ON THE $\mathbb{Q}$-DIVISOR METHOD AND ITS APPLICATION

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Abstract. For a smooth projective 3-fold of general type, we prove that the relative canonical stability $\mu_s(3) \leq 8$. This is induced by our improved result of Kollár: the m-canonical map of X is birational onto its image whenever $m \geq 5k+6$, provided $P_k(X) \geq 2$. The $\mathbb{Q}$-divisor method is intensively developed to prove our results.

Introduction

Let X be a smooth projective threefold with big K_X. Many authors have studied the pluricanonical systems or the pluricanonical maps of X(see [1], [4], [8], [11], [16], [18], [19], [20], [26], [30] etc). Suppose $h^0(X, kK_X) \geq 2$. J. Kollár([16], Corollary 4.8) first proved that the $(11k+5)$-canonical map is birational. Then, in [5], it has been proved that either the $(7k+3)$-canonical map or the $(7k+5)$-canonical map is birational. We denote by ϕ_m the m-canonical map of X. Since it's still unclear whether the birationality of ϕ_m does imply the birationality of ϕ_{m+1}, we hope to find certain "stable property" of ϕ_m. In this paper, we modify the $\mathbb{Q}$-divisor method and then present much better results than in [16] and [5, 6]. The $\mathbb{Q}$-divisor method was originally developed by Kawamata, Reid, Shokorov and others in connection with the minimal model program initiated by Mori. It was also exploited much effectively, for various purpose, by many authors such as Ein, Kollár, Lazarsfeld, Viehweg and so on. As far as our method can tell here, the results are as follows.

Theorem 0.1. *Let X be a smooth projective 3-fold of general type. Suppose the k-th plurigenus $P_k(X) \geq 2$. Then the m-canonical map is birational onto its image for all $m \geq 5k+6$.*

Theorem 0.2. *Let X be a smooth projective 3-fold of general type. Suppose $p_g(X) \geq 2$. Then ϕ_8 is birational onto its image.*

The base field is always supposed to be algebraically closed of characteristic 0.

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For readers' convenience, let us recall the definition of the so-called *relative canonical stability* as follows.

Let X be a nonsingular projective variety of general type of dimension n. We define

$k_0(X) := min\{ k|\ P_k(X) \geq 2\}$;

$k_s(X) := min\{ k|\ \phi_m$ is birational for $m \geq k\}$, which is called *the canonical stability of* X;

$\mu_s(X) := \frac{k_s(X)}{k_0(X)}$, which is called *the relative canonical stability* of X.

$\mu_s(n) := \sup\{\mu_s(X)|X$ is a smooth projective n-fold of general type$\}$, which is referred to as *the n-th relative canonical stability.*

It's well-known that $\mu_s(1) = 3$ and $\mu_s(2) = 5$. A conjecture concerning μ_s is that $\mu_s(n) = 2n+1$. Theorem 0.1 and Theorem 0.2 imply the following

Corollary 0.3. *$\mu_s(3) \leq 8$.*

Remark 0.4. $\mu_s(n)$ is very interesting for $n \geq 3$. For example, if $\mu_s(4) < +\infty$, then the following is true:

there is a constant m_0 such that ϕ_{m_0} is birational for all smooth projective 3-fold X of general type with $p_g(X) > 0$.

In fact, taking the product of X with a curve of genus ≥ 2, one can realize the above statement very easily. Unfortunately, few information found on $\mu_s(4)$ yet.

1. Key lemmas

Let S be a smooth projective surface and D a divisor on S. We would like to know when the system $|K_S + D|$ gives a birational rational map onto its image. It is well-known that Reider ([25]) gave much effective results when D is nef and big. Because, in our cases, D may be not nef, we hope to find an effective sufficient condition in order to treat our situation.

Suppose M is a divisor on S with $h^0(S, M) \geq 2$ and $|M|$ is movable. Taking necessary blow-ups $\pi : S' \longrightarrow S$, along the indeterminency of the system, such that the movable part of $|\pi^*(M)|$ is basepoint free, we can get a morphism $g := \Phi_M \circ \pi : S' \longrightarrow W$ where W is the image of S' in $\mathbb{P}^{h^0(S,M)-1}$. If $\dim(W) = 1$, we can take the Stein factorization $g : S' \xrightarrow{f} B \longrightarrow W$, where B is a smooth projective curve and f is a fibration. Denote $b := g(B)$.

Definition 1.1. If $b = 0$, we say that $|M|$ is *composed of a rational pencil of curves.* Otherwise, $|M|$ is *composed of an irrational pencil of curves.*

We can write $\pi^*(M) \sim_{\text{lin}} M_0 + Z_0$ where M_0 is the movable part and Z_0 the fixed one. According to Bertini's theorem, a general member $C \in |M_0|$ is a smooth irreducible curve if $|M_0|$ is not composed of pencil of curves. Otherwise, we can write $M_0 \sim_{\mathrm{lin}} \sum_{i=1}^{m} C_i$ where the C_i's are fibers of the fibration f for all i. Denote by C a general fiber of f in the latter case. Then C is still an irreducible smooth curve.

Definition 1.2. In the above setting, C is called *a generic irreducible element of* $|M_0|$. Meanwhile, $\pi(C)$ is called *a generic irreducible element of* $|M|$. Note that, however, $\pi(C)$ may be a singular curve.

Now we can set up the following lemma.

Lemma 1.3. *Let S be a smooth projective surface and D a divisor on S. The rational map Φ_{K_S+D} is birational onto its image if the following conditions hold.*

(i) There is a divisor M on S with $h^0(S, M) \geq 2$ and $|M|$ movable such that $K_S + D \geq M$. If $|M|$ is composed of an irrational pencil of curves, denoting by C its generic irreducible element, $D - 2C \geq \ulcorner A_0 \urcorner$ holds for certain nef and big $\mathbb{Q}$-divisor A_0 on S (with $A_0 \cdot C > 1$ in the case $\kappa(S) = -\infty$).

(ii) $D - C \geq \ulcorner A \urcorner$ holds for certain nef and big $\mathbb{Q}$-divisor A on S with $A \cdot C > 2$, where C is a generic irreducible element of $|M|$.

Proof. It's clear that one may suppose that $|M|$ is base point free. Taking the Stein-factorization of Φ_M, we can get

$$\Phi_M : X \xrightarrow{f} W \xrightarrow{s} \mathbb{P}^{h^0(M)-1},$$

where f has connected fibers. A generic irreducible element of $|M|$ is a smooth projective curve.

If $|M|$ is not composed of a pencil of curves, then it's sufficient to verify the birationality of $\Phi_{K_S+D}|_C$ by virtue of Tankeev's principle ([27], Lemma 2).

If $|M|$ is composed of a pencil of curves, then f is a fibration onto the smooth curve W. In this case, C is a general fiber of f. When W is a rational curve, $|K_S + D|$ can distinguish different fibers of f since $K_S + D \geq C$. When W is irrational, suppose C_1 and C_2 are two general fibers of f. By assumption, we have

$$D - C_1 - C_2 = \ulcorner A_0 \urcorner + F,$$

where F is an effective divisor on S and A_0 is a nef and big $\mathbb{Q}$-divisor. We consider the system $|K_S+D-F|$. According to Kawamata-Viehweg vanishing theorem, we have the surjective map

$$H^0(S, K_S + D - F) \longrightarrow H^0(C_1, K_{C_1} + G_1) \oplus H^0(C_2, K_{C_2} + G_2).$$

By (i), we have $h^0(C_1, K_{C_1} + G_1) > 0$ and $h^0(C_2, K_{C_2} + G_2) > 0$. Thus $|K_S + D - F|$ can distinguish C_1 and C_2, so can $|K_S + D|$. Therefore it's also sufficient to verify the birationality of $\Phi_{K_S+D}|_C$.

By (ii), we can write $D-C = \ulcorner A \urcorner + F_1$ where F_1 is an effective divisor and A is a nef and big $\mathbb{Q}$-divisor. We consider the system $|K_S+D-F_1|$.

By vanishing theorem, we have the surjective map

$$H^0(S, K_S + D - F_1) \longrightarrow H^0(C, K_C + G)$$

where G is a divisor on C with $\deg(G) \geq 3$. Thus $\Phi_{K_S+D-F_1}|_C$ is an embedding, so is $\Phi_{K_S+D}|_C$. $\square$

Lemma 1.4. *Let S be a smooth projective surface of general type. Then $K_S + \ulcorner A \urcorner + D$ is effective if A is a nef and big $\mathbb{Q}$-divisor and if $h^0(S, D) \geq 2$.*

Proof. We may suppose that $|D|$ is basepoint free. Denote by C a generic irreducible element of $|D|$. Then the vanishing theorem gives the exact sequence

$$H^0(S, K_S + \ulcorner A \urcorner + C) \longrightarrow H^0(C, K_C + H) \longrightarrow 0,$$

where $H := \ulcorner A \urcorner|_C$ is a divisor of positive degree. It is obvious that $h^0(C, K_C + H) \geq 2$ since C is a curve of genus ≥ 2. The proof is completed. $\square$

Lemma 1.5. *([6], Lemma 2.7) Let X be a smooth projective variety of dimension ≥ 2. Let D be a divisor on X, $h^0(X, \mathcal{O}_X(D)) \geq 2$ and S be a smooth irreducible divisor on X such that S is not a fixed component of $|D|$. Denote by M the movable part of $|D|$ and by N the movable part of $|D|_S|$ on S. Suppose the natural restriction map*

$$H^0(X, \mathcal{O}_X(D)) \xrightarrow{\theta} H^0(S, \mathcal{O}_S(D|_S))$$

is surjective. Then $M|_S \geq N$ and thus

$$h^0(S, \mathcal{O}_S(M|_S)) = h^0(S, \mathcal{O}_S(N)) = h^0(S, \mathcal{O}_S(D|_S)).$$

2. The case with $k_0 \geq 2$

Sometimes, for simplicity, we denote $k_0(X)$ and $k_s(X)$ by k_0 and k_s respectively.

Proposition 2.1. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $\dim \phi_{k_0}(X) \geq 2$, then $P_m(X) \geq 2$ for all $m \geq 2k_0$.*

Proof. First we take a birational modification $\pi : X' \longrightarrow X$, according to Hironaka, such that

(1) X' is smooth;

(2) the movable part of $|k_0 K_{X'}|$ defines a morphism;

(3) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

Denote by $S_0 := S_{k_0}$ a generic irreducible element of the movable part of $|k_0 K_{X'}|$. Then S_0 is a smooth projective surface of general type by Bertini's theorem. By the vanishing theorem, we have the exact sequence

$$\begin{aligned} &H^0(X', K_{X'} + \ulcorner (t+k_0)\pi^*(K_X)\urcorner + S_0) \\ \longrightarrow &H^0(S_0, K_{S_0} + \ulcorner (t+k_0)\pi^*(K_X)\urcorner|_{S_0}) \longrightarrow 0, \end{aligned}$$

where $t \geq 0$ is a given integer and

$$\ulcorner (t+k_0)\pi^*(K_X)\urcorner|_{S_0} \geq \ulcorner t\pi^*(K_X)|_{S_0}\urcorner + D_0,$$

$D_0 := S_0|_{S_0}$ has the property $h^0(S_0, D_0) \geq 2$ according to the assumption.

If $t = 0$, then

$$P_{2k_0+1}(X) \geq h^0(S_0, K_{S_0} + D_0) \geq 2$$

by Lemma 1.2 of [5].

If $t > 0$, we still have the following exact sequence

$$H^0(S_0, K_{S_0} + \ulcorner t\pi^*(K_X)|_{S_0}\urcorner + C) \longrightarrow H^0(K_C + G) \longrightarrow 0,$$

where C is a generic irreducible element of the movable part of $|D_0|$ and $G := \ulcorner t\pi^*(K_X)|_{S_0}\urcorner|_C$ is a divisor of positive degree on C. Since C is a curve of genus ≥ 2, we have $h^0(C, K_C + G) \geq 2$. Thus we can see that $P_{2k_0+t+1} \geq 2$. The proof is completed. □

Corollary 2.2. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If ϕ_{k_0} is birational, then $k_s \leq 3k_0$.*

Proof. This is obvious according to Proposition 2.1. □

Theorem 2.3. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $\dim \phi_{k_0}(X) = 3$ and ϕ_{k_0} is not birational, then $k_s \leq 3k_0 + 2$.*

Proof. Taking the same modification $\pi : X' \longrightarrow X$ as in the proof of Proposition 2.1, we still denote by S_0 the general member of the movable part of $|k_0K_{X'}|$. Note that both $|k_0K_{X'}|$ and $|\ulcorner k_0\pi^*(K_X)\urcorner|$ have the same movable part. For a given integer $t > 0$, we have

$$K_{X'} + \ulcorner (t+2k_0)\pi^*(K_X)\urcorner + S_0 \leq (t + 3k_0 + 1)K_{X'}.$$

It is sufficient to prove the birationality of the rational map defined by

$$|K_{X'} + \ulcorner (t+2k_0)\pi^*(K_X)\urcorner + S_0|.$$

Because $K_{X'} + \ulcorner (t+2k_0)\pi^*(K_X)\urcorner$ is effective by the proof of Proposition 2.1, we have

$$K_{X'} + \ulcorner (t+2k_0)\pi^*(K_X)\urcorner + S_0 \geq S_0.$$

Thus we only need to prove the birationality of

$$\Phi_{K_{X'} + \ulcorner (t+2k_0)\pi^*(K_X)\urcorner + S_0}\ \big|_{S_0}\ .$$

We have the following exact sequence by the vanishing theorem

$$\begin{aligned}&H^0(X, K_{X'} + \ulcorner(t+2k_0)\pi^*(K_X)\urcorner + S_0)\\ \longrightarrow &H^0(S_0, K_{S_0} + \ulcorner(t+2k_0)\pi^*(K_X)\urcorner|_{S_0}) \longrightarrow 0,\end{aligned}$$

which means

$$|K_{X'} + \ulcorner(t+2k_0)\pi^*(K_X)\urcorner + S_0|\ \Big|_{S_0} = |K_{S_0} + \ulcorner(t+2k_0)\pi^*(K_X)\urcorner|_{S_0}|.$$

Noting that

$$K_{S_0} + \ulcorner(t+2k_0)\pi^*(K_X)\urcorner|_{S_0} \geq K_{S_0} + \ulcorner t\pi^*(K_X)|_{S_0}\urcorner + 2L_0,$$

where $L_0 := S_0|_{S_0}$, we want to show that $\Phi_{K_{S_0}+\ulcorner t\pi(K_X)|_{S_0}\urcorner+2L_0}$ is birational. Because $|L_0|$ gives a generically finite map, we see from Lemma 1.4 that $K_{S_0} + \ulcorner t\pi^*(K_X)|_{S_0}\urcorner + L_0$ is effective. On the other hand, let C be a generic irreducible element of $|L_0|$, then $\dim\Phi_{L_0}(C) = 1$. So $L_0 \cdot C \geq 2$ and thus $(t\pi^*(K_X)|_{S_0} + L_0)\cdot C > 2$. By Lemma 1.3, $|K_{S_0} + \ulcorner t\pi^*(K_X)|_{S_0}\urcorner + 2L_0|$ gives a birational map. The proof is completed. □

Theorem 2.4. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $\dim\phi_{k_0}(X) = 2$, then $k_s \leq 4k_0 + 4$.*

Proof. First we take the same modification $\pi : X' \longrightarrow X$ as in the proof of Proposition 2.1. We also suppose that S_0 is the movable part of $|k_0K_{X'}|$. For a given integer $t > 0$, we obviously have

$$K_{X'} + \ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner + 2S_0 \leq (t+4k_0+3)K_{X'}.$$

Thus it is sufficient to verify the birationality of the rational map defined by

$$|K_{X'} + \ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner + 2S_0|.$$

By Proposition 2.1,

$$K_{X'} + \ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner + S_0$$

is effective. Thus we only have to prove the birationality of the restriction

$$\Phi_{K_{X'}+\ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner+2S_0}\ \Big|_{S_0}$$

for the general S_0. The vanishing theorem gives the exact sequence

$$\begin{aligned}&H^0(X', K_{X'} + \ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner + 2S_0)\\ \longrightarrow &H^0(S_0, K_{S_0} + \ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner\ \Big|_{S_0} + S_0|_{S_0}) \longrightarrow 0.\end{aligned}$$

This means

$$\Phi_{K_{X'}+\ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner+2S_0}\ \Big|_{S_0} = \Phi_{K_{S_0}+\ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner|_{S_0}+S_0|_{S_0}}.$$

Suppose M_{2k_0+2} is the movable part of $|(2k_0+2)K_{X'}|$. We have to study some property of $\Big|\ M_{2k_0+2}|_{S_0}\ \Big|$. Note that M_{2k_0+2} is also the movable part of

$$|\ulcorner(2k_0+2)\pi^*(K_X)\urcorner|.$$

We have $K_{X'}+\ulcorner\pi^*(K_X)\urcorner+2S_0\le(2k_0+2)K_{X'}$. The vanishing theorem gives the exact sequence

$$H^0(X',K_{X'}+\ulcorner\pi^*(K_X)\urcorner+2S_0)\xrightarrow{\alpha}H^0(S_0,K_{S_0}+\ulcorner\pi^*(K_X)\urcorner|_{S_0}+L_0)\longrightarrow 0,$$

where $L_0:=S_0|_{S_0}$. Denote by M'_{2k_0+2} the movable part of $|K_{X'}+\ulcorner\pi^*(K_X)\urcorner+2S_0|$ and by G the movable part of $|K_{S_0}+\ulcorner\pi^*(K_X)\urcorner|_{S_0}+L_0|$. By Lemma 2.3, we have $G\le M'_{2k_0+2}|_{S_0}\le M_{2k_0+2}|_{S_0}$. Noting that $|L_0|$ is a free pencil, we can suppose C is a generic irreducible element of $|L_0|$. Now the key step is to show that $\dim\Phi_G(C)=1$. In fact, the vanishing theorem gives

$$|K_{S_0}+\ulcorner\pi^*(K_X)|_{S_0}\urcorner+L_0|\ \big|_C=|K_C+D|,$$

where $D:=\ulcorner\pi^*(K_X)|_{S_0}\urcorner\ \big|_C$ is a divisor of positive degree. Because C is a curve of genus ≥ 2, $|K_C+D|$ gives a finite map. This shows

$$\dim\Phi_{K_{S_0}+\ulcorner\pi^*(K_X)|_{S_0}\urcorner+L_0}(C)=1,$$

thus $\dim\Phi_G(C)=1$. Therefore $\dim\Phi_{M_{2k_0+2}|_{S_0}}(C)=1$. and so $M_{2k_0+2}|_{S_0}\cdot C\ge 2$. Noting that $h^0(S_0,M_{2k_0+2}|_{S_0})\ge h^0(S_0,G)\ge 2$, we get from Lemma 1.4 that $K_{S_0}+\ulcorner t\pi^*(K_X)|_{S_0}\urcorner+M_{2k_0+2}|_{S_0}$ is effective. Now Lemma 1.3 implies the birationality of the rational map defined by $|K_{S_0}+\ulcorner t\pi^*(K_X)|_{S_0}\urcorner+M_{2k_0+2}|_{S_0}+L_0|$. Because

$$\begin{aligned}&|K_{S_0}+\ulcorner t\pi^*(K_X)|_{S_0}\urcorner+M_{2k_0+2}|_{S_0}+L_0|\\ \subset&|K_{S_0}+\ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner|_{S_0}+L_0|,\end{aligned}$$

$\Phi_{K_{S_0}+\ulcorner(t+2k_0+2)\pi^*(K_X)\urcorner|_{S_0}+S_0|_{S_0}}$ is birational. We have proved the theorem. ☐

From now on, we suppose that $\dim\phi_{k_0}(X)=1$. We can take the same modification $\pi: X'\longrightarrow X$ as in the proof of Proposition 2.1. Let $g:=\phi_{k_0}\circ\pi$ be the morphism from X' onto $W\subset\mathbb{P}^{P_{k_0}-1}$, where W is the Zariski closure of the image of X through ϕ_{k_0}. Let $g: X'\xrightarrow{f}Q\longrightarrow W$ be the Stein-factorization. Then Q is a smooth projective curve. Denote $b:=g(Q)$, the genus of Q.

2.5. If $b>0$, we have already known from [5] that $k_s\le 2k_0+4$.

In the rest of this section, we mainly study the case when Q is the rational curve $\mathbb{P}^1$. We have a derived fibration $f: X'\longrightarrow\mathbb{P}^1$. Let S be a general fiber of the fibration. Then S is a smooth projective surface of general type. Note that S is also the generic irreducible element of the movable part of the system $|k_0K_{X'}|$. Let $\sigma: S\longrightarrow S_0$ be the contraction onto the minimal model.

Theorem 2.6. *Let X be a minimal projective 3-fold of general type with only $\mathbb{Q}$-factorial terminal singularities. If $\dim\phi_{k_0}(X)=1$ and $b=0$, then $k_s\le 5k_0+6$.*

Proof. For all $i > 0$, denote by M_i the movable part of $|iK_{X'}|$. By Kollár's method ([16] or see [5], 2.2), we have $M_{9k_0+4}|_S \geq 4\sigma^*(K_{S_0})$.

By the vanishing theorem, one has

$$\begin{aligned}&|K_{X'} + \ulcorner(9k_0+4)\pi^*(K_X)\urcorner + S||_S = |K_S + \ulcorner(9k_0+4)\pi^*(K_X)\urcorner_S|\\ &\supset |K_S + M_{9k_0+4}|_S| \supset |5\sigma^*(K_{S_0})|.\end{aligned}$$

By Lemma 1.5, we see that $M_{10k_0+5}|_S \geq 5\sigma^*(K_{S_0})$. Repeatedly performing this process, one has

$$M_{9k_0+4+m(k_0+1)}|_S \geq (m+4)\sigma^*(K_{S_0})$$

for all integer $m > 0$. This means that we can write

$$(5k_0 + (m+4)(k_0+1))\pi^*(K_X)|_S \sim_{\mathbb{Q}} (m+4)\sigma^*(K_{S_0}) + E_{\mathbb{Q}}^{(m)},$$

where $E_{\mathbb{Q}}^{(m)}$ is an effective $\mathbb{Q}$-divisor only relating to m. Thus

$$(\frac{5k_0}{m+4} + (k_0+1))\pi^*(K_X)|_S \sim_{\text{num}} \sigma^*(K_{S_0}) + \frac{1}{m+4}E_{\mathbb{Q}}^{(m)}. \qquad (2.6.1)$$

We can write $2\sigma^*(K_{S_0}) \sim_{\text{lin}} C + Z$, where C is the movable part and Z the fixed one. According to Xiao([31]), $|C|$ is composed of a pencil of curves if and only if $K_{S_0}^2 = 1$ and $p_g(S) = 0$. We prove this theorem step by step.

Step 1. $tK_{X'}$ is effective for all $t \geq 3k_0 + 4$.

Denote by $\alpha \geq 2k_0 + 3$ a positive integer. We have

$$|(\alpha + k_0 + 1)K_{X'}| \supset |K_{X'} + \ulcorner\alpha\pi^*(K_X)\urcorner + S|.$$

By the vanishing theorem, one has

$$|K_{X'} + \ulcorner\alpha\pi^*(K_X)\urcorner + S||_S = |K_S + \ulcorner\alpha\pi^*(K_X)\urcorner|_S| \supset |K_S + \ulcorner\alpha\pi^*(K_X)|_S\urcorner|.$$

Now we consider a sub-system

$$|K_S + \ulcorner\alpha\pi^*(K_X)|_S - Z - \frac{2}{m+4}E_{\mathbb{Q}}^{(m)}\urcorner|.$$

Because

$$\alpha\pi^*(K_X)|_S - Z - \frac{2}{m+4}E_{\mathbb{Q}}^{(m)} - C \sim_{\text{num}} t_0\pi^*(K_X)|_S$$

where $t_0 := \alpha - \frac{10k_0}{m+4} - 2k_0 - 2 > 0$ for big m. Thus we have

$$|K_S + \ulcorner\alpha\pi^*(K_X)|_S - Z - \frac{2}{m+4}E_{\mathbb{Q}}^{(m)}\urcorner||_S = |K_C + D|,$$

where D is divisor on C with $\deg(D) > 0$. Therefore $P_{\alpha+k_0+1}(X) \geq h^0(C, K_C + D) \geq 2$ since $g(C) \geq 2$.

Step 2. The birationality.

Denote by $\beta \geq 4k_0 + 5$ a positive integer. Considering the system $|K_{X'} + \ulcorner\beta\pi^*(K_X)\urcorner + S|$, we have

$$|K_{X'} + \ulcorner\beta\pi^*(K_X)\urcorner + S||_S \supset |K_S + \ulcorner\beta\pi^*(K_X)|_S\urcorner|.$$

By Step 1, it's sufficient to verify the birationality of $\Phi_{K_S+\ulcorner\beta\pi^*(K_X)|_S\urcorner}$. It's obvious that

$$K_S+\ulcorner\beta\pi^*(K_X)|_S\urcorner \geq K_S+\ulcorner\beta\pi^*(K_X)|_S - Z - \frac{4}{m+4}E^{(m)}_{\mathbb{Q}}\urcorner.$$

Denote by $A := \ulcorner\beta\pi^*(K_X)|_S - Z - \frac{4}{m+4}E^{(m)}_{\mathbb{Q}}\urcorner - 2\sigma^*(K_{S_0})$. We have

$$A \sim_{\text{num}} (\beta - 4k_0 - 4 - \frac{20k_0}{m+4})\pi^*(K_X)|_S.$$

So A is also nef and big for big m. Now we have

$$|K_S+\ulcorner\beta\pi^*(K_X)|_S - Z - \frac{4}{m+4}E^{(m)}_{\mathbb{Q}}\urcorner| = |K_S+\ulcorner A\urcorner + 2\sigma^*(K_{S_0}) + C|.$$

By Lemmas 1.3 and 1.4, one can easily see that the above system defines a birational map onto its image. The theorem follows. ☐

3. The case with $k_0 = 1$

From now on, we only consider the case with $p_g(X) \geq 2$. We always suppose X is a minimal projective 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. The first effective result was obtained by Kollár who proved that ϕ_{16} is birational. In [6], it has been proved that ϕ_9 is birational. Here we would like to prove the birationality of ϕ_8.

By virtue of Theorem 2.3, Theorem 2.4 and (2.5), we only have to consider the situation with $\dim\phi_1(X) = 1$ and $b = 0$. We have a derived fibration $f : X' \longrightarrow C$. A general fiber S is a projective smooth surface of general type. We note that $p_g(S) > 0$ in this case. In order to formulate our proof, we classify S into the following types:

(I) $K^2_{S_0} = 1$, $p_g(S) = 2$;
(II) $K^2_{S_0} = 2$, $p_g(S) = 3$;
(III) $K^2_{S_0} = 2$, $p_g(S) = 2$;
(IV) $K^2_{S_0} = 1$, $p_g(S) = 1$;
(V) $K^2_{S_0} \geq 3$;
(VI) $K^2_{S_0} = 2$, $p_g(S) = 1$.

Claim 3.1. If S is of type (I), then Φ_{8K_X} is birational.

Proof. Denote by C a generic irreducible element of the movable part of $|K_S|$. Then it's well-known that C is a smooth curve of genus 2. According to Claim in Proposition 5.3 of [6], we have $\xi := \pi^*(K_X)\cdot C \geq \frac{3}{5}$. For a positive integer t, we have

$$|K_{X'}+\ulcorner t\pi^*(K_X)\urcorner+S||_S = |K_S+\ulcorner t\pi^*(K_X)\urcorner|_S| \supset |K_S+\ulcorner t\pi^*(K_X)|_S\urcorner|.$$

Since $p_g(X) > 0$, taking $t = 1$ and applying Lemma 1.5, one has $M_3|_S \geq C$. Taking $t = 3$ and applying Lemma 1.5 once more, one has $M_5|_S \geq 2C$. This means

$$5\pi^*(K_X)|_S \sim_{\mathbb{Q}} 2C + E^{(5)}_{\mathbb{Q}},$$

where $E_{\mathbb{Q}}^{(5)}$ is an effective $\mathbb{Q}$-divisor. Thus we have

$$\frac{5}{2}\pi^*(K_X)|_S \sim_{\text{num}} C + \frac{1}{2}E_{\mathbb{Q}}^{(5)}.$$

Now $t\pi^*(K_X)|_S - C - \frac{1}{2}E_{\mathbb{Q}}^{(5)} \sim_{\text{num}} (t-\frac{5}{2})\pi^*(K_X)|_S$. It is easy to see that, for $t \geq 6$, $(t-\frac{5}{2})\pi^*(K_X)|_S \cdot C > 2$. Lemma 1.3 implies that $\Phi_{K_S+\ulcorner 6\pi^*(K_X)|_S\urcorner}$ is birational and so is Φ_{8K_X}. □

Claim 3.2. If S is of type (II), then Φ_{7K_X} is birational.

Proof. We still denote by C a generic irreducible element of the movable part of $|K_S|$. It's well-known that $|C|$ defines a generically finite map and C is a smooth curve of genus 3. By a parallel argument as in the proof of Claim 3.1, we have $M_{2m+1}|_S \geq mC$ for any positive integer m. This means

$$(2m+1)\pi^*(K_X)|_S \sim_{\mathbb{Q}} mC + E_{\mathbb{Q}}^{(m)},$$

where $E_{\mathbb{Q}}^{(m)}$ is an effective $\mathbb{Q}$-divisor depending on m. Thus we have

$$\frac{2m+1}{m}\pi^*(K_X)|_S \sim_{\text{num}} C + \frac{1}{m}E_{\mathbb{Q}}^{(m)}.$$

Therefore $\eta := \pi^*(K_X)|_S \cdot C \geq \frac{m}{2m+1}C^2 \geq \frac{2m}{2m+1}$ for all $m > 0$. So $\eta \geq 1$. We want to verify the birationality of $|K_S + \ulcorner t\pi^*(K_X)|_S\urcorner|$ for certain t. Because

$$t\pi^*(K_X)|_S - C - \frac{1}{m}E_{\mathbb{Q}}^{(m)} \sim_{\text{num}} (t - \frac{2m+1}{m})\pi^*(K_X)|_S =: A,$$

Fix a big m, one can see that $A \cdot C > 2$ for $t \geq 5$. Thus, by Lemma 1.3, $\Phi_{K_S+\ulcorner 5\pi^*(K_X)|_S\urcorner}$ is birational and so is Φ_{7K_X}. □

Claim 3.3. If S is of type (III), then Φ_{7K_X} is birational.

Proof. Denote by C a generic irreducible element of the movable part of $|K_S|$. Recall that $\sigma : S \longrightarrow S_0$ is the contraction onto the minimal model. $C_1 := \sigma_*(C)$ is the movable part of $|K_{S_0}|$. It's easy to see that C_1 has two types:

(3.3.1) $|K_{S_0}| = |C_1| + Z$, where $C_1^2 = 0$ and C_1 is smooth curve of genus 2.

(3.3.2) $|K_{S_0}| = |C_1|$, where C_1 is a smooth curve of genus 3.

In either cases, we always have $\sigma^*(K_{S_0}) \cdot C = K_{S_0} \cdot C_1 = 2$. By Theorem 3.1 of [7], $|mK_{S_0}|$ is basepoint free for $m \geq 2$. Now Kollár's technique gives $7\pi^*(K_X)|_S \geq 2\sigma^*(K_{S_0})$ and so $M_7|_S \geq 2\sigma^*(K_{S_0})$. By a parallel argument as in the proof of Claim 3.1, we get

$$(2m+3)\pi^*(K_X)|_S \geq M_{2m+3}|_S \geq m\sigma^*(K_{S_0})$$

for all positive integer m. Thus

$$\pi^*(K_X)|_S \cdot C \geq \frac{m}{2m+3}\sigma^*(K_{S_0}) \cdot C \geq \frac{2m}{2m+3}.$$

So $\pi^*(K_{S_0})\cdot C\geq 1$. Now by the same argument as in the proof of Claim 4.2, one can easily obtain the birationality of Φ_{7K_X}. $\square$

Claim 3.4. If S is of type (IV), then Φ_{8K_X} is birational.

Proof. We consider the natural map

$$H^0(X',M_2)\xrightarrow{\alpha_2}\Lambda_2\subset H^0(S,M_2|_S)\subset H^0(S,2K_S),$$

where Λ_2 is the image of α_2. Since $P_2(S)=3$, $0<\dim_{\mathbb{C}}\Lambda_2\leq 3$.

(3.4.1) $\dim_{\mathbb{C}}\Lambda_2=3$. In this case, Λ_2 defines the bicanonical map of S. By [7], $|2K_{S_0}|$ is base point free. Thus we see that $M_2|_S\geq 2\sigma^*(K_{S_0})$. We can write $2\pi^*(K_X)|_S\sim 2\sigma^*(K_{S_0})+E_{\mathbb{Q}}$, where $E_{\mathbb{Q}}$ is an effective $\mathbb{Q}$-divisor. Denote by C a general member of $|2\sigma^*(K_{S_0})|$. Now we have $4\pi^*(K_X)|_S-C-E_{\mathbb{Q}}\sim_{\rm num}2\pi^*(K_X)|_S$ and $2\pi^*(K_X)|_S\cdot C\geq 4$. By Lemma 1.3, $\Phi_{K_S+\ulcorner 4\pi^*(K_X)|_S\urcorner}$ is birational and so is Φ_{6K_X}.

(3.4.2) $\dim_{\mathbb{C}}\Lambda_2=2$. Because Λ_2 defines a morphism, the movable part of Λ_2 forms a complete linear pencil. The pencil is rational because $q(S)=0$. Denote by C_1 a generic irreducible element of the movable part of Λ_2. By Lemma 3.7 below, $\sigma^*(K_{S_0})\cdot C_1\geq 2$. Because $M_2|_S\geq C_1$, we can write $2\pi^*(K_X)|_S\sim_{\rm num}C_1+E'_{\mathbb{Q}}$, where $E'_{\mathbb{Q}}$ is an effective $\mathbb{Q}$-divisor. Now $5\pi^*(K_X)|_S-C_1-E'_{\mathbb{Q}}\sim_{\rm num}3\pi^*(K_X)|_S$ and, by (3.6.1), $\pi^*(K_X)|_S\cdot C_1\geq\frac{1}{2}\sigma^*(K_{S_0})\cdot C_1\geq 1$. According to Lemma 1.3, $\Phi_{K_S+\ulcorner 5\pi^*(K_X)|_S\urcorner}$ is birational and so is Φ_{7K_X}.

(3.4.3) $\dim_{\mathbb{C}}\Lambda_2=1$. In this case, $|2K_X|$ is composed of a pencil of surfaces. One can see that $q(X)=h^2(\mathcal{O}_X)=0$, whence $\chi(\mathcal{O}_X)\leq -1$. Applying Reid's R-R formula([24]), one has $P_2(X)\geq 4$. We can write $2\pi^*(K_X)\sim_{\rm num}aS+E'$,where $a\geq 3$ and E' is an effective divisor. We hope to prove the birationality of Φ_{8K_X}. Because $p_g(X)>0$, it's sufficient to prove the birationality of $\Phi_{8K_X}|_S$. By virtue of Kollár's method, one can see that $|7K_{X'}||_S\supset|2\sigma^*(K_{S_0})|$. Therefore we are reduced to prove the birationality of $(\Phi_{8K_{X'}}|_S)|_C$, where C is a general member of $|2\sigma^*(K_{S_0})|$.

By the vanishing theorem, one has $|K_{X'}+\ulcorner 7\pi^*(K_X)-\frac{1}{a}E'\urcorner||_S\supset|K_S+\ulcorner L\urcorner|$ where $L\sim_{\rm num}(7-\frac{2}{a})\pi^*(K_X)|_S$. (3.5.1) below is still true when S is of type (IV), i.e.

$$\frac{2(2m+3)}{m}\pi^*(K_X)\sim_{\rm num}C+E_{\mathbb{Q}}^{(m)},$$

where $E_{\mathbb{Q}}^{(m)}$ is an effective $\mathbb{Q}$-divisor. Thus $|K_S+\ulcorner L-E_{\mathbb{Q}}^{(m)}\urcorner||_C=|K_C+D|$, where $D:=\ulcorner L-E_{\mathbb{Q}}^{(m)}\urcorner|_C$. Because $L-C-E_{\mathbb{Q}}^{(m)}\sim_{\rm num}(3-\frac{2}{3}-\frac{6}{m})\pi^*(K_X)|_S$ for all $m>0$, we see that $\deg(D)>2$ whenever m is large. Thus $\Phi_{K_S+\ulcorner L\urcorner}|_C$ is birational and so is Φ_{8K_X}. $\square$

Claim 3.5. If S is of type (V), then Φ_{7K_X} is birational.

Proof. By [7], $|mK_{S_0}|$ is basepoint free for all $m\geq 2$. Denote by C the movable part of $|2K_S|$. Then $C\sim_{\rm lin}2\sigma^*(K_{S_0})$. It's sufficient to prove the birationality of $\Phi_{7K_X}|_S$. By Kollár's method, $|7K_{X'}||_S \supset |2\sigma^*(K_{S_0})| = |C|$. We only need to verify the birationality of $(\Phi_{7K_{X'}}|_S)|_C$. The vanishing theorem gives $|K_{X'}+\ulcorner 7\pi^*(K_X)\urcorner+S||_S \supset |K_S+\ulcorner 7\pi^*(K_X)|_S\urcorner|$. Applying Lemma 1.5, we get $M_9|_S \geq 3\sigma^*(K_{S_0})$. Repeatedly proceeding the above process while replacing "7" by "9, 11, $\cdots$", one can obtain $M_{2m+3}|_S \geq m\sigma^*(K_{S_0})$ and so

$$(2m+3)\pi^*(K_X)|_S \sim_{\text{num}} m\sigma^*(K_{S_0}) + E_{\mathbb{Q}}^{(m)},$$

where m is a positive integer. Thus we have

$$\frac{2(2m+3)}{m}\pi^*(K_X)|_S \sim_{\text{num}} 2\sigma^*(K_{S_0}) + \frac{2}{m}E_{\mathbb{Q}}^{(m)}. \tag{3.5.1}$$

It's easy to see that $\pi^*(K_X)|_S \cdot C \geq 3$.

Now taking a very big m, one has

$$|K_S + \ulcorner 5\pi^*(K_X)|_S - \frac{2}{m}E_{\mathbb{Q}}^{(m)}\urcorner||_C = |K_C + D|,$$

where $D := \ulcorner 5\pi^*(K_X)|_S - \frac{2}{m}E_{\mathbb{Q}}^{(m)}\urcorner|_C$ and $\deg(D) \geq (1-\frac{6}{m})\pi^*(K_X)|_S \cdot C > 2$. Therefore we have proved that $(\Phi_{7K_{X'}}|_S)|_C$ is birational. So Φ_{7K_X} is birational. □

Claim 3.6. If S is of type (VI), then Φ_{8K_X} is birational.

Proof. We keep the same notations as in the proof of Claim 3.5. The proof is almost the same except that we have here $\pi^*(K_X)|_S \cdot C \geq 2$. Thus we can prove that $|K_S + \ulcorner 6\pi^*(K_X)|_S - \frac{2}{m}E_{\mathbb{Q}}^{(m)}\urcorner||_C$ is birational by the same argument. This, in turn, proves the birationality of Φ_{8K_X}. We conclude the claim. □

Claims 3.1 through 3.6 imply Theorem 0.2.

Lemma 3.7. *Let S be a smooth projective surface of general type. Let $\sigma : S \longrightarrow S_0$ be the contraction onto the minimal model. Suppose we have an effective irreducible curve C on S such that $C \leq \sigma^*(2K_{S_0})$ and $h^0(S,C) = 2$. If $K_{S_0}^2 = p_g(S) = 1$, then $C \cdot \sigma^*(K_{S_0}) \geq 2$.*

Proof. We can suppose $|C|$ is a free pencil. Otherwise, we can blow-up S at base points of $|C|$. Denote $C_1 := \sigma(C)$. Then $h^0(S_0, C_1) \geq 2$. Suppose $C \cdot \sigma^*(K_{S_0}) = 1$. Then $C_1 \cdot K_{S_0} = 1$. Because $p_a(C_1) \geq 2$, we can see that $C_1^2 > 0$. From $K_{S_0}(K_{S_0} - C_1) = 0$, we get $(K_{S_0} - C_1)^2 \leq 0$, i.e. $C_1^2 \leq 1$. Thus $C_1^2 = 1$ and $K_{S_0} \sim_{\text{num}} C_1$. This means $K_{S_0} \sim_{\text{lin}} C_1$ by virtue of [3], which is impossible because $p_g(S) = 1$. So $C \cdot \sigma^*(K_{S_0}) \geq 2$. □

Remark 3.8. Slightly modifying our method, one can even prove the following statements:

Let X be a minimal projective 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Suppose $p_g(X) \geq 2$. Then

(1) ϕ_7 is birational whenever $p_g(X) \neq 2$.

(2) ϕ_6 is birational whenever $p_g(X) \neq 2,\ 3$.
(3) ϕ_5 is birational whenever $p_g(X) \neq 2,\ 3,\ 4$.
We don't give the explicit proof since we think that the calculation is more complicated. However it's not difficult for a reader to verify these statements once he understands our method.

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