathbf{w}_q$. At any point $s \in [0, \ell]$. Let $\widetilde{\nabla}$ be the covariant derivatives with respect to $g(T)$. For any $t \ge T$, and for any s, choose an unitary frame near $\gamma(s)$ such that $g_{\alpha\bar\beta}(\gamma(s), T) = \delta_{\alpha\beta}$ and $g_{\alpha\bar\beta}(\gamma(s), t) = \lambda_\alpha \delta_{\alpha\beta}$. In the following, we write $g = g(t)$ and $\widetilde{g} = g(T)$. Then

$$\begin{aligned} \left|\widetilde{\nabla}_{\gamma'}\left(g_{\alpha\bar\beta} w^\alpha w^{\bar\beta}\right)\right|^2 &= \left(\widetilde{\nabla}_{\gamma'} g_{\alpha\bar\beta}\right)\left(\widetilde{\nabla}_{\gamma'} g_{\gamma\bar\delta}\right) w^\alpha w^{\bar\beta} w^\gamma w^{\bar\delta} \\ &\le \left(g_{\alpha\bar\beta} w^\alpha w^{\bar\beta}\right)^2 \sum_{\alpha,\beta} \lambda_\alpha^{-1}\lambda_\beta^{-1} |\widetilde{\nabla}_{\mathbf{u}} g_{\alpha\bar\beta}|^2 \end{aligned} \tag{4.4}$$

where we have used the facts that $g_{\alpha\bar\beta} w^\alpha w^{\bar\beta} = \sum_\alpha \lambda_\alpha |w^\alpha|^2$ and that $\mathbf{w}$ is parallel with respect to $g(T)$, and the Schwarz inequality. On the other hand,

$$\begin{aligned} &Q^2(\gamma(s), t; T) \\ &\ge g^{\alpha\bar\delta}(\gamma(s), t) g^{\gamma\bar\beta}(\gamma(s), t) g^{\xi\bar\zeta}(\gamma(s), T) \widetilde{\nabla}_\xi g_{\alpha\bar\beta}(\gamma(s), t) \widetilde{\nabla}_{\bar\zeta} g_{\gamma\bar\delta}(\gamma(s), t) \\ &= \sum_{\alpha,\beta,\xi} \lambda_\alpha^{-1}\lambda_\beta^{-1} |\widetilde{\nabla}_\xi g_{\alpha\bar\beta}|^2. \end{aligned} \tag{4.5}$$

Combining (4.3), (4.4), (4.5) and the fact that $|\gamma'|_T = 1$, we have

$$\left|\widetilde{\nabla}_{\gamma'}\left(g_{\alpha\bar\beta}(\gamma(s), t_k) w^\alpha w^{\bar\beta}\right)\right|^2 \le C_1 \left(g_{\alpha\bar\beta} w^\alpha w^{\bar\beta}\right)^2$$

for some constant C_1 which is independent of t, T, q $\mathbf{w}_p$, and $\mathbf{w}_q$. Hence and

$$\left|\frac{\partial}{\partial s}\left(g_{\alpha\bar\beta}(\gamma(s), t_k) w^\alpha w^{\bar\beta}\right)\right|^2 \le C_1 \left(g_{\alpha\bar\beta} v^\alpha v^{\bar\beta}\right)^2.$$

Integrating from $s = 0$ to $s = \ell$, we have

$$\left|\log \frac{g_{\alpha\bar\beta}(t) v^\alpha(\ell) w^{\bar\beta}(\ell)}{g_{\alpha\bar\beta}(t) w^\alpha(0) w^{\bar\beta}(0)}\right| \le C_3 \ell \le C_3 R \tag{4.6}$$

for some constant C_3 independent of t, T, q and $\mathbf{w}_q$ and $\mathbf{w}_p$. In particular, if we take $t = 0$, using the fact that the holomorphic bisectional curvature of

$g(x,0)$ is positive and hence the holonomy group is transitive [13], we may prove as in [11] that

$$\left|\log \frac{g_{\alpha\bar\beta}(t)u_1^\alpha u_1^{\bar\beta}}{g_{\alpha\bar\beta}(t)u_2^\alpha u_2^{\bar\beta}}\right| \le C_4 \tag{4.7}$$

for some constant C_4, for all $\mathbf{u}_1,\ \mathbf{u}_2 \in T_p^{1,0}(M)$ such that $g_{\alpha\bar\beta}(0)u_1^\alpha u_1^{\bar\beta} = g_{\alpha\bar\beta}(0)u_2^\alpha u_2^{\bar\beta}$. Now if $\mathbf{u}_1$, $\mathbf{u}_2$ are such that $g_{\alpha\bar\beta}(T)u_1^\alpha u_1^{\bar\beta} = g_{\alpha\bar\beta}(T)u_2^\alpha u_2^{\bar\beta}$, then by (4.7), we have

$$\frac{g_{\alpha\bar\beta}(t)u_1^\alpha u_1^{\bar\beta}}{g_{\alpha\bar\beta}(t)u_2^\alpha u_2^{\bar\beta}}\frac{g_{\alpha\bar\beta}(0)u_2^\alpha u_2^{\bar\beta}}{g_{\alpha\bar\beta}(0)u_1^\alpha u_1^{\bar\beta}} \le \exp(C_4)$$

and

$$\frac{g_{\alpha\bar\beta}(T)u_2^\alpha u_2^{\bar\beta}}{g_{\alpha\bar\beta}(T)u_1^\alpha u_1^{\bar\beta}}\frac{g_{\alpha\bar\beta}(0)u_1^\alpha u_1^{\bar\beta}}{g_{\alpha\bar\beta}(0)u_2^\alpha w_2^{\bar\beta}} \le \exp(C_4)$$

and hence we have

$$\frac{g_{\alpha\bar\beta}(t)u_1^\alpha u_1^{\bar\beta}}{g_{\alpha\bar\beta}(t)u_2^\alpha u_2^{\bar\beta}} \le \exp(2C_4)$$

for all $t \ge T$. Combining this to (4.6), using the fact that

$$\begin{aligned} g_{\alpha\bar\beta}(T)w_p^\alpha w_p^{\bar\beta} &= g_{\alpha\bar\beta}(T)w_q^\alpha w_q^{\bar\beta} \\ &= g_{\alpha\bar\beta}(T)w^\alpha(\ell)w^{\bar\beta}(\ell) \end{aligned} \tag{4.8}$$

the lemma is proved. □

Proof. (Theorem 4.2) Let $h(t) = \frac{1}{|\mathbf{v}_p|^2}g(t)$. By the proof of completeness in Lemma 3.5, because of Lemma 4.1 and Theorem 4.1 it is sufficient to prove the existence a limit for $h(t_k)$. For this it is sufficient to show that in a fixed coordinate neighborhood $U \subset B_0(R)$ the Christoffel symbols of $h(t)$ are uniformly bounded. This can be proved as in Lemma 3.4. In this case, using Theorem 4.1(3), Lemma 4.1, and the fact that $g(t)$ is nonincreasing, we can conclude as in the proof of Lemma 3.4 that

$$\frac{\partial}{\partial t}||A||_0^2 \le C_1(1+t)^{-\frac{3}{2}}||A||_0 \tag{4.9}$$

where A is defined as in Lemma 3.4, which is the difference between the Christoffel symbols of $g(t)$ and $g(0)$ and $||A||_0$ is the norm of A in $g(0)$. Here C_1 is a constant depending only on $g(0)$, R and the constant C in the assumption (4.3) in the theorem. From this it is easy to see that $||A||_0$ is uniformly bounded in $U \times [0,\infty)$. Hence the theorem is true. □

In the second part of this section, we will prove the following:

Theorem 4.3. *There exists a constant $C(n)$ depending only on n such that if M^n is a complete noncompact Kähler manifold satisfying the conditions in 4.1 and the following:*

(i)
$$\frac{1}{V_x(r)}\int_{B_x(r)} R \le \frac{C(n)}{1+r^2}$$
for all $x\in M$ and for all $r>0$; and

(ii) *there exist a point $p\in M$ and a sequence $t_k\to\infty$ such that $\frac{1}{|\mathbf{v}_p|^2_{t_k}}g(p,t_k)$ are uniformly equivalent to $g(p,0)$, where $\mathbf{v}_p$ is a fixed vector in $T^{1,0}_p(M)$ with $|\mathbf{v}_p|_0=1$.*

Then the metrics $\frac{1}{|\mathbf{v}_p|^2_{t_k}}g(x,t_k)$ subconverge uniformly in the C^∞ topology in compact sets to a complete Kähler flat metric on M. In particular, the universal covering space of M is biholomorphic to $\mathbb{C}^n$.

In order to prove the theorem, we need several lemma.

Lemma 4.2. *Let (M^n,g) be a complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature. Suppose there exists a constant $a>0$ such that*

$$\frac{1}{V_x(r)}\int_{B_x(r)} R \le \frac{a}{1+r^2} \tag{4.10}$$

for all $x\in M$ for all r. Let $g_{\alpha\bar\beta}(x,t)$ be the long time solution of (4.1). Then there exist constants C_1 depending only on n and C_2 depending only on a and n such that

$$\int_0^t R(x,\tau)d\tau \le aC_1\log(1+t)+C_2 \tag{4.11}$$

for all $x\in M$ and for all t, where $R(x,t)$ is the scalar curvature of $g(t)$ at x.

Proof. For fixed t, the scalar curvature $R(x,\tau)$ of $g(\tau)$ is uniformly bounded on $M\times[0,t]$. Let

$$\mathfrak{M}(t)=\max_{x\in M}\int_0^t R(x,\tau)d\tau. \tag{4.12}$$

By [10, Corollary2.1], there exist positive constants C_3 and C_4 depending only on n such that if $r^2=C_4t(1+\mathfrak{M}(t))$, then

$$\begin{aligned}\mathfrak{M}(t)&\le C_3\int_0^r \frac{as}{1+s^2}ds\\ &=\frac{aC_3}{2}\log\left(1+C_4t\left(1+\mathfrak{M}(t)\right)\right)\\ &\le \frac{aC_3}{2}\left[\log(1+C_4)+\log(1+t)+\log(1+\mathfrak{M}(t))\right].\end{aligned} \tag{4.13}$$

Suppose $\mathfrak{M}(t) \geq aC_3$, then

$$\log(1+\mathfrak{M}(t)) \leq \frac{1}{aC_3}\mathfrak{M}(t) + \log(1+aC_3). \tag{4.14}$$

By (4.13), we have

$$\mathfrak{M}(t) \leq aC_3\left[\log(1+C_4) + \log(1+t) + \log(1+aC_3)\right]. \tag{4.15}$$

Hence we have

$$\mathfrak{M}(t) \leq aC_3\left[\log(1+C_4) + \log(1+t) + \log(1+aC_3) + 1\right]. \tag{4.16}$$

From this the lemma follows. □

Lemma 4.3. *Let M^n be a complete noncompact Kähler manifold satisfying the conditions in Theorem 4.1 and let $g(t)$ be the solution in (4.1). Then for any $R > 0$ there is a constant C_R such that for any $T \geq 1$, $q \in B_T(R)$ and any $t \geq T$,*

$$Rc_{\max}(q,t) - Rc_{\min}(q,t) \leq Rc_{\max}(p,t) - Rc_{\min}(p,t) + C_R(1+t)^{-\frac{3}{2}}. \tag{4.17}$$

Here as before, p is a fixed point and $B_T(R)$ is the geodesic ball of radius R with center at p in $g(T)$.

Proof. For $t \geq T$, et $\mathbf{v}_q$ and $\mathbf{w}_q$ in $T^{1,0}(M)$ such that $|\mathbf{v}_q|_t = |\mathbf{w}_q|_t = 1$ and $R_{\alpha\bar{\beta}}(q,t)v_q^\alpha v_q^{\bar{\beta}} = Rc_{\max}(q,t)$, $R_{\alpha\bar{\beta}}(q,t)w_q^\alpha w_q^{\bar{\beta}} = Rc_{\min}(q,t)$. Let $\gamma(s)$ be a minimal geodesic from p to q in $g(t)$ with length ℓ which is no greater than R because $B_T(R) \subset B_t(R)$. Let $\mathbf{v}(s)$ and $\mathbf{w}(s)$ be parallel vector fields along γ in $g(t)$ so that $\mathbf{v}(\ell) = \mathbf{v}_q$ and $\mathbf{w}(\ell) = \mathbf{w}_q$. Then

$$\begin{aligned}
&Rc_{\max}(q,t) - Rc_{\min}(q,t)\\
&= R_{\alpha\bar{\beta}}(q,t)v_q^\alpha v_q^{\bar{\beta}} - R_{\alpha\bar{\beta}}(q,t)w_q^\alpha w_q^{\bar{\beta}}\\
&= R_{\alpha\bar{\beta}}(q,t)v(\ell)^\alpha v(\ell)^{\bar{\beta}} - R_{\alpha\bar{\beta}}(q,t)w(\ell)^\alpha w(\ell)^{\bar{\beta}}\\
&= R_{\alpha\bar{\beta}}(p,t)v(0)^\alpha v(0)^{\bar{\beta}} - R_{\alpha\bar{\beta}}(p,t)w(0)^\alpha w(0)^{\bar{\beta}}\\
&\quad + \int_0^\ell \frac{\partial}{\partial s}\left[R_{\alpha\bar{\beta}}(\gamma(s),t)v(s)^\alpha v(s)^{\bar{\beta}} - R_{\alpha\bar{\beta}}(\gamma(s),t)w(s)^\alpha w(s)^{\bar{\beta}}\right] ds\\
&= R_{\alpha\bar{\beta}}(p,t)v(0)^\alpha v(0)^{\bar{\beta}} - R_{\alpha\bar{\beta}}(p,t)w(0)^\alpha w(0)^{\bar{\beta}}\\
&\quad + \int_0^\ell \left(\nabla_{\gamma'(s)}R_{\alpha\bar{\beta}}(\gamma(s),t)\right) v(s)^\alpha v(s)^{\bar{\beta}} ds\\
&\quad - \int_0^\ell \left(\nabla_{\gamma'(s)}R_{\alpha\bar{\beta}}(\gamma(s),t)\right) w(s)^\alpha w(s)^{\bar{\beta}} ds\\
&\leq Rc_{\max}(p,t) - Rc_{\min}(p,t) + CR(1+t)^{-\frac{3}{2}}
\end{aligned} \tag{4.18}$$

where C is a constant depending only on $g(0)$, where we have used Theorem 4.1. This completes the proof of the lemma. □

Lemma 4.4. *With the same assumptions as in Lemma 4.2, suppose* $\epsilon = aC_1 < 1$ *in (4.11). For any* $R > 0$, $t \geq T \geq 1$, $q \in B_T(R)$, $\mathbf{v}_q$, $\mathbf{w}_q \in T^{1,0}(M)$ *such that* $|\mathbf{v}_q|_T = |\mathbf{w}_q|_T$, *we have*

$$\frac{|\mathbf{v}_q|_t}{|\mathbf{w}_q|_t} \leq C_R\,(1+t)^{\frac{1}{2}\epsilon}. \tag{4.19}$$

where C_R *is a constant independent of* $T,\ t,\ q,\ \mathbf{v}_q,\ \mathbf{w}_q$.

Proof. By the proof of Lemma 4.3 we have for $q \in B_T(R)$ and $t \geq T$,

$$Rc_{\max}(q,t) - Rc_{\min}(q,t) \leq Rc_{\max}(p,t) - Rc_{\min}(p,t) + C_1(1+t)^{-\frac{3}{2}}, \tag{4.20}$$

where C_1 is a constant independent of $T,\ t,\ q$. Hence for $t \geq T$,

$$\begin{aligned} \frac{\partial}{\partial t}\log\frac{|\mathbf{v}_q|_t}{|\mathbf{w}_q|_t} &\leq \frac{1}{2}\left(-\frac{R_{\alpha\bar\beta}(q,t)v_q^{\alpha}v_q^{\bar\beta}}{g_{\alpha\bar\beta}(q,t)v_q^{\alpha}v_q^{\bar\beta}} + \frac{R_{\alpha\bar\beta}(q,t)w_q^{\alpha}w_q^{\bar\beta}}{g_{\alpha\bar\beta}(q,t)w_q^{\alpha}w_q^{\bar\beta}}\right) \\ &\leq \frac{1}{2}\left(Rc_{\max}(p,t) - Rc_{\min}(p,t) + C_R(1+t)^{-\frac{3}{2}}\right) \end{aligned} \tag{4.21}$$

Integrating from T to t and using (4.11), we have

$$\log\frac{|\mathbf{v}_q|_t}{|\mathbf{w}_q|_t} \leq \frac{1}{2}\epsilon\log(1+t) + C_2, \tag{4.22}$$

where C_2 is a constant independent of $T,\ t,\ q,\ \mathbf{v}_q,\ \mathbf{w}_q$. Hence the lemma is true. □

As before, let $A_{\alpha\beta}^{\gamma} = \Gamma_{\alpha\beta}^{\gamma} - \widetilde\Gamma_{\alpha\beta}^{\gamma}$, where $\Gamma_{\alpha\beta}^{\gamma}$ and $\widetilde\Gamma_{\alpha\beta}^{\gamma}$ are Christoffel symbols of $g(t)$ and $\widetilde g = g(T)$ respectively. Consider the norm of A in $g(T)$. Namely

$$||A||_T^2 = \widetilde g_{\gamma\bar\tau}\widetilde g^{\alpha\bar\delta}\widetilde g^{\xi\bar\zeta}A_{\alpha\xi}^{\gamma}A_{\bar\delta\bar\zeta}^{\bar\tau}. \tag{4.23}$$

Lemma 4.5. *With the same assumptions as in Lemma 4.2, suppose* $\epsilon = aC_1 < 1$ *in (4.11). Then for any* $R > 0$, *there is a constant* C_R *such that for any* $T \geq 1$ *and for any* $t \geq T$,

$$||A||_T \leq C_R \tag{4.24}$$

in $B_T(R)$.

Proof. As in (3.10), we have

$$\frac{\partial}{\partial t}||A||_T^2 = -\widetilde g_{\gamma\bar\delta}\widetilde g^{\alpha\bar\beta}\widetilde g^{\xi\bar\zeta}\left[g^{\gamma\bar\sigma}\nabla_\alpha R_{\xi\bar\sigma}\overline{A_{\beta\zeta}^{\delta}} + A_{\alpha\xi}^{\gamma}\overline{g^{\delta\bar\sigma}\nabla_\beta R_{\zeta\bar\sigma}}\right] \tag{4.25}$$

Choose coordinates such that $\widetilde{g}_{\alpha\bar\beta}=\delta_{\alpha\beta}$, $g_{\alpha\bar\beta}=\lambda_\alpha\delta_{\alpha\beta}$. Then in $B_T(R)$,

(4.26)

$$\begin{aligned}\left|\widetilde{g}_{\gamma\bar\delta}\widetilde{g}^{\alpha\bar\beta}\widetilde{g}^{\xi\bar\zeta}g^{\gamma\bar\sigma}\nabla_\alpha R_{\xi\bar\sigma}\overline{A^{\delta}_{\beta\zeta}}\right| &\le C(n)\sum_{\gamma,\alpha,\xi}\lambda_\gamma^{-1}\left|\nabla_\alpha R_{\xi\bar\gamma}\right|\,||A||_T\\ &= C(n)\sum_{\gamma,\alpha,\xi}\lambda_\alpha^{\frac12}\lambda_\xi^{\frac12}\lambda_\gamma^{-\frac12}\lambda_\gamma^{-\frac12}\lambda_\xi^{-\frac12}\lambda_\alpha^{-\frac12}\left|\nabla_\alpha R_{\xi\bar\gamma}\right|\,||A||_T\\ &\le C_2\,(1+t)^{\frac12\epsilon}\sum_{\gamma,\alpha,\xi}\lambda_\gamma^{-\frac12}\lambda_\xi^{-\frac12}\lambda_\alpha^{-\frac12}\left|\nabla_\alpha R_{\xi\bar\gamma}\right|\,||A||_T\\ &\le C_3(1+t)^{-\frac32+\frac12\epsilon}||A||_T\end{aligned}$$

for some constants C_2, C_3 independent of t and T, where we have used Lemma 4.4, the fact that $\lambda_\alpha\le 1$ and the estimates for $||\nabla Rc||$. Combining this with (4.25), since $\epsilon<1$, and $||A||_T(T)=0$, it is easy to see that the lemma is true. □

Lemma 4.6. *With the same assumptions as in Lemma 4.2, suppose $\epsilon = aC_1<1$ in (4.11). For any $R>0$ there is a constant C_R such that if $t\ge T\ge 1$, then*

$$C_R^{-1}\le\frac{g^{\alpha\bar\beta}(x,T)g_{\alpha\bar\beta}(x,t)}{g^{\alpha\bar\beta}(p,T)g_{\alpha\bar\beta}(p,t)}\le C_R \tag{4.27}$$

and

$$C_R^{-1}\le\frac{g^{\alpha\bar\beta}(x,t)g_{\alpha\bar\beta}(x,T)}{g^{\alpha\bar\beta}(p,t)g_{\alpha\bar\beta}(p,T)}\le C_R \tag{4.28}$$

for $x\in B_T(R)$.

Proof. We only prove (4.27) as the proof of (4.28) is similar. We want to estimate $\left|\widetilde\nabla\log\left[g^{\alpha\bar\beta}(x,T)g_{\alpha\bar\beta}(x,t)\right]\right|$ in $B_T(R)$, where $\widetilde\nabla$ is the covariant derivative of $g(T)$. At a point, choose a normal coordinates so that $g_{\alpha\bar\beta}(T)=\delta_{\alpha\bar\beta}$ and $g_{\alpha\bar\beta}(t)=\lambda_\alpha\delta_{\alpha\beta}$. Then

$$\begin{aligned}\frac{\partial}{\partial\xi}\log\left[g^{\alpha\bar\beta}(x,T)g_{\alpha\bar\beta}(x,t)\right] &= \frac{g^{\alpha\bar\beta}(x,T)\frac{\partial}{\partial\xi}g_{\alpha\bar\beta}(x,t)}{g^{\alpha\bar\beta}(x,T)g_{\alpha\bar\beta}(x,t)}\\ &= \frac{g^{\alpha\bar\beta}(x,T)\widetilde\nabla_\xi g_{\alpha\bar\beta}(x,t)}{g^{\alpha\bar\beta}(x,T)g_{\alpha\bar\beta}(x,t)}\\ &= \frac{g^{\alpha\bar\beta}(x,T)A^{\tau}_{\alpha\xi}g_{\tau\bar\beta}(x,t)}{g^{\alpha\bar\beta}(x,T)g_{\alpha\bar\beta}(x,t)}.\end{aligned} \tag{4.29}$$

Hence for x in $B_T(R)$,

$$\left|\frac{\partial}{\partial\xi}\log\left[g^{\alpha\bar\beta}(x,T)g_{\alpha\bar\beta}(x,t)\right]\right| \le \frac{\sum_\alpha |A^\alpha_{\alpha\xi}|\lambda_\alpha}{\sum_\alpha \lambda_\alpha} \le C_R, \tag{4.30}$$

where we have used Lemma 4.5. Hence

$$\left|\widetilde\nabla\log\left[g^{\alpha\bar\beta}(x,T)g_{\alpha\bar\beta}(x,t)\right]\right| \le C_R \tag{4.31}$$

in $B_T(R)$. Integrating (4.31) along a minimal geodesic in $g(T)$ from x to p, (4.27) follows. □

Lemma 4.7. *With the same assumptions as in Lemma 4.2, suppose $\epsilon = aC_1 < 1$ in (4.11) and suppose there exist $t_k \to \infty$, $t_k \ge 1$, such that $\frac{1}{|\mathbf{v}_p|^2_{t_k}}g(p,t_k)$ are uniformly equivalent to $g(p,0)$, where $\mathbf{v}_p$ is a fixed vector in $T^{1,0}_p(M)$ with $|\mathbf{v}_p|_0 = 1$. Then for any $R > 0$ there is a constant C_R independent of k and k_0 such that*

$$\frac{|\mathbf{u}_q|_{t_k}}{|\mathbf{w}_q|_{t_k}} \le C_R$$

for all $q \in B_{t_{k_0}}(R)$, $k \ge k_0$ and $\mathbf{u}_q$, $\mathbf{w}_q \in T^{1,0}_q(M)$ with $|\mathbf{u}_q|_{t_{k_0}} = |\mathbf{w}_q|_{t_{k_0}}$.

Proof. By the assumption, there is a constant $C > 0$ independent of k and k_0 such that

$$g^{\alpha\bar\beta}(p,t_k)g_{\alpha\bar\beta}(p,t_{k_0})g^{\gamma\bar\delta}(p,t_{k_0})g_{\gamma\bar\delta}(p,t_k) \le C.$$

From this and Lemma 4.6, the result follows. □

Proof. (Theorem 4.3) By Theorem 4.1, Lemmas 4.5, 4.7, one can proceed as in the proof of Theorem 4.2 to conclude that Theorem 4.3 is true. □

References

1. H.-D. Cao: Limits of solutions to the Kähler-Ricci flow. J. Differential Geom. **45** (1997), 257–272.
2. H.-D. Cao: Existence of gradient Kähler-Ricci solitons. Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), 1–16, A K Peters, Wellesley, MA, 1996.
3. Chen, B.L. and Zhu, X.P., *On complete noncompact Khler manifolds with positive bisectional curvature .* , to appear in Math.Ann.
4. Cao, Huai-Dong and Hamilton, Richard S., *Gradient Khler-Ricci solitons and periodic orbits.* Comm. Anal. Geom. **8** (2000), no. 3, 517–529.
5. Greene, R. E. and Wu, H., *C^∞ convex functions and manifolds of positive curvature.* Acta Math. **137** (1976), no. 3-4, 209–245.
6. Greene, R. E. and Wu, H.,*Analysis on noncompact Khler manifolds*, Proc. Sympos. Pure Math., **30** Part 2 (1977), 69-100.
7. Gilbarg, David and Trudinger, Neil S, *Elliptic partial differential equations of second order*, second edition, Springer-Verlag, (1983).
8. Hamilton, Richard S., *Three manifolds with positive Ricci curvature*, J. of Differential Geometry. **17** (1982), no. 2, 255-306.
9. Hamilton, Richard S., *Formation of Singularities in the Ricci Flow*, Contemporary Mathematics. **71** (1988), 237-261.

10. Ni, L. and Tam, L.F., *Kähler-Ricci flow and the Poincare-Lelong equation*, to appear in Comm. Anal. Geom.
11. Shi, Wan-Xiong, *Ricci deformation of the metric on complete noncompact Kähler manifolds*, PhD thesis, Harvard University, 1990.
12. Shi, Wan-Xiong, *Ricci Flow and the uniformization on complete non compact Kähler manifolds*, J. of Differential Geometry. **45** (1997), no. 1, 94-220.
13. Simons, J., *On the transitivity of holonomy systems*, Ann. of Math. **76** (1962), 213-234.
14. Siu, Yum Tong, *Pseudoconvexity and the problem of Levi*, Bull. Amer. Math. Soc. **84** (1978), 481–512.
15. Yau, Shing-Tung, *A review of complex differential geometry*, Proc. Sympos. Pure Math., **52** Part 2 (1991), 619–625.

Harvard University, Department of Mathematics, One Oxford Street, Cambridge, MA 02138, USA

E-mail address: `chau@math.harvard.edu`

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China.

E-mail address: `lftam@math.cuhk.edu.hk`